EX. SHEET 3

EX. 1

X has possible value $\{1, 2, 3, 4, 5\}$
and prob. mass function

$p_X(1) = \frac{1}{7} \quad p_X(2) = \frac{1}{14} \quad p_X(3) = \frac{3}{14}$

$p_X(4) = p_X(5) = 2/7$

a) $\mathbb{P}(X \leq 3) = F_X(3) = \frac{1}{7} + \frac{1}{14} + \frac{3}{14} = \frac{3}{7}$

CUMULATIVE DISTRIBUTION FUNCTION
OF X:

$F_X(t) = \mathbb{P}(X \leq t)$

b) $\mathbb{P}(X < 3) = \frac{1}{7} + \frac{1}{14} = \frac{3}{14}$

c) $\mathbb{P}(X < 4.12 \mid X > 1.638) =$

$$= \frac{\mathbb{P}(1.638 < X < 4.12)}{\mathbb{P}(X > 1.638)} =$$

$$= \frac{\frac{1}{14} + \frac{3}{14} + \frac{2}{7}}{1 - \frac{1}{7}} = \frac{\frac{4}{7}}{\frac{6}{7}} = \frac{2}{3}$$

EX. 3

X cont.

density:

$$f(x) = \begin{cases} 3e^{-3x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Verify that f is a density function.

$f(x) \geq 0$ ✓ $\qquad \int_{-\infty}^{+\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{+\infty} f(x)\,dX = \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{=0}\,dx}_{=0} + \int_{0}^{+\infty} f(x)\,dx =$$

$$= \int_{0}^{+\infty} 3e^{-3x}\,dx = 3 \cdot \left[\frac{e^{-3x}}{-3}\right]_{0}^{+\infty} = 3\left(0 - \left(-\frac{1}{3}\right)\right) = 1$$

b) $\mathbb{P}(-1 < x < 1) = \int_{-1}^{1} f(x)\,dx =$

$$= \int_{0}^{1} f(x)\,dx + \int_{-1}^{0} \underbrace{f(x)}_{=0}\,dx$$

$$= \int_{0}^{1} 3e^{-3x}\,dx = 3\left[\frac{e^{-3x}}{-3}\right]_{0}^{1}$$

$$= 3\left[\frac{e^{-3}}{-3} - \frac{1}{-3}\right]$$

$$= 1 - e^{-3}.$$

$$X \sim \mathrm{Exp}(\underset{\lambda}{\overset{}{\underset{\|}{3}}}) \qquad f_X(t) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(-1 < X < 1) = ? \qquad F_X(t) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

$\|$ → if X continuous

$$P(a < X < b)$$

$$\lambda = 3$$

$$P(-1 < X < 1) \overset{X\text{ cont.}}{=} F_X(1) - F_X(-1) = \overset{\uparrow}{1} - e^{-3} - 0$$

d) $P(X < 5) = \int_{-\infty}^{5} f_X(x)\,dx \qquad F_X(t) = P(X \leq t)$

$$= F_X(5) = 1 - e^{-15}.$$

since X is cont.

$$P(2 < X < 4 \mid X < 5) = \frac{P(2 < X < 4,\ X < 5)}{P(X < 5)} =$$

$$= \frac{P(2 < X < 4)}{P(X < 5)} = \frac{F_X(4) - F_X(2)}{F_X(5)}$$

$$= \frac{1 - e^{-12} - (1 - e^{-6})}{1 - e^{-15}} = \frac{e^{-6} - e^{-12}}{1 - e^{-15}}$$

$$E[X] = 1/\lambda$$

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \underbrace{\int_{-\infty}^{0} x \cdot 0\,dx}_{0}$$

$$+ \int_{0}^{+\infty} 3x e^{-3x}\,dx =$$

$$= 3\int_0^{+\infty} \underbrace{x}_{g}\underbrace{e^{-3x}}_{f'}\,dx = 3\left(\left[\underbrace{x}_{g}\cdot \underbrace{\frac{e^{-3x}}{-3}}_{f}\right]_0^{+\infty} - \int_0^{+\infty} \underbrace{1}_{g'}\cdot \underbrace{\frac{e^{-3x}}{-3}}_{f}\,dx\right)$$

$g'(x)=1 \quad f(x)=\frac{e^{-3x}}{-3}$

$$= 3\left((0-0) + \frac{1}{3}\int_0^{+\infty} e^{-3x}\,dx\right)$$

$$= \int_0^{+\infty} e^{-3x}\,dx = \left[\frac{e^{-3x}}{-3}\right]_0^{+\infty} = \left(0 - \frac{1}{-3}\right) = \frac{1}{3}$$

$$\mathbb{E}\left[e^{2X}\right] = ?$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)\cdot f_X(x)\,dx$$

$$\mathbb{E}\left[e^{2X}\right] = \int_{-\infty}^{+\infty} e^{2x}\cdot f_X(x)\,dx =$$

$$= \int_0^{+\infty} e^{2X}\cdot 3e^{-3x}\,dx + \underbrace{\int_{-\infty}^{0} e^{2X}\cdot 0\,dx}_{0}$$

$$= \int_0^{+\infty} 3e^{-x}\,dx =$$

$$= 3\left[\frac{e^{-x}}{-1}\right]_0^{+\infty} = 3(0-(-1)) = 3.$$

Ex. 9

Ex.7 X cont. rand. var. with cumulative distr. func.

$$F_X(x) = \begin{cases} 0 & \text{if } x < \sqrt{2} \\ x^2 - 2 & \text{if } \sqrt{2} \leq x < \sqrt{3} \\ 1 & \text{if } x \geq \sqrt{3} \end{cases}$$

$\mathbb{P}(X = 1.6) = 0$ since X is cont.

$$\mathbb{P}(1 \leq X \leq 3/2) = F_X\left(\frac{3}{2}\right) - F_X(1) = \left(\frac{9}{4} - 2\right) - 0 = \frac{1}{4}$$

$$f_X(x) \underset{\substack{\downarrow \\ X \text{ cont.}}}{=} \frac{d}{dx} F_X(x) = \begin{cases} 2x & \text{if } x \in (\sqrt{2}, \sqrt{3}) \\ 0 & \text{otherwise} \end{cases}$$

Ex. 40

Let X be discrete with values in $\mathbb{N}$.

We know that X has cumulative distrib.

such that $F(n) = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}_{>0}$

Write the probability mass function.

$$F(n) = \mathbb{P}(X \leq n)$$

$$F(m-1) = \mathbb{P}(X \leq m-1)$$

X assumes values in $\mathbb{N}$

$$\mathbb{P}(X=m) = \underbrace{\mathbb{P}(X\leq m)}_{1-\frac{1}{m}} - \underbrace{\mathbb{P}(X \leq m-1)}_{1-\frac{1}{m-1}} = \frac{1}{m-1} - \frac{1}{m}$$

So for $m \in \mathbb{N}$

$$\mathbb{P}(X=m) = \begin{cases} \frac{1}{m-1} - \frac{1}{m} & \text{if } m \in \mathbb{N},\ m \geq 2 \\ 0 & \text{if } m = 0,1 \end{cases}$$

You know

$\mathbb{E}[X] = 3$, $\mathbb{E}[X^2] = 13$

X r.v.

$a, b \in \mathbb{R}$

$\mathbb{E}[aX+b] =$

$= a\,\mathbb{E}[X] + b$.

X, Y r.v.

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\mathbb{E}[3X+2] = 3\,\mathbb{E}[X] + 2$$
$$= 11$$

$$\mathbb{E}[(2X+3)^2] = \mathbb{E}[4X^2 + 9 + 12X]$$
$$= 4\,\mathbb{E}[X^2] + 9 + 12\,\mathbb{E}[X] =$$
$$= 4 \cdot 13 + 9 + 12 \cdot 3 = 52 + 9 + 36 = 97$$