$P(X=1) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} = \frac{1}{4}$, $P(X=2) = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} = \frac{1}{4}$

$E(X) = \sum_{k=1}^{13} k \frac{\binom{4}{k}\binom{48}{13-k}}{\binom{52}{13}} = \frac{13}{4}$

$F(a) + P(a < X \le b) = F(b)$

$\{X \le a\} \cup \{a < X \le b\} = \{X \le b\}$

$P(a < X \le b) = P(X \le b) - P(X \le a)$

$P(R-\epsilon \le X \le R) = F(R) - F(R-\epsilon) = \frac{R^d}{R^d} - \frac{(R-\epsilon)^d}{R^d}, \frac{R^d - (R-\epsilon)^d}{R^d} = 1 - \left(\frac{R-\epsilon}{R}\right)^d = 1 - \left(1 - \frac{\epsilon}{R}\right)^d$

$P(R-\epsilon \le X \le R) = 1 - \left(1 - \frac{\epsilon}{R}\right)^d$

$d \to \infty$?

$\epsilon < \frac{1}{2}$

$P(\text{landing in edge region}) = 1 - (1-2\epsilon)^d$

in d-dimensions $1 - (1-2\epsilon)^d \to 1$ as $d \to \infty$

✓ Expectations are not necessarily finite, and do not exist

St. Petersburg "Paradox"

Consider a sequence of fair coin tosses, all independent

H	TH	TTH	Payoff is 2^n when H comes up for the 1st time on the nth toss
+2	+4	+8	
1/2	1/4	1/8	

H	TH	TTH
+2	-4	+8
1/2	1/4	1/8

$1 - 1 + 1 - 1 \cdots$ expectation does not exist

$X \in \{2, 4, 8, \ldots\}$

$P(X = 2^n) = 2^{-n}$, $n = 1, 2, 3$

$\mathbb{E}[X] = \infty$

TT T T

(rare)

Law of large numbers

$X_1, X_2, \ldots$ $\frac{X_1 + X_2 + \cdots + X_n}{n} \to \infty$ $\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty$

+1 Heads

+0 Tails

$E(X) = \frac{1}{2}$

[redacted]

$\mathbb{E}[X] = \infty$

Given $X \in \{1, 2, 3, 4, 5, 6\}$ uniform

x	g(x)
1	-7
2	0
3	-5
4	0
5	0
6	0

$\mathbb{E}[g(X)] = \mathbb{E}[Y]$

$Y = g(X)$

$Y \in \{-5, -7, 0\}$ $P(Y=-5) = \frac{1}{3}$, $P(Y=-7) = \frac{1}{6}$, $P(Y=0) = \frac{1}{2}$

$\mathbb{E}[Y] = (-5)\left(\frac{1}{3}\right) + (-7)\left(\frac{1}{6}\right) + 0\left(\frac{1}{2}\right)$

$\mathbb{E}[g(X)] = g(1)P(X=1) + g(2)P(X=2) + \ldots g(6)P(X=6) = (-7)\frac{1}{6} + (-5)\frac{1}{3}$

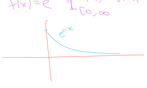
$\boxed{\text{Fact: } \mathbb{E}[g(X)] = \sum_{a \in \mathcal{A}} g(a)P(X=a)}$

$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

$\int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = e^{-x}\Big|_\infty^0 = e^0 = 1$

Exponential Distribution

$f(x) = e^{-x} \mathbb{1}_{[0,\infty)}(x)$ is a density



$\frac{1}{2} e^{-|x|}$

Laplace distribution

X has density $\frac{1}{2}e^{-|x|}$ $|x| = \begin{cases} -x & x < 0 \\ x & x \ge 0 \end{cases}$ double exponential

Compute $\mathbb{E}[|X|] = \int_{-\infty}^{\infty} |x| \frac{1}{2} e^{-|x|}\, dx = \frac{1}{2}\int_{-\infty}^{0} (-x) e^{x}\, dx + \frac{1}{2}\int_0^\infty x e^{-x}\, dx$

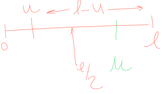
$\int u\, dv = uv - \int v \frac{du}{dx}$

$u = x$, $dv = e^{-x}$

Stick of length l is broken at a uniform place. What is the expected value of the larger piece?

$g(u) = \begin{cases} l - u, & u \le l/2 \\ u & u > l/2 \end{cases}$

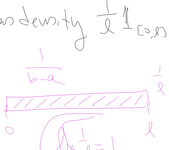
$U \sim \mathcal{U}[0, l]$



$\mathbb{E}[g(U)] = \frac{1}{l}\int_0^l g(u)\, du$

U has density $\frac{1}{l}\mathbb{1}_{[0,l]}$

$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx = \int_0^l g(x) \frac{1}{l}$



$\int_0^l \frac{1}{l}\, dx = 1$

$= \frac{1}{l}\int_0^{l/2} (l - u)\, du + \frac{1}{l}\int_{l/2}^{l} u\, du$

$= \frac{1}{l}\left(\left(lu - \frac{u^2}{2}\right)\Big|_0^{l/2} + \frac{1}{2}u^2\Big|_{l/2}^{l}\right)$

$= \frac{1}{l}\left(l\frac{l}{2} - \frac{(l/2)^2}{2} + \frac{1}{2}\left(l^2 - \left(\frac{l}{2}\right)^2\right)\right)$

$= \frac{1}{l}\left(\frac{1}{2}l^2 - \frac{1}{8}l^2\right) + \frac{1}{2}\left(l^2 - \frac{1}{4}l^2\right)$

$= l\left(\frac{1}{2} - \frac{1}{8} + \frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}l > \frac{1}{2}l$

$\mathbb{E}[g(X)]$

$\mathbb{E}[X]$ $\boxed{\mu_n = \mathbb{E}[X^n], \; n \ge 1}$

μ_1 1st moment

n^{th} moment

$\mu_1, \mu_2, \mu_3, \ldots$

$X \sim \text{Ber}(p)$ $P(X=1) = p$, $P(X=0) = 1 - p$

$X \in \{0, 1\}$ $\mathbb{E}[X] = p$

$X = X^n \; \forall n \ge 1$, $p, p, p,$

$\mu_n = \mathbb{E}[X^n] = \mathbb{E}[X] = p$

$X \in \{-1, 1\}$ $P(X=1) = P(X=-1) = \frac{1}{2}$

Rademacher

$\mathbb{E}[X^n] = 1^n P(X=1) + (-1)^n P(X=-1)$

$= \frac{1}{2}\left(1 + (-1)^n\right)$

$n=1$ $\mu_1 = 0$ all odd moments

$n=2$ $\mu_2 = 1$ all even moments

$0, 0, 1, 0, 1, \ldots$

n odd $(-1)^n = -1$

n even $(-1)^n = 1$