$\sin x = \frac{\text{opp}}{\text{hyp}}$ (sin = opp/hyp)

$\cos x = \frac{\text{adj}}{\text{hyp}}$ (cos = adj/hyp)

$\tan x = \frac{\text{opp}}{\text{adj}}$ (tan = opp/adj)

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\sin^2 x + \cos^2 x = 1$

$\sin(2x) = 2\sin x \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$

Distribution

Discrete: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}$ (finite, countably infinite) $\sum P(X=a) = 1$

Continuous: $P(X=a) = 0 \ \forall a \in \mathbb{R}$, $\int_{-\infty}^{\infty} P(X=x)\,dx = 1$

Example: Normal, Poisson, Binomial, Geometric, Discrete Uniform $\{1, 2, 3, 4, 5, 6\}$

$P(X=1) = \frac{1}{6} \ \forall a \in \mathbb{R}$

Uniform $[a, b]$: $P(X=x) = \frac{1}{b-a}$

Normal, Gaussian

Distribution that is neither discrete nor continuous

Car insurance

Discrete 500 ←

Cantor: $\frac{\text{Number of deaths}}{\text{Number of people}}$

X = amount of money $\in \{0, 10{,}000\}$

$P(X=500) = P(500 \le X \le 10{,}000)$

After all, 500 not continuous $= \frac{10{,}000}{9{,}500}$

"Should return $[0, 500]$ not discrete

Go back to sec 4 slides

$\mathbb{Z}$ money or chad $\mathbb{Q}$ on a line just (Exponential)

$\mathbb{R}$ time and / not discrete

$P(T=2) > 0$ not continuous

Mixture?

Distribution of X

measure of center, eg. expectation

Discrete

$P(X \in \mathbb{R}) = 1$

$\mathbb{E}[X] = \sum_{a \in \mathbb{R}} a P(X=a)$ weighted average of the atoms

$P(X=a) = 1$

$\mathbb{E}[X] = a \cdot 1 = a$

Bern(p)

$P(X=0) = 1-p$, $P(X=1) = p$

$p^1 q^0$

$\mathbb{E}[X] = 0(1-p) + 1p = p$

$X \sim Bin(n, p)$, $X \in \{0, 1, \ldots, n\}$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \frac{1}{(n-k)!} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} \frac{1}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$

$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$

$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} p^k (1-p)^{n-k}$

$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$

$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-1-(k-1)}$

$j = k-1$

$k=1 \Rightarrow j=0$

$k=n \Rightarrow j=n-1$

$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$

$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$

$= np(p + 1 - p)^{n-1}$

$= np$

Conclusion $\mathbb{E}[X] = np$

Expectation of Geometric $X \sim Geo(p)$

$0 < p < 1$

$P(X=k) = (1-p)^{k-1} p \quad k \in \{1, 2, \ldots\}$

$1 - p = q$

$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$

$= \sum_{k=1}^{\infty} k q^{k-1} p = p \sum_{k=1}^{\infty} k q^{k-1}$

Calculus power rule

$\frac{1}{dq} q^k = k q^{k-1}$

$= p \sum_{k=1}^{\infty} \frac{1}{dq} q^k$

$= p \frac{1}{dq} \sum_{k=1}^{\infty} q^k$

$\frac{1}{dq}(q^2 + q^3)$

$= \frac{1}{dq} q^2 + \frac{1}{dq} q^3$

$= p \frac{1}{dq} \left(\frac{q}{1-q}\right)$

$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

$= p\left(\frac{1 \cdot (1-q) - q \cdot (-1)}{(1-q)^2}\right) = \frac{p}{(1-q)^2}$

$= \frac{p}{p^2} = \frac{1}{p}$

$p = 10^{-6}$, $\mathbb{E}[X] = 1{,}000{,}000$

Bernoulli
Binomial
Geometric

Poisson

$X \sim Pois(\lambda)$ $\lambda > 0$ $X \in \{0, 1, 2, \ldots\}$

$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$

$e^{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1$

$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda} \lambda^k}{k!}$

$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k \lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$

$= \lambda e^{-\lambda} \boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = \lambda e^{-\lambda} e^{\lambda} = \lambda$

$\lim_{n \to \infty} a_n = L$

$\lim_{n \to \infty} a_{n+1} = L$

$1 + \lambda$

$k = 1 + \frac{\lambda}{1}$

$\mathbb{E}[X] = \lambda$

$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$

$P(X=0) = e^{-\lambda}$

$\lambda = 1$ $e^{-1} = 1 - e^{-1}$

0.69314... $\log 2$ $e^{\log 2} = \frac{1}{2}$ 0.69315

Birthday Problem

np

$\lambda = \binom{23}{2} \frac{1}{365}$

$= \frac{23!}{2!\,21!} \frac{1}{365}$

$= \frac{23 \times 22}{2} \frac{1}{365} = \frac{23(11)}{365} = \frac{253}{365}$

Discrete expectation $\mathbb{E}[X] = \sum_{a \in \mathbb{R}} a P(X=a)$

Draw one uniformly

Get \$1,000 if Green

0 if red

X winnings $\mathbb{E}[X] = 1000 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 500$

Urn U, 2,280 Balls

1,000 Green 1280 red

Selects two without replacement, put in Urn A

(without balls seen by you)

You pick an urn of your choice

Withdraw a ball (Seen)

replace

repeat once

X your winnings

Maximize $\mathbb{E}[X]$?

Expectation for continuous variables

X has density f

discrete

$\mathbb{E}[X] = \sum_{a \in \mathbb{R}} a P(X=a)$

$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

Suppose $X \sim U[a, b]$

$\mathbb{E}[X]$

$\mathbb{E}[X] = \int_a^b x \cdot \frac{1}{b-a}\,dx$

$= \frac{1}{b-a} \int_a^b x\,dx$

$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$

$= \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2}\right)$

$= \frac{1}{2(b-a)} (b^2 - a^2)$

$= \frac{1}{2(b-a)} (b+a)(b-a)$

$= \frac{a+b}{2}$

Dartboard radius R, hit with uniformity on boards

X = distance from center

$\mathbb{E}[X]$ $\frac{\pi R^2}{\pi R^2}$ $\frac{\pi r^2}{\pi R^2}$ $0 \le r \le R$

Density $\frac{2r}{R^2}$

$\mathbb{E}[X] = \int_0^R \frac{2r}{R^2} \cdot r\,dr = \frac{2}{R^2} \int_0^R r^2\,dr = \frac{2}{R^2} \left[\frac{r^3}{3}\right]_0^R = \frac{2}{3R^2}(R^3 - 0)$

$= \frac{2}{3} R$