#21, Chapter 2

4 problems on exam $B(n,p)$ X number answered correctly

need 3 or more for an A, $A=\{X\geq 3\}$, $X\sim B(4,0.8)$

Prob of 0.80 answering correctly, independent

a) $P(A)$ $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$

$P(X\geq 3)=P(X=3, X=4)=P(X=3)+P(X=4)$

$=\binom{4}{3}(.8)^3(1-.8)+\binom{4}{4}(.8)^4$

$=4\,.8^3(.2)+(.8)^4$

$A\subset B \Rightarrow A\cap B=A$

$X=X_1+X_2+X_3+X_4$

b) $P(X\geq 3\mid X_1=1)=\dfrac{P(X\geq 3, X_1=1)}{P(X_1=1)}$

#11, Chapter 2

C Careful

R Reckless

A accident

$P(A\mid R)=0.04$ $P(C)=0.80$

$P(A\mid C)=0.01$

$P(\Omega\mid C)=1$

$P(A\cup A^c\mid C)=1$

$P(A\mid C)+P(A^c\mid C)=1$

$Q: P(C\mid A)$

$=\dfrac{P(A\mid C)P(C)}{P(A\mid C)P(C)+P(A\mid R)P(R)}=\dfrac{(0.01)(0.80)}{(0.01)(0.80)+(0.04)(0.20)}=\dfrac{8}{8+8}=\dfrac{1}{2}$

$P(X\geq 3\mid X_1=1)=\dfrac{P(X_1+X_2+X_3+X_4\geq 3, X_1=1)}{P(X_1=1)}$

$X=X_1+X_2+X_3+X_4$

$\dfrac{P(X_2+X_3+X_4\geq 2, X_1=1)}{P(X_1=1)}=\dfrac{P(X_2+X_3+X_4\geq 2)\cancel{P(X_1=1)}}{\cancel{P(X_1=1)}}$

#19, Chapter 2

Urn $\{1,2,..7\}$ $\left\{\begin{matrix}1, 2 & 5, 6\\ 3, 4, & 7\end{matrix}\right\}$

2 balls are drawn X_1 number on 1st, X_2 on 2nd

$P(X_1=4, X_2=5)$

$[n]=\{1,2,\dots,n\}$

a) drawn with replacement

$\Omega=\{(\omega_1,\omega_2):\omega_1\in\{1,..,7\}, \omega_2\in\{1,..,7\}\}=[7]^2$

$|\Omega|=7\times 7=49$ $A=\{(4,5)\}$ $P(A)=\dfrac{|A|}{|\Omega|}=\dfrac{1}{49}=\dfrac{1}{7}\times\dfrac{1}{7}=P(X_1=4)P(X_2=5)$

$P(C\cap D)=P(C)P(D)$ therefore $C \perp\!\!\!\perp D$

b) $\Omega=[7]^2$? no $(4,4)$ impossible under sampling without replacement

Yes take $[7]^2$ as Ω, but now its not uniform!

Want to still keep uniform

$\Omega=\{(\omega_1,\omega_2):\omega_1\in\{1,..,7\}, \omega_2\in\{1,..,7\}, \omega_1\neq\omega_2\}$

$|\Omega|=7\cdot 6=42$

$A=\{(4,5)\}$ $|A|=1$

$P(A)=\dfrac{1}{42}>\dfrac{1}{49}=P(X_1=4)P(X_2=5)$

$P(X_1=4, X_2=5)$ $\neq$

$\therefore X_1$ and X_2 are NOT independent

$P(X_1=4, X_2=5)>P(X_1=4)P(X_2=5)$ events are positively associated

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$\{X\geq t\}=\{X-a\geq t-a\}$

$1-\left(\frac{4}{9}\right)^6$ Show you work

Repeated trials, $X_1, X_2, X_3, \dots$ independent, same distributions

Example, drawing from an urn with replacement

$\text{Bern}(p)$ $p\in[0,1]$

Coin toss $(0,0,1,1,0,1,0)$ $X=3$, $p^3(1-p)^4$

$P(0)=1-p$, $P(1)=p$

n repeated trials $X=X_1+\dots+X_n\sim \text{Bin}(n,p)$

$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$

Distribution Function $P(X\leq k)=\sum_{j=0}^{k}P(X=j)=\sum_{j=0}^{k}\binom{n}{j}p^j(1-p)^{n-j}$

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toss a die $\{1,2,..,6\}$, want to get the first 6

$X=$ # trials until 1st 6

trial $\underline{1}\,\underline{3}\,\underline{2}\,\underline{6}$ $X=4$

$1\underline{6}62$ $X=2$

$P(X=4)=P(\text{1st trial not 6, 2nd trial not 6, 3rd trial not 6, 4th trial is 6})$

$=P(\text{1st not 6})\dots P(\text{3rd not 6})P(\text{4th is 6})$

$\overset{\text{independence}}{=}\dfrac{5}{6}\cdot\dfrac{5}{6}\,\dfrac{5}{6}\,\dfrac{1}{6}=\left(\dfrac{5}{6}\right)^{\textcircled{3}}\dfrac{1}{6}$ $3=4-1$

$X\sim \text{Geom}(p)$, $p\in[0,1]$

$P(X=n)=(1-p)^{n-1}p$, $n=1,2,3,\dots$

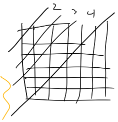
Y : # failures $Y = X - 1$

$$P(Y=n) = P(X-1=n) = P(X=n+1) = (1-p)^n p, \quad n=0,1,2\ldots$$

Roll a pair of 2 dice, uniform model, sum of dice

repeated trials. $X_1, X_2, X_3, \ldots$

$$P(\text{we get 5 before getting 7}) \overset{?}{<} \frac{1}{2}$$



$$\{5 \text{ before } 7\} = \bigcup_{n=1}^{\infty} A_n \qquad n=1,2,\ldots$$

5 .. $X=1$

$\textcircled{4}$ 5 $X=2$

$\textcircled{24}$ 5 .. $X=3$

$A_n = \{\text{neither 5 nor 7 on roll } n-1, \text{ 5 on roll } n\}$

$$P(5 \text{ before } 7) = \sum_{n=1}^{\infty} P(A_n)$$

$$P(5 \text{ or } 7) = P(5) + P(7) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36} = \frac{5}{18}$$

$$P(A_n) = \left(\frac{13}{18}\right)^{n-1} \frac{4}{36}$$

$$P(\text{other than } \{5,7\}) = 1 - \frac{5}{18} = \frac{13}{18}$$

$$\frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} = \frac{1}{9} \frac{1}{1 - \frac{13}{18}}$$

$$\frac{\text{1st term}}{1 - \text{ratio between terms}}$$

$$= \frac{1}{9} \frac{1}{5/18} = \frac{18}{9} \frac{1}{5} = 2/5 < \frac{1}{2}$$

Coincidence?

Toss two dice once, X sum

$$P(X=5 \mid X \in \{5,7\}) = \frac{P(X=5)}{P(X=5) + P(X=7)} = \frac{4/36}{4/36 + 6/36} = \frac{4}{4+6} = \frac{4}{10} = 2/5$$

$\{5\} \cap \{5,7\} = \{5\}$

$X \sim \text{Pois}(\lambda)$ $\lambda > 0$ Poisson

$X \in \{0,1,2,\ldots\}$

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

a) non-negative ✓

b) sums to 1

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = e^{-\lambda} e^{\lambda} = 1$$

occurence of rare events

$\text{Bin}(n, \lambda/n)$, $n \to \infty$ obtain

$$(n)_k = n(n-1)(n-2)\cdots(n-k+1)$$

k term

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{(n)_k}{k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$$\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$$

compound interest

$(1+r)$, $n=1$ $\quad (1+r/2)(1+\frac{r}{2}) = (1+\frac{r}{2})^2$

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1^{-k} = 1$$

$\to 1$

$$\lim_{n\to\infty} a_n = a \qquad \lim_{n\to\infty} g(a_n) = g(a)$$

$$\frac{(n)_k}{n^k} = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-k+1}{n}\right)$$

k

$$\lim \frac{n-1}{n} = \lim \left(1 - \frac{1}{n}\right) = 1$$

$$\frac{n-j}{n} = 1 - j/n \to 1$$

$$X \sim B(n, \lambda/n)$$

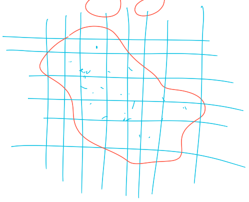
$$\lim_{n\to\infty} P_n(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k=0,\ldots,n \quad \text{Poisson}$$

$n \to \infty$ many trials

Law of Rare Events

$p = \lambda/n$ probability of each trial is small

Bombing of London in WW2



Hits	Quadrants hit	Predicted via Poisson
0	229	227
1	211	211.39
2	93	98.54
3	35	30.62
4	7	7.14
5+	1	1.57

$\lambda = 0.93$

Deaths by horsekicks in the Prussian Army

Sampling without replacement from urn

N total items, N_A special, sample n items

X number of special items in sample

Hypergeometric Distribution

$$P(X=k) = \frac{\binom{N_A}{k}\binom{N-N_A}{n-k}}{\binom{N}{n}}$$

g groups (different colors)

N_1 type 1, N_2 type 2, , N_g type g

$N_1 + N_2 + \dots + N_g = N$ Sample n without replacement

X_i number of items of type i

$X_1 + X_2 + \dots + X_g = n$

$$P(X_1=k_1, X_2=k_2, \, X_g=k_g) = \frac{\binom{N_1}{k_1}\binom{N_2}{k_2}\dots\binom{N_g}{k_g}}{\binom{N}{n}}$$

Random Variables

and their distributions

Discrete random variables, $\mathcal{A}$ in a discrete set

$X \in \mathcal{A}$ (finite, or countably infinite)

Binomial $\uparrow$ $X \in \{0, 1, .., n\}$

Poisson $\{0, 1, 2, \dots, \}$

p.m.f. $P(X=a) = p_X(a)$

$B \subset \mathcal{A}$

$P(X \in B) = \sum p_X(a)$

$a \in B$

Cumulative Distribution Function C.D.F.

$F_X(t) = P(X \le t)$

$B(2, 1/3)$

$P(X=0) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

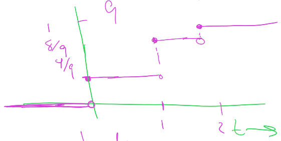
$P(X=1) = \binom{2}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^1$, $P(X=2) = \binom{2}{2}\left(\frac{1}{3}\right)^2$

$\frac{2 \times 2}{9} = \frac{4}{9}$ $\frac{1}{9}$ $4+4+1=9$

$P(X \le 0) = P(X=0)$

$P(X \le \frac{1}{2}) = P(X=0)$

$P(X \le t) =$ $P(X \le 1) = P(0) + P(1)$



piecewise constant

jumps of size $P(X=a)$ at atoms $a \in \mathbb{Q}$.

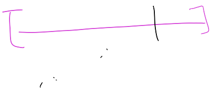
$P(X=a) > 0$

a : atom

X continuous random variable

$P(X=a) = 0 \quad \forall a \in \mathbb{R}$

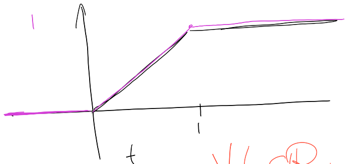
$X \sim U[0,1]$, uniform on unit interval



$P(X \le \boxed{\tfrac{1}{2}}) = \boxed{\tfrac{1}{2}}$ $P(X \le t) = t$

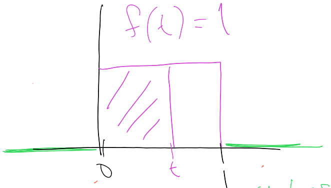
$P(X \le \boxed{4/5}) = \boxed{\tfrac{4}{5}}$ $0 \le t \le 1$

X has CDF $F_X(t) = t \quad 0 \le t \le 1$



$\forall t \in \mathbb{R}$

$f(t) = 1$



$F_X(t) = t = \int_{-\infty}^{t} f(u)\,du$

~~$0 \le t \le 1$~~

$f(t) = \mathbb{1}_{[0,1]}(t) = \begin{cases} 1 & t \in [0,1] \\ 0 & t \notin [0,1] \end{cases}$

Density Function

$F_X(t) = \int_{-\infty}^{t} f(u)\,du$