SHEET 1

Ex. 16

Flip a fair coin 5 times

For every heads you pay 1$
For " tails I pay you 1$

X = my net winnings after 5 flips.
Write the possible values for X and its mass function.

$$X = \begin{cases} 5 & \text{if } 5 \text{ heads} \\ 3 & \text{if } 4 \text{ heads} \\ 1 & \text{if } 3 \text{ heads} \\ -1 & \text{if } 2 \text{ heads} \\ -3 & \text{if } 1 \text{ heads} \\ -5 & \text{if } 0 \text{ heads} \end{cases}$$

(H, T, H, H, H)
(H, H, H, H, T)

$$\mathbb{P}(X=5) = \left(\frac{1}{2}\right)^5 \qquad \mathbb{P}(X=3) = \left(\frac{1}{2}\right)^5 \cdot 5$$

$$\binom{5}{1} = \binom{5}{4}$$

$$\mathbb{P}(X=1) = \left(\frac{1}{2}\right)^5 \cdot \binom{5}{2} = \frac{10}{32}$$

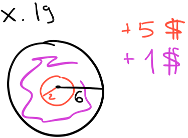
$$\binom{5}{2} \;\|\; \binom{5}{3} = \frac{5 \cdot 4}{2} = 10$$

$$\mathbb{P}(X=-1) = \left(\frac{1}{2}\right)^5 \cdot \underbrace{\binom{5}{2}}_{10} = \frac{10}{2^5}$$

$$\mathbb{P}(X=-3) = \left(\frac{1}{2}\right)^5 \cdot \binom{5}{1} = \frac{5}{32}$$

$$\mathbb{P}(X=-5) = \left(\frac{1}{2}\right)^5 \cdot \binom{5}{0} = \frac{1}{32}$$

Ex. 19



X = money won

$$X = \begin{cases} 1 & \text{prob. } 1-p = \frac{8}{9} \\ 5 & \text{prob. } p = \frac{1}{9} \end{cases}$$

$$p = \frac{\text{Red Area}}{\text{Area of circle with radius } 6} = \frac{\pi \cdot 4}{\pi \cdot 36} = \frac{1}{9}.$$

Ex. 32 5 cards from a deck of 52 cards.

$$\mathbb{P}(\text{Full house}) = ?$$

3♡ 3◇ 3♤ 2♡ 2♧

$$\mathbb{P}(\text{Full house}) = \frac{13 \cdot 12 \binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

n° of ways to choose the face value of the 3 cards

ways to choose the face value for the 2 cards

suits for 3 cards

suits for 2 cards.

$$\simeq 0.014$$

Ex. 41

We repeat a game 4 times. Each time only one player can win. 3 players in total with same winning probab.

$$\mathbb{P}(\underbrace{\exists \text{ at least one player that wins no game}}_{A})$$

$$A = A_1 \cup A_2 \cup A_3$$

$$A_i = \{\text{player } i \text{ wins no game}\}$$

$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) + \mathbb{P}(A_1 \cap A_2 \cap A_3)$$

$$\mathbb{P}(A_1) = \left(\frac{2}{3}\right)^4 = \mathbb{P}(A_2) = \mathbb{P}(A_3)$$

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(\text{player 3 wins all the 4 games}) = \left(\frac{1}{3}\right)^4 = \mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_2 \cap A_3)$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) \overset{\text{Someone must win!}}{=} 0$$

$$\mathbb{P}(A) = \left(\frac{2}{3}\right)^4 \cdot 3 - \left(\frac{1}{3}\right)^4 \cdot 3 =$$

$$= \cancel{3}\left(\frac{1}{3}\right)^{\overset{3}{\cancel{4}}}\left[2^4 - 1\right] = \frac{15}{27} = \frac{5}{9}$$

SHEET 2

Ex. 2) TOSS A COIN 3 TIMES

Prob. of having tails at the 2nd tossing knowing that there is at least one tails among the 3 tossings.

SAMPLE SPACE $\Omega = \{(x,y,z) \mid x,y,z \in \{H,T\}\}$

$$\#\Omega = 2\cdot 2\cdot 2 = \underset{8}{\underset{\shortparallel}{2^3}}$$

$A = \{2^{nd} \text{ tossing is } T\}$

$B = \{\text{at least one } T\}$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A\cap B) \underset{A\subseteq B}{=} \mathbb{P}(A) = \frac{4 = \#A}{8 = \#\Omega} = \frac{1}{2}$$

$A = \{(H,T,H), (T,T,T), (T,T,H), (H,T,T)\}$

$$\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \frac{1}{8} = \frac{7}{8}$$

no tails

$B^c = \{(H,H,H)\}$

$$\mathbb{P}(A|B) = \frac{1/2}{7/8} = \frac{1}{2}\cdot\frac{8}{7} = \frac{4}{7}.$$

Ex. 5 2 Urns

$U_1 = \{①, ②, ③\}$

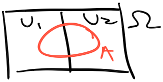
$U_2 = \{②, ③, ④, ⑤\}$

Prob. to choose U_1 : $1/5$

Prob. to choose U_2: $4/5$

After having chosen an urn, we draw one ball.

$$\mathbb{P}(\underbrace{\text{Draw ball ②}}_{A}) = ?$$



$$\mathbb{P}(A) = \overset{\mathbb{P}(A|U_1)\mathbb{P}(U_1)}{\mathbb{P}(A\cap U_1)} + \mathbb{P}(A\cap U_2)$$

$$= \mathbb{P}(A|U_1)\mathbb{P}(U_1) + \mathbb{P}(A|U_2)\mathbb{P}(U_2)$$

$$\mathbb{P}(A|U_1) = 1/3 \qquad = \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{4}\cdot\frac{4}{5}$$

$$\mathbb{P}(A|U_2) = 1/4 \qquad = \frac{1}{15} + \frac{1}{5} = \frac{4}{15}$$

Ex. 9 Suppose that you $\overbrace{\text{have drawn ball ③}}^{A}$

$\underbrace{\text{Prob. to have drawn from } U_2}_{U_2}$.

$$\mathbb{P}(U_2|A) = \frac{\mathbb{P}(U_2\cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|U_2)\mathbb{P}(U_2)}{\mathbb{P}(A)}$$

$$= \frac{1/5}{4/15} = \frac{3}{4}$$

Ex. 19 $U = \{①, \dots, ⑦\}$

We draw 2 balls.

X_1 = label of the 1st ball

X_2 = label of the 2nd ball

$\mathbb{P}(X_1=4)$, $\mathbb{P}(X_2=5)$, $\mathbb{P}(X_1=4 \overset{\cap}{,} X_2=5)$

if

a) balls are drawn one by one with replacement

$\mathbb{P}(X_1=4)=\frac{1}{7}$

$\mathbb{P}(X_2=5)=\frac{1}{7}$ → the urn has the same config.

$$\mathbb{P}(\underbrace{X_1=4}_{A},\underbrace{X_2=5}_{B})=\mathbb{P}(X_1=4)\,\mathbb{P}(X_2=5)=\frac{1}{7}\cdot\frac{1}{7}=\frac{1}{49}$$

$$\mathbb{P}(A\cap B)=\begin{cases}\mathbb{P}(A)\mathbb{P}(B) & \text{if } A\perp\!\!\!\perp B\\ \underbrace{\mathbb{P}(B|A)}_{\mathbb{P}(B) \text{ if } A\perp\!\!\!\perp B}\mathbb{P}(A) & \text{in general}\end{cases}$$

b) without replacement

$\mathbb{P}(X_1=4)=1/7$

$$\mathbb{P}(X_2=5)=\overbrace{\mathbb{P}(X_2=5|X_1=5)}^{=0}\cdot\mathbb{P}(X_1=5)+$$

$$+\underbrace{\mathbb{P}(X_2=5|X_1\neq 5)}_{1/6}\,\underbrace{\mathbb{P}(X_1\neq 5)}_{6/7}=$$

$$=1/7$$

$$\mathbb{P}(\underbrace{X_1=4}_{A_1} \underset{\cap}{,} \underbrace{X_2=5}_{A_2}) = \underset{\overset{\parallel}{\frac{1}{6}}}{\mathbb{P}(A_2|A_1)} \cdot \underset{\overset{\parallel}{\frac{1}{7}}}{\mathbb{P}(A_1)} = \frac{1}{42}$$

$$\left[\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_3|A_1 \cap A_2)\mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_1)\right]$$

- $\{X_1=4\}, \{X_2=5\}$ are indep?

$$\underset{\frac{1}{42}}{\mathbb{P}(X_1=4, X_2=5)} \overset{\parallel}{=} \underset{\overset{\parallel}{1/7}}{\mathbb{P}(X_1=4)} \, \underset{\overset{\parallel}{1/7}}{\mathbb{P}(X_2=5)} \;?$$

$$\Rightarrow \frac{1}{42} \neq \frac{1}{49} \Rightarrow \{X_1=4\},\{X_2=5\} \text{ are not indep.}$$