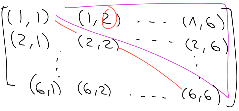
EX. 1

ROLL A FAIR DIE TWICE.

$A = \{$ THE 2nd ROLL IS HIGHER THAN THE 1st ROLL $\}$

SAMPLE SPACE Ω:

$$\Omega = \left\{ \begin{matrix} (1,1), (1,2), \ldots (1,6) \\ (2,1), (2,2) \ldots (2,6) \\ \vdots \\ (6,1) \ldots \ldots (6,6) \end{matrix} \right\}$$



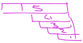
$A = \{ 2^{nd}$ roll $> 1^{st}$ roll $\} = \{$ ◹ $\}$

Since all outcomes are equally likely, we can compute

$$\mathbb{P}(A) = \frac{\overset{\to \text{n° of elements}}{\#A}}{\#\Omega} \qquad \#A = 15 \qquad \#\Omega = 36$$

$\frac{36-6}{2}$

$\frac{m^2 - m}{2}$



EX. 2

$\overset{C}{\text{cereal}}, \overset{E}{\text{eggs}}, \overset{F}{\text{fruit}}$

Bob chooses two items from

$\Omega = \{ (C,E), (C,F), (E,F) \}$

$A = \{$ Bob's breakfast includes cereals $\}$
$= \{ (C,E), (C,F) \}$

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{2}{3}.$$

EX. 8:

A bag of scrabble tiles:
$\{5E, 4A, 3N, 2B\}$

We draw 4 tiles

$C = \{$ we get $2E, 1A, 1N \}$

Compute $\mathbb{P}(C)$ if:

a) the tiles are drawn one by one as ordered sample (without replacement)

b) the tiles are drawn all at once as an unordered sample.

Solution:

a) We think all letters different

$\mathcal{S} = \{E_1, \dots, E_5, A_1, \dots, A_4, N_1, \dots, N_3, B_1, B_2\}$

$\mathbb{P}(C) = ?$ $\qquad C = \{$ we have $2E, 1A, 1N \}$

SAMPLE SPACE: all ordered sequences Ω of 4 symbols from $\mathcal{S}$.

$$\#\Omega = 14 \cdot 13 \cdot 12 \cdot 11 = (14)_4$$

$$\begin{array}{cccc} (x_1, & x_2, & x_3, & x_4) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 14 & 13 & 12 & 11 \end{array}$$

$\#C$? $\qquad C = \{2E, 1A, 1N\}$

(x_1, x_2, x_3, x_4)

5E in the bag and we have to draw 2:

$$\binom{5}{2} \rightarrow E_1\, E_3$$

Choose 1A out of 4:

$$\binom{4}{1} = 4 \rightarrow A_2$$

Choose 1N out of 3:

$$\binom{3}{1} = 3 \rightarrow N_3$$

$\Rightarrow E_1, E_3 A_2 N_3$ $\qquad$ # of ways: $\binom{5}{2} \cdot \binom{4}{1} \cdot \binom{3}{1}$

$A_2 N_3 E_1 E_3$

↳ 4! possible orders

$$\#C = 4! \cdot \binom{5}{2}\binom{4}{1}\binom{3}{1}$$

$$\#\Omega = (14)_4 = 14 \cdot 13 \cdot 12 \cdot 11$$

$$\mathbb{P}(C) = \frac{4! \cdot 10 \cdot \cancel{4} \cdot \cancel{3}}{14 \cdot 13 \cdot \cancel{12} \cdot 11} = \frac{120}{1001}$$

$$\binom{5}{2} = \frac{5 \cdot 4}{2}$$

$$\binom{5}{2} = \frac{5!}{2! \cdot (5-2)!}$$

b) Draw the tiles all at once:

$\Omega = \{$ all unordered sequences of 4 elements drawn from $\mathcal{S}$ $\}$

$$\#\Omega = \binom{14}{4} = \frac{\boxed{14!}}{4! \cdot 10!} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4!}$$

→ $14! = 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10!$

$$\#C = \binom{5}{2}\binom{4}{1}\binom{3}{1}$$

2E out of 5

1A out of 4

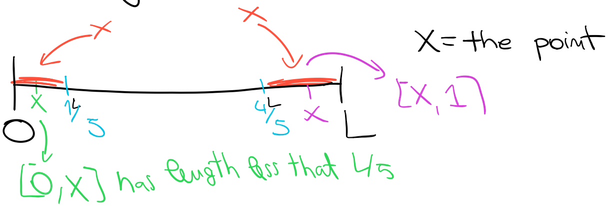
→ 1N out of 3.

$$\mathbb{P}(C) = \frac{\#C}{\#\Omega} = \frac{10 \cdot 4 \cdot 3}{\frac{14 \cdot 13 \cdot 12 \cdot 11}{4!}} = \frac{4! \cdot 10 \cdot 4 \cdot 3}{14 \cdot 13 \cdot 12 \cdot 11}$$

$$= \frac{120}{1001}$$

Ex. 9 We break a stick at a point chosen uniformly at random.

$\mathbb{P}$(shortest piece is less than $\frac{1}{5}$-th of the original one)



$$\mathbb{P}(A) = \frac{\text{length of } [0, \frac{L}{5}] \cup [\frac{4L}{5}, 1]}{\text{length of } [0, L)} = \frac{\frac{L}{5} + \frac{L}{5}}{L}$$

$$= 2/5.$$

Ex. 12 Roll a die until we see 4.

$$\mathbb{P}(\underbrace{\text{we need at most 3 rolls}}_{A}) =$$

$$= 1 - \mathbb{P}(A^c)$$

$$A^c = \{\text{we need more than 3 rolls}\} =$$

$$= \{\text{the first 3 rolls are not 4}\}$$

$$\mathbb{P}(A^c) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{5}{6}\right)^3$$

$$\mathbb{P}(A) = 1 - \left(\frac{5}{6}\right)^3$$

$\therefore \mathbb{P}(\underset{\overset{\shortparallel}{A}}{\text{We}} \text{ need an even number of rolls})$

$$\mathbb{P}(\text{we need } n \text{ rolls}) = \underbrace{\frac{5}{6}\cdot\frac{5}{6}\cdots\frac{5}{6}}_{\substack{n-1 \text{ times} \\ (\text{not a } 4)}} \cdot \underbrace{\frac{1}{6}}_{\text{we have the 4}} =$$

$$= \left(\frac{5}{6}\right)^{n-1}\cdot\frac{1}{6}$$

$$A = A_2 \cup A_4 \cup A_6 \cup \ldots$$

$$A_i = \{\text{we need } i \text{ rolls}\}$$

$$\mathbb{P}(A) = \sum_{i \text{ even}} \mathbb{P}(A_i)$$

$2k$ is an even number for $k \in \mathbb{N}$

$$\mathbb{P}(A) = \sum_{k=1}^{\infty} \boxed{\mathbb{P}(A_{2k})} = \sum_{k=1}^{\infty} \left(\frac{5}{6}\right)^{2k-1}\cdot\frac{1}{6} =$$

$$= \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{2k-1}\cdot\left(\frac{5}{6}\right)\cdot\frac{6}{5}$$

$$= \frac{1}{\cancel{6}}\cdot\frac{\cancel{6}}{5}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{2k} =$$

$$= \frac{1}{5}\sum_{k=\underset{\shortparallel a}{1}}^{\infty}\Bigg(\underset{\shortparallel r}{\frac{25}{36}}\Bigg)^{k}$$

Geometric sum: $\sum^{\infty} r^k = \frac{r^a}{\quad}$

$K = a$ $1 - r$

$$D = \frac{1}{5}\cdot\frac{\frac{25}{36}}{1-\frac{25}{36}} = \frac{1}{5}\cdot\frac{\frac{25}{36}}{\frac{11}{36}} =$$

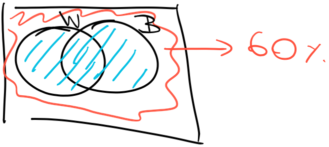
$$= \frac{1}{\cancel{5}}\cdot\frac{\overset{5}{\cancel{25}}}{\cancel{36}}\cdot\frac{\cancel{36}}{11} = \frac{5}{11}$$

Ex. 13

- 25% of students that wear a watch
- 30% of students that wear a bracelet
- 60% wear neither a watch nor a bracelet

One student at random:

a) P(student wears watch or bracelet).



$$= 1 - \frac{60}{100} = \frac{40}{100}$$

P(wearing both) $\frac{25}{100} + \frac{30}{100} - \frac{40}{100} = \frac{15}{100}$

$$P(W \cap B) = P(W) + P(B) - P(W \cup B)$$

Ex. 21: $U = \{3\,\text{Black}, 2\,\text{Red}, 2\,\text{Green}\}$

We draw 3 chips without replacement.

$A = \{$all chips are ~~of~~ different color$\}$

$\mathbb{P}(A) = ?$

a) We draw the chips one by one as an ordered sample

b) // // all at once as an unordered sample.

$$\mathbb{P}(A) = \frac{\overset{\text{black}}{\binom{3}{1}}\overset{\text{red}}{\binom{2}{1}}\overset{\text{green}}{\binom{2}{1}} \cdot 3!}{(7)_3}$$

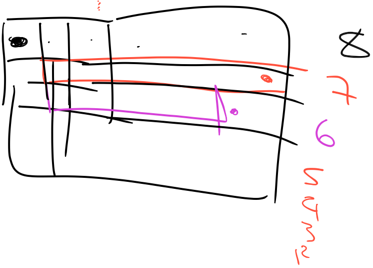
$(\overset{7}{x_1}, \overset{6}{x_2}, \overset{5}{x_3})$

7 chips

b) $$\mathbb{P}(A) = \frac{\binom{3}{1}\binom{2}{1}\binom{2}{1}}{\binom{7}{3}}$$

Ex. 30 8 rooks placed randomly on a chess board $\to 8 \times 8$

$\mathbb{P}($no two $\overset{=A}{}$ chosen rooks share a row or a column$) = ?$



8! positions for the 8 rooks satisfying the event.

$$\mathbb{P}(A) = \frac{8!}{\binom{64}{8}}$$