Birthday Problem

Scheme one

Scheme two

Finite sample spaces

Ω infinite $\Omega = \{1, 2, 3, \ldots\}$ Discrete, countably infinite

Given the current probability $p \in [0,1]$

Flip a coin until Heads appear

$P(\text{we obtain the first head at } k)$, $n = 1, 2, \ldots$

$P(\text{first heads on } k) = p$

$P(\text{first} \ldots \text{ } 2) = P(TH) = (1-p)p$

$P(\ldots \quad 3) = (1-p)(1-p)p = (1-p)^2 p$

$P(\ldots \quad n) = (1-p)^{n-1} p$ $\quad P(TTH)$ $\quad A_1 \cap A_2 \cap \cdots \cap A_n$

Axioms 3) $A_1, A_2, \ldots$ events, pairwise disjoint

then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ countable additivity

$A_i = \{\text{Heads eventually appear}\} = \bigcup_{n=1}^{\infty} A_i$

$A_i = \{\text{heads appear for the first time on } i^{th} \text{ toss}\}$

$P(A_i) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} (1-p)^{i-1} p = p \sum_{i=1}^{\infty} (1-p)^{i-1} = p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$

$r \in (0,1)$ Geometric series

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

$P(A_i) = (1-p)^{i-1} p$ $\quad P_i = (1-p)^{i-1} p$, $i = 1, 2, \ldots$ probability mass function

P(need at least n tosses to obtain the 1st Heads)

B_n $\quad B_n = \bigcup_{i=n}^{\infty} A_i$ $\quad$ 1st heads at i^{th} toss

$P(B_n) = \sum_{i=n}^{\infty} P(A_i) = \sum_{i=n}^{\infty} (1-p)^{i-1} p = p \sum_{i=n}^{\infty} (1-p)^{i-1}$

$\sum_{i=n}^{\infty} (1-p)^{i-1} p = p \sum_{i=n}^{\infty} (1-p)^{i-1} = p \frac{(1-p)^{n-1}}{1-(1-p)} = (1-p)^{n-1}$

$\Omega = [0,1]$ $\quad$ Finite, uniform $P(A) = \frac{|A|}{|\Omega|}$

Uniform probability $\quad |A|$ length of A

$P(\text{uniform number is in } [0, \tfrac{1}{2}]) = \frac{1}{2}$

$P(\text{uniform number is } 0.782) = 0$

Uniform measure $P(x) = P(y)$

$[0,1] = \bigcup_{x \in [0,1]} \{x\}$ $\quad P(x) > 0$

$1 = P([0,1]) = \sum_{x \in [0,1]} P(x) = \infty$

Suppose $P(x) > 0 \Rightarrow P(y) = P(x) > 0$ all y, $P(y) > 0$

$P(\{\frac{1}{100}, \frac{2}{100}, \ldots \frac{100}{100}\}) = \sum_{k=1}^{100} P(x_k) = \underbrace{100}_{n P(x) > 1} P(x)$ $\quad n > \frac{1}{P(x)}$

Axioms 1) $\forall A \in \mathcal{F}$, $P(A) \in [0,1]$

2) $P(\Omega) = 1$

3) $A_1, A_2, \ldots$ pairwise disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

De Morgan's law

$(A \cup B)^c = A^c \cap B^c$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$ $\quad 1_C 1_D = 1_{C \cap D}$ $\quad 1_{C^c} = 1 - 1_C$

$1_{A^c} 1_{B^c} = (1 - 1_A)(1 - 1_B) = 1 - 1_A - 1_B + 1_A 1_B = 1 - 1_A - 1_B + 1_{A \cap B}$

$1_{A^c} 1_{B^c} = 1$?

$1_{A^c \cap B^c} = 1_{(A \cup B)^c} = 1 - 1_{A \cup B}$

$1_{A^c} = 1$, $1_{B^c} = 1$

$1 - 1_A = 1$, $1 - 1_B = 1$

$(1 - 1_A)(1 - 1_B) = 1$

$1_{A \cup B} = 1_A + 1_B - 1_{A \cap B}$

$1_{A_1^c \cap \cdots \cap A_n^c} = \prod_{i=1}^{n} (1 - 1_{A_i}) = 1 - \sum 1_{A_i} + \sum_{i,j} 1_{A_i \cap A_j} - \sum 1_{A_i \cap A_j \cap A_k}$

$= 1_{(A_1 \cup \cdots \cup A_n)^c} = 1 - 1_{A_1 \cup \cdots \cup A_n}$

$1_{A_1 \cup \cdots \cup A_n} = \sum 1_{A_i} - \sum_{i,j} 1_{A_i \cap A_j} + \sum_{i,j,k} 1_{A_i \cap A_j \cap A_k} + \cdots (-1)^{n-1} 1_{A_1 \cap \cdots \cap A_n}$

$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i,j} P(A_i \cap A_j) + \cdots + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n)$

Hat matching problem

Envelope stuffing

n individuals, all check in their hat

recieve a hat at random when they depart

Question: What is the probability that all recive their own hat

$n!$ ways hats can be distributed

$P(\text{all recive their own hat}) = \frac{1}{n!}$

Question: Probability that no one recieves their own hat

A_j: individual j recives their own hat

$\bigcup_{j=1}^{n} A_j$ $\quad$ at least one person recieves their own hat

$\left(\bigcup_{j=1}^{n} A_j\right)^c$ no one

$P(A_i) = P(A_j) = P(A_1)$

$P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} P(A_j) - \sum_{i,j} P(A_i \cap A_j) + \cdots$

$P(A_i \cap A_j) = P(A_1 \cap A_2)$

$P(A_j) = \frac{1}{n}$ $\quad P(A_j \cap A_k) = \frac{1}{n(n-1)}$, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{n(n-1)(n-2)}$

$P\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \frac{1}{n} - \sum_{i,j} \frac{1}{n(n-1)} + \sum_{i,j,k} \frac{1}{n(n-1)}$