Math 126 vs. IB Math HL Calculus Option

Section 5.6: Inverse Trigonometric functions	**Section 3.5 (Page 36 of PDF)**: The inverse functions, their domains and ranges, their graphs **Section 3.6 (Page 36)**: Algebraic and graphical methods of solving trigonometric functions **Section 6.2 (Page 44)**: Derivatives of arcsinx, arccosx and arctanx
Section 5.7: Hyperbolic functions	**Section 2.5 (Page 33)**: Polynomial functions and their graphs. The factor and remainder theorems. **Section 2.6 (Page 33)**: Solving quadratic equations. Use of discriminant. Solving polynomial equations graphically and algebraically. **Section 2.7 (Page 34)**: Solutions of inequalities
Section 5.8" L'Hospital's Rule	**Section 9.7 (Page 60)**: Using L'Hopital's rule or the Taylor series to evaluate the limits of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ and $\lim_{x\to\infty}\frac{f(x)}{g(x)}$
Section 6.1-6.6: Techniques of Integration	**Section 6.4 (Page 45)**: Indefinite integration of x^n, e^x, a^x, trigonometric functions, reciprocal trig functions, and inverse trig functions **Section 6.7 (Page 46)**: Integration by substitution and by parts. **Section 9.4 (Page 58)**: The integral as a limit of a sum; lower and upper Reimann sums. Form $\frac{d}{dx}\left[\int_a^x f(y)dy\right] = f(x)$ **Section 9.5 (Page 59)**: Integration using Euler's method, variables separable, and homogenous differential equations. Solution of $y' + P(x)y = Q(x)$ using the integrating factor
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Sections 9.3-9.4: Polar coordinates	Polar coordinates covered as a part of complex numbers. **Section 1.6 (Page 30):** Modulus-argument (polar) form $z = r(cos\theta + isin\theta) = rcis\theta = re^{i\theta}$. The complex plane/Argand diagram. Convert Cartesian form to polar form.

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Diploma Programme

Mathematics HL guide

First examinations 2014

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IB mission statement

The International Baccalaureate aims to develop inquiring, knowledgeable and caring young people who help to create a better and more peaceful world through intercultural understanding and respect.

To this end the organization works with schools, governments and international organizations to develop challenging programmes of international education and rigorous assessment.

These programmes encourage students across the world to become active, compassionate and lifelong learners who understand that other people, with their differences, can also be right.

IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.

IB learners strive to be:

Inquirers	They develop their natural curiosity. They acquire the skills necessary to conduct inquiry and research and show independence in learning. They actively enjoy learning and this love of learning will be sustained throughout their lives.
Knowledgeable	They explore concepts, ideas and issues that have local and global significance. In so doing, they acquire in-depth knowledge and develop understanding across a broad and balanced range of disciplines.
Thinkers	They exercise initiative in applying thinking skills critically and creatively to recognize and approach complex problems, and make reasoned, ethical decisions.
Communicators	They understand and express ideas and information confidently and creatively in more than one language and in a variety of modes of communication. They work effectively and willingly in collaboration with others.
Principled	They act with integrity and honesty, with a strong sense of fairness, justice and respect for the dignity of the individual, groups and communities. They take responsibility for their own actions and the consequences that accompany them.
Open-minded	They understand and appreciate their own cultures and personal histories, and are open to the perspectives, values and traditions of other individuals and communities. They are accustomed to seeking and evaluating a range of points of view, and are willing to grow from the experience.
Caring	They show empathy, compassion and respect towards the needs and feelings of others. They have a personal commitment to service, and act to make a positive difference to the lives of others and to the environment.
Risk-takers	They approach unfamiliar situations and uncertainty with courage and forethought, and have the independence of spirit to explore new roles, ideas and strategies. They are brave and articulate in defending their beliefs.
Balanced	They understand the importance of intellectual, physical and emotional balance to achieve personal well-being for themselves and others.
Reflective	They give thoughtful consideration to their own learning and experience. They are able to assess and understand their strengths and limitations in order to support their learning and personal development.

Contents

Purpose of this document

This publication is intended to guide the planning, teaching and assessment of the subject in schools. Subject teachers are the primary audience, although it is expected that teachers will use the guide to inform students and parents about the subject.

This guide can be found on the subject page of the online curriculum centre (OCC) at http://occ.ibo.org, a password-protected IB website designed to support IB teachers. It can also be purchased from the IB store at http://store.ibo.org.

Additional resources

Additional publications such as teacher support materials, subject reports, internal assessment guidance and grade descriptors can also be found on the OCC. Specimen and past examination papers as well as markschemes can be purchased from the IB store.

Teachers are encouraged to check the OCC for additional resources created or used by other teachers. Teachers can provide details of useful resources, for example: websites, books, videos, journals or teaching ideas.

First examinations 2014

The Diploma Programme

The Diploma Programme is a rigorous pre-university course of study designed for students in the 16 to 19 age range. It is a broad-based two-year course that aims to encourage students to be knowledgeable and inquiring, but also caring and compassionate. There is a strong emphasis on encouraging students to develop intercultural understanding, open-mindedness, and the attitudes necessary for them to respect and evaluate a range of points of view.

The Diploma Programme hexagon

The course is presented as six academic areas enclosing a central core (see figure 1). It encourages the concurrent study of a broad range of academic areas. Students study: two modern languages (or a modern language and a classical language); a humanities or social science subject; an experimental science; mathematics; one of the creative arts. It is this comprehensive range of subjects that makes the Diploma Programme a demanding course of study designed to prepare students effectively for university entrance. In each of the academic areas students have flexibility in making their choices, which means they can choose subjects that particularly interest them and that they may wish to study further at university.

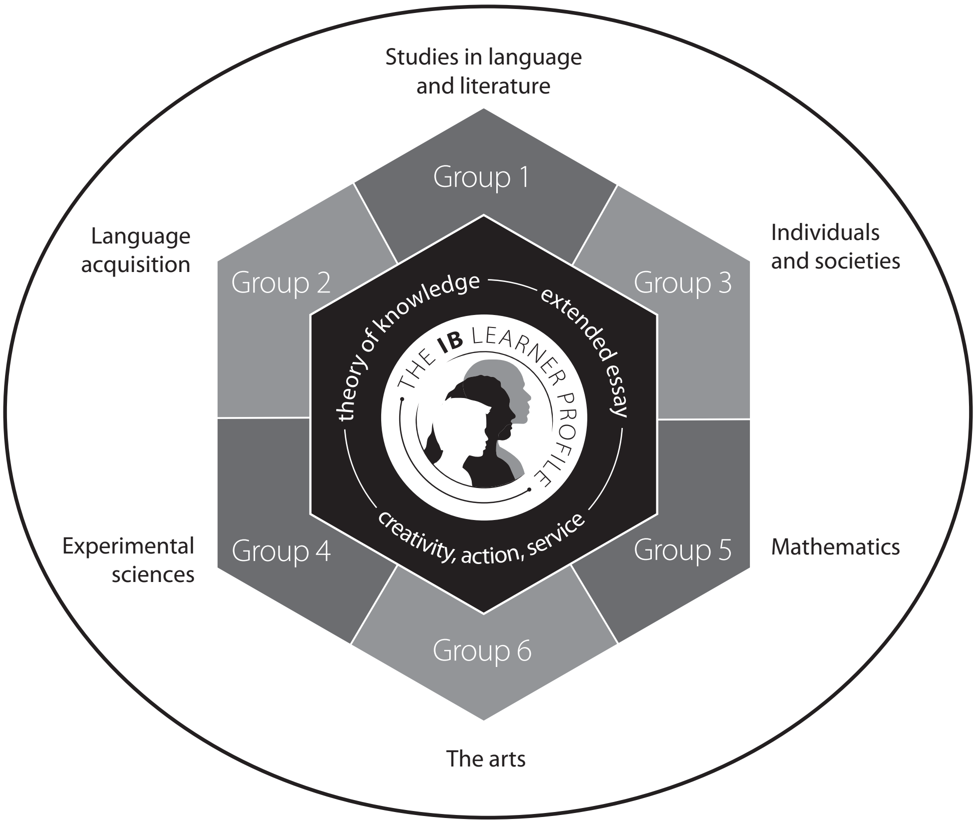

Figure 1

Diploma Programme model

Choosing the right combination

Students are required to choose one subject from each of the six academic areas, although they can choose a second subject from groups 1 to 5 instead of a group 6 subject. Normally, three subjects (and not more than four) are taken at higher level (HL), and the others are taken at standard level (SL). The IB recommends 240 teaching hours for HL subjects and 150 hours for SL. Subjects at HL are studied in greater depth and breadth than at SL.

At both levels, many skills are developed, especially those of critical thinking and analysis. At the end of the course, students' abilities are measured by means of external assessment. Many subjects contain some element of coursework assessed by teachers. The courses are available for examinations in English, French and Spanish, with the exception of groups 1 and 2 courses where examinations are in the language of study.

The core of the hexagon

All Diploma Programme students participate in the three course requirements that make up the core of the hexagon. Reflection on all these activities is a principle that lies at the heart of the thinking behind the Diploma Programme.

The theory of knowledge course encourages students to think about the nature of knowledge, to reflect on the process of learning in all the subjects they study as part of their Diploma Programme course, and to make connections across the academic areas. The extended essay, a substantial piece of writing of up to 4,000 words, enables students to investigate a topic of special interest that they have chosen themselves. It also encourages them to develop the skills of independent research that will be expected at university. Creativity, action, service involves students in experiential learning through a range of artistic, sporting, physical and service activities.

The IB mission statement and the IB learner profile

The Diploma Programme aims to develop in students the knowledge, skills and attitudes they will need to fulfill the aims of the IB, as expressed in the organization's mission statement and the learner profile. Teaching and learning in the Diploma Programme represent the reality in daily practice of the organization's educational philosophy.

Nature of the subject

Introduction

The nature of mathematics can be summarized in a number of ways: for example, it can be seen as a well-defined body of knowledge, as an abstract system of ideas, or as a useful tool. For many people it is probably a combination of these, but there is no doubt that mathematical knowledge provides an important key to understanding the world in which we live. Mathematics can enter our lives in a number of ways: we buy produce in the market, consult a timetable, read a newspaper, time a process or estimate a length. Mathematics, for most of us, also extends into our chosen profession: visual artists need to learn about perspective; musicians need to appreciate the mathematical relationships within and between different rhythms; economists need to recognize trends in financial dealings; and engineers need to take account of stress patterns in physical materials. Scientists view mathematics as a language that is central to our understanding of events that occur in the natural world. Some people enjoy the challenges offered by the logical methods of mathematics and the adventure in reason that mathematical proof has to offer. Others appreciate mathematics as an aesthetic experience or even as a cornerstone of philosophy. This prevalence of mathematics in our lives, with all its interdisciplinary connections, provides a clear and sufficient rationale for making the study of this subject compulsory for students studying the full diploma.

Summary of courses available

Because individual students have different needs, interests and abilities, there are four different courses in mathematics. These courses are designed for different types of students: those who wish to study mathematics in depth, either as a subject in its own right or to pursue their interests in areas related to mathematics; those who wish to gain a degree of understanding and competence to understand better their approach to other subjects; and those who may not as yet be aware how mathematics may be relevant to their studies and in their daily lives. Each course is designed to meet the needs of a particular group of students. Therefore, great care should be taken to select the course that is most appropriate for an individual student.

In making this selection, individual students should be advised to take account of the following factors:

- their own abilities in mathematics and the type of mathematics in which they can be successful
- their own interest in mathematics and those particular areas of the subject that may hold the most interest for them
- their other choices of subjects within the framework of the Diploma Programme
- their academic plans, in particular the subjects they wish to study in future
- their choice of career.

Teachers are expected to assist with the selection process and to offer advice to students.

Mathematical studies SL

This course is available only at standard level, and is equivalent in status to mathematics SL, but addresses different needs. It has an emphasis on applications of mathematics, and the largest section is on statistical techniques. It is designed for students with varied mathematical backgrounds and abilities. It offers students opportunities to learn important concepts and techniques and to gain an understanding of a wide variety of mathematical topics. It prepares students to be able to solve problems in a variety of settings, to develop more sophisticated mathematical reasoning and to enhance their critical thinking. The individual project is an extended piece of work based on personal research involving the collection, analysis and evaluation of data. Students taking this course are well prepared for a career in social sciences, humanities, languages or arts. These students may need to utilize the statistics and logical reasoning that they have learned as part of the mathematical studies SL course in their future studies.

Mathematics SL

This course caters for students who already possess knowledge of basic mathematical concepts, and who are equipped with the skills needed to apply simple mathematical techniques correctly. The majority of these students will expect to need a sound mathematical background as they prepare for future studies in subjects such as chemistry, economics, psychology and business administration.

Mathematics HL

This course caters for students with a good background in mathematics who are competent in a range of analytical and technical skills. The majority of these students will be expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology. Others may take this subject because they have a strong interest in mathematics and enjoy meeting its challenges and engaging with its problems.

Further mathematics HL

This course is available only at higher level. It caters for students with a very strong background in mathematics who have attained a high degree of competence in a range of analytical and technical skills, and who display considerable interest in the subject. Most of these students will expect to study mathematics at university, either as a subject in its own right or as a major component of a related subject. The course is designed specifically to allow students to learn about a variety of branches of mathematics in depth and also to appreciate practical applications. It is expected that students taking this course will also be taking mathematics HL.

Note: Mathematics HL is an ideal course for students expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering or technology. It should not be regarded as necessary for such students to study further mathematics HL. Rather, further mathematics HL is an optional course for students with a particular aptitude and interest in mathematics, enabling them to study some wider and deeper aspects of mathematics, but is by no means a necessary qualification to study for a degree in mathematics.

Mathematics HL—course details

The course focuses on developing important mathematical concepts in a comprehensible, coherent and rigorous way. This is achieved by means of a carefully balanced approach. Students are encouraged to apply their mathematical knowledge to solve problems set in a variety of meaningful contexts. Development of each topic should feature justification and proof of results. Students embarking on this course should expect to develop insight into mathematical form and structure, and should be intellectually equipped to appreciate the links between concepts in different topic areas. They should also be encouraged to develop the skills needed to continue their mathematical growth in other learning environments.

The internally assessed component, the exploration, offers students the opportunity for developing independence in their mathematical learning. Students are encouraged to take a considered approach to various mathematical activities and to explore different mathematical ideas. The exploration also allows students to work without the time constraints of a written examination and to develop the skills they need for communicating mathematical ideas.

This course is a demanding one, requiring students to study a broad range of mathematical topics through a number of different approaches and to varying degrees of depth. Students wishing to study mathematics in a less rigorous environment should therefore opt for one of the standard level courses, mathematics SL or mathematical studies SL. Students who wish to study an even more rigorous and demanding course should consider taking further mathematics HL in addition to mathematics HL.

Prior learning

Mathematics is a linear subject, and it is expected that most students embarking on a Diploma Programme (DP) mathematics course will have studied mathematics for at least 10 years. There will be a great variety of topics studied, and differing approaches to teaching and learning. Thus students will have a wide variety of skills and knowledge when they start the mathematics HL course. Most will have some background in arithmetic, algebra, geometry, trigonometry, probability and statistics. Some will be familiar with an inquiry approach, and may have had an opportunity to complete an extended piece of work in mathematics.

At the beginning of the syllabus section there is a list of topics that are considered to be prior learning for the mathematics HL course. It is recognized that this may contain topics that are unfamiliar to some students, but it is anticipated that there may be other topics in the syllabus itself that these students have already encountered. Teachers should plan their teaching to incorporate topics mentioned that are unfamiliar to their students.

Links to the Middle Years Programme

The prior learning topics for the DP courses have been written in conjunction with the Middle Years Programme (MYP) mathematics guide. The approaches to teaching and learning for DP mathematics build on the approaches used in the MYP. These include investigations, exploration and a variety of different assessment tools.

A continuum document called *Mathematics: The MYP–DP continuum* (November 2010) is available on the DP mathematics home pages of the OCC. This extensive publication focuses on the alignment of mathematics across the MYP and the DP. It was developed in response to feedback provided by IB World Schools, which expressed the need to articulate the transition of mathematics from the MYP to the DP. The publication also highlights the similarities and differences between MYP and DP mathematics, and is a valuable resource for teachers.

Mathematics and theory of knowledge

The *Theory of knowledge guide* (March 2006) identifies four ways of knowing, and it could be claimed that these all have some role in the acquisition of mathematical knowledge. While perhaps initially inspired by data from sense perception, mathematics is dominated by reason, and some mathematicians argue that their subject is a language, that it is, in some sense, universal. However, there is also no doubt that mathematicians perceive beauty in mathematics, and that emotion can be a strong driver in the search for mathematical knowledge.

As an area of knowledge, mathematics seems to supply a certainty perhaps missing in other disciplines. This may be related to the "purity" of the subject that makes it sometimes seem divorced from reality. However, mathematics has also provided important knowledge about the world, and the use of mathematics in science and technology has been one of the driving forces for scientific advances.

Despite all its undoubted power for understanding and change, mathematics is in the end a puzzling phenomenon. A fundamental question for all knowers is whether mathematical knowledge really exists independently of our thinking about it. Is it there "waiting to be discovered" or is it a human creation?

Students' attention should be drawn to questions relating theory of knowledge (TOK) and mathematics, and they should be encouraged to raise such questions themselves, in mathematics and TOK classes. This includes questioning all the claims made above. Examples of issues relating to TOK are given in the "Links" column of the syllabus. Teachers could also discuss questions such as those raised in the "Areas of knowledge" section of the TOK guide.

Mathematics and the international dimension

Mathematics is in a sense an international language, and, apart from slightly differing notation, mathematicians from around the world can communicate within their field. Mathematics transcends politics, religion and nationality, yet throughout history great civilizations owe their success in part to their mathematicians being able to create and maintain complex social and architectural structures.

Despite recent advances in the development of information and communication technologies, the global exchange of mathematical information and ideas is not a new phenomenon and has been essential to the progress of mathematics. Indeed, many of the foundations of modern mathematics were laid many centuries ago by Arabic, Greek, Indian and Chinese civilizations, among others. Teachers could use timeline websites to show the contributions of different civilizations to mathematics, but not just for their mathematical content. Illustrating the characters and personalities of the mathematicians concerned and the historical context in which they worked brings home the human and cultural dimension of mathematics.

The importance of science and technology in the everyday world is clear, but the vital role of mathematics is not so well recognized. It is the language of science, and underpins most developments in science and technology. A good example of this is the digital revolution, which is transforming the world, as it is all based on the binary number system in mathematics.

Many international bodies now exist to promote mathematics. Students are encouraged to access the extensive websites of international mathematical organizations to enhance their appreciation of the international dimension and to engage in the global issues surrounding the subject.

Examples of global issues relating to international-mindedness (**Int**) are given in the "Links" column of the syllabus.

Aims

Group 5 aims

The aims of all mathematics courses in group 5 are to enable students to:

1. enjoy mathematics, and develop an appreciation of the elegance and power of mathematics
2. develop an understanding of the principles and nature of mathematics
3. communicate clearly and confidently in a variety of contexts
4. develop logical, critical and creative thinking, and patience and persistence in problem-solving
5. employ and refine their powers of abstraction and generalization
6. apply and transfer skills to alternative situations, to other areas of knowledge and to future developments
7. appreciate how developments in technology and mathematics have influenced each other
8. appreciate the moral, social and ethical implications arising from the work of mathematicians and the applications of mathematics
9. appreciate the international dimension in mathematics through an awareness of the universality of mathematics and its multicultural and historical perspectives
10. appreciate the contribution of mathematics to other disciplines, and as a particular "area of knowledge" in the TOK course.

Assessment objectives

Problem-solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. Having followed a DP mathematics HL course, students will be expected to demonstrate the following.

1. **Knowledge and understanding**: recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.
2. **Problem-solving**: recall, select and use their knowledge of mathematical skills, results and models in both real and abstract contexts to solve problems.
3. **Communication and interpretation**: transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation.
4. **Technology**: use technology, accurately, appropriately and efficiently both to explore new ideas and to solve problems.
5. **Reasoning**: construct mathematical arguments through use of precise statements, logical deduction and inference, and by the manipulation of mathematical expressions.
6. **Inquiry approaches**: investigate unfamiliar situations, both abstract and real-world, involving organizing and analysing information, making conjectures, drawing conclusions and testing their validity.

Syllabus outline

Syllabus component	Teaching hours
	HL
All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as prior learning.	
Topic 1 Algebra	30
Topic 2 Functions and equations	22
Topic 3 Circular functions and trigonometry	22
Topic 4 Vectors	24
Topic 5 Statistics and probability	36
Topic 6 Calculus	48
Option syllabus content Students must study all the sub-topics in one of the following options as listed in the syllabus details. **Topic 7** Statistics and probability **Topic 8** Sets, relations and groups **Topic 9** Calculus **Topic 10** Discrete mathematics	48
Mathematical exploration Internal assessment in mathematics HL is an individual exploration. This is a piece of written work that involves investigating an area of mathematics.	10
Total teaching hours	**240**

Approaches to the teaching and learning of mathematics HL

Throughout the DP mathematics HL course, students should be encouraged to develop their understanding of the methodology and practice of the discipline of mathematics. The processes of **mathematical inquiry**, **mathematical modelling and applications** and the **use of technology** should be introduced appropriately. These processes should be used throughout the course, and not treated in isolation.

Mathematical inquiry

The IB learner profile encourages learning by experimentation, questioning and discovery. In the IB classroom, students should generally learn mathematics by being active participants in learning activities rather than recipients of instruction. Teachers should therefore provide students with opportunities to learn through mathematical inquiry. This approach is illustrated in figure 2.

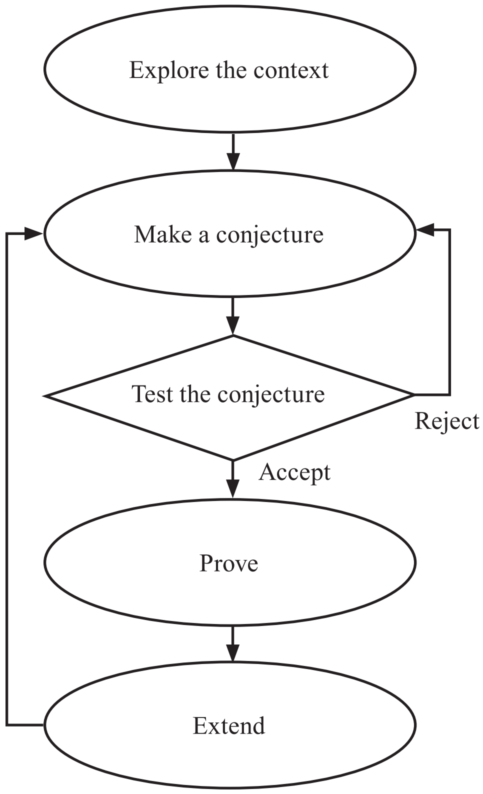

Figure 2

Mathematical modelling and applications

Students should be able to use mathematics to solve problems in the real world. Engaging students in the mathematical modelling process provides such opportunities. Students should develop, apply and critically analyse models. This approach is illustrated in figure 3.

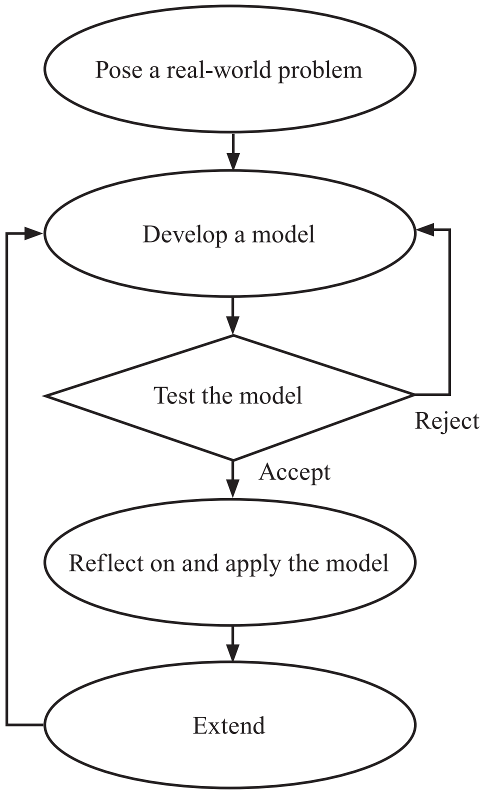

Figure 3

Technology

Technology is a powerful tool in the teaching and learning of mathematics. Technology can be used to enhance visualization and support student understanding of mathematical concepts. It can assist in the collection, recording, organization and analysis of data. Technology can increase the scope of the problem situations that are accessible to students. The use of technology increases the feasibility of students working with interesting problem contexts where students reflect, reason, solve problems and make decisions.

As teachers tie together the unifying themes of **mathematical inquiry**, **mathematical modelling and applications** and the **use of technology**, they should begin by providing substantial guidance, and then gradually encourage students to become more independent as inquirers and thinkers. IB students should learn to become strong communicators through the language of mathematics. Teachers should create a safe learning environment in which students are comfortable as risk-takers.

Teachers are encouraged to relate the mathematics being studied to other subjects and to the real world, especially topics that have particular relevance or are of interest to their students. Everyday problems and questions should be drawn into the lessons to motivate students and keep the material relevant; suggestions are provided in the "Links" column of the syllabus. The mathematical exploration offers an opportunity to investigate the usefulness, relevance and occurrence of mathematics in the real world and will add an extra dimension to the course. The emphasis is on communication by means of mathematical forms (for example, formulae, diagrams, graphs and so on) with accompanying commentary. Modelling, investigation, reflection, personal engagement and mathematical communication should therefore feature prominently in the DP mathematics classroom.

For further information on “Approaches to teaching a DP course”, please refer to the publication *The Diploma Programme: From principles into practice* (April 2009). To support teachers, a variety of resources can be found on the OCC and details of workshops for professional development are available on the public website.

Format of the syllabus

- **Content**: this column lists, under each topic, the sub-topics to be covered.
- **Further guidance**: this column contains more detailed information on specific sub-topics listed in the content column. This clarifies the content for examinations.
- **Links**: this column provides useful links to the aims of the mathematics HL course, with suggestions for discussion, real-life examples and ideas for further investigation. **These suggestions are only a guide for introducing and illustrating the sub-topic and are not exhaustive.** Links are labelled as follows.

Appl	real-life examples and links to other DP subjects
Aim 8	moral, social and ethical implications of the sub-topic
Int	international-mindedness
TOK	suggestions for discussion

Note that any syllabus references to other subject guides given in the “Links” column are correct for the current (2012) published versions of the guides.

Notes on the syllabus

- Formulae are only included in this document where there may be some ambiguity. All formulae required for the course are in the mathematics HL and further mathematics HL formula booklet.
- The term “technology” is used for any form of calculator or computer that may be available. However, there will be restrictions on which technology may be used in examinations, which will be noted in relevant documents.
- The terms “analysis” and “analytic approach” are generally used when referring to an approach that does not use technology.

Course of study

The content of all six topics and one of the option topics in the syllabus must be taught, although not necessarily in the order in which they appear in this guide. Teachers are expected to construct a course of study that addresses the needs of their students and includes, where necessary, the topics noted in prior learning.

Integration of the mathematical exploration

Work leading to the completion of the exploration should be integrated into the course of study. Details of how to do this are given in the section on internal assessment and in the teacher support material.

Time allocation

The recommended teaching time for higher level courses is 240 hours. For mathematics HL, it is expected that 10 hours will be spent on work for the exploration. The time allocations given in this guide are approximate, and are intended to suggest how the remaining 230 hours allowed for the teaching of the syllabus might be allocated. However, the exact time spent on each topic depends on a number of factors, including the background knowledge and level of preparedness of each student. Teachers should therefore adjust these timings to correspond to the needs of their students.

Use of calculators

Students are expected to have access to a graphic display calculator (GDC) at all times during the course. The minimum requirements are reviewed as technology advances, and updated information will be provided to schools. It is expected that teachers and schools monitor calculator use with reference to the calculator policy. Regulations covering the types of calculators allowed in examinations are provided in the *Handbook of procedures for the Diploma Programme*. Further information and advice is provided in the *Mathematics HL/SL: Graphic display calculators teacher support material* (May 2005) and on the OCC.

Mathematics HL and further mathematics HL formula booklet

Each student is required to have access to a clean copy of this booklet during the examination. It is recommended that teachers ensure students are familiar with the contents of this document from the beginning of the course. It is the responsibility of the school to download a copy from IBIS or the OCC, check that there are no printing errors, and ensure that there are sufficient copies available for all students.

Teacher support materials

A variety of teacher support materials will accompany this guide. These materials will include guidance for teachers on the introduction, planning and marking of the exploration, and specimen examination papers and markschemes.

Command terms and notation list

Teachers and students need to be familiar with the IB notation and the command terms, as these will be used without explanation in the examination papers. The “Glossary of command terms” and “Notation list” appear as appendices in this guide.

Prior learning topics

As noted in the previous section on prior learning, it is expected that all students have extensive previous mathematical experiences, but these will vary. It is expected that mathematics HL students will be familiar with the following topics before they take the examinations, because questions assume knowledge of them. Teachers must therefore ensure that any topics listed here that are unknown to their students at the start of the course are included at an early stage. They should also take into account the existing mathematical knowledge of their students to design an appropriate course of study for mathematics HL. This table lists the knowledge, together with the syllabus content, that is essential to successful completion of the mathematics HL course.

Students must be familiar with SI (*Système International*) units of length, mass and time, and their derived units.

Topic	Content
Number	Routine use of addition, subtraction, multiplication and division, using integers, decimals and fractions, including order of operations. Rational exponents. Simplification of expressions involving roots (surds or radicals), including rationalizing the denominator. Prime numbers and factors (divisors), including greatest common divisors and least common multiples. Simple applications of ratio, percentage and proportion, linked to similarity. Definition and elementary treatment of absolute value (modulus), $\lvert a \rvert$. Rounding, decimal approximations and significant figures, including appreciation of errors. Expression of numbers in standard form (scientific notation), that is, $a \times 10^k$, $1 \le a < 10$, $k \in \mathbb{Z}$.
Sets and numbers	Concept and notation of sets, elements, universal (reference) set, empty (null) set, complement, subset, equality of sets, disjoint sets. Operations on sets: union and intersection. Commutative, associative and distributive properties. Venn diagrams. Number systems: natural numbers; integers, $\mathbb{Z}$; rationals, $\mathbb{Q}$, and irrationals; real numbers, $\mathbb{R}$. Intervals on the real number line using set notation and using inequalities. Expressing the solution set of a linear inequality on the number line and in set notation. Mappings of the elements of one set to another; sets of ordered pairs.

Topic	Content
Algebra	Manipulation of linear and quadratic expressions, including factorization, expansion, completing the square and use of the formula. Rearrangement, evaluation and combination of simple formulae. Examples from other subject areas, particularly the sciences, should be included. Linear functions, their graphs, gradients and y-intercepts. Addition and subtraction of simple algebraic fractions. The properties of order relations: $<$, $\leq$, $>$, $\geq$. Solution of linear equations and inequalities in one variable, including cases with rational coefficients. Solution of quadratic equations and inequalities, using factorization and completing the square. Solution of simultaneous linear equations in two variables.
Trigonometry	Angle measurement in degrees. Compass directions. Right-angle trigonometry. Simple applications for solving triangles. Pythagoras' theorem and its converse.
Geometry	Simple geometric transformations: translation, reflection, rotation, enlargement. Congruence and similarity, including the concept of scale factor of an enlargement. The circle, its centre and radius, area and circumference. The terms arc, sector, chord, tangent and segment. Perimeter and area of plane figures. Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapeziums (trapezoids); compound shapes. Volumes of cuboids, pyramids, spheres, cylinders and cones. Classification of prisms and pyramids, including tetrahedra.
Coordinate geometry	Elementary geometry of the plane, including the concepts of dimension for point, line, plane and space. The equation of a line in the form $y = mx + c$. Parallel and perpendicular lines, including $m_1 = m_2$ and $m_1 m_2 = -1$. The Cartesian plane: ordered pairs (x, y), origin, axes. Mid-point of a line segment and distance between two points in the Cartesian plane.
Statistics and probability	Descriptive statistics: collection of raw data, display of data in pictorial and diagrammatic forms, including frequency histograms, cumulative frequency graphs. Obtaining simple statistics from discrete and continuous data, including mean, median, mode, quartiles, range, interquartile range and percentiles. Calculating probabilities of simple events.

Syllabus content

Topic 1—Core: Algebra 30 hours

The aim of this topic is to introduce students to some basic algebraic concepts and applications.

	Content	Further guidance	Links
1.1	Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series. Sigma notation. Applications.	Sequences can be generated and displayed in several ways, including recursive functions. Link infinite geometric series with limits of convergence in 6.1. Examples include compound interest and population growth.	**Int:** The chess legend (Sissa ibn Dahir). **Int:** Aryabhatta is sometimes considered the “father of algebra”. Compare with al-Khawarizmi. **Int:** The use of several alphabets in mathematical notation (eg first term and common difference of an arithmetic sequence). **TOK:** Mathematics and the knower. To what extent should mathematical knowledge be consistent with our intuition? **TOK:** Mathematics and the world. Some mathematical constants (π , e, ϕ , Fibonacci numbers) appear consistently in nature. What does this tell us about mathematical knowledge? **TOK:** Mathematics and the knower. How is mathematical intuition used as a basis for formal proof? (Gauss’ method for adding up integers from 1 to 100.) *(continued)*

	Content	Further guidance	Links
			(see notes above) **Aim 8:** Short-term loans at high interest rates. How can knowledge of mathematics result in individuals being exploited or protected from extortion? **Appl:** Physics 7.2, 13.2 (radioactive decay and nuclear physics).
1.2	Exponents and logarithms. Laws of exponents; laws of logarithms. Change of base.	Exponents and logarithms are further developed in 2.4.	**Appl:** Chemistry 18.1, 18.2 (calculation of pH and buffer solutions). **TOK:** The nature of mathematics and science. Were logarithms an invention or discovery? (This topic is an opportunity for teachers and students to reflect on “the nature of mathematics”.)
1.3	Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{N}$. **Not required:** Permutations where some objects are identical. Circular arrangements. Proof of binomial theorem.	The ability to find $\binom{n}{r}$ and ${}^nP_r$ using both the formula and technology is expected. Link to 5.4. Link to 5.6, binomial distribution.	**TOK:** The nature of mathematics. The unforeseen links between Pascal’s triangle, counting methods and the coefficients of polynomials. Is there an underlying truth that can be found linking these? **Int:** The properties of Pascal’s triangle were known in a number of different cultures long before Pascal (eg the Chinese mathematician Yang Hui). **Aim 8:** How many different tickets are possible in a lottery? What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

	Content	Further guidance	Links
1.4	Proof by mathematical induction.	Links to a wide variety of topics, for example, complex numbers, differentiation, sums of series and divisibility.	**TOK:** Nature of mathematics and science. What are the different meanings of induction in mathematics and science? **TOK:** Knowledge claims in mathematics. Do proofs provide us with completely certain knowledge? **TOK:** Knowledge communities. Who judges the validity of a proof?
1.5	Complex numbers: the number $\mathrm{i} = \sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument. Cartesian form $z = a + \mathrm{i}b$. Sums, products and quotients of complex numbers.	When solving problems, students may need to use technology.	**Appl:** Concepts in electrical engineering. Impedance as a combination of resistance and reactance; also apparent power as a combination of real and reactive powers. These combinations take the form $z = a + \mathrm{i}b$. **TOK:** Mathematics and the knower. Do the words imaginary and complex make the concepts more difficult than if they had different names? **TOK:** The nature of mathematics. Has “i” been invented or was it discovered? **TOK:** Mathematics and the world. Why does “i” appear in so many fundamental laws of physics?

	Content	Further guidance	Links
1.6	Modulus–argument (polar) form $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{cis}\,\theta = r\mathrm{e}^{\mathrm{i}\theta}$. The complex plane.	$r\mathrm{e}^{\mathrm{i}\theta}$ is also known as Euler's form. The ability to convert between forms is expected. The complex plane is also known as the Argand diagram.	**Appl:** Concepts in electrical engineering. Phase angle/shift, power factor and apparent power as a complex quantity in polar form. **TOK:** The nature of mathematics. Was the complex plane already there before it was used to represent complex numbers geometrically? **TOK:** Mathematics and the knower. Why might it be said that $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$ is beautiful?
1.7	Powers of complex numbers: de Moivre's theorem. n^{th} roots of a complex number.	Proof by mathematical induction for $n \in \mathbb{Z}^+$.	**TOK:** Reason and mathematics. What is mathematical reasoning and what role does proof play in this form of reasoning? Are there examples of proof that are not mathematical?
1.8	Conjugate roots of polynomial equations with real coefficients.	Link to 2.5 and 2.7.	
1.9	Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases where there is a unique solution, an infinity of solutions or no solution.	These systems should be solved using both algebraic and technological methods, eg row reduction. Systems that have solution(s) may be referred to as consistent. When a system has an infinity of solutions, a general solution may be required. Link to vectors in 4.7.	**TOK:** Mathematics, sense, perception and reason. If we can find solutions in higher dimensions, can we reason that these spaces exist beyond our sense perception?

Topic 2—Core: Functions and equations

22 hours

The aims of this topic are to explore the notion of function as a unifying theme in mathematics, and to apply functional methods to a variety of mathematical situations. It is expected that extensive use will be made of technology in both the development and the application of this topic.

	Content	Further guidance	Links
2.1	Concept of function $f : x \mapsto f(x)$: domain, range; image (value).		**Int:** The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries. How did the notation we use today become internationally accepted?
	Odd and even functions.		
	Composite functions $f \circ g$.	$(f \circ g)(x) = f(g(x))$. Link with 6.2.	**TOK:** The nature of mathematics. Is mathematics simply the manipulation of symbols under a set of formal rules?
	Identity function.		
	One-to-one and many-to-one functions.	Link with 3.4.	
	Inverse function f^{-1}, including domain restriction. Self-inverse functions.	Link with 6.2.	

	Content	Further guidance	Links
2.2	The graph of a function; its equation $y = f(x)$. Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes and symmetry, and consideration of domain and range. The graphs of the functions $y = \lvert f(x) \rvert$ and $y = f(\lvert x \rvert)$. The graph of $y = \dfrac{1}{f(x)}$ given the graph of $y = f(x)$.	 Use of technology to graph a variety of functions.	**TOK:** Mathematics and knowledge claims. Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically (analytically)? **Appl:** Sketching and interpreting graphs; Geography SL/HL (geographic skills); Chemistry 11.3.1. **Int:** Bourbaki group analytical approach versus Mandlebrot visual approach.
2.3	Transformations of graphs: translations; stretches; reflections in the axes. The graph of the inverse function as a reflection in $y = x$.	Link to 3.4. Students are expected to be aware of the effect of transformations on both the algebraic expression and the graph of a function.	**Appl:** Economics SL/HL 1.1 (shift in demand and supply curves).
2.4	The rational function $x \mapsto \dfrac{ax+b}{cx+d}$, and its graph. The function $x \mapsto a^x$, $a > 0$, and its graph. The function $x \mapsto \log_a x$, $x > 0$, and its graph.	The reciprocal function is a particular case. Graphs should include both asymptotes and any intercepts with axes. Exponential and logarithmic functions as inverses of each other. Link to 6.2 and the significance of e. Application of concepts in 2.1, 2.2 and 2.3.	 **Appl:** Geography SL/HL (geographic skills); Physics SL/HL 7.2 (radioactive decay); Chemistry SL/HL 16.3 (activation energy); Economics SL/HL 3.2 (exchange rates).

	Content	Further guidance	Links
2.5	Polynomial functions and their graphs. The factor and remainder theorems. The fundamental theorem of algebra.	The graphical significance of repeated factors. The relationship between the degree of a polynomial function and the possible numbers of x-intercepts.	
2.6	Solving quadratic equations using the quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$ to determine the nature of the roots. Solving polynomial equations both graphically and algebraically. Sum and product of the roots of polynomial equations. Solution of $a^x = b$ using logarithms. Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.	May be referred to as roots of equations or zeros of functions. Link the solution of polynomial equations to conjugate roots in 1.8. For the polynomial equation $\sum_{r=0}^{n} a_r x^r = 0$, the sum is $\frac{-a_{n-1}}{a_n}$, the product is $\frac{(-1)^n a_0}{a_n}$.	**Appl:** Chemistry 17.2 (equilibrium law). **Appl:** Physics 2.1 (kinematics). **Appl:** Physics 4.2 (energy changes in simple harmonic motion). **Appl:** Physics (HL only) 9.1 (projectile motion). **Aim 8:** The phrase “exponential growth” is used popularly to describe a number of phenomena. Is this a misleading use of a mathematical term?

	Content	Further guidance	Links
2.7	Solutions of $g(x) \geq f(x)$. Graphical or algebraic methods, for simple polynomials up to degree 3. Use of technology for these and other functions.		

Topic 3—Core: Circular functions and trigonometry — 22 hours

The aims of this topic are to explore the circular functions, to introduce some important trigonometric identities and to solve triangles using trigonometry.

On examination papers, radian measure should be assumed unless otherwise indicated, for example, by $x \mapsto \sin x^{\circ}$.

	Content	Further guidance	Links
3.1	The circle: radian measure of angles. Length of an arc; area of a sector.	Radian measure may be expressed as multiples of π, or decimals. Link with 6.2.	**Int:** The origin of degrees in the mathematics of Mesopotamia and why we use minutes and seconds for time. **TOK:** Mathematics and the knower. Why do we use radians? (The arbitrary nature of degree measure versus radians as real numbers and the implications of using these two measures on the shape of sinusoidal graphs.) **TOK:** Mathematics and knowledge claims. If trigonometry is based on right triangles, how can we sensibly consider trigonometric ratios of angles greater than a right angle? **Int:** The origin of the word "sine". **Appl:** Physics SL/HL 2.2 (forces and dynamics). **Appl:** Triangulation used in the Global Positioning System (GPS). **Int:** Why did Pythagoras link the study of music and mathematics? **Appl:** Concepts in electrical engineering. Generation of sinusoidal voltage.
3.2	Definition of $\cos\theta$, $\sin\theta$ and $\tan\theta$ in terms of the unit circle. Exact values of sin, cos and tan of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples. Definition of the reciprocal trigonometric ratios $\sec\theta$, $\csc\theta$ and $\cot\theta$. Pythagorean identities: $\cos^2\theta + \sin^2\theta = 1$; $1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$.		
3.3	Compound angle identities. Double angle identities. **Not required:** Proof of compound angle identities.	Derivation of double angle identities from compound angle identities. Finding possible values of trigonometric ratios without finding θ, for example, finding $\sin 2\theta$ given $\sin\theta$.	

(continued)

	Content	Further guidance	Links
3.4	Composite functions of the form $f(x) = a\sin(b(x+c)) + d$. Applications.		*(see notes above)* **TOK:** Mathematics and the world. Music can be expressed using mathematics. Does this mean that music is mathematical, that mathematics is musical or that both are reflections of a common "truth"? **Appl:** Physics SL/HL 4.1 (kinematics of simple harmonic motion).
3.5	The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.		
3.6	Algebraic and graphical methods of solving trigonometric equations in a finite interval, including the use of trigonometric identities and factorization. **Not required:** The general solution of trigonometric equations.		**TOK:** Mathematics and knowledge claims. How can there be an infinite number of discrete solutions to an equation?
3.7	The cosine rule The sine rule including the ambiguous case. Area of a triangle as $\frac{1}{2}ab\sin C$.		**TOK:** Nature of mathematics. If the angles of a triangle can add up to less than 180°, 180° or more than 180°, what does this tell us about the "fact" of the angle sum of a triangle and about the nature of mathematical knowledge?
	Applications.	Examples include navigation, problems in two and three dimensions, including angles of elevation and depression.	**Appl:** Physics SL/HL 1.3 (vectors and scalars); Physics SL/HL 2.2 (forces and dynamics). **Int:** The use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

Topic 4—Core: Vectors

24 hours

The aim of this topic is to introduce the use of vectors in two and three dimensions, and to facilitate solving problems involving points, lines and planes.

	Content	Further guidance	Links
4.1	Concept of a vector. Representation of vectors using directed line segments. Unit vectors; base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. Components of a vector: $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$.		**Aim 8:** Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb. **Appl:** Physics SL/HL 1.3 (vectors and scalars); Physics SL/HL 2.2 (forces and dynamics). **TOK:** Mathematics and knowledge claims. You can perform some proofs using different mathematical concepts. What does this tell us about mathematical knowledge?
	Algebraic and geometric approaches to the following: • the sum and difference of two vectors; • the zero vector $\boldsymbol{0}$, the vector $-\boldsymbol{v}$; • multiplication by a scalar, $k\boldsymbol{v}$; • magnitude of a vector, $\lvert\boldsymbol{v}\rvert$; • position vectors $\overrightarrow{OA} = \boldsymbol{a}$.	Proofs of geometrical properties using vectors.	
	$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$	Distance between points A and B is the magnitude of $\overrightarrow{AB}$.	

	Content	Further guidance	Links
4.2	The definition of the scalar product of two vectors. Properties of the scalar product: $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$; $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$; $(k\boldsymbol{v}) \cdot \boldsymbol{w} = k(\boldsymbol{v} \cdot \boldsymbol{w})$; $\boldsymbol{v} \cdot \boldsymbol{v} = \lvert\boldsymbol{v}\rvert^2$. The angle between two vectors. Perpendicular vectors; parallel vectors.	$\boldsymbol{v} \cdot \boldsymbol{w} = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Link to 3.6. For non-zero vectors, $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ is equivalent to the vectors being perpendicular. For parallel vectors, $\lvert\boldsymbol{v} \cdot \boldsymbol{w}\rvert = \lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert$.	**Appl:** Physics SL/HL 2.2 (forces and dynamics). **TOK:** The nature of mathematics. Why this definition of scalar product?
4.3	Vector equation of a line in two and three dimensions: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$. Simple applications to kinematics. The angle between two lines.	Knowledge of the following forms for equations of lines. Parametric form: $x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$. Cartesian form: $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$.	**Appl:** Modelling linear motion in three dimensions. **Appl:** Navigational devices, eg GPS. **TOK:** The nature of mathematics. Why might it be argued that vector representation of lines is superior to Cartesian?
4.4	Coincident, parallel, intersecting and skew lines; distinguishing between these cases. Points of intersection.		

	Content	Further guidance	Links
4.5	The definition of the vector product of two vectors. Properties of the vector product: $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$; $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$; $(k\boldsymbol{v}) \times \boldsymbol{w} = k(\boldsymbol{v} \times \boldsymbol{w})$; $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$. Geometric interpretation of $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$.	$\boldsymbol{v} \times \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta \, \boldsymbol{n}$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ and $\boldsymbol{n}$ is the unit normal vector whose direction is given by the right-hand screw rule. Areas of triangles and parallelograms.	**Appl:** Physics SL/HL 6.3 (magnetic force and field).
4.6	Vector equation of a plane $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$. Use of normal vector to obtain the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$. Cartesian equation of a plane $ax + by + cz = d$.		
4.7	Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.	Link to 1.9. Geometrical interpretation of solutions.	**TOK:** Mathematics and the knower. Why are symbolic representations of three-dimensional objects easier to deal with than visual representations? What does this tell us about our knowledge of mathematics in other dimensions?

Topic 5—Core: Statistics and probability

36 hours

The aim of this topic is to introduce basic concepts. It may be considered as three parts: manipulation and presentation of statistical data (5.1), the laws of probability (5.2–5.4), and random variables and their probability distributions (5.5–5.7). It is expected that most of the calculations required will be done on a GDC. The emphasis is on understanding and interpreting the results obtained. Statistical tables will no longer be allowed in examinations.

	Content	Further guidance	Links
5.1	Concepts of population, sample, random sample and frequency distribution of discrete and continuous data. Grouped data: mid-interval values, interval width, upper and lower interval boundaries. Mean, variance, standard deviation. **Not required:** Estimation of mean and variance of a population from a sample.	For examination purposes, in papers 1 and 2 data will be treated as the population. In examinations the following formulae should be used: $\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n}$, $\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$.	**TOK:** The nature of mathematics. Why have mathematics and statistics sometimes been treated as separate subjects? **TOK:** The nature of knowing. Is there a difference between information and data? **Aim 8:** Does the use of statistics lead to an overemphasis on attributes that can easily be measured over those that cannot? **Appl:** Psychology SL/HL (descriptive statistics); Geography SL/HL (geographic skills); Biology SL/HL 1.1.2 (statistical analysis). **Appl:** Methods of collecting data in real life (census versus sampling). **Appl:** Misleading statistics in media reports.

	Content	Further guidance	Links
5.2	Concepts of trial, outcome, equally likely outcomes, sample space (U) and event. The probability of an event A as $\mathrm{P}(A)=\frac{n(A)}{n(U)}$. The complementary events A and A' (not A). Use of Venn diagrams, tree diagrams, counting principles and tables of outcomes to solve problems.		**Aim 8:** Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (eg economics)? **Int:** The development of the mathematical theory of probability in 17th century France.
5.3	Combined events; the formula for $\mathrm{P}(A \cup B)$. Mutually exclusive events.		
5.4	Conditional probability; the definition $\mathrm{P}(A \mid B)=\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$. Independent events; the definition $\mathrm{P}(A \mid B)=\mathrm{P}(A)=\mathrm{P}(A \mid B')$. Use of Bayes' theorem for a maximum of three events.	Use of $\mathrm{P}(A \cap B)=\mathrm{P}(A)\mathrm{P}(B)$ to show independence.	**Appl:** Use of probability methods in medical studies to assess risk factors for certain diseases. **TOK:** Mathematics and knowledge claims. Is independence as defined in probabilistic terms the same as that found in normal experience?

	Content	Further guidance	Links
5.5	Concept of discrete and continuous random variables and their probability distributions. Definition and use of probability density functions. Expected value (mean), mode, median, variance and standard deviation. Applications.	For a continuous random variable, a value at which the probability density function has a maximum value is called a mode. Examples include games of chance.	**TOK:** Mathematics and the knower. To what extent can we trust samples of data? **Appl:** Expected gain to insurance companies.
5.6	Binomial distribution, its mean and variance. Poisson distribution, its mean and variance. **Not required:** Formal proof of means and variances.	Link to binomial theorem in 1.3. Conditions under which random variables have these distributions.	**TOK:** Mathematics and the real world. Is the binomial distribution ever a useful model for an actual real-world situation?
5.7	Normal distribution. Properties of the normal distribution. Standardization of normal variables.	Probabilities and values of the variable must be found using technology. The standardized value (z) gives the number of standard deviations from the mean. Link to 2.3.	**Appl:** Chemistry SL/HL 6.2 (collision theory); Psychology HL (descriptive statistics); Biology SL/HL 1.1.3 (statistical analysis). **Aim 8:** Why might the misuse of the normal distribution lead to dangerous inferences and conclusions? **TOK:** Mathematics and knowledge claims. To what extent can we trust mathematical models such as the normal distribution? **Int:** De Moivre's derivation of the normal distribution and Quetelet's use of it to describe *l'homme moyen*.

Topic 6—Core: Calculus

48 hours

The aim of this topic is to introduce students to the basic concepts and techniques of differential and integral calculus and their application.

	Content	Further guidance	Links
6.1	Informal ideas of limit, continuity and convergence. Definition of derivative from first principles $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. The derivative interpreted as a gradient function and as a rate of change. Finding equations of tangents and normals. Identifying increasing and decreasing functions. The second derivative. Higher derivatives.	Include result $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. Link to 1.1. Use of this definition for polynomials only. Link to binomial theorem in 1.3. Both forms of notation, $\frac{dy}{dx}$ and $f'(x)$, for the first derivative. Use of both algebra and technology. Both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$, for the second derivative. Familiarity with the notation $\frac{d^ny}{dx^n}$ and $f^{(n)}(x)$. Link with induction in 1.4.	**TOK:** The nature of mathematics. Does the fact that Leibniz and Newton came across the calculus at similar times support the argument that mathematics exists prior to its discovery? **Int:** How the Greeks' distrust of zero meant that Archimedes' work did not lead to calculus. **Int:** Investigate attempts by Indian mathematicians (500–1000 CE) to explain division by zero. **TOK:** Mathematics and the knower. What does the dispute between Newton and Leibniz tell us about human emotion and mathematical discovery? **Appl:** Economics HL 1.5 (theory of the firm); Chemistry SL/HL 11.3.4 (graphical techniques); Physics SL/HL 2.1 (kinematics).

	Content	Further guidance	Links
6.2	Derivatives of x^n, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$. Differentiation of sums and multiples of functions. The product and quotient rules. The chain rule for composite functions. Related rates of change. Implicit differentiation. Derivatives of $\sec x$, $\csc x$, $\cot x$, a^x, $\log_a x$, $\arcsin x$, $\arccos x$ and $\arctan x$.		**Appl:** Physics HL 2.4 (uniform circular motion); Physics 12.1 (induced electromotive force (emf)). **TOK:** Mathematics and knowledge claims. Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work. What does this tell us about the importance of proof and the nature of mathematics? **TOK:** Mathematics and the real world. The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats, such as getting a man on the Moon. What does this tell us about the links between mathematical models and physical reality?
6.3	Local maximum and minimum values. Optimization problems. Points of inflexion with zero and non-zero gradients. Graphical behaviour of functions, including the relationship between the graphs of f, f' and f''. **Not required:** Points of inflexion, where $f''(x)$ is not defined, for example, $y = x^{1/3}$ at $(0,0)$.	Testing for the maximum or minimum using the change of sign of the first derivative and using the sign of the second derivative. Use of the terms "concave up" for $f''(x) > 0$, "concave down" for $f''(x) < 0$. At a point of inflexion, $f''(x) = 0$ and changes sign (concavity change).	

	Content	Further guidance	Links
6.4	Indefinite integration as anti-differentiation. Indefinite integral of x^n, $\sin x$, $\cos x$ and e^x. Other indefinite integrals using the results from 6.2. The composites of any of these with a linear function.	Indefinite integral interpreted as a family of curves. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$. Examples include $\int (2x-1)^5\,dx$, $\int \frac{1}{3x+4}\,dx$ and $\int \frac{1}{x^2+2x+5}\,dx$.	
6.5	Anti-differentiation with a boundary condition to determine the constant of integration. Definite integrals. Area of the region enclosed by a curve and the x-axis or y-axis in a given interval; areas of regions enclosed by curves. Volumes of revolution about the x-axis or y-axis.	The value of some definite integrals can only be found using technology.	**Appl:** Industrial design.

	Content	Further guidance	Links
6.6	Kinematic problems involving displacement s, velocity v and acceleration a. Total distance travelled.	$v = \frac{\mathrm{d}s}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = v\frac{\mathrm{d}v}{\mathrm{d}s}$. Total distance travelled $= \int_{t_1}^{t_2} \lvert v \rvert \,\mathrm{d}t$.	**Appl:** Physics HL 2.1 (kinematics). **Int:** Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage? Who decides what is mathematics?
6.7	Integration by substitution Integration by parts.	On examination papers, non-standard substitutions will be provided. Link to 6.2. Examples: $\int x \sin x \,\mathrm{d}x$ and $\int \ln x \,\mathrm{d}x$. Repeated integration by parts. Examples: $\int x^2 \mathrm{e}^x \,\mathrm{d}x$ and $\int \mathrm{e}^x \sin x \,\mathrm{d}x$.	

Topic 7—Option: Statistics and probability 48 hours

The aims of this option are to allow students the opportunity to approach statistics in a practical way; to demonstrate a good level of statistical understanding; and to understand which situations apply and to interpret the given results. It is expected that GDCs will be used throughout this option, and that the minimum requirement of a GDC will be to find probability distribution function (pdf), cumulative distribution function (cdf), inverse cumulative distribution function, *p*-values and test statistics, including calculations for the following distributions: binomial, Poisson, normal and *t*. Students are expected to set up the problem mathematically and then read the answers from the GDC, indicating this within their written answers. Calculator-specific or brand-specific language should not be used within these explanations.

	Content	Further guidance	Links
7.1	Cumulative distribution functions for both discrete and continuous distributions.		
	Geometric distribution.		
	Negative binomial distribution.		**Int:** Also known as Pascal's distribution.
	Probability generating functions for discrete random variables.	$G(t) = \mathrm{E}(t^X) = \sum_x P(X = x)t^x$.	
	Using probability generating functions to find mean, variance and the distribution of the sum of *n* independent random variables.		**Aim 8:** Statistical compression of data files.
7.2	Linear transformation of a single random variable.	$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$,	
	Mean of linear combinations of *n* random variables.	$\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.	
	Variance of linear combinations of *n* independent random variables.		
	Expectation of the product of independent random variables.	$\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$.	

	Content	Further guidance	Links
7.3	Unbiased estimators and estimates. Comparison of unbiased estimators based on variances. $\overline{X}$ as an unbiased estimator for μ. S^2 as an unbiased estimator for σ^2.	T is an unbiased estimator for the parameter θ if $\mathrm{E}(T)=\theta$. T_1 is a more efficient estimator than T_2 if $\mathrm{Var}(T_1)<\mathrm{Var}(T_2)$. $\overline{X}=\sum_{i=1}^{n}\frac{X_i}{n}$. $S^2=\sum_{i=1}^{n}\frac{\left(X_i-\overline{X}\right)^2}{n-1}$.	**TOK**: Mathematics and the world. In the absence of knowing the value of a parameter, will an unbiased estimator always be better than a biased one?
7.4	A linear combination of independent normal random variables is normally distributed. In particular, $X\sim \mathrm{N}(\mu,\sigma^2)\Rightarrow \overline{X}\sim \mathrm{N}\left(\mu,\frac{\sigma^2}{n}\right)$. The central limit theorem.		**Aim 8/TOK:** Mathematics and the world. "Without the central limit theorem, there could be no statistics of any value within the human sciences." **TOK:** Nature of mathematics. The central limit theorem can be proved mathematically (formalism), but its truth can be confirmed by its applications (empiricism).

	Content	Further guidance	Links
7.5	Confidence intervals for the mean of a normal population.	Use of the normal distribution when σ is known and use of the *t*-distribution when σ is unknown, regardless of sample size. The case of matched pairs is to be treated as an example of a single sample technique.	**TOK:** Mathematics and the world. Claiming brand A is "better" on average than brand B can mean very little if there is a large overlap between the confidence intervals of the two means. **Appl:** Geography.
7.6	Null and alternative hypotheses, H_0 and H_1. Significance level. Critical regions, critical values, *p*-values, one-tailed and two-tailed tests. Type I and II errors, including calculations of their probabilities. Testing hypotheses for the mean of a normal population.	Use of the normal distribution when σ is known and use of the t-distribution when σ is unknown, regardless of sample size. The case of matched pairs is to be treated as an example of a single sample technique.	**TOK:** Mathematics and the world. In practical terms, is saying that a result is significant the same as saying that it is true? **TOK:** Mathematics and the world. Does the ability to test only certain parameters in a population affect the way knowledge claims in the human sciences are valued? **Appl:** When is it more important not to make a Type I error and when is it more important not to make a Type II error?

	Content	Further guidance	Links
7.7	Introduction to bivariate distributions.	Informal discussion of commonly occurring situations, eg marks in pure mathematics and statistics exams taken by a class of students, salary and age of teachers in a certain school. The need for a measure of association between the variables and the possibility of predicting the value of one of the variables given the value of the other variable.	**Appl:** Geographic skills. **Aim 8:** The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.
	Covariance and (population) product moment correlation coefficient ρ.	$\mathrm{Cov}(X,Y)=\mathrm{E}[(X-\mu_x)(Y-\mu_y)]$ $=\mathrm{E}(XY)-\mu_x\mu_y$, where $\mu_x=\mathrm{E}(X),\ \mu_y=\mathrm{E}(Y)$. $\rho=\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$.	**Appl:** Using technology to fit a range of curves to a set of data.
	Proof that $\rho = 0$ in the case of independence and ± 1 in the case of a linear relationship between X and Y.	The use of ρ as a measure of association between X and Y, with values near 0 indicating a weak association and values near +1 or near –1 indicating a strong association.	**TOK:** Mathematics and the world. Given that a set of data may be approximately fitted by a range of curves, where would we seek for knowledge of which equation is the "true" model?
	Definition of the (sample) product moment correlation coefficient R in terms of n paired observations on X and Y. Its application to the estimation of ρ.	$R=\dfrac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2\sum_{i=1}^{n}(Y_i-\bar{Y})^2}}$ $=\dfrac{\sum_{i=1}^{n}X_iY_i-n\bar{X}\bar{Y}}{\sqrt{\left(\sum_{i=1}^{n}X_i^2-n\bar{X}^2\right)\left(\sum Y_i^2-n\bar{Y}^2\right)}}$.	**Aim 8:** The physicist Frank Oppenheimer wrote: "Prediction is dependent only on the assumption that observed patterns will be repeated." This is the danger of extrapolation. There are many examples of its failure in the past, eg share prices, the spread of disease, climate change. *(continued)*

	Content	Further guidance	Links
	Informal interpretation of r, the observed value of R. Scatter diagrams.	Values of r near 0 indicate a weak association between X and Y, and values near ± 1 indicate a strong association.	*(see notes above)*
	The following topics are based on the assumption of bivariate normality.	It is expected that the GDC will be used wherever possible in the following work.	
	Use of the t-statistic to test the null hypothesis $\rho = 0$.	$R\sqrt{\frac{n-2}{1-R^2}}$ has the student's t-distribution with $(n-2)$ degrees of freedom.	
	Knowledge of the facts that the regression of X on Y $\left(\mathrm{E}(X) \mid Y = y\right)$ and Y on X $\left(\mathrm{E}(Y) \mid X = x\right)$ are linear. Least-squares estimates of these regression lines (proof not required). The use of these regression lines to predict the value of one of the variables given the value of the other.	$x-\bar{x} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(y_i-\bar{y})^2}\right)(y-\bar{y})$ $= \left(\frac{\sum_{i=1}^{n}x_i y_i - n\bar{x}\,\bar{y}}{\sum_{i=1}^{n}y_i^2 - n\bar{y}^2}\right)(y-\bar{y})$, $y-\bar{y} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)(x-\bar{x})$ $= \left(\frac{\sum_{i=1}^{n}x_i y_i - n\bar{x}\,\bar{y}}{\sum_{i=1}^{n}x_i^2 - n\bar{x}^2}\right)(x-\bar{x})$.	

Topic 8—Option: Sets, relations and groups 48 hours

The aims of this option are to provide the opportunity to study some important mathematical concepts, and introduce the principles of proof through abstract algebra.

	Content	Further guidance	Links
8.1	Finite and infinite sets. Subsets.		**TOK:** Cantor theory of transfinite numbers, Russell's paradox, Godel's incompleteness theorems.
	Operations on sets: union; intersection; complement; set difference; symmetric difference.		
	De Morgan's laws: distributive, associative and commutative laws (for union and intersection).	Illustration of these laws using Venn diagrams. Students may be asked to prove that two sets are the same by establishing that $A \subseteq B$ and $B \subseteq A$.	**Appl:** Logic, Boolean algebra, computer circuits.
8.2	Ordered pairs: the Cartesian product of two sets.		
	Relations: equivalence relations; equivalence classes.	An equivalence relation on a set forms a partition of the set.	**Appl**, **Int:** Scottish clans.
8.3	Functions: injections; surjections; bijections.	The term codomain.	
	Composition of functions and inverse functions.	Knowledge that the function composition is not a commutative operation and that if f is a bijection from set A to set B then f^{-1} exists and is a bijection from set B to set A.	

	Content	Further guidance	Links
8.4	Binary operations. Operation tables (Cayley tables).	A binary operation $*$ on a non-empty set S is a rule for combining any two elements $a, b \in S$ to give a unique element c. That is, in this definition, a binary operation on a set is not necessarily closed.	
8.5	Binary operations: associative, distributive and commutative properties.	The arithmetic operations on $\mathbb{R}$ and $\mathbb{C}$. Examples of distributivity could include the fact that, on $\mathbb{R}$, multiplication is distributive over addition but addition is not distributive over multiplication.	**TOK:** Which are more fundamental, the general models or the familiar examples?
8.6	The identity element e. The inverse a^{-1} of an element a. Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse. Proofs of the uniqueness of the identity and inverse elements.	Both the right-identity $a * e = a$ and left-identity $e * a = a$ must hold if e is an identity element. Both $a * a^{-1} = e$ and $a^{-1} * a = e$ must hold.	

	Content	Further guidance	Links
8.7	The definition of a group $\{G, *\}$. The operation table of a group is a Latin square, but the converse is false.	For the set G under a given operation $*$: • G is closed under $*$; • $*$ is associative; • G contains an identity element; • each element in G has an inverse in G.	**Appl:** Existence of formula for roots of polynomials. **Appl:** Galois theory for the impossibility of such formulae for polynomials of degree 5 or higher.
	Abelian groups.	$a * b = b * a$, for all $a, b \in G$.	
8.8	Examples of groups: • $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition; • integers under addition modulo n; • non-zero integers under multiplication, modulo p, where p is prime;		**Appl:** Rubik's cube, time measures, crystal structure, symmetries of molecules, strut and cable constructions, Physics H2.2 (special relativity), the 8–fold way, supersymmetry.
	symmetries of plane figures, including equilateral triangles and rectangles; invertible functions under composition of functions.	The composition $T_2 \circ T_1$ denotes T_1 followed by T_2.	
8.9	The order of a group. The order of a group element.		
	Cyclic groups. Generators. Proof that all cyclic groups are Abelian.		**Appl:** Music circle of fifths, prime numbers.

	Content	Further guidance	Links
8.10	Permutations under composition of permutations. Cycle notation for permutations. Result that every permutation can be written as a composition of disjoint cycles. The order of a combination of cycles.	On examination papers: the form $p=\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$ or in cycle notation (132) will be used to represent the permutation $1\to 3$, $2\to 1$, $3\to 2$.	**Appl:** Cryptography, campanology.
8.11	Subgroups, proper subgroups.	A proper subgroup is neither the group itself nor the subgroup containing only the identity element.	
	Use and proof of subgroup tests.	Suppose that $\{G,*\}$ is a group and H is a non-empty subset of G. Then $\{H,*\}$ is a subgroup of $\{G,*\}$ if $a*b^{-1}\in H$ whenever $a,b\in H$. Suppose that $\{G,*\}$ is a finite group and H is a non-empty subset of G. Then $\{H,*\}$ is a subgroup of $\{G,*\}$ if H is closed under $*$.	
	Definition and examples of left and right cosets of a subgroup of a group.		
	Lagrange's theorem. Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)		**Appl:** Prime factorization, symmetry breaking.

	Content	Further guidance	Links
8.12	Definition of a group homomorphism.	Infinite groups as well as finite groups. Let $\{G,*\}$ and $\{H,\circ\}$ be groups, then the function $f:G\to H$ is a homomorphism if $f(a*b)=f(a)\circ f(b)$ for all $a,b\in G$.	
	Definition of the kernel of a homomorphism. Proof that the kernel and range of a homomorphism are subgroups.	If $f:G\to H$ is a group homomorphism, then $\mathrm{Ker}(f)$ is the set of $a\in G$ such that $f(a)=e_H$.	
	Proof of homomorphism properties for identities and inverses.	Identity: let e_G and e_H be the identity elements of $(G,*)$ and $(H,\circ)$, respectively, then $f(e_G)=e_H$. Inverse: $f(a^{-1})=\left(f(a)\right)^{-1}$ for all $a\in G$.	
	Isomorphism of groups.	Infinite groups as well as finite groups. The homomorphism $f:G\to H$ is an isomorphism if f is bijective.	
	The order of an element is unchanged by an isomorphism.		

Topic 9—Option: Calculus

48 hours

The aims of this option are to introduce limit theorems and convergence of series, and to use calculus results to solve differential equations.

	Content	Further guidance	Links
9.1	Infinite sequences of real numbers and their convergence or divergence.	Informal treatment of limit of sum, difference, product, quotient; squeeze theorem. Divergent is taken to mean not convergent.	**TOK:** Zeno's paradox, impact of infinite sequences and limits on our understanding of the physical world.
9.2	Convergence of infinite series. Tests for convergence: comparison test; limit comparison test; ratio test; integral test.	The sum of a series is the limit of the sequence of its partial sums. Students should be aware that if $\lim\limits_{x \to \infty} x_n = 0$ then the series is not necessarily convergent, but if $\lim\limits_{x \to \infty} x_n \neq 0$, the series diverges.	**TOK:** Euler's idea that $1-1+1-1+\ldots=\frac{1}{2}$. Was it a mistake or just an alternative view?
	The *p*-series, $\sum \frac{1}{n^p}$.	$\sum \frac{1}{n^p}$ is convergent for $p>1$ and divergent otherwise. When $p=1$, this is the harmonic series.	
	Series that converge absolutely. Series that converge conditionally.	Conditions for convergence.	
	Alternating series. Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.	The absolute value of the truncation error is less than the next term in the series.	

	Content	Further guidance	Links
9.3	Continuity and differentiability of a function at a point. Continuous functions and differentiable functions.	Test for continuity: $\lim\limits_{x\to a-} f(x) = f(a) = \lim\limits_{x\to a+} f(x)$. Test for differentiability: f is continuous at a and $\lim\limits_{h\to 0-} \dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0+} \dfrac{f(a+h)-f(a)}{h}$ exist and are equal. Students should be aware that a function may be continuous but not differentiable at a point, eg $f(x)=\lvert x\rvert$ and simple piecewise functions.	
9.4	The integral as a limit of a sum; lower and upper Riemann sums. Fundamental theorem of calculus. Improper integrals of the type $\int_a^\infty f(x)\,\mathrm{d}x$.	 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\int_a^x f(y)\,\mathrm{d}y\right] = f(x)$.	**Int:** How close was Archimedes to integral calculus? **Int:** Contribution of Arab, Chinese and Indian mathematicians to the development of calculus. **Aim 8:** Leibniz versus Newton versus the "giants" on whose shoulders they stood—who deserves credit for mathematical progress? **TOK:** Consider $f(x)=\dfrac{1}{x}$, $1 \le x \le \infty$. An infinite area sweeps out a finite volume. Can this be reconciled with our intuition? What does this tell us about mathematical knowledge?

	Content	Further guidance	Links
9.5	First-order differential equations. Geometric interpretation using slope fields, including identification of isoclines. Numerical solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ using Euler's method. Variables separable. Homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$ using the substitution $y = vx$. Solution of $y' + P(x)y = Q(x)$, using the integrating factor.	$y_{n+1} = y_n + hf(x_n, y_n)$, $x_{n+1} = x_n + h$, where h is a constant.	**Appl:** Real-life differential equations, eg Newton's law of cooling, population growth, carbon dating.
9.6	Rolle's theorem. Mean value theorem. Taylor polynomials; the Lagrange form of the error term. Maclaurin series for e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $(1+x)^p$, $p \in \mathbb{Q}$. Use of substitution, products, integration and differentiation to obtain other series. Taylor series developed from differential equations.	Applications to the approximation of functions; formula for the error term, in terms of the value of the $(n+1)^{\text{th}}$ derivative at an intermediate point. Students should be aware of the intervals of convergence.	**Int, TOK:** Influence of Bourbaki on understanding and teaching of mathematics. **Int:** Compare with work of the Kerala school.

	Content	Further guidance	Links
9.7	The evaluation of limits of the form $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ and $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$.	The indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.	
	Using l'Hôpital's rule or the Taylor series.	Repeated use of l'Hôpital's rule.	

Topic 10—Option: Discrete mathematics 48 hours

The aim of this option is to provide the opportunity for students to engage in logical reasoning, algorithmic thinking and applications.

	Content	Further guidance	Links
10.1	Strong induction. Pigeon-hole principle.	For example, proofs of the fundamental theorem of arithmetic and the fact that a tree with n vertices has $n-1$ edges.	**TOK:** Mathematics and knowledge claims. The difference between proof and conjecture, eg Goldbach's conjecture. Can a mathematical statement be true before it is proven? **TOK:** Proof by contradiction.
10.2	$a \mid b \Rightarrow b = na$ for some $n \in \mathbb{Z}$.		
	The theorem $a \mid b$ and $a \mid c \Rightarrow a \mid (bx \pm cy)$ where $x, y \in \mathbb{Z}$.	The division algorithm $a = bq + r$, $0 \le r < b$.	
	Division and Euclidean algorithms. The greatest common divisor, $\gcd(a,b)$, and the least common multiple, $\operatorname{lcm}(a,b)$, of integers a and b. Prime numbers; relatively prime numbers and the fundamental theorem of arithmetic.	The Euclidean algorithm for determining the greatest common divisor of two integers.	**Int:** Euclidean algorithm contained in Euclid's *Elements*, written in Alexandria about 300 BCE. **Aim 8:** Use of prime numbers in cryptography. The possible impact of the discovery of powerful factorization techniques on internet and bank security.
10.3	Linear Diophantine equations $ax + by = c$.	General solutions required and solutions subject to constraints. For example, all solutions must be positive.	**Int:** Described in Diophantus' *Arithmetica* written in Alexandria in the 3rd century CE. When studying *Arithmetica*, a French mathematician, Pierre de Fermat (1601–1665) wrote in the margin that he had discovered a simple proof regarding higher-order Diophantine equations—Fermat's last theorem.

	Content	Further guidance	Links
10.4	Modular arithmetic. The solution of linear congruences. Solution of simultaneous linear congruences (Chinese remainder theorem).		**Int:** Discussed by Chinese mathematician Sun Tzu in the 3rd century CE.
10.5	Representation of integers in different bases.	On examination papers, questions that go beyond base 16 will not be set.	**Int:** Babylonians developed a base 60 number system and the Mayans a base 20 number system.
10.6	Fermat's little theorem.	$a^p = a \pmod{p}$, where p is prime.	**TOK:** Nature of mathematics. An interest may be pursued for centuries before becoming "useful".

	Content	Further guidance	Links
10.7	Graphs, vertices, edges, faces. Adjacent vertices, adjacent edges. Degree of a vertex, degree sequence. Handshaking lemma.	Two vertices are adjacent if they are joined by an edge. Two edges are adjacent if they have a common vertex.	**Aim 8:** Symbolic maps, eg Metro and Underground maps, structural formulae in chemistry, electrical circuits. **TOK:** Mathematics and knowledge claims. Proof of the four-colour theorem. If a theorem is proved by computer, how can we claim to know that it is true?
	Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs; trees; weighted graphs, including tabular representation. Subgraphs; complements of graphs.	It should be stressed that a graph should not be assumed to be simple unless specifically stated. The term adjacency table may be used.	**Aim 8:** Importance of planar graphs in constructing circuit boards.
	Euler's relation: $v-e+f=2$; theorems for planar graphs including $e \le 3v-6$, $e \le 2v-4$, leading to the results that κ_5 and $\kappa_{3,3}$ are not planar.	If the graph is simple and planar and $v \ge 3$, then $e \le 3v-6$. If the graph is simple, planar, has no cycles of length 3 and $v \ge 3$, then $e \le 2v-4$.	**TOK:** Mathematics and knowledge claims. Applications of the Euler characteristic $(v-e+f)$ to higher dimensions. Its use in understanding properties of shapes that cannot be visualized.
10.8	Walks, trails, paths, circuits, cycles.		
	Eulerian trails and circuits.	A connected graph contains an Eulerian circuit if and only if every vertex of the graph is of even degree.	**Int:** The "Bridges of Königsberg" problem.
	Hamiltonian paths and cycles.	Simple treatment only.	
10.9	Graph algorithms: Kruskal's; Dijkstra's.		

	Content	Further guidance	Links
10.10	Chinese postman problem. **Not required:** Graphs with more than four vertices of odd degree.	To determine the shortest route around a weighted graph going along each edge at least once.	**Int:** Problem posed by the Chinese mathematician Kwan Mei-Ko in 1962.
	Travelling salesman problem. Nearest-neighbour algorithm for determining an upper bound. Deleted vertex algorithm for determining a lower bound.	To determine the Hamiltonian cycle of least weight in a weighted complete graph.	**TOK:** Mathematics and knowledge claims. How long would it take a computer to test all Hamiltonian cycles in a complete, weighted graph with just 30 vertices?
10.11	Recurrence relations. Initial conditions, recursive definition of a sequence.		**TOK:** Mathematics and the world. The connections of sequences such as the Fibonacci sequence with art and biology.
	Solution of first- and second-degree linear homogeneous recurrence relations with constant coefficients.	Includes the cases where auxiliary equation has equal roots or complex roots.	
	The first-degree linear recurrence relation $u_n = au_{n-1} + b$.		
	Modelling with recurrence relations.	Solving problems such as compound interest, debt repayment and counting problems.	

Glossary of terminology: Discrete mathematics

Introduction

Teachers and students should be aware that many different terminologies exist in graph theory, and that different textbooks may employ different combinations of these. Examples of these are: vertex/node/junction/point; edge/route/arc; degree/order of a vertex; multiple edges/parallel edges; loop/self-loop.

In IB examination questions, the terminology used will be as it appears in the syllabus. For clarity, these terms are defined below.

Terminology

Bipartite graph	A graph whose vertices can be divided into two sets such that no two vertices in the same set are adjacent.
Circuit	A walk that begins and ends at the same vertex, and has no repeated edges.
Complement of a graph *G*	A graph with the same vertices as G but which has an edge between any two vertices if and only if G does not.
Complete bipartite graph	A bipartite graph in which every vertex in one set is joined to every vertex in the other set.
Complete graph	A simple graph in which each pair of vertices is joined by an edge.
Connected graph	A graph in which each pair of vertices is joined by a path.
Cycle	A walk that begins and ends at the same vertex, and has no other repeated vertices.
Degree of a vertex	The number of edges joined to the vertex; a loop contributes two edges, one for each of its end points.
Disconnected graph	A graph that has at least one pair of vertices not joined by a path.
Eulerian circuit	A circuit that contains every edge of a graph.
Eulerian trail	A trail that contains every edge of a graph.
Graph	Consists of a set of vertices and a set of edges.
Graph isomorphism between two simple graphs *G* and *H*	A one-to-one correspondence between vertices of G and H such that a pair of vertices in G is adjacent if and only if the corresponding pair in H is adjacent.
Hamiltonian cycle	A cycle that contains all the vertices of the graph.
Hamiltonian path	A path that contains all the vertices of the graph.
Loop	An edge joining a vertex to itself.

Minimum spanning tree	A spanning tree of a weighted graph that has the minimum total weight.
Multiple edges	Occur if more than one edge joins the same pair of vertices.
Path	A walk with no repeated vertices.
Planar graph	A graph that can be drawn in the plane without any edge crossing another.
Simple graph	A graph without loops or multiple edges.
Spanning tree of a graph	A subgraph that is a tree, containing every vertex of the graph.
Subgraph	A graph within a graph.
Trail	A walk in which no edge appears more than once.
Tree	A connected graph that contains no cycles.
Walk	A sequence of linked edges.
Weighted graph	A graph in which each edge is allocated a number or weight.
Weighted tree	A tree in which each edge is allocated a number or weight.

Assessment in the Diploma Programme

General

Assessment is an integral part of teaching and learning. The most important aims of assessment in the Diploma Programme are that it should support curricular goals and encourage appropriate student learning. Both external and internal assessment are used in the Diploma Programme. IB examiners mark work produced for external assessment, while work produced for internal assessment is marked by teachers and externally moderated by the IB.

There are two types of assessment identified by the IB.

- Formative assessment informs both teaching and learning. It is concerned with providing accurate and helpful feedback to students and teachers on the kind of learning taking place and the nature of students' strengths and weaknesses in order to help develop students' understanding and capabilities. Formative assessment can also help to improve teaching quality, as it can provide information to monitor progress towards meeting the course aims and objectives.
- Summative assessment gives an overview of previous learning and is concerned with measuring student achievement.

The Diploma Programme primarily focuses on summative assessment designed to record student achievement at or towards the end of the course of study. However, many of the assessment instruments can also be used formatively during the course of teaching and learning, and teachers are encouraged to do this. A comprehensive assessment plan is viewed as being integral with teaching, learning and course organization. For further information, see the IB *Programme standards and practices* document.

The approach to assessment used by the IB is criterion-related, not norm-referenced. This approach to assessment judges students' work by their performance in relation to identified levels of attainment, and not in relation to the work of other students. For further information on assessment within the Diploma Programme, please refer to the publication *Diploma Programme assessment: Principles and practice*.

To support teachers in the planning, delivery and assessment of the Diploma Programme courses, a variety of resources can be found on the OCC or purchased from the IB store (http://store.ibo.org). Teacher support materials, subject reports, internal assessment guidance, grade descriptors, as well as resources from other teachers, can be found on the OCC. Specimen and past examination papers as well as markschemes can be purchased from the IB store.

Methods of assessment

The IB uses several methods to assess work produced by students.

Assessment criteria

Assessment criteria are used when the assessment task is open-ended. Each criterion concentrates on a particular skill that students are expected to demonstrate. An assessment objective describes what students should be able to do, and assessment criteria describe how well they should be able to do it. Using assessment criteria allows discrimination between different answers and encourages a variety of responses. Each criterion comprises a set of hierarchically ordered level descriptors. Each level descriptor is worth one or more marks. Each criterion is applied independently using a best-fit model. The maximum marks for each criterion may differ according to the criterion's importance. The marks awarded for each criterion are added together to give the total mark for the piece of work.

Markbands

Markbands are a comprehensive statement of expected performance against which responses are judged. They represent a single holistic criterion divided into level descriptors. Each level descriptor corresponds to a range of marks to differentiate student performance. A best-fit approach is used to ascertain which particular mark to use from the possible range for each level descriptor.

Markschemes

This generic term is used to describe analytic markschemes that are prepared for specific examination papers. Analytic markschemes are prepared for those examination questions that expect a particular kind of response and/or a given final answer from the students. They give detailed instructions to examiners on how to break down the total mark for each question for different parts of the response. A markscheme may include the content expected in the responses to questions or may be a series of marking notes giving guidance on how to apply criteria.

Assessment outline

First examinations 2014

Assessment component	Weighting
External assessment (5 hours)	**80%**
Paper 1 (2 hours) No calculator allowed. (120 marks) *Section A* Compulsory short-response questions based on the core syllabus. *Section B* Compulsory extended-response questions based on the core syllabus.	**30%**
Paper 2 (2 hours) Graphic display calculator required. (120 marks) *Section A* Compulsory short-response questions based on the core syllabus. *Section B* Compulsory extended-response questions based on the core syllabus.	**30%**
Paper 3 (1 hour) Graphic display calculator required. (60 marks) Compulsory extended-response questions based mainly on the syllabus options.	**20%**
Internal assessment This component is internally assessed by the teacher and externally moderated by the IB at the end of the course. **Mathematical exploration** Internal assessment in mathematics HL is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. (20 marks)	**20%**

External assessment

General

Markschemes are used to assess students in all papers. The markschemes are specific to each examination.

External assessment details

Papers 1, 2 and 3

These papers are externally set and externally marked. Together, they contribute 80% of the final mark for the course. These papers are designed to allow students to demonstrate what they know and what they can do.

Calculators

Paper 1

Students are not permitted access to any calculator. Questions will mainly involve analytic approaches to solutions, rather than requiring the use of a GDC. The paper is not intended to require complicated calculations, with the potential for careless errors. However, questions will include some arithmetical manipulations when they are essential to the development of the question.

Papers 2 and 3

Students must have access to a GDC at all times. However, not all questions will necessarily require the use of the GDC. Regulations covering the types of GDC allowed are provided in the *Handbook of procedures for the Diploma Programme*.

Mathematics HL and further mathematics HL formula booklet

Each student must have access to a clean copy of the formula booklet during the examination. It is the responsibility of the school to download a copy from IBIS or the OCC and to ensure that there are sufficient copies available for all students.

Awarding of marks

Marks may be awarded for method, accuracy, answers and reasoning, including interpretation.

In paper 1 and paper 2, full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations (in the form of, for example, diagrams, graphs or calculations). Where an answer is incorrect, some marks may be given for correct method, provided this is shown by written working. All students should therefore be advised to show their working.

Paper 1

Duration: 2 hours
Weighting: 30%

- This paper consists of section A, short-response questions, and section B, extended-response questions.
- Students are not permitted access to any calculator on this paper.

Syllabus coverage

- Knowledge of **all** core topics is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **120** marks, representing **30%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily each be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

Section A

- This section consists of compulsory short-response questions based on the core syllabus. It is worth 60 marks.
- The intention of this section is to test students' knowledge and understanding across the breadth of the syllabus. However, it should not be assumed that the separate topics are given equal emphasis.

Question type

- A small number of steps is needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Section B

- This section consists of a small number of compulsory extended-response questions based on the core syllabus. It is worth 60 marks.
- Individual questions may require knowledge of more than one topic.
- The intention of this section is to test students' knowledge and understanding of the core in depth. The range of syllabus topics tested in this section may be narrower than that tested in section A.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions will develop a single theme.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem-solving.

Paper 2

Duration: 2 hours
Weighting: 30%

- This paper consists of section A, short-response questions, and section B, extended-response questions.
- A GDC is required for this paper, but not every question will necessarily require its use.

Syllabus coverage

- Knowledge of **all** core topics is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **120** marks, representing **30%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily each be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

Section A

- This section consists of compulsory short-response questions based on the core syllabus. It is worth 60 marks.
- The intention of this section is to test students' knowledge and understanding across the breadth of the syllabus. However, it should not be assumed that the separate topics are given equal emphasis.

Question type

- A small number of steps is needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Section B

- This section consists of a small number of compulsory extended-response questions based on the core syllabus. It is worth 60 marks.
- Individual questions may require knowledge of more than one topic.
- The intention of this section is to test students' knowledge and understanding of the core in depth. The range of syllabus topics tested in this section may be narrower than that tested in section A.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions will develop a single theme.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem-solving.

Paper 3

Duration: 1 hour
Weighting: 20%

- This paper consists of a small number of compulsory extended-response questions based on the option chosen.
- Where possible, the first part of each question will be on core material leading to the option topic. When this is not readily achievable, as, for example, with the discrete mathematics option, the level of difficulty of the earlier part of a question will be comparable to that of the core questions.

Syllabus coverage

- Students must answer **all** questions.
- Knowledge of the entire content of the option studied, as well as the core material, is required for this paper.

Mark allocation

- This paper is worth **60** marks, representing **20%** of the final mark.
- Questions may be unequal in terms of length and level of difficulty. Therefore, individual questions may not be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of each question.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions will develop a single theme or be divided into unconnected parts. Where the latter occur, the unconnected parts will be clearly labelled as such.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem-solving.

Internal assessment

Purpose of internal assessment

Internal assessment is an integral part of the course and is compulsory for all students. It enables students to demonstrate the application of their skills and knowledge, and to pursue their personal interests, without the time limitations and other constraints that are associated with written examinations. The internal assessment should, as far as possible, be woven into normal classroom teaching and not be a separate activity conducted after a course has been taught.

Internal assessment in mathematics HL is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. It is marked according to five assessment criteria.

Guidance and authenticity

The exploration submitted for internal assessment must be the student's own work. However, it is not the intention that students should decide upon a title or topic and be left to work on the exploration without any further support from the teacher. The teacher should play an important role during both the planning stage and the period when the student is working on the exploration. It is the responsibility of the teacher to ensure that students are familiar with:

- the requirements of the type of work to be internally assessed
- the IB academic honesty policy available on the OCC
- the assessment criteria—students must understand that the work submitted for assessment must address these criteria effectively.

Teachers and students must discuss the exploration. Students should be encouraged to initiate discussions with the teacher to obtain advice and information, and students must not be penalized for seeking guidance. However, if a student could not have completed the exploration without substantial support from the teacher, this should be recorded on the appropriate form from the *Handbook of procedures for the Diploma Programme*.

It is the responsibility of teachers to ensure that all students understand the basic meaning and significance of concepts that relate to academic honesty, especially authenticity and intellectual property. Teachers must ensure that all student work for assessment is prepared according to the requirements and must explain clearly to students that the exploration must be entirely their own.

As part of the learning process, teachers can give advice to students on a **first draft** of the exploration. This advice should be in terms of the way the work could be improved, but this first draft must not be heavily annotated or edited by the teacher. The next version handed to the teacher after the first draft must be the final one.

All work submitted to the IB for moderation or assessment must be authenticated by a teacher, and must not include any known instances of suspected or confirmed malpractice. Each student must sign the coversheet for internal assessment to confirm that the work is his or her authentic work and constitutes the final version of that work. Once a student has officially submitted the final version of the work to a teacher (or the coordinator) for internal assessment, together with the signed coversheet, it cannot be retracted.

Authenticity may be checked by discussion with the student on the content of the work, and scrutiny of one or more of the following:

- the student's initial proposal
- the first draft of the written work
- the references cited
- the style of writing compared with work known to be that of the student.

The requirement for teachers and students to sign the coversheet for internal assessment applies to the work of all students, not just the sample work that will be submitted to an examiner for the purpose of moderation. If the teacher and student sign a coversheet, but there is a comment to the effect that the work may not be authentic, the student will not be eligible for a mark in that component and no grade will be awarded. For further details refer to the IB publication *Academic honesty* and the relevant articles in the *General regulations: Diploma Programme*.

The same piece of work cannot be submitted to meet the requirements of both the internal assessment and the extended essay.

Group work

Group work should not be used for explorations. Each exploration is an individual piece of work based on different data collected or measurements generated.

It should be made clear to students that all work connected with the exploration, including the writing of the exploration, should be their own. It is therefore helpful if teachers try to encourage in students a sense of responsibility for their own learning so that they accept a degree of ownership and take pride in their own work.

Time allocation

Internal assessment is an integral part of the mathematics HL course, contributing 20% to the final assessment in the course. This weighting should be reflected in the time that is allocated to teaching the knowledge, skills and understanding required to undertake the work as well as the total time allocated to carry out the work.

It is expected that a total of approximately 10 teaching hours should be allocated to the work. This should include:

- time for the teacher to explain to students the requirements of the exploration
- class time for students to work on the exploration
- time for consultation between the teacher and each student
- time to review and monitor progress, and to check authenticity.

Using assessment criteria for internal assessment

For internal assessment, a number of assessment criteria have been identified. Each assessment criterion has level descriptors describing specific levels of achievement together with an appropriate range of marks. The level descriptors concentrate on positive achievement, although for the lower levels failure to achieve may be included in the description.

Teachers must judge the internally assessed work against the criteria using the level descriptors.

- The aim is to find, for each criterion, the descriptor that conveys most accurately the level attained by the student.
- When assessing a student's work, teachers should read the level descriptors for each criterion, starting with level 0, until they reach a descriptor that describes a level of achievement that has not been reached. The level of achievement gained by the student is therefore the preceding one, and it is this that should be recorded.
- Only whole numbers should be recorded; partial marks, that is fractions and decimals, are not acceptable.
- Teachers should not think in terms of a pass or fail boundary, but should concentrate on identifying the appropriate descriptor for each assessment criterion.
- The highest level descriptors do not imply faultless performance but should be achievable by a student. Teachers should not hesitate to use the extremes if they are appropriate descriptions of the work being assessed.
- A student who attains a high level of achievement in relation to one criterion will not necessarily attain high levels of achievement in relation to the other criteria. Similarly, a student who attains a low level of achievement for one criterion will not necessarily attain low achievement levels for the other criteria. Teachers should not assume that the overall assessment of the students will produce any particular distribution of marks.
- It is expected that the assessment criteria be made available to students.

Internal assessment details

Mathematical exploration

Duration: 10 teaching hours
Weighting: 20%

Introduction

The internally assessed component in this course is a mathematical exploration. This is a short report written by the student based on a topic chosen by him or her, and it should focus on the mathematics of that particular area. The emphasis is on mathematical communication (including formulae, diagrams, graphs and so on), with accompanying commentary, good mathematical writing and thoughtful reflection. A student should develop his or her own focus, with the teacher providing feedback via, for example, discussion and interview. This will allow the students to develop areas of interest to them without a time constraint as in an examination, and allow all students to experience a feeling of success.

The final report should be approximately 6 to 12 pages long. It can be either word processed or handwritten. Students should be able to explain all stages of their work in such a way that demonstrates clear understanding. While there is no requirement that students present their work in class, it should be written in such a way that their peers would be able to follow it fairly easily. The report should include a detailed bibliography, and sources need to be referenced in line with the IB academic honesty policy. Direct quotes must be acknowledged.

The purpose of the exploration

The aims of the mathematics HL course are carried through into the objectives that are formally assessed as part of the course, through either written examination papers, or the exploration, or both. In addition to testing the objectives of the course, the exploration is intended to provide students with opportunities to increase their understanding of mathematical concepts and processes, and to develop a wider appreciation of mathematics. These are noted in the aims of the course, **in particular, aims 6–9 (applications, technology, moral, social**

and ethical implications, and the international dimension). It is intended that, by doing the exploration, students benefit from the mathematical activities undertaken and find them both stimulating and rewarding. It will enable students to acquire the attributes of the IB learner profile.

The specific purposes of the exploration are to:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete a piece of mathematical work over an extended period of time
- enable students to experience the satisfaction of applying mathematical processes independently
- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- encourage students, where appropriate, to discover, use and appreciate the power of technology as a mathematical tool
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of their work
- provide opportunities for students to show, with confidence, how they have developed mathematically.

Management of the exploration

Work for the exploration should be incorporated into the course so that students are given the opportunity to learn the skills needed. Time in class can therefore be used for general discussion of areas of study, as well as familiarizing students with the criteria. Further details on the development of the exploration are included in the teacher support material.

Requirements and recommendations

Students can choose from a wide variety of activities, for example, modelling, investigations and applications of mathematics. To assist teachers and students in the choice of a topic, a list of stimuli is available in the teacher support material. However, students are not restricted to this list.

The exploration should not normally exceed 12 pages, including diagrams and graphs, but excluding the bibliography. However, it is the quality of the mathematical writing that is important, not the length.

The teacher is expected to give appropriate guidance at all stages of the exploration by, for example, directing students into more productive routes of inquiry, making suggestions for suitable sources of information, and providing advice on the content and clarity of the exploration in the writing-up stage.

Teachers are responsible for indicating to students the existence of errors but should not explicitly correct these errors. It must be emphasized that students are expected to consult the teacher throughout the process.

All students should be familiar with the requirements of the exploration and the criteria by which it is assessed. Students need to start planning their explorations as early as possible in the course. Deadlines should be firmly established. There should be a date for submission of the exploration topic and a brief outline description, a date for the submission of the first draft and, of course, a date for completion.

In developing their explorations, students should aim to make use of mathematics learned as part of the course. The mathematics used should be commensurate with the level of the course, that is, it should be similar to that suggested by the syllabus. It is not expected that students produce work that is outside the mathematics HL syllabus—however, this is not penalized.

Internal assessment criteria

The exploration is internally assessed by the teacher and externally moderated by the IB using assessment criteria that relate to the objectives for mathematics HL.

Each exploration is assessed against the following five criteria. The final mark for each exploration is the sum of the scores for each criterion. The maximum possible final mark is 20.

Students will not receive a grade for mathematics HL if they have not submitted an exploration.

Criterion A	Communication
Criterion B	Mathematical presentation
Criterion C	Personal engagement
Criterion D	Reflection
Criterion E	Use of mathematics

Criterion A: Communication

This criterion assesses the organization and coherence of the exploration. A well-organized exploration includes an introduction, has a rationale (which includes explaining why this topic was chosen), describes the aim of the exploration and has a conclusion. A coherent exploration is logically developed and easy to follow.

Graphs, tables and diagrams should accompany the work in the appropriate place and not be attached as appendices to the document.

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	The exploration has some coherence.
2	The exploration has some coherence and shows some organization.
3	The exploration is coherent and well organized.
4	The exploration is coherent, well organized, concise and complete.

Criterion B: Mathematical presentation

This criterion assesses to what extent the student is able to:

- use appropriate mathematical language (notation, symbols, terminology)
- define key terms, where required
- use multiple forms of mathematical representation, such as formulae, diagrams, tables, charts, graphs and models, where appropriate.

Students are expected to use mathematical language when communicating mathematical ideas, reasoning and findings.

Students are encouraged to choose and use appropriate ICT tools such as graphic display calculators, screenshots, graphing, spreadsheets, databases, drawing and word-processing software, as appropriate, to enhance mathematical communication.

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is some appropriate mathematical presentation.
2	The mathematical presentation is mostly appropriate.
3	The mathematical presentation is appropriate throughout.

Criterion C: Personal engagement

This criterion assesses the extent to which the student engages with the exploration and makes it their own. Personal engagement may be recognized in different attributes and skills. These include thinking independently and/or creatively, addressing personal interest and presenting mathematical ideas in their own way.

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited or superficial personal engagement.
2	There is evidence of some personal engagement.
3	There is evidence of significant personal engagement.
4	There is abundant evidence of outstanding personal engagement.

Criterion D: Reflection

This criterion assesses how the student reviews, analyses and evaluates the exploration. Although reflection may be seen in the conclusion to the exploration, it may also be found throughout the exploration.

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	There is evidence of limited or superficial reflection.
2	There is evidence of meaningful reflection.
3	There is substantial evidence of critical reflection.

Criterion E: Use of mathematics

This criterion assesses to what extent and how well students use mathematics in the exploration.

Students are expected to produce work that is commensurate with the level of the course. The mathematics explored should either be part of the syllabus, or at a similar level or beyond. It should not be completely based on mathematics listed in the prior learning. If the level of mathematics is not commensurate with the level of the course, a maximum of two marks can be awarded for this criterion.

The mathematics can be regarded as correct even if there are occasional minor errors as long as they do not detract from the flow of the mathematics or lead to an unreasonable outcome.

Sophistication in mathematics may include understanding and use of challenging mathematical concepts, looking at a problem from different perspectives and seeing underlying structures to link different areas of mathematics.

Rigour involves clarity of logic and language when making mathematical arguments and calculations.

Precise mathematics is error-free and uses an appropriate level of accuracy at all times.

Achievement level	Descriptor
0	The exploration does not reach the standard described by the descriptors below.
1	Some relevant mathematics is used. Limited understanding is demonstrated.
2	Some relevant mathematics is used. The mathematics explored is partially correct. Some knowledge and understanding are demonstrated.
3	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct. Good knowledge and understanding are demonstrated.
4	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and reflects the sophistication expected. Good knowledge and understanding are demonstrated.
5	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is correct and reflects the sophistication and rigour expected. Thorough knowledge and understanding are demonstrated.
6	Relevant mathematics commensurate with the level of the course is used. The mathematics explored is precise and reflects the sophistication and rigour expected. Thorough knowledge and understanding are demonstrated.

Glossary of command terms

Command terms with definitions

Students should be familiar with the following key terms and phrases used in examination questions, which are to be understood as described below. Although these terms will be used in examination questions, other terms may be used to direct students to present an argument in a specific way.

Calculate	Obtain a numerical answer showing the relevant stages in the working.
Comment	Give a judgment based on a given statement or result of a calculation.
Compare	Give an account of the similarities between two (or more) items or situations, referring to both (all) of them throughout.
Compare and contrast	Give an account of the similarities and differences between two (or more) items or situations, referring to both (all) of them throughout.
Construct	Display information in a diagrammatic or logical form.
Contrast	Give an account of the differences between two (or more) items or situations, referring to both (all) of them throughout.
Deduce	Reach a conclusion from the information given.
Demonstrate	Make clear by reasoning or evidence, illustrating with examples or practical application.
Describe	Give a detailed account.
Determine	Obtain the only possible answer.
Differentiate	Obtain the derivative of a function.
Distinguish	Make clear the differences between two or more concepts or items.
Draw	Represent by means of a labelled, accurate diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Estimate	Obtain an approximate value.
Explain	Give a detailed account, including reasons or causes.
Find	Obtain an answer, showing relevant stages in the working.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.
Identify	Provide an answer from a number of possibilities.

Integrate	Obtain the integral of a function.
Interpret	Use knowledge and understanding to recognize trends and draw conclusions from given information.
Investigate	Observe, study, or make a detailed and systematic examination, in order to establish facts and reach new conclusions.
Justify	Give valid reasons or evidence to support an answer or conclusion.
Label	Add labels to a diagram.
List	Give a sequence of brief answers with no explanation.
Plot	Mark the position of points on a diagram.
Predict	Give an expected result.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show	Give the steps in a calculation or derivation.
Show that	Obtain the required result (possibly using information given) without the formality of proof. "Show that" questions do not generally require the use of a calculator.
Sketch	Represent by means of a diagram or graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.
Solve	Obtain the answer(s) using algebraic and/or numerical and/or graphical methods.
State	Give a specific name, value or other brief answer without explanation or calculation.
Suggest	Propose a solution, hypothesis or other possible answer.
Verify	Provide evidence that validates the result.
Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.

Notation list

Of the various notations in use, the IB has chosen to adopt a system of notation based on the recommendations of the International Organization for Standardization (ISO). This notation is used in the examination papers for this course without explanation. If forms of notation other than those listed in this guide are used on a particular examination paper, they are defined within the question in which they appear.

Because students are required to recognize, though not necessarily use, IB notation in examinations, it is recommended that teachers introduce students to this notation at the earliest opportunity. Students are **not** allowed access to information about this notation in the examinations.

Students must always use correct mathematical notation, not calculator notation.

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \mid x \in \mathbb{Q}, x > 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \mid x \in \mathbb{R}, x > 0\}$
$\mathbb{C}$	the set of complex numbers, $\{a + \mathrm{i}b \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathrm{Re}\, z$	the real part of z
$\mathrm{Im}\, z$	the imaginary part of z
$\mathrm{cis}\,\theta$	$\cos\theta + \mathrm{i}\sin\theta$
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid \quad\}$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union

$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$A \times B$	the Cartesian product of sets A and B (that is, $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$)
$a \mid b$	a divides b
$a^{1/n}$, $\sqrt[n]{a}$	a to the power of $\frac{1}{n}$, n^{th} root of a (if $a \geq 0$ then $\sqrt[n]{a} \geq 0$)
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geq 0,\ x \in \mathbb{R} \\ -x \text{ for } x < 0,\ x \in \mathbb{R} \end{cases}$
$\equiv$	identity
$\approx$	is approximately equal to
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	implies and is implied by
$[a, b]$	the closed interval $a \leq x \leq b$
$]a, b[$	the open interval $a < x < b$
u_n	the n^{th} term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + \ldots + u_n$
S_∞	the sum to infinity of a sequence, $u_1 + u_2 + \ldots$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + \ldots + u_n$
$\prod_{i=1}^{n} u_i$	$u_1 \times u_2 \times \ldots \times u_n$

$\binom{n}{r}$	$\dfrac{n!}{r!(n-r)!}$
$f: A \to B$	f is a function under which each element of set A has an image in set B
$f: x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the n^{th} derivative of y with respect to x
$f^{(n)}(x)$	the n^{th} derivative of $f(x)$ with respect to x
$\int y \,\mathrm{d}x$	the indefinite integral of y with respect to x
$\int_a^b y \,\mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	the exponential function of x
$\log_a x$	the logarithm to the base a of x
$\ln x$	the natural logarithm of x, $\log_{\mathrm{e}} x$
sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
csc, sec, cot	the reciprocal circular functions
$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
$[\mathrm{AB}]$	the line segment with end points A and B

AB	the length of $[\mathrm{AB}]$
(AB)	the line containing points A and B
$\hat{A}$	the angle at A
$\mathrm{C\hat{A}B}$	the angle between $[\mathrm{CA}]$ and $[\mathrm{AB}]$
$\Delta\mathrm{ABC}$	the triangle whose vertices are A, B and C
$\boldsymbol{v}$	the vector $\boldsymbol{v}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\boldsymbol{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\boldsymbol{a}\rvert$	the magnitude of $\boldsymbol{a}$
$\lvert\overrightarrow{\mathrm{AB}}\rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\boldsymbol{v}\cdot\boldsymbol{w}$	the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v}\times\boldsymbol{w}$	the vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\mathbf{I}$	the identity matrix
$\mathrm{P}(A)$	the probability of event A
$\mathrm{P}(A')$	the probability of the event "not A"
$\mathrm{P}(A \mid B)$	the probability of the event A given B
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
P_x	the probability distribution function $\mathrm{P}(X = x)$ of the discrete random variable X
$f(x)$	the probability density function of the continuous random variable X
$F(x)$	the cumulative distribution function of the continuous random variable X
$\mathrm{E}(X)$	the expected value of the random variable X
$\mathrm{Var}(X)$	the variance of the random variable X
μ	population mean
σ^2	population variance, $\sigma^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
σ	population standard deviation

$\overline{x}$	sample mean
s_n^2	sample variance, $s_n^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance, $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n-1}$, where $n = \sum_{i=1}^{k} f_i$
$\mathrm{B}(n, p)$	binomial distribution with parameters n and p
$\mathrm{Po}(m)$	Poisson distribution with mean m
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$X \sim \mathrm{B}(n, p)$	the random variable X has a binomial distribution with parameters n and p
$X \sim \mathrm{Po}(m)$	the random variable X has a Poisson distribution with mean m
$X \sim \mathrm{N}(\mu, \sigma^2)$	the random variable X has a normal distribution with mean μ and variance σ^2
Φ	cumulative distribution function of the standardized normal variable with distribution $\mathrm{N}(0, 1)$
ν	number of degrees of freedom
$A \setminus B$	the difference of the sets A and B (that is, $A \setminus B = A \cap B' = \{x \mid x \in A \text{ and } x \notin B\}$)
$A \Delta B$	the symmetric difference of the sets A and B (that is, $A \Delta B = (A \setminus B) \cup (B \setminus A)$)
κ_n	a complete graph with n vertices
$\kappa_{n,m}$	a complete bipartite graph with one set of n vertices and another set of m vertices
$\mathbb{Z}_p$	the set of equivalence classes $\{0, 1, 2, \ldots, p-1\}$ of integers modulo p
$\gcd(a, b)$	the greatest common divisor of integers a and b
$\mathrm{lcm}(a, b)$	the least common multiple of integers a and b
$\boldsymbol{A}_G$	the adjacency matrix of graph G
$\boldsymbol{C}_G$	the cost adjacency matrix of graph G