Assignment 5

(1) Consider an iid sample $X_1, \dots, X_n$ from the Cauchy location family $\mathcal{C}(\mu, 1)$, where each observation has distribution

$$p(x;\mu) = \frac{1}{\pi}\frac{1}{1+(x-\mu)^2}, \quad \mu \in \mathbb{R}.$$

(a) Suppose the median is used as an estimate of μ. Derive its asymptotic distribution. For this part you can use the material in Chapter 13 of Ferguson on Order Statistics, in particular, the Corollary on page 91.

(b) Recall, or show now if not previously presented, the result of problem 6 of Assignment 4, how one obtains the limiting normal distribution for the properly centered and scaled MLE, and specify its parameters.

(c) Compute the ratio of the variances in the two asymptotic distributions in the previous two parts. This quantity is known as the Asymptotic Relative Efficiency, or ARE, of one estimator when compared to another.

(d) If statistician A has n observations, and uses the MLE to estimate μ, and statistician B uses the median, how large a sample size will statistician B require to have the same variance as A? Does statistician B have any advantages in using the median?

(e) Make the same comparison between the MLE and the median for the parameter μ in the family $\mathcal{N}(\mu, \sigma^2)$, taking σ^2 known if desired.

(2) Let L be a bounded linear function on the Hilbert space $\mathbb{H}$. Show that the null space

$$\mathcal{N}(L) = \{h : L(h) = 0\}$$

is closed. (Hint: Consider the complement.)

(3) Show if S is a closed subspace of a Hilbert space $\mathbb{H}$ then $\mathbb{H} = S \oplus S^{\perp}$.

(4) Show that a linear functional on the Hilbert space $\mathbb{H}$ is bounded if and only if it is continuous.

(5) Given a symmetric, non-negative definite (Kernel) function $K(x,y)$ on $\chi \times \chi$, show that the inner product defined on linear combinations by

$$\langle \sum_{i=1}^{n} \alpha_i K(x_i, \cdot), \sum_{i=1}^{n} \beta_i K(y_i, \cdot) \rangle = \sum_{i,j=}^{n} \alpha_i \beta_j K(x_i, y_j)$$

is well defined.

(6) Show that the kernel for an RKHS is unique, that is, if the kernels K and $\widetilde{K}$ both have the representer property for point evaluation in $\mathbb{H}$, then they must be equal.

(7) Show that if P is a probability measure on χ, and $\int K^2(x,y)dP(x)dP(y) < \infty$ then $T_K(f) = \int K(x,y)f(y)dP$ is a bounded linear operator from $\mathbb{H}$ to itself.

(8) Let $\mathbb{H}$ be a reproducing kernel Hilbert space. Show that convergence in norm implies pointwise convergence, that is, if for some sequence $f_n \in \mathbb{H}$ and an $f \in \mathbb{H}$ we have $\lim_{n\to\infty} \|f_n - f\|_{\mathbb{H}} = 0$ then $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \chi$. (Hint: Use the reproducing property combined with the Cauchy-Schwarz inequality.)

(9) Let $\mathcal{H}$ be an RKHS, and for $n \geq 0$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$, where $x_i \in \chi, i = 1, \ldots, n$, define

$$\mathbb{L}_{n,\boldsymbol{x}} = \left\{ \sum_{i=1}^{n} \alpha_i K(x_i, \dot{}), \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.$$

Let $y_1, \ldots, y_n$ be arbitrary real numbers. Prove that if f and g are elements of $\mathbb{L}_{n,\boldsymbol{x}}$ whose evaluations at each of $x_1, \ldots, x_n$ agree, then f and g are equal when evaluated at any $x \in \chi$.

(10) Consider the Cauchy location family

$$p(x;\theta) = \frac{1}{\pi(1 + (x-\theta)^2)}, \quad \theta \in \mathbb{R}.$$

Construct the Neyman Pearson test for $H_0 : \theta = 0$ versus $H_1 : \theta = 1$ based on one observation X, and show for some values of α the rejection region may be disconnected, and in particular, not of the form $[t, \infty)$ for $t \in \mathbb{R}$.

(11) Construct the Neyman Pearson tests for the parameters of the classical families of distributions, such as the Poisson, Gamma, Beta, Uniform, and Binomial. For any of these families with more than one parameter, you may construct tests for a single parameter when the other parameters are fixed, or when all parameters are fixed under either hypothesis; that is, you may consider either version of the simple vs simple hypothesis test.

(12) Compute the Type I error, and power, of the randomized NP test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ with $\theta_0 < \theta_1$ when observing $X_1, \ldots, X_n$ iid from

$\mathcal{U}[0, \theta], \theta > 0$, which rejects H_0 with probability 1 when $\theta_0 < X_{(n)} \leq \theta_1$ and with probability α when $X_{(n)} \leq \theta_0$.