Assignment 1

1. Give two (or more) equivalent definitions of independence for two random variables.
2. Write down the univariate and multivariate normal densities, and compute their moment generating functions.
3. State the following inequalities: Cauchy-Schwarz, Markov, Chebyshev, Hölder, Jensen. Find which ones on the list are consequences of another, and provide a proof.
4. For X and Y random variables with finite second moment, define the Covariance $\text{Cov}(X, Y)$ and Correlation $\text{Cor}(X, Y)$ between X and Y. Prove that $|\text{Cor}(X, Y)|$ is at most one, and determine a necessary and sufficient condition for this quantity to equal 1.
5. Prove that if X and Y are independent and have finite second moments then $\text{Cov}(X, Y) = 0$. Regarding the converse, find an example, or prove impossible: There exist random variables X and Y with finite second moments and positive variances such that $\text{Cov}(X, Y) = 0$ with Y a deterministic function of X.
6. Define convergence in: probability, almost sure, and in distribution. What relations, such as implications, exist between these type of convergence?
7. State the Weak Law of Large Numbers. Provide an easy proof of the conclusion under a set of strong assumptions.
8. State the Central Limit Theorem in one dimension. State how this result is typically proved. Demonstrate how the one dimensional result can be used to extend the conclusion to higher dimensions.
9. Carefully state and prove a version of Taylor's theorem in one dimension with integral remainder, and show how it implies one of the more commonly used versions. Use the one dimensional version to prove a result for a real valued function with domain in Euclidean space.
10. Define the singular values of a real valued, possibly non-square matrix. State the singular value decomposition for matrices, and give the key points on which its proof depends. State the consequence of this result for symmetric matrices, and the further special case for projection matrices P, which

are both symmetric and idempotent, that is, $P^T = P$, and $P^2 = P$. (Hint: It may help to first prove that if a matrix $A \in \mathbb{R}^{n\times n}$ satisfies $q(A) = 0$ for some polynomial q, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ (which may be repeated) satisfy $q(\lambda_i) = 0, i = 1, \ldots, n.$)

11. With tr standing for the trace, for two square matrices of the same dimension, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Prove or find a counterexample to: For three square matrices of the same dimension, $\mathrm{tr}(ABC) = \mathrm{tr}(ACB)$