MAXIMIZING EXPECTED VALUE WITH TWO STAGE STOPPING RULES

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Let $X_n, \ldots, X_1$ be i.i.d. random variables with distribution function F and finite expectation. A statistician, knowing F, observes the X values sequentially and is given two chances to choose X's using stopping rules. The statistician's goal is to select a value of X as large as possible. Let V_n^2 equal the expectation of the larger of the two values chosen by the statistician when proceeding optimally. We obtain the asymptotic behavior of the sequence V_n^2 for a large class of F's belonging to the domain of attraction (for the maximum) $\mathcal{D}(G_{II}^\alpha)$, where $G_{II}^\alpha(x) = \exp(-x^{-\alpha})\mathbf{I}(x > 0)$ with $\alpha > 1$. The results are compared with those for the asymptotic behavior of the classical one choice value sequence V_n^1, as well as with the "prophet value" sequence $E(\max\{X_n, \ldots, X_1\})$, and indicate that substantial improvement is obtained when given two chances to stop, rather than one.

Some key words:

Optimal Stopping, Prophet value, Two-choice, Domains of attraction, Asymptotic value.

1. INTRODUCTION

Kennedy and Kertz (1991) study the asymptotic behavior of the value sequence of a one choice optimal stopping problem, as $n \to \infty$, where one observes $X_n, \ldots, X_1$ independent, identically distributed random variables with known distribution F, and the payoff is the random variable at which one stops. The goal is to maximize the expected payoff. The value of such a sequence is therefore $\sup EX_t = V_n^1$, where the supremum is over all stopping times t, which stop after at most n observations, with probability

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one. They show that the asymptotic behavior depends on which of the three well-known domains of attraction (for the maximum) F belongs to.

Recently Assaf and Samuel-Cahn (2000) and Assaf, Goldstein and Samuel-Cahn (2002) study stopping problems in which the optimal stopper is given more than one choice. There "Prophet Inequalities" are derived for general finite sequences of independent, non-negative, not necessarily identically distributed, random variables.

The present paper focuses on a different aspect of multiple choices. We consider the situation where the observations are i.i.d. random variables, and the stopper is given two choices. His payoff is the larger of the two values chosen, and the goal is to maximize the expected return. As an example, the situation we consider here may correspond to a situation in which you put your first selected item (perhaps a house or a job offer) "on hold" as a guaranteed fallback value. You then proceed sequentially to select a second item (which should be of greater value than the first, unless it is the last one) and finish by taking the better of the two items selected.

Let $\mathcal{D}(G_{II}^{\alpha})$ be the domain of attraction for the maximum to

$$G_{II}^{\alpha}(x) = \exp(-x^{-\alpha})\mathbf{I}(x > 0),$$

(see notation in Kennedy and Kertz (1991), or p.4 of Leadbetter, Lindgren and Rootzen, (1983)), and $V_n^2 = \sup_{1 \le t_1 \le t_2 \le n} E[\max\{X_{t_1}, X_{t_2}\}]$. We study the asymptotic behavior of V_n^2 for $F \in \mathcal{D}(G_{II}^{\alpha})$. The case $F \in \mathcal{D}(G_{III}^{\alpha})$ has recently been considered in Assaf, Goldstein and Samuel-Cahn (2003), henceforth referred to as AGS. Because the distributions whose maxima are attracted to G_{III}^{α} are bounded above, it was more natural there to consider the goal to be the minimization of the expected value of the minimum of the two values chosen. The results obtained there about the minimum were then translated back to results about the maximum; see Remark 7.4 of AGS.

A necessary and sufficient condition for $F \in \mathcal{D}(G_{II}^{\alpha})$ (see Theorem 1.6.2 of Leadbetter, Lindgren and Rootzen, 1983) is $x_F = \sup\{x : F(x) < 1\} = \infty$, and that for some $\alpha > 0$,

$$\lim_{t\to\infty} [1 - F(tx)]/[1 - F(t)] = x^{-\alpha} \quad \text{for all } x > 0,$$

which can also be written as

$$1 - F(x) = x^{-\alpha}L(x) \quad \text{as } x \to \infty \tag{1}$$

where $L(x)$ is slowly varying at ∞. We will treat the case where

$$\lim_{x\to\infty} L(x) = \mathcal{L} \in (0, \infty). \tag{2}$$

Throughout we will consider the case $\alpha > 1$; the case $0 < \alpha \leq 1$ is uninteresting, as then the expectation of X is infinite. It is known that for $X_n, \ldots, X_1$ i.i.d. with distribution $F \in \mathcal{D}(G_{II}^{\alpha})$ with finite expectation, for $M_n = \max(X_n, \ldots X_1)$ we have

$$\lim_{n\to\infty} n[1 - F(EM_n)] = [\Gamma\left(1 - \frac{1}{\alpha}\right)]^{-\alpha}. \tag{3}$$

(See proposition 2.1 of Resnick, 1987). Let V_n^1 be the optimal one-choice value, when the goal is to maximize the expected value of the item chosen. Then by Kennedy and Kertz (1991),

$$\lim_{n\to\infty} n[1 - F(V_n^1)] = \frac{\alpha - 1}{\alpha}. \tag{4}$$

When (2) is satisfied, then taking $\mathcal{L} = 1$ for convenience, the limits in (3) and (4) can be rewritten, respectively, as

$$\lim_{n\to\infty} n^{-1/\alpha} EM_n = \Gamma(1 - 1/\alpha)$$

$$\text{and} \quad \lim_{n\to\infty} n^{-1/\alpha} V_n^1 = \left(\frac{\alpha}{\alpha - 1}\right)^{1/\alpha}. \tag{5}$$

Thus if we denote by $V_n = V_n^2$ the optimal two choice value, it is reasonable to expect that $\lim_{n\to\infty} n^{-1/\alpha} V_n = d_\alpha$, where $\left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} < d_\alpha < \Gamma(1 - 1/\alpha)$. We prove the following

Theorem 1.1. *Let $X_n, X_{n-1}, \ldots, X_1$ be i.i.d. random variables with finite expectation, having distribution function F satisfying $x_F = \infty$, $1 - F(x) = x^{-\alpha} L(x)$ where $\lim_{x\to\infty} L(x) = \mathcal{L} \in (0, \infty)$. Define*

$$h(u) = \left(\frac{\alpha}{\alpha - 1} + \frac{1}{u}\right)^{1/\alpha} \quad \text{for } 0 < u < \infty$$

and let β_α be the unique solution y to

$$\int_0^y h(u)du - \left(\frac{1}{\alpha} + y\right) h(y) = 0. \tag{6}$$

Then

$$\lim_{n\to\infty} n[1 - F(V_n^2)] = [h(\beta_\alpha)]^{-\alpha},$$

so in particular, for $\mathcal{L} = 1$

$$\lim_{n\to\infty} n^{-1/\alpha} V_n^2 = h(\beta_\alpha) = d_\alpha.$$

The proof of Theorem 1.1 is given in Section 6.

The paper is organized as follows. In Section 2 we derive the optimality equations and give a heuristic argument leading to Theorem 1.1. Section 3 contains some lemmas. In Section 4 we derive a general theorem for the convergence of recursions. These are applied in Section 5 to the family of Pareto distributions, and are then, in Section 6, generalized to all distributions considered in Theorem 1.1. Section 7 contains a table of numerical evaluations and comparisons of the limiting scaled values, as n tends to infinity, of the optimal one and two stop values, and EM_n. It is seen that considerable improvement is obtained, through the possibility of a second choice. For example, numerical computations indicate that the limit, as n tends to infinity, of the ratio $[V_n^2 - V_n^1]/[EM_n - V_n^1]$ is never less than 78%.

After the present work was accepted, the paper of Kühne and Rüschendorf (2002) came to our attention. That paper treats the same problem as the one here, using Poisson-approximations, and in full generality, basing their results on their earlier detailed paper Kühne and Rüschendorf (2000). Detailed results are given for the domain of attraction $\mathcal{D}(e^{-e^{-x}})$, only. Our approach in the present paper is more direct, and our results for the present domain of attraction are more explicit than theirs, when (2) holds.

2. PRELIMINARIES AND HEURISTICS

For X an integrable random variable with distribution function F, let

$$g(x) = g_1(x) = E[X \vee x] \quad \text{and} \quad g_{n+1}(x) = g(g_n(x)), \quad n \geq 1.$$

The function $g_n(x)$ equals the optimal one-choice value when there are n observations to be made, and one is guaranteed the value x; for this reason when stressing this interpretation we denote $g_n(x)$ by $V_n^1(x)$. On the other hand, for notational clarity, we will drop the superscript on V_n^2, denoting it V_n. Clearly $g_n(x)$ is increasing in x for all n. The optimality equations for the two choice values are

$$\begin{aligned} V_2 &= E[X_1 \vee X_2] \\ V_{n+1} &= E[V_n \vee V_n^1(X_{n+1})] \quad n \geq 2. \end{aligned} \tag{7}$$

Note that we 'reverse' index our sequence of variables $X_n, \ldots, X_1$ for convenience, so that X_k denotes the k^{th} variable from the end of the horizon. The first term in the expectation (7) corresponds to passing up X_{n+1} and keeping two choices on the remaining n variables; the second term corresponds to choosing X_{n+1} and continuing with one choice on the remaining variables with the value X_{n+1} guaranteed.

We now present an outline of the argument which yields our result and an accompanying simple heuristic. From (7), letting b_n be defined through

$$V_n = g_n(b_n), \tag{8}$$

we see that if $X_{n+1} > b_n$ then it should be chosen and otherwise passed up. Write (7) in the form

$$V_{n+1} = V_n F(b_n) + \int_{b_n}^{\infty} g_n(x) dF(x). \tag{9}$$

As a representative of $F \in \mathcal{D}(G_{II}^{\alpha})$ we first consider, for $\alpha > 1$ fixed, the Pareto distribution with

$$F_\alpha(x) = [1 - x^{-\alpha}]\mathbf{I}(x \geq 1) \quad \text{and density} \quad \alpha x^{-(\alpha+1)}\mathbf{I}(x \geq 1). \tag{10}$$

For this family

$$g(x) = x + \frac{1}{\alpha - 1} x^{-(\alpha-1)} \qquad \text{for } x \geq 1 \tag{11}$$

$$\text{so} \quad g_{n+1}(x) = g_n(x) + \frac{1}{\alpha - 1} g_n(x)^{-(\alpha-1)} \qquad \text{for } x \geq 1. \tag{12}$$

Since $P(X \geq 1) = 1$ we have $V_n^1 = g_n(1)$.

In particular, for the Pareto family, (9) becomes

$$V_{n+1} = V_n[1 - b_n^{-\alpha}] + \alpha \int_{b_n}^{\infty} g_n(x) x^{-(\alpha+1)} dx. \tag{13}$$

For $y \geq 1/n$ let

$$f_n(y) = n^{-1/\alpha} g_n((ny)^{1/\alpha}), \quad \text{that is,} \quad g_n(x) = n^{1/\alpha} f_n(x^\alpha / n) \tag{14}$$

and

$$W_n^1 = n^{-1/\alpha} V_n, \quad \text{and} \quad B_n = n^{-1} b_n^\alpha,$$

thus, from (8),

$$W_n = f_n(B_n). \tag{15}$$

Note that $g_n(x)$ is strictly increasing for $x \geq 1$, and hence $f_n(x)$ is strictly increasing for $x \geq 1/n$. Rewriting (13) in this notation and multiplying by $n^{-1/\alpha}$ yields

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1} = W_n \left[1 - \frac{1}{nB_n}\right] + \alpha \int_{b_n}^{\infty} f_n\left(\frac{x^\alpha}{n}\right) x^{-(\alpha+1)} dx,$$

and making the change of variable $y = x^\alpha/n$, we obtain

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1} = W_n \left[1 - \frac{1}{nB_n}\right] + n^{-1} \int_{B_n}^{\infty} f_n(y) y^{-2} dy. \quad (16)$$

When the change of variable is done directly in (7), this expression can also be written as

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1} = n^{-1} \int_{1/n}^{\infty} [W_n \vee f_n(y)] y^{-2} dy. \quad (17)$$

Since $((n+1)/n)^{1/\alpha} = 1 + 1/(\alpha n) + O(n^{-2})$, if we multiply (16) by n we have

$$n(W_{n+1} - W_n) = -\frac{1}{\alpha} W_{n+1} - \frac{1}{B_n} W_n + \int_{B_n}^{\infty} f_n(y) y^{-2} dy + O(n^{-1}). \quad (18)$$

If $W_n \to W$ such that $W_n = W + a/n + O(n^{-2})$, then $n(W_{n+1} - W_n) \to 0$ and if also $B_n \to B_\alpha$ and $f_n(y) \to f(y)$ then taking limits in (18) yields

$$0 = -\left(\frac{1}{\alpha} + \frac{1}{B_\alpha}\right) W + \int_{B_\alpha}^{\infty} f(y) y^{-2} dy \quad (19)$$

and by (15)

$$W = f(B_\alpha). \quad (20)$$

To find f we use the following heusistics: set, for short $\rho = \alpha/(\alpha - 1)$. By (5)

$$V_n^1 = g_n(1) \approx (n\rho)^{1/\alpha}.$$

Suppose for a given x we can find k such that

$$g_n(x) \approx g_{n+k}(1) \approx ((n+k)\rho)^{1/\alpha}.$$

But

$$g_{n+k}(1) = g_n(g_k(1)), \text{ thus } x \approx g_k(1) \approx (k\rho)^{1/\alpha},$$

i.e.

$$k \approx x^\alpha/\rho,$$

and thus

$$g_n(x) \approx ((n + x^\alpha/\rho)\rho)^{1/\alpha} = (n\rho + x^\alpha)^{1/\alpha}.$$

Set $y = x^\alpha/n$ to get

$$g_n((ny)^{1/\alpha}) \approx n^{1/\alpha}(\rho + y)^{1/\alpha}.$$

Thus by (14)

$$f_n(y) \approx \left(\frac{\alpha}{\alpha-1} + y\right)^{1/\alpha}.$$

Let

$$f(y) = \left(\frac{\alpha}{\alpha-1} + y\right)^{1/\alpha}. \tag{21}$$

Note that by (5) and (21)

$$\lim_{y\to 0} f(y) = \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} = \lim_{n\to\infty} n^{-1/\alpha} V_n^1.$$

Also, (20) translates to

$$W = \left(\frac{\alpha}{\alpha-1} + B_\alpha\right)^{1/\alpha}. \tag{22}$$

Set $y = 1/u$ and $h(u) = f(1/u)$ in (22) and (21), and write $\beta_\alpha = 1/B_\alpha$ to obtain

$$h(u) = \left(\frac{\alpha}{\alpha-1} + \frac{1}{u}\right)^{1/\alpha}, \tag{23}$$

and

$$W = h(\beta_\alpha).$$

Now substituting into (19)

$$\int_0^{\beta_\alpha} h(u)du - \left(\frac{1}{\alpha} + \beta_\alpha\right) h(\beta_\alpha) = 0$$

which explains the theorem.

3. SOME LEMMAS

In this section we determine the limiting behavior of the sequence of functions $f_n, n = 1, 2, \ldots$ given in (14), for determining the asymptotic behavior of the two stop value for an i.i.d. sequence with distribution (10), a representative of $F \in \mathcal{D}(G_{II}^\alpha)$.

Lemma 3.1. *Let $f_n(y)$ and $f(y)$ be given in (14) and (21), respectively. Then*

$$f_n(y) > f(y) \text{ for all } y \geq 1/n. \tag{24}$$

Proof. We prove (24) by induction. For $n = 1$ by (11)

$$f_1(y) = g_1(y^{1/\alpha}) = y^{1/\alpha} + \frac{1}{\alpha - 1} y^{-1+1/\alpha} = y^{1/\alpha}\left(1 + \frac{1}{(\alpha-1)y}\right), y \geq 1$$

while

$$f(y) = y^{1/\alpha}\left(1 + \frac{\alpha}{(\alpha-1)y}\right)^{1/\alpha}.$$

But $1/\alpha < 1$, therefore $(1+x)^{1/\alpha} < 1 + x/\alpha$, thus $f(y) < y^{1/\alpha}\left(1 + \frac{1}{(\alpha-1)y}\right) = f_1(y)$, and (24) holds for $n = 1$. Now assume (24) holds for some $n \geq 1$. Then by (14), for $x \geq 1$,

$$g_n(x) = n^{1/\alpha} f_n\left(\frac{x^\alpha}{n}\right) > n^{1/\alpha} f\left(\frac{x^\alpha}{n}\right) = x\left(\frac{\alpha n}{(\alpha-1)x^\alpha} + 1\right)^{1/\alpha} \quad (25)$$

and thus, since g is increasing

$$\begin{aligned} g_{n+1}(x) = g(g_n(x)) &> x\left(\frac{\alpha n}{(\alpha-1)x^\alpha} + 1\right)^{1/\alpha}\left\{1 + \frac{1}{\alpha-1}\left[x^\alpha\left(\frac{\alpha n}{(\alpha-1)x^\alpha} + 1\right)\right]^{-1}\right\} \\ &= x\left(\frac{\alpha n}{(\alpha-1)x^\alpha} + 1\right)^{1/\alpha}\left[1 + \frac{1}{\alpha n + (\alpha-1)x^\alpha}\right]. \end{aligned} \quad (26)$$

Thus, similar to (25), it suffices to show that for $x \geq 1$ the right hand side of (26) is greater than

$$x\left(\frac{\alpha(n+1)}{(\alpha-1)x^\alpha} + 1\right)^{1/\alpha},$$

i.e. that

$$\frac{\alpha n + (\alpha-1)x^\alpha}{(\alpha-1)x^\alpha}\left(1 + \frac{1}{\alpha n + (\alpha-1)x^\alpha}\right)^\alpha > \frac{\alpha(n+1) + (\alpha-1)x^\alpha}{(\alpha-1)x^\alpha}.$$

Put over the common factor $(\alpha-1)x^\alpha$, the numerator on the left hand side is greater than $(\alpha n + (\alpha-1)x^\alpha)\left(1 + \frac{\alpha}{\alpha n + (\alpha-1)x^\alpha}\right) = \alpha(n+1) + (\alpha-1)x^\alpha$ which is the numerator of the right hand side. ■

Lemma 3.2. *Let $\varepsilon_n(y) = f_n(y) - f(y)$. Then for $n \geq 1$*

$$\varepsilon_n(y) < \frac{y^{1/\alpha - 2}}{2(\alpha-1)n} \quad \text{for } y \geq 1/n. \quad (27)$$

Proof: Consider $(1+t)^\delta$ where $0<\delta<1$, $t>0$. Then by Taylor expansion, for some $0<\theta<1$

$$(1+t)^\delta = 1+\delta t - \frac{\delta(1-\delta)t^2}{2(1+\theta t)^{2-\delta}} > 1+\delta t - \frac{\delta(1-\delta)t^2}{2}.$$

Hence for $t=\alpha/((\alpha-1)y)$ and $\delta=1/\alpha$,

$$f(y) = y^{1/\alpha}\left(1+\frac{\alpha}{(\alpha-1)y}\right)^{1/\alpha} > y^{1/\alpha}\left[1+\frac{1}{(\alpha-1)y}-\frac{1}{2(\alpha-1)y^2}\right]. \tag{28}$$

We prove (27) by induction. For $n=1$ we have $f_1(y)=y^{1/\alpha}(1+\frac{1}{(\alpha-1)y})$, for $y\geq 1$. Thus (27) follows immediately for $n=1$ by (28). Now suppose (27) holds for some $n\geq 1$. Set $a_n=n/(n+1)$. Then by (14) and (12) it follows, after setting $y=x^\alpha/n$, that

$$a_n^{-1/\alpha} f_{n+1}(a_n y) = f_n(y) + \frac{1}{(\alpha-1)n} f_n(y)^{-(\alpha-1)}. \tag{29}$$

We want to show that $\varepsilon_{n+1}(y) < \frac{y^{1/\alpha-2}}{2(\alpha-1)(n+1)}$ for $y\geq 1/(n+1)$. This is equivalent to

$$\varepsilon_{n+1}(a_n y) < \frac{a_n^{1/\alpha-2}y^{1/\alpha-2}}{2(\alpha-1)(n+1)} = \frac{a_n^{1/\alpha}y^{1/\alpha-2}(n+1)}{2(\alpha-1)n^2} \text{ for } y\geq 1/n. \tag{30}$$

Now, by (29)

$$\begin{aligned}\varepsilon_{n+1}(a_n y) &= f_{n+1}(a_n y) - f(a_n y) \qquad (31)\\ &= a_n^{1/\alpha}\left\{f_n(y) + \frac{1}{(\alpha-1)n} f_n(y)^{-(\alpha-1)} - a_n^{-1/\alpha} f(a_n y)\right\}.\end{aligned}$$

Note that

$$f'(y) = \frac{1}{\alpha} f(y)^{-(\alpha-1)} > 0 \text{ and } f''(y) = \frac{-(\alpha-1)}{\alpha^2} f(y)^{-(2\alpha-1)} < 0.$$

Thus by Taylor expansion we can write, for some $0<\theta<1$,

$$f(x+\Delta) = f(x) + \frac{\Delta}{\alpha} f(x)^{-(\alpha-1)} - \frac{\Delta^2(\alpha-1)}{2\alpha^2} f(x+\theta\Delta)^{-(2\alpha-1)}.$$

Thus with $\Delta = \frac{\alpha}{n(\alpha-1)}$, we have

$$a_n^{-1/\alpha} f(a_n y) = \left(\frac{n+1}{n}\frac{\alpha}{\alpha-1}+y\right)^{1/\alpha} = f\left(y+\frac{\alpha}{n(\alpha-1)}\right) \tag{32}$$

$$= f(y) + \frac{1}{n(\alpha-1)} f(y)^{-(\alpha-1)} - \frac{1}{2n^2(\alpha-1)} f(y+\frac{\theta\alpha}{n(\alpha-1)})^{-(2\alpha-1)}.$$

Substituting (32) into (31) we obtain $\varepsilon_{n+1}(a_n y)$ is $a_n^{1/\alpha}$ times

$$\varepsilon_n(y) + \frac{1}{n(\alpha-1)} \; (f_n(y)^{-(\alpha-1)} - f(y)^{-(\alpha-1)}) \\ + \; \frac{1}{2n^2(\alpha-1)} f(y + \frac{\theta\alpha}{n(\alpha-1)})^{-(2\alpha-1)}. \tag{33}$$

Now by Lemma 3.1, since $\alpha > 1$, we have $f_n(y)^{-(\alpha-1)} < f(y)^{-(\alpha-1)}$. Thus the second term in (33) is negative. Also $f(y)$ is increasing, thus $f(y)^{-(2\alpha-1)}$ is decreasing, and

$$f^{-(2\alpha-1)}\left(y + \frac{\theta\alpha}{n(\alpha-1)}\right) < f^{-(2\alpha-1)}(y) = \frac{1}{\left(\frac{\alpha}{\alpha-1} + y\right)^{2-1/\alpha}} < y^{1/\alpha-2}.$$

Thus from (33) and the induction hypothesis, for $y \geq 1/n$, and therefore for $a_n y \geq 1/(n+1)$, we have

$$\begin{aligned} \varepsilon_{n+1}(a_n y) &< a_n^{1/\alpha}\{\frac{y^{1/\alpha-2}}{2(\alpha-1)n} + \frac{1}{2n^2(\alpha-1)} y^{1/\alpha-2}\} \\ &= a_n^{1/\alpha} y^{1/\alpha-2}(n+1)/[2(\alpha-1)n^2] \end{aligned}$$

which shows (30). ■

The following Corollary is an immediate consequence of Lemmas 3.1 and 3.2.

Corollary 3.1. *Let*

$$h_n(u) = f_n\left(\frac{1}{u}\right) \quad 0 < u \leq n, \tag{34}$$

where f_n is defined in (14). Then for all $y > 0$, as $n \to \infty$,

$$f_n(y) \to f(y) = \left(\frac{\alpha}{\alpha-1} + y\right)^{1/\alpha} \quad \textit{and}$$

$$h_n(y) \to h(y) = \left(\frac{\alpha}{\alpha-1} + \frac{1}{y}\right)^{1/\alpha}.$$

Note that

$$\lim_{y\to\infty} h(y) = \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} = \lim_{n\to\infty} n^{1/\alpha} V_n^1,$$

and that (2) holds for (10) with $\mathcal{L} = 1$.

With the notation (34) we can rewrite (17) as

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1} = n^{-1} \int_0^n [h_n(u) \vee W_n] du \tag{35}$$

and if we let $\beta_n = 1/B_n$, then, from (15)

$$W_n = h_n(\beta_n).$$

4. CONVERGENCE OF RECURSIONS

We consider functions q and Q satisfying the following condition. We allow $q(0)$ to take the value infinity.

Condition 4.1. There exist positive numbers A and a such that $q(y)$ is nonnegative for $y \geq 0$, strictly monotone decreasing and differentiable for $0 < y < A$, non-increasing for $y \geq A$ and $\int_0^a q(u)du < \infty$. Further, $Q(A) > 0$ where $Q(y)$ is defined by

$$Q(y) = \int_0^y q(u)du - (1/\alpha + y)\, q(y). \tag{36}$$

Lemma 4.1. *Under Condition 4.1, the function $Q(y)$ is strictly increasing in $(0, A)$, non-decreasing for $y > A$ and there exists a unique value b for which $Q(b) = 0$ and $b < A$.*

Proof. Let $0 \leq y_1 < y_2$. Elementary calculations yield

$$Q(y_2) - Q(y_1) \geq \left(\frac{1}{\alpha} + y_1\right)(q(y_1) - q(y_2)),$$

and monotonicity now follows from noting the latter expression is positive if $y_1 < A$, and non-negative otherwise. Since $Q(0) < 0$ and $Q(A) > 0$ the claim on the root b follows. ■

Our aim is to prove

Theorem 4.1. *Let Condition 4.1 be satisfied, and let $m \geq 1$ and c be arbitrary. Define*

$$Z_m = c \text{ and } \left(\frac{n+1}{n}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\int_0^n (q(u) \vee Z_n) du \quad \text{for } n \geq m. \tag{37}$$

Then the limit of Z_n exists and satisfies

$$\lim_{n\to\infty} Z_n = d = q(b)$$

where b is the unique root of $Q(y) = 0$.

The crux of the proof of Theorem 4.1 is the following Lemma.

Lemma 4.2. *Let Condition 4.1 be satisfied, and let $m \geq 1$ be any integer and c any constant, and suppose that Z_n for $n \geq m$ is defined by (37). Then for every $\delta \in (0, \min\{q(0) - d, d - q(A)\})$ there exist $\Delta > 0$ and n_0 such that for all $n \geq n_0$,*

$$\text{if } Z_n > d + \delta \text{ then } Z_{n+1} \leq (1 - \Delta/n) Z_n, \tag{38}$$

$$\text{if } Z_n < d - \delta \text{ then } Z_{n+1} \geq (1 + \Delta/n) Z_n, \tag{39}$$

$$\text{if } Z_n > d \text{ then } Z_{n+1} > d, \text{ and} \tag{40}$$

$$\text{if } |Z_n - d| \leq \delta \text{ then } |Z_{n+1} - d| \leq \delta. \tag{41}$$

Proof: We have

$$(\frac{n}{n+1})^{1/\alpha} = 1 - \frac{1}{\alpha n} + \frac{1}{\alpha}(\frac{1}{\alpha} + 1)\frac{1}{2n^2} + O_\alpha(n^{-3}),$$

and hence for any $\rho \neq 0$

$$(\frac{n}{n+1})^{1/\alpha}(1+\frac{1}{\rho n}) = 1+\frac{1}{n}(\frac{1}{\rho} - \frac{1}{\alpha}) + \frac{1}{n^2}\left(\frac{1}{2\alpha}(\frac{1}{\alpha} + 1) - \frac{1}{\alpha\rho}\right) + O_{\alpha,\rho}(n^{-3}), \tag{42}$$

where we write $O_\lambda(c_n)$ to indicate a sequence bounded in absolute value by c_n times a constant depending only on λ, a collection of parameters.

Define

$$M(t) = \int_0^{q^{-1}(t)} \left(\frac{q(y)}{t} - 1\right) dy \quad \text{for } q(A) < t \leq q(0).$$

From (36), $Q(b) = 0$ and $d = q(b)$, we have

$$M(d) = 1/\alpha.$$

It is not hard to see that $M(t)$ is strictly decreasing over its range. Hence, setting $\Delta_2 = (M(d-\delta) - 1/\alpha)/2$ and $\Delta_1 = (1/\alpha - M(d+\delta))/2$ we have $\Delta = \min\{\Delta_1, \Delta_2\} > 0$.

Consider the function

$$r_n(t) = \frac{1}{n}\int_0^n \left(\frac{q(y)}{t} \vee 1\right) dy.$$

We first show that $Z_n < q(0)$ for all n sufficiently large. This is obvious when $q(0) = \infty$. If $q(0) < \infty$ then note first that $Z_n < q(0)$ implies

$Z_{n+1} < q(0)$. Now assuming that $Z_n > q(0)$ for all n sufficiently large gives $Z_{n+1} \leq (n/(n+1))^{1/\alpha} Z_n$ and the contradiction that $Z_n \to 0$ by $\prod_k (k/(k+1))^{1/\alpha} = 0$.

Since $Z_{m+1} > 0$ we have $Z_n > 0$ for all $n \geq m+1$, and now by (37)

$$Z_{n+1}/Z_n = \left(\frac{n}{n+1}\right)^{1/\alpha} r_n(Z_n). \tag{43}$$

By definition, for $q(A) < t \leq q(0)$

$$r_n(t) = 1 + \frac{1}{n}\int_0^{q^{-1}(t)\wedge n} \left(\frac{q(y)}{t} - 1\right) dy = 1 + \frac{1}{n} M(t) \quad \text{when } n \geq q^{-1}(t).$$

To prove (38), assume $Z_n > d + \delta$. Since r_n is decreasing, using (43) and (42), we have for all $n > q^{-1}(d+\delta)$,

$$\begin{aligned}
Z_{n+1} &\leq Z_n(\frac{n}{n+1})^{1/\alpha} r_n(d+\delta) \\
&= Z_n(\frac{n}{n+1})^{1/\alpha}(1 + \frac{1}{n}M(d+\delta)) \\
&= \left[1 + \frac{1}{n}(M(d+\delta) - \frac{1}{\alpha}) + O_{\alpha,d-\delta}(n^{-2})\right] Z_n \\
&\leq (1 - \frac{\Delta_1}{n})Z_n \leq (1 - \frac{\Delta}{n})Z_n
\end{aligned}$$

for all n sufficiently large, showing (38).

Next we prove (39). When $Z_n < d - \delta$, we have similarly that for $n > q^{-1}(d-\delta)$,

$$\begin{aligned}
Z_{n+1} &\geq Z_n(\frac{n}{n+1})^{1/\alpha} r_n(d-\delta) \\
&= Z_n(\frac{n}{n+1})^{1/\alpha}(1 + \frac{1}{n}M(d-\delta)) \\
&= \left[1 + \frac{1}{n}(M(d-\delta) - \frac{1}{\alpha}) + O_{\alpha,d-\delta}(n^{-2})\right] Z_n \\
&\geq (1 + \frac{\Delta_2}{n})Z_n \geq (1 + \frac{\Delta}{n})Z_n
\end{aligned}$$

for all n sufficiently large.

Turning now to (40) and (41), for $Z_n \geq d-\delta$, since $d-\delta > q(A)$, $\beta_n < A$ is well defined by

$$q(\beta_n) = Z_n.$$

Now by (37) and (36), for $n \geq A$,

$$\left(\frac{n+1}{n}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\left(\int_0^{\beta_n} q(y)dy + (n-\beta_n)q(\beta_n)\right) = \frac{1}{n}Q(\beta_n)+(1+\frac{1}{\alpha n})Z_n;$$

thus

$$Z_{n+1} = \left(\frac{n}{n+1}\right)^{1/\alpha} \frac{1}{n}Q(\beta_n) + R_n Z_n \tag{44}$$

where

$$R_n = (\frac{n}{n+1})^{1/\alpha}(1+\frac{1}{\alpha n}). \tag{45}$$

For $u > q(A)$ consider

$$Q(q^{-1}(u)) = \int_0^{q^{-1}(u)} q(y)dy - (1/\alpha + q^{-1}(u))u.$$

Since $q^{-1}(u)$ is differentiable for $u > q(A)$, for such u

$$\frac{d}{du}Q(q^{-1}(u)) = -\left(1/\alpha + q^{-1}(u)\right).$$

Hence, evaluating $Q(q^{-1}(u))$ by a Taylor expansion around d, and using $Q(b) = Q(q^{-1}(d)) = 0$, we obtain that there exists some ξ_{Z_n} between d and Z_n such that

$$Q(\beta_n) = Q(q^{-1}(Z_n)) = -(Z_n - d)(1/\alpha + q^{-1}(\xi_{Z_n})). \tag{46}$$

Subtracting d from both sides of (44) and using (46) we obtain

$$Z_{n+1}-d = \left\{1 - (\frac{n}{n+1})^{1/\alpha}\frac{1}{n}(\frac{1}{\alpha} + q^{-1}(\xi_{Z_n}))\right\}(Z_n-d)+[R_n-1]Z_n. \tag{47}$$

Take n_1 such that for all $n \geq n_1$

$$(\frac{n}{n+1})^{1/\alpha}\frac{1}{n}(\frac{1}{\alpha} + q^{-1}(d-\delta)) < 1 \quad \text{and} \quad 0 < R_n - 1,$$

where we use (45) and (42) with $\rho = \alpha > 1$ for the second inequality.

Take now $Z_n > d$. Then $\xi_{Z_n} > d$, $q^{-1}(\xi_{Z_n}) < q^{-1}(d) < q^{-1}(d-\delta)$, and for $n \geq n_1$

$$0 < \left\{1 - (\frac{n}{n+1})^{1/\alpha}\frac{1}{n}(\frac{1}{\alpha} + q^{-1}(\xi_{Z_n}))\right\},$$

so that the first term on the right hand side of (47) is strictly positive. For $n \geq n_1$ the second term on the right hand side is also positive, and the sum of these two terms is therefore positive. This proves (40).

Turning to (41), suppose that $|Z_n - d| \leq \delta$. Then $|\xi_{Z_n} - d| \leq \delta$, and therefore

$$q^{-1}(d+\delta) \leq q^{-1}(\xi_{Z_n}) \leq q^{-1}(d-\delta),$$

Hence, letting $\Delta_3 = (1/\alpha + q^{-1}(d+\delta))/2 > 0$ we have $(1/\alpha + q^{-1}(\xi_{Z_n}))/2 \geq \Delta_3$ and

$$0 \leq \left\{1 - (\frac{n}{n+1})^{1/\alpha}\frac{1}{n}(\frac{1}{\alpha} + q^{-1}(\xi_{Z_n}))\right\} \leq 1 - \frac{\Delta_3}{n.} \tag{48}$$

Further, from (45), again using (42) with $\rho = \alpha$, there exists K_α such that

$$|R_n - 1| \leq \frac{K_\alpha}{n^2}.$$

Then for all n so large that

$$\frac{K_\alpha}{n}(d+\delta) \leq \Delta_3 \delta$$

we have, using (47) and (48),

$$\begin{aligned} |Z_{n+1} - d| &\leq (1 - \frac{\Delta_3}{n})|Z_n - d| + |R_n - 1|Z_n \\ &\leq (1 - \frac{\Delta_3}{n})\delta + \frac{K_\alpha}{n^2}(d+\delta) \\ &\leq \delta. \end{aligned}$$

This proves (41). ■

Proof of Theorem 4.1: Let $0 < \delta < d - q(A)$, and $n \geq n_0$.

Case I: $Z_{n_0} < d - \delta$. If $Z_n < d - \delta$ for all $n \geq n_0$ then by (39) we would have

$$Z_{n+1} \geq \prod_{j=n_0}^{n} (1 + \frac{\Delta}{j}) Z_{n_0} \to \infty,$$

a contradiction. Hence for some $n_1 \geq n_0$ we have $Z_{n_1} \geq d - \delta$, and we would therefore be in Case II or Case III.

Case II: $Z_{n_1} > d + \delta$ for some $n_1 \geq n_0$. If $Z_n > d + \delta$ for all $n \geq n_1$ we would have, by (38), that

$$Z_{n+1} \leq \prod_{j=n_1}^{n} (1 - \frac{\Delta}{j}) Z_{n_1} \to 0,$$

again a contradiction. Hence there exists $n_2 \geq n_1$ such that $Z_{n_2} \leq d + \delta$. By (40), $Z_{n_2} > d$, reducing to Case III.

Case III: $|Z_{n_1} - d| \leq \delta$ for some $n_1 \geq n_0$. In this case $|Z_n - d| \leq \delta$ for all $n \geq n_1$ by (41). Since δ can be taken arbitrarily small, the Theorem is complete. ■.

5. THE PARETO FAMILY

Let H be to h in (23) as Q is to q in (36). We first show

Lemma 5.1. *There exists a unique value β_α such that $H(\beta_\alpha) = 0$.*

Proof. Note that $h(y)$ is strictly decreasing and differentiable for $0 < y < \infty$ and that $\int_0^a h(u) < \infty$ for all $a > 0$ since $\alpha > 1$. This Lemma therefore follows from Lemma 4.1 if we show that $H(y)$ is positive for some y.

Now

$$H'(y) = -\left(\frac{1}{\alpha} + y\right) h'(y) = \frac{1}{\alpha}\left(\frac{1}{\alpha} + y\right) y^{-2} h(y)^{-(\alpha-1)}.$$

Since $\lim_{y\to\infty} h(y) = (\alpha/(\alpha-1))^{\frac{1}{\alpha}} < \infty$ it follows that $\int_a^\infty H'(y)dy = \infty$, thus $\lim_{y\to\infty} H(y) = \infty$ and the Lemma follows. ■

By Lemmas 3.1, 3.2 and (34), for $j = 1, 2, \ldots$ and $0 < y \leq j$

$$h(y) < h_j(y) < h(y) + \frac{y^{2-\frac{1}{\alpha}}}{2(\alpha-1)j} := \tilde{h}_j(y). \tag{49}$$

Recall that h and h_j are strictly monotone decreasing in their respective ranges.

The derivative of $\tilde{h}_j$ as defined in (49) is

$$\frac{d\tilde{h}_j(y)}{dy} = -\frac{1}{\alpha} y^{-2} h(y)^{-(\alpha-1)} + \frac{2\alpha - 1}{2\alpha(\alpha-1)j} y^{1-\frac{1}{\alpha}}.$$

Fix $A > \beta_\alpha$. It follows that for all $j > j_0(A)$ the function $\tilde{h}_j(y)$ is strictly monotone decreasing in $y \in (0, A]$. For $j > j_0(A)$ let

$$\tilde{h}_j^A(y) = \begin{cases} \tilde{h}_j(y) & \text{for } 0 < y \leq A \\ \tilde{h}_j(A) & \text{for } A < y < \infty. \end{cases}$$

Since h_j is strictly decreasing it follows from (49) that for $0 < y \leq j$

$$h_j(y) \leq \tilde{h}_j^A(y). \tag{50}$$

Note also that the sequence $\tilde{h}_j^A(y)$ is monotone decreasing in j, that is, if $j < n$

$$\tilde{h}_n^A(y) < \tilde{h}_j^A(y) \text{ for all } 0 < y < \infty. \tag{51}$$

Lemma 5.2. *Let $\tilde{H}_j^A(y)$ be defined for $\tilde{h}_j^A(y)$ through (36). Then for all $j > j_1(A)$ there exists a value $\beta_{j,\alpha}$ such that $\tilde{H}_j^A(\beta_{j,\alpha}) = 0$, and with $d_\alpha = h(\beta_\alpha)$ and $d_{j,\alpha} = \tilde{h}_j^A(\beta_{j,\alpha})$,*

$$\lim_{j\to\infty} \beta_{j,\alpha} = \beta_\alpha \quad \textit{and} \quad \lim_{j\to\infty} d_{j,\alpha} = d_\alpha. \tag{52}$$

Proof: Since $\tilde{H}_j^A(y) \to H(y)$ uniformly on $[0, A]$ as $j \to \infty$ it follows that $\lim \tilde{H}_j^A(A) = H(A) > H(\beta_\alpha) = 0$. Hence for all $j > j_1(A)$ the value $\beta_{j,\alpha}$ exists. Now (52) follows from the uniform convergence of $\tilde{h}_j^A$ and $\tilde{H}_j^A$ to h and H, respectively. ■

It will become convenient to consider value and scaled value sequences arising from stopping on the independent variables $X_n, \dots, X_{m+1}, Y_m, \dots, Y_1$. The scaled value sequence W_n for this problem satisfies (35) for $n \geq m$ with starting value $W_m = m^{-1/\alpha} V_m(Y_m, \dots, Y_1)$. Note that for any m and c there exists $Y_m, \dots, Y_1$ such that $c = m^{-1/\alpha} V_m(Y_m, \dots, Y_1)$; the simplest construction is obtained by letting $Y_j = cm^{1/\alpha}$ for $1 \leq j \leq m$. Our suppression of the dependence of W_n on m and c is justified by Theorem 5.1, which states that the limiting value of W_n is the same for all such sequences.

Lemma 5.3. *Let $m \geq 1$ be any integer and c be any constant. Let $W_m = c$ and for $n > m$ let W_n be determined by the recursion (35). Let*

$$Z_m^- = c \textit{ and } \left(\frac{n+1}{n}\right)^{1/\alpha} Z_{n+1}^- = \frac{1}{n}\int_0^n (h(y) \vee Z_n^-) dy \textit{ for } n \geq m. \tag{53}$$

For $j > j_1(A)$ fixed let $m_j = \max\{m, j\}$ and define the sequence $Z_{j,n}^+$ through

$$Z_{j,m_j}^+ = W_{m_j} \textit{ and}$$
$$\left(\frac{n+1}{n}\right)^{1/\alpha} Z_{j,n+1}^+ = \frac{1}{n}\int_0^n (\tilde{h}_j^A(y) \vee Z_{j,n}^+) dy, \quad n \geq m_j. \tag{54}$$

Then for all $n \geq m_j$

$$Z_n^- \leq W_n \leq Z_{j,n}^+ \tag{55}$$

and

$$\lim_{n\to\infty} Z_n^- = d_\alpha = h(\beta_\alpha) \text{ and } \lim_{n\to\infty} Z_{j,n}^+ = d_{j,\alpha} = \tilde{h}_j^A(\beta_{j,\alpha}) \tag{56}$$

Proof. Since by (49), (51) and (50) for all $n > j > j_1(A)$

$$h(y) < h_n(y) \le \tilde{h}_n^A(y) < \tilde{h}_j^A(y), \quad 0 < y \le n,$$

we obtain (55) directly by a comparison of the definitions in (35), (53) and (54). The conclusions in (56) are immediate from Theorem 4.1. ■

Theorem 5.1. *Let $m \ge 1$ be any integer, let $X_n, \ldots, X_{m+1}, Y_m, \ldots, Y_1$ be independent random variables, where $X_i \sim F_\alpha$ of (10) and $Y_m, \ldots, Y_1$ have finite expectation. For $n > m$, let*

$$V_{n,m} = V_n(X_n, \ldots, X_{m+1}, Y_m, \ldots, Y_1)$$

be the optimal two choice value. Then $W_n = n^{-1/\alpha} V_{n,m}$ satisfies

$$\lim_{n\to\infty} W_n = h(\beta_\alpha) = d_\alpha \tag{57}$$

where β_α is defined through (6). In particular, the optimal two stop value V_n for the sequence of i.i.d. r.v.'s with distribution function F_α of (10) satisfies

$$\lim_{n\to\infty} n^{-1} V_n^\alpha = h(\beta_\alpha)^\alpha,$$

that is, Theorem 1.1 holds for the family of distributions F_α.

Proof. Applying Lemma 5.3 with $c = m^{-1/\alpha} V_m(Y_m, \ldots, Y_1)$, for all $j > j_1(A)$

$$d_\alpha \le \liminf_{n\to\infty} W_n \le \limsup_{n\to\infty} W_n \le d_{j,\alpha}.$$

Now let $j \to \infty$ and use (52) to get (57). Clearly the values W_n for the i.i.d. sequence with distribution F_α are generated by recursion (35), $m = 2$ and $c = 2^{-1/\alpha} E[X_1 \vee X_2]$. ■

6. EXTENSION TO GENERAL DISTRIBTUIONS

Let $F \in \mathcal{D}(G_{II}^\alpha)$. By Proposition 2.1 of Resnick (1987),

$$\text{if for some integer } 0 < k < \alpha, \int_{-\infty}^0 |x|^k dF(x) < \infty, \tag{58}$$

$$\text{then} \quad \lim_{n\to\infty} E[n^{-1/\alpha} M_n]^k = \Gamma(1 - \alpha^{-1} k).$$

Since we are considering random variables with finite expectation, it follows that F satisfies (58) with $k = 1$.

It suffices to prove Theorem 1.1 for positive random variables. Indeed, let X be a random variable with finite mean but otherwise arbitrary, X^+ be the positive part of X, and V_n and V_n^+ the corresponding two stop values. Clearly we have

$$V_n \leq V_n^+. \tag{59}$$

For an inequality in the other direction, note that when we apply the optimal rules on the X^+ sequence, if the first variable selected is at time $t_1 < n$ it was because the positive threshold value b_n was exceeded, so that

$$X_{t_1}^+ \quad \text{is positive on the event} \quad \{t_1 < n\}.$$

Hence, applying the optimal X^+ rules on the X sequence, which may not be optimal for it, we obtain

$$X_{t_1} = X_{t_1}^+ \quad \text{on the event} \quad \{t_1 < n\},$$

and moreover that

$$X_{t_1} \vee X_{t_2} = X_{t_1}^+ \vee X_{t_2}^+ \quad \text{on the event} \quad \{t_1 < n\},$$

yielding

$$V_n \geq V_n^+ P(t_1 < n) - E[\max(0, -X)]P(t_1 = n).$$

Since $E|X| < \infty$ and $P(t_1 < n) \to 1$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} n^{-1/\alpha} V_n \geq \lim_{n\to\infty} n^{-1/\alpha} V_n^+$$

which combined with (59) gives $\lim_{n\to\infty} n^{-1/\alpha} V_n = \lim_{n\to\infty} n^{-1/\alpha} V_n^+$. Thus, without loss of generality, we henceforth assume that $X \geq 0$.

We consider F satisfying (1), and (2). First note that without loss of generality we may assume that (2) holds with $\mathcal{L} = 1$, and prove Theorem 1.1 for this case only. This follows since if X is such that (1) and (2) hold, then for $Y = X/\mathcal{L}^{1/\alpha}$ we have $1 - F_Y(y) = y^{-\alpha}\hat{L}(y)$ for $\hat{L}(y) \to 1$ as $y \to \infty$, and $1 - F_Y(V_n^Y) = 1 - F_X(V_n^X)$, where V_n^X and V_n^Y are the optimal two choice values of iid sequences of length n from the F_X and F_Y distributions, respectively.

Lemma 6.1. *Let $X_\alpha \sim F_\alpha$, where F_α is given in (10) and let $X \geq 0$ with $X \sim F$ where $1 - F(x) = x^{-\alpha}L(x)$ and $\lim_{x\to\infty} L(x) = 1$. Then there exists a bounded function $L^*(x)$ satisfying $\lim_{x\to\infty} L^*(x) = 1$ such that*

$$X =_d X_\alpha L^*(X_\alpha). \tag{60}$$

Proof: Let

$$F^{-1}(u) = \sup\{x : F(x) < u\} \quad \text{for } u \in (0,1],$$

and

$$L^*(y) = F^{-1}(1 - y^{-\alpha})/y \quad \text{for } y > 1.$$

It is well known (see e.g. Lemma 6.4 of AGS) that for $U \sim \mathcal{U}[0,1]$,

$$X =_d F^{-1}(U);$$

since $F_\alpha(X_\alpha) = 1 - X_\alpha^{-\alpha} \sim \mathcal{U}[0,1]$, (60) follows. Now writing

$$F^{-1}(u) = \sup\{x : 1 - x^{-\alpha} L(x) < u\},$$

we have

$$F^{-1}(1 - y^{-\alpha}) = \sup\{x : 1 - x^{-\alpha} L(x) < 1 - y^{-\alpha}\} = \sup\{x : x L^{-1/\alpha}(x) < y\}.$$

Hence $\lim_{y\to\infty} L^*(y) = 1$ is equivalent to

$$\lim_{y\to\infty} \sup\{\frac{x}{y} : \frac{x}{y} < L^{1/\alpha}(x)\} = 1. \tag{61}$$

Since for every fixed x, $\lim_{y\to\infty} x/y = 0$, it follows that $x \to \infty$ as $y \to \infty$. But $\lim_{x\to\infty} L^{1/\alpha}(x) = 1$, and (61) follows. Let B be such that for $y \geq B$ we have $L^*(y) \leq 2$, say. Using $X \geq 0$, on $y \in (1, B]$ we have $0 \leq F^{-1}(0^+) \leq F^{-1}(1 - y^{-\alpha}) \leq F^{-1}(1 - B^{-\alpha})$. Hence the function $L^*(y)$ is bounded on its domain $(1, \infty)$. ■

Proof of Theorem 1.1 By (60) we write

$$X_i = X_{\alpha,i} L^*(X_{\alpha,i}) \quad \text{a.s.}$$

where $X_n, \ldots, X_1$ are i.i.d. with distribution satisfying the conditions of the Theorem with $\mathcal{L} = 1$, and $X_{\alpha,i}$ are distributed with distribution F_α of (10). Let X_{t_n} and $X_{\alpha,t_n(\alpha)}$ be the optimally stopped two stop random variables on the i.i.d. sequences $X_n, \ldots, X_1$ and $X_{n,\alpha}, \ldots, X_{1,\alpha}$, respectively, where t_n and $t_n(\alpha)$ denote the respective times corresponding to the optimal values. Let $\epsilon > 0$ be given and let c^+ be such that $L^*(x) < 1 + \epsilon$

for all $x \geq c^+$. Then

$$\limsup_{n\to\infty} E[n^{-1/\alpha} X_{t_n}] = \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n} L^*(X_{\alpha,t_n})]$$
$$= \limsup_{n\to\infty} \Big(E[n^{-1/\alpha} X_{\alpha,t_n} L^*(X_{\alpha,t_n}) \mathbf{I}(X_{\alpha,t_n} \geq c^+)]$$
$$+ E[n^{-1/\alpha} X_{\alpha,t_n} L^*(X_{\alpha,t_n}) \mathbf{I}(X_{\alpha,t_n} < c^+)] \Big)$$
$$= \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n} L^*(X_{\alpha,t_n}) \mathbf{I}(X_{\alpha,t_n} \geq c^+)] \quad (62)$$
$$\leq \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n} (1+\epsilon) \mathbf{I}(X_{\alpha,t_n} \geq c^+)] \quad (63)$$
$$= (1+\epsilon) \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n} \mathbf{I}(X_{\alpha,t_n} \geq c^+)]$$
$$= (1+\epsilon) \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n}]$$
$$\leq (1+\epsilon) \limsup_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)}] \quad (64)$$
$$= (1+\epsilon) h(\beta_\alpha),$$

where to obtain (62) we have used Lemma (6.1) to conclude that $X_{\alpha,t_n} L^*(X_{\alpha,t_n}) \mathbf{I}(X_{\alpha,t_n} < c^+)$ is bounded, and therefore the second expectation on the line above (62) has limit zero as $n \to \infty$; the last equality follows from Theorem 5.1.

Since (63) holds for any $\epsilon > 0$ we have

$$\limsup_{n\to\infty} n^{-1/\alpha} V_n(X_n, \ldots, X_1) \leq h(\beta_\alpha). \quad (65)$$

Now let c^- be such that $L^*(x) > 1 - \epsilon$ for all $x \geq c^-$. Consider using the rule $t_n(\alpha)$ on the sequence $X_n, \ldots, X_1$. Since this rule may not be optimal for that sequence, we have

$$\liminf_{n\to\infty} [n^{-1/\alpha} V_n(X_n, \ldots, X_1)] \geq \liminf_{n\to\infty} E[n^{-1/\alpha} X_{t_n(\alpha)}]$$
$$= \liminf_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)} L^*(X_{\alpha,t_n(\alpha)})]$$
$$= \liminf_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)} L^*(X_{\alpha,t_n(\alpha)}) \mathbf{I}(X_{\alpha,t_n(\alpha)} \geq c^-)]$$
$$\geq \liminf_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)} (1-\epsilon) \mathbf{I}(X_{\alpha,t_n(\alpha)} \geq c^-)] \quad (66)$$
$$= (1-\epsilon) \liminf_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)} \mathbf{I}(X_{\alpha,t_n(\alpha)} \geq c^-)]$$
$$= (1-\epsilon) \liminf_{n\to\infty} E[n^{-1/\alpha} X_{\alpha,t_n(\alpha)}] = (1-\epsilon) h(\beta_\alpha).$$

Since (66) is true for every $\epsilon > 0$ we get, by (65),

$$h(\beta_\alpha) \geq \limsup_{n\to\infty} n^{-1/\alpha} V_n(X_n, \ldots, X_1) \geq \liminf_{n\to\infty} n^{-1/\alpha} V_n(X_n, \ldots, X_1) \geq h(\beta_\alpha),$$

and Theorem 1.1 follows. ■

7. NUMERICAL EVALUATIONS AND REMARKS

In Table 1, for $\alpha = 1.1, 1.2, \ldots, 2, 3, \ldots, 10$, the values in column (1), we tabulate for $\mathcal{L} = 1$ the following quantities in the columns indicated

(2) β_α
(3) $\lim_{n\to\infty} n^{-1/\alpha}\, V_n^1$
(4) $\lim_{n\to\infty} n^{-1/\alpha}\, V_n^2$
(5) $\lim_{n\to\infty} n^{-1/\alpha}\, EM_n$

In columns (6), (7) and (8) we tabulate the ratios (4)/(3), (5)/(4) and (5)/(3) respectively. The final column, column (9), of Table 1 represents the relative (limiting) improvement attained by using two stops rather than one, as compared to the reference value of the prophet, i.e.

$$\lim_{n\to\infty} (V_n^2 - V_n^1)/(EM_n - V_n^1). \tag{67}$$

The ratio in (67) has a minimum value of 0.788041 attained for $\alpha \approx 2.32$. Thus the limiting improvement when using two choices rather than one is never below 78.8%.

The following limiting statements can be shown to hold.

(i) For $\alpha \to \infty$,

$$\begin{aligned}
&\lim_{\alpha\to\infty} \beta_\alpha = e - 1 \\
&\lim_{\alpha\to\infty} \lim_{n\to\infty} n(1 - F(V_n^1)) = 1 \\
&\lim_{\alpha\to\infty} \lim_{n\to\infty} n(1 - F(V_n^2)) = 1 - e^{-1} \\
&\lim_{\alpha\to\infty} \lim_{n\to\infty} n(1 - F(EM_n)) = e^{-\gamma} \\
\text{and}\quad &\lim_{\alpha\to\infty} \lim_{n\to\infty} \frac{V_n^2 - V_n^1}{EM_n - V_n^1} = [1 - \log(e-1)]/\gamma = .7946\ldots,
\end{aligned}$$

where $\gamma = .5772\ldots$ is Euler's constant.

(ii) For $\alpha \to 1$,

$$\begin{aligned}
&\lim_{\alpha\to 1} \beta_\alpha = 0 \\
&\lim_{\alpha\to 1} \lim_{n\to\infty} n(1 - F(c_n)) = 0 \quad \text{for } c_n = V_n^1, V_n^2 \text{ and } EM_n, \\
\text{but}\quad &\lim_{\alpha\to 1} \lim_{n\to\infty} (V_n^2 - V_n^1)/(EM_n - V_n^1) = 1,
\end{aligned}$$

so the limiting relative improvement for this case is 100%.

Remark 7.1. The present approach can easily be applied to obtain the asymptotic behavior for the one-choice value (obtained in Kennedy and Kertz (1991) by a different method) when $F(x)$ satisfies (1) and $\lim_{x\to\infty} L(x) = \mathcal{L} \in (0,\infty)$. First assume $X \sim F_\alpha$ of (10). Then for the one choice value V_n^1 we have $V_n^1 = EX$ and

$$V_{n+1}^1 = E[X_{n+1} \vee V_n^1] = \alpha \int_1^\infty [x \vee V_n^1] x^{-(\alpha+1)} dx. \qquad (68)$$

Set $W_n^1 = n^{-1/\alpha} V_n^1$. Multiply (68) by $n^{-1/\alpha}$ to obtain

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1}^1 = \alpha \int_1^\infty [n^{-1/\alpha} x \vee W_n^1] x^{-(\alpha+1)} dx. \qquad (69)$$

Substituting $u = nx^{-\alpha}$, (69) can be rewritten as

$$\left(\frac{n+1}{n}\right)^{1/\alpha} W_{n+1}^1 = \frac{1}{n} \int_0^n [u^{-1/\alpha} \vee W_n^1] du.$$

Now $q(u) = u^{-1/\alpha}$ satisfies Condition 4.1 and one can thus apply Theorem 4.1 directly to show that

$$\lim_{n\to\infty} W_n^1 = d = b^{-1/\alpha},$$

where b solves $Q(y) = 0$ with

$$Q(y) = \int_0^y u^{-1/\alpha} du - \left(\frac{1}{\alpha} + y\right) y^{-1/\alpha}$$

that is, that

$$\frac{\alpha}{\alpha-1} y^{-1/\alpha+1} - \left(\frac{1}{\alpha} + y\right) y^{-1/\alpha} = 0,$$

from which it follows immediately that $b = (\alpha-1)/\alpha$ and $\lim_{n\to\infty} W_n^1 = \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha}$. The general result for the wider class of distributions satisfying (1) with $\lim L(x) = \mathcal{L} \in (0,\infty)$ now follows much in the same way as the arguments in Section 6.

Remark 7.2. Hill and Kertz (1982) and Kertz (1986) study one-choice prophet inequalities for non-negative i.i.d. random variables. They show that these prophet inequalities are n-dependent, and for each fixed n the extremal ratio can be obtained by a random variable with $n+1$ atoms, (thus these variables do not belong to any domain of attraction for the maximum). As $n \to \infty$ the ratio tends to $1.34\ldots$. It may therefore be of interest to note, by comparison, that the extremal ratio of $\lim[EM_n/V_1^n]$ as $n \to \infty$, for the family studied here is $1.2882\ldots$, attained for $\alpha = 1.4628\ldots$.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
α	β_α	$\lim \frac{V_n^1}{n^{1/\alpha}}$	$\lim \frac{V_n^2}{n^{1/\alpha}}$	$\lim \frac{M_n}{n^{1/\alpha}}$	(4)/(3)	(5)/(4)	(5)/(3)	$\frac{[(4)-(3)]}{[(5)-(3)]}$
1.1	0.54315	8.84546	10.18170	10.50590	1.15107	1.03184	1.18771	0.80477
1.2	0.68106	4.45102	5.34177	5.56632	1.20012	1.04204	1.25057	0.79867
1.3	0.78076	3.08932	3.77026	3.94584	1.22042	1.04657	1.27725	0.79501
1.4	0.85971	2.44692	3.00352	3.14912	1.22747	1.04848	1.28697	0.79266
1.5	0.92489	2.08008	2.55383	2.67894	1.22775	1.04899	1.28790	0.79109
1.6	0.98008	1.84599	2.26032	2.37044	1.22445	1.04872	1.28410	0.79003
1.7	1.02764	1.68530	2.05476	2.15338	1.21922	1.04800	1.27774	0.78931
1.8	1.06915	1.56912	1.90340	1.99289	1.21303	1.04702	1.27007	0.78881
1.9	1.10577	1.48182	1.78769	1.86974	1.20641	1.04590	1.26179	0.78848
2	1.13836	1.41421	1.69660	1.77245	1.19968	1.04471	1.25331	0.78827
3	1.33839	1.14471	1.30982	1.35412	1.14423	1.03382	1.18293	0.78846
4	1.43534	1.07457	1.19365	1.22542	1.11081	1.02662	1.14038	0.78939
5	1.49277	1.04564	1.13935	1.16423	1.08962	1.02184	1.11341	0.79017
6	1.53080	1.03085	1.10830	1.12879	1.07512	1.01849	1.09500	0.79077
7	1.55784	1.02227	1.08833	1.10577	1.06463	1.01602	1.08168	0.79124
8	1.57806	1.01683	1.07448	1.08965	1.05669	1.01412	1.07162	0.79161
9	1.59375	1.01317	1.06432	1.07776	1.05048	1.01263	1.06375	0.79191
10	1.60628	1.01059	1.05657	1.06863	1.04549	1.01142	1.05743	0.79215

8. FINAL REMARKS

The last two authors are very saddened to announce that our invaluable colleague and friend David Assaf passed away most suddenly on December 23^{rd} 2003. On that very day, in a last email from Prof. Assaf to us he wrote that he had some ideas and 'I will say more on this in a few days.' We regret on many levels that our work on two stage stopping can now only remain more or less in its current form, without the benefit of those further comments, now forever lost, which would have certainly greatly improved it.

References

1. ASSAF, D. & SAMUEL-CAHN, E. (2000) Simple ratio prophet inequalities for a mortal with multiple choices. *J. Appl. Prob.*, **37**, pp. 1084-1091.
2. ASSAF, D., GOLDSTEIN, L.& SAMUEL-CAHN, E. (2002). Ratio prophet inequalities when the mortal has several choices. *Ann. Appl. Prob.*, **12**, pp. 972-984.
3. ASSAF, D., GOLDSTEIN, L.& SAMUEL-CAHN, E. (2004). Two Choice Optimal Stopping. *Adv. Appl. Prob.* **36**, pp. 1116-1147
4. HILL, T.P. & KERTZ, R.P. (1982). Comparisons of stop rule and supremum expectations of i.i.d. random variables. *Ann. of Prob.*, **10**, pp. 336-345.
5. KENNEDY, D. P. & KERTZ, R.P. (1991). The asymptotic behavior of the reward sequence in the optimal stopping of i.i.d random variables. *Ann. Prob.* **19**, pp. 329-341.
6. KERTZ, R.P. (1986). Stop rule and supremum expectations of i.i.d. random variables: a complete comparison by conjugate duality. *J. Multivariate Analysis*, **19**, pp. 88-112.
7. KÜHNE, R. & RÜSCHENDORF, L. (2002) On optimal two-stopping problems. *Limit theorems in probability and statistics*, Vol. II, eds. Berkes, I, Csáki, E. & Csörgö, M, pp. 261-271, János Bolyai Math. Soc., Budapest.
8. KÜHNE, R. & RÜSCHENDORF, L. (2000) Approximation of optimal stopping problems. *Stochastic Process. Appl.*, **90**, pp. 301-325.
9. LEADBETTER, M. R., LINDGREN, G. & ROOTZÉN, H.(1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New York, Heidelberg, Berlin
10. RESNICK, S. I. (1987). *Extreme Values, Regular variation, and Point Processes.* Springer. N.Y.