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Optimal Two-Choice Stopping on an Exponential Sequence

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Abstract: Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with distribution function F. A statistician may choose two X values from the sequence by means of two stopping rules t_1, t_2, with the goal of maximizing $E(X_{t_1} \vee X_{t_2})$. We describe the optimal stopping rules and the asymptotic behavior of the optimal expected stopping values, V_n^2, as $n \to \infty$, when F is the exponential distribution. Specifically, we show that $\lim_{n\to\infty} n(1 - F(V_n^2)) = 1 - e^{-1}$, and conjecture that this same limit obtains for any F in the (Type I) domain of attraction of $\exp(-e^{-x})$.

Keywords: Domains of attraction; Multiple-choice stopping rules; Prophet value.

Subject Classifications: 60G40.

1. INTRODUCTION AND SUMMARY

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables from a known distribution F, where n is a fixed horizon. We consider the situation where the aim of a statistician (optimal stopper) is to sequentially pick as large an X value as possible, but unlike the classical case, where only one choice is permitted, the statistician here is permitted two choices, and the second choice may depend on the first. The value to the statistician for using stopping rules t_1 and t_2 is the maximum of X_{t_1} and X_{t_2}, leading to the goal of maximizing $E[X_{t_1} \vee X_{t_2}]$ over all stopping rules t_1, t_2 satisfying $1 \leq t_1 \leq t_2 \leq n$. The statistician's first choice, X_{t_1}, can be thought of as a guaranteed fallback value. A situation as described may arise,

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e.g., if one is interested in buying a house, and while inspecting houses, and only one house is needed, one may put one other house on hold. One interesting aspect of this problem is that the usual backward induction does not apply directly, and a two-stage backward induction is needed.

Consider first a one-choice situation, where a fixed value x has already been promised. Let $V_n^1(x)$ denote the optimal return for this situation. Then, clearly, with $X \sim F$,

$$V_0^1(x) = x \quad \text{and} \quad V_n^1(x) = E\big[X \vee V_{n-1}^1(x)\big] \text{ for all } n \geq 1, \tag{1.1}$$

and the optimal stopping rule in the n-horizon problem is

$$\tau_n = \min\{i \leq n : X_i \geq V_{n-i}^1(x)\} \wedge n. \tag{1.2}$$

If we denote $g(x) = E[X \vee x]$ then the recursion in (1.1) can be written as

$$V_n^1(x) = g(g_{n-1}(x)) = g_n(x). \tag{1.3}$$

The usual one-choice value, V_n^1, is clearly $V_n^1(-\infty)$, or, if $X \geq 0$, then also equal to $V_n^1(0)$.

Let V_n^2 denote the optimal value attainable in the two-choice situation, for horizon n. Then, similar to the form of (1.1), we have the following backward induction:

$$V_2^2 = E[X_1 \vee X_2] \quad \text{and} \quad V_n^2 = E\big[V_{n-1}^2 \vee V_{n-1}^1(X)\big] \text{ for all } n > 1. \tag{1.4}$$

The interpretation of (1.4) is as follows: If the current observation (which we have denoted as just X) is large enough, take it, and then continue optimally as in the one-choice situation, where the value X is guaranteed. If X is not chosen, continue optimally with the two-choice situation and horizon $n-1$. The optimal strategy is, therefore, for a first choice use

$$\tau_n^1 = \min\{i \leq n-1 : V_{n-i}^1(X_i) \geq V_{n-i}^2\} \wedge (n-1);$$

for a second choice, apply the rule of (1.2) adapted to the time τ_n^1 of the first choice, and to the value $X_{\tau_n^1}$ chosen, that is, use

$$\tau_n^2 = \min\{i : \tau_n^1 < i \leq n, X_i \geq V_{n-i}^1(X_{\tau_n^1})\} \wedge n.$$

Our interest lies in finding the asymptotic behavior, as $n \to \infty$, of the sequence V_n^2 of (1.4). It is well known that there are three types of asymptotic distributions for the maximum (see Leadbetter et al., 1983, p. 4), corresponding to three domains of attraction. The asymptotic behavior of V_n^1 for the one-choice situation has been studied by Kennedy and Kertz (1991), who show that the limiting behavior of V_n^1 depends upon the domain of attraction to which F belongs. This will therefore clearly also be the case for the two-choice value sequence, V_n^2. It is evident that, because of the much more complicated structure of the value sequence V_n^2 over V_n^1, as given in (1.4) and (1.1) respectively, the study of the asymptotic behavior for two choices will be more involved.

In the present article, we study the case where $F(x) = 1 - e^{-\theta x}$, i.e., the exponential distribution. This distribution belongs to the domain of attraction of $\exp(-e^{-x})$. Examples of the asymptotic behavior of V_n^2 for the two-choice stopping problem on i.i.d. sequences with distributions belonging to the two other domains of attraction are studied in Assaf et al. (2004, 2006). Our main result here, for the canonical representative of the distributions in the Type I domain of attraction, is the following.

Theorem 1.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. exponentially distributed random variables. Then*

$$\lim_{n\to\infty} n\big(1 - F(V_n^2)\big) = 1 - e^{-1}, \tag{1.5}$$

where V_n^2 is the optimal two choice value.

Let $M_n = \max\{X_1, \ldots, X_n\}$. The corresponding asymptotic values for V_n^1 and EM_n are

$$\lim_{n\to\infty} n(1 - F(EM_n)) = e^{-\gamma}, \tag{1.6}$$

where γ is Euler's constant (see, e.g., Leadbetter et al., 1983), and, as obtained by Kennedy and Kertz (1991),

$$\lim_{n\to\infty} n\big(1 - F\big(V_n^1\big)\big) = 1. \tag{1.7}$$

Limit (1.6) for the maximum and limit (1.7) for the optimal one-choice rule hold for any F belonging to the domain of attraction of $\exp(-e^{-x})$. In addition, the two corresponding limits over the other two domains of attraction behave in this same fashion, as do all known limits for the optimal two-choice rule (see Assaf et al., 2004, 2006). Based on this evidence, we conjecture that (1.5) holds for all F in the Type I domain of attraction.

For the exponential distribution, we may assume without loss of generality that $\theta = 1$, for which (1.5)–(1.7) are easily seen to be equivalent to

$$\lim_{n\to\infty} \big(V_n^2 - \log n\big) = 1 - \log(e-1) = 0.4586\ldots \tag{1.8}$$

and

$$\lim_{n\to\infty} (EM_n - \log n) = \gamma = 0.5772\ldots \quad \text{and} \quad \lim_{n\to\infty} \big(V_n^1 - \log n\big) = 0, \tag{1.9}$$

respectively.

2. PRELIMINARIES AND HEURISTICS

For $F(x) = 1 - e^{-x}$ we have $V_1^1(x) = g_1(x) = E[x \vee X] = x + e^{-x}$. To simplify notation, we write V_n instead of V_n^2. Using (1.3), the recursion in (1.4) can be written as

$$V_{n+1} = \int_0^\infty [g_n(x) \vee V_n] e^{-x} dx, \tag{2.1}$$

or, if we let $b_n > 0$ denote the unique value such that $g_n(b_n) = V_n$ (also called the *indifference value*), then (2.1) can be rewritten as

$$V_{n+1} = (1 - e^{-b_n})V_n + \int_{b_n}^{\infty} g_n(x)e^{-x}dx. \tag{2.2}$$

Set

$$h_n(x) = g_n(x + \log n) - \log n, \quad a_n = \log((n+1)/n), \tag{2.3}$$

$$B_n = b_n - \log n \quad \text{and} \quad W_n = V_n - \log n = h_n(B_n). \tag{2.4}$$

Then, (2.2) can be rewritten as

$$W_{n+1} = \left(1 - \frac{1}{n}e^{-B_n}\right)W_n + \frac{1}{n}\int_{B_n}^{\infty} h_n(x)e^{-x}dx - a_n \tag{2.5}$$

or

$$n(W_{n+1} - W_n) = -e^{-B_n}W_n + \int_{B_n}^{\infty} h_n(x)e^{-x}dx - na_n. \tag{2.6}$$

To motivate our result, consider the following heuristics. Assume that for some B and h,

$$B_n \to B \quad \text{and} \quad h_n(y) \to h(y) \text{ as } n \to \infty.$$

Then, under regularity, using (2.4),

$$W_n = h_n(B_n) \to h(B) = W. \tag{2.7}$$

But now, should $n(W_{n+1} - W_n)$ converge to a nonzero constant A, W_n would grow like $A\sum k^{-1} \sim A\log n$, giving a contradiction. Hence $n(W_{n+1} - W_n)$ must tend to zero, and taking limits in (2.6) yields

$$0 = -We^{-B} + \int_{B}^{\infty} h(x)e^{-x}dx - 1. \tag{2.8}$$

Substituting $h(B) = W$ from (2.7) into (2.8) gives an equation for the unknown B, thus yielding W if h were known.

Here is a heuristic for determining h: By (1.9),

$$\lim_{n\to\infty}\left[V_n^1 - \log(n+1)\right] = 0.$$

Since V_n^1 is the value when nothing is guaranteed, we have $V_n^1 = g_n(0)$, and thus

$$g_n(0) \approx \log(n+1). \tag{2.9}$$

Suppose that for large enough n and a fixed guaranteed value x, there is t such that

$$g_n(x) = g_{n+t}(0) = g_n(g_t(0)). \tag{2.10}$$

That is, there is some number of extra observations t such that the statistician is indifferent to having $n+t$ variables from which to chose, or the guaranteed x and n variables.

Equation (2.10) implies $x = g_t(0) \approx \log(t+1)$, yielding $t+1 \approx e^x$. But on the other hand, $g_{n+t}(0) \approx \log(n+t+1) \approx \log(n+e^x) \approx g_n(x)$. Using (2.3), we have

$$h_n(x) \approx \log(n + e^{x+\log n}) - \log n = \log(1+e^x).$$

This suggests

$$\lim_{n\to\infty} h_n(x) = h(x) = \log(1+e^x) = x + \log(1+e^{-x}), \quad -\infty < x < \infty. \tag{2.11}$$

From (2.8), (2.7), and (2.11), B solves

$$1 = -\log\left(1+e^B\right)e^{-B} + e^{-B}\int_0^\infty \log\left(1+e^{B+u}\right)e^{-u}du. \tag{2.12}$$

Letting $s = e^{-u}$, the integral in (2.12) can be evaluated as

$$\int_0^\infty \log\left(1+e^{B+u}\right)e^{-u}du = \int_0^1 \log\left(1+\frac{e^B}{s}\right)ds = e^B\big(\left(1+e^{-B}\right)\log(1+e^B) - B\big),$$

and now substitution back into (2.12) yields $1+B = \log(1+e^B)$, the unique solution of which is

$$B = -\log(e-1) = -0.54132\ldots,$$

and now, from (2.7) and (2.11), $W = 1 - \log(e-1) = 0.45867\ldots$, which is equivalent to conclusion (1.5) of Theorem 1.1. A rigorous proof of the theorem is given in Section 4.

3. PROPERTIES OF h_n AND THE LIMITING h

Lemma 3.1. *$h_n(x)$ is strictly monotone increasing for $-\log n \le x < \infty$.*

Proof. We have that $g(x)$, and hence $g_n(x)$, are strictly monotone increasing for $x \ge 0$, and now the result follows by (2.3). □

Lemma 3.2. *Let $h(x)$ be given in* (2.11). *Then, $h_n(x) > h(x)$ for $x \ge -\log n$, $n = 1, 2, \ldots$.*

Proof. For $n=1$ the claim is simply that $h_1(x) = x + e^{-x} > x + \log(1+e^{-x}) = h(x)$ for all $x \ge 0$, which is immediate. Now suppose the claim holds for n. We show that it holds for $n+1$. By the induction hypothesis,

$$\begin{aligned} g_n(x) &= \log n + h_n(x - \log n) > \log n + h(x - \log n) \\ &= \log n + \log(1 + e^{x - \log n}) = \log(n + e^x). \end{aligned} \tag{3.1}$$

Thus, since g is increasing,

$$g_{n+1}(x) = g(g_n(x)) > g(\log(n+e^x)) = \log(n+e^x) + e^{-\log(n+e^x)}$$
$$= \log(n+e^x) + \frac{1}{n+e^x}. \quad (3.2)$$

Thus, similar to (3.1) it suffices to show that the right-hand side of (3.2) is greater than $\log(n+1+e^x)$. The latter statement is equivalent to $\frac{1}{n+e^x} > \log\left(1+\frac{1}{n+e^x}\right)$, which clearly holds. □

Lemma 3.3. *Let* $\varepsilon_n(x) = h_n(x) - h(x)$. *Then,* $\varepsilon_n(x) < e^{-x}/\sqrt{n}$, *for* $x \geq -\log n$.

Proof. For $n = 1$ we have $\varepsilon_1(x) = e^{-x} - \log(1+e^{-x})$, so clearly the statement holds for $n = 1$. Now, using (2.3),

$$h_{n+1}(x) = g_{n+1}(x + \log(n+1)) - \log(n+1)$$
$$= g_n(x+\log(n+1)) + e^{-g_n(x+\log(n+1))} - \log(n+1)$$
$$= h_n(x+a_n) - a_n + e^{-[h_n(x+a_n)+\log n]}$$
$$= h_n(x+a_n) + \frac{1}{n}e^{-h_n(x+a_n)} - a_n. \quad (3.3)$$

In particular, for $n = 1$,

$$h_2(x) = h_1(x+a_1) + e^{-h_1(x+a_2)} - a_1 = x + a_1 + e^{-(x+a_1)} + e^{-(x+a_1+e^{-(x+a_1)})} - a_1.$$

We shall show directly that the lemma is true for $n = 2$, for which

$$\varepsilon_2(x) = \frac{1}{2}e^{-x}\left(1 + e^{-\frac{1}{2}e^{-x}}\right) - \log(1+e^{-x}). \quad (3.4)$$

For $-\log 2 \leq x \leq 0$ we shall show

$$\varepsilon_2(x) - \frac{e^{-x}}{\sqrt{2}} < 0,$$

that is,

$$\frac{1}{2}e^{-x}\left(1 - \sqrt{2} + e^{-\frac{1}{2}e^{-x}}\right) - \log(1+e^{-x}) < 0. \quad (3.5)$$

Differentiation shows that the left-hand side of (3.5) is increasing in x for $x \leq 0$. Thus we shall show that for $x = 0$ inequality (3.5) holds, that is, that $\frac{1}{2}(1-\sqrt{2}+e^{-\frac{1}{2}}) - \log 2 < 0$, which is equivalent to $1 - \sqrt{2} + e^{-\frac{1}{2}} - \log 4 < 0$, which clearly holds.

Now, for $x > 0$ the inequality $\log(1+e^{-x}) > e^{-x} - \frac{e^{-2x}}{2}$ holds. Substituting this in (3.4) we have

$$\varepsilon_2(x) < \frac{1}{2}e^{-x}\left(-1 + e^{-\frac{1}{2}e^{-x}} + e^{-x}\right). \quad (3.6)$$

We shall show that the right-hand side of (3.6) is less than $e^{-x}/\sqrt{2}$, which is equivalent to

$$-1 + e^{-\frac{1}{2}e^{-x}} + e^{-x} < \sqrt{2}. \quad (3.7)$$

Now the left-hand side of (3.7) is decreasing in x: thus it suffices to show (3.7) for $x = 0$, where the inequality simplifies to $-1 + e^{-\frac{1}{2}} + 1 < \sqrt{2}$, which clearly holds. Thus the lemma holds for $n = 2$.

Suppose the lemma holds for $n \geq 2$. We shall show that it holds for $n + 1$. By (3.3), for $x \geq -\log n$

$$h_{n+1}(x - a_n) = h_n(x) + \frac{1}{n} e^{-h_n(x)} - a_n. \tag{3.8}$$

We show that, for $x \geq -\log n$, $\varepsilon_{n+1}(x - a_n) < \frac{n+1}{n} e^{-x}/\sqrt{n+1} = \frac{\sqrt{n+1}}{n} e^{-x}$ by a Taylor expansion of $h(x - a_n)$. Note that

$$h'(x) = e^x/(1 + e^x), \quad h''(x) = e^x/(1 + e^x)^2 > 0;$$

thus, for some $\theta \in (0, 1)$,

$$h(x - a_n) = h(x) - a_n \frac{e^{x - \theta a_n}}{1 + e^{x - \theta a_n}} > h(x) - a_n \frac{e^x}{1 + e^x}. \tag{3.9}$$

Thus, by (3.8) and (3.9),

$$\begin{aligned}
\varepsilon_{n+1}(x - a_n) &< \varepsilon_n(x) + \frac{1}{n} e^{-h_n(x)} - a_n + a_n \frac{e^x}{1 + e^x} \\
&< \varepsilon_n(x) + \frac{1}{n(1 + e^x)} - \frac{a_n}{1 + e^x} \\
&< \varepsilon_n(x) + \frac{1}{n(1 + e^x)} - \left(\frac{1}{n} - \frac{1}{2n^2}\right) \frac{1}{1 + e^x} \\
&= \varepsilon_n(x) + \frac{1}{2n^2(1 + e^x)} \\
&< \frac{e^{-x}}{\sqrt{n}} + \frac{e^{-x}}{2n^2(1 + e^{-x})},
\end{aligned}$$

where the second inequality uses $h_n(x) > h(x)$ by Lemma 3.2, the third inequality uses $\log(1 + y) > y - \frac{y^2}{2}$ for $0 < y < 1$, and the last inequality uses the induction hypothesis. Thus we must show that for $x \geq -\log n$ we have $\frac{1}{\sqrt{n}} + \frac{1}{2n^2(1+e^{-x})} < \frac{\sqrt{n+1}}{n}$, and hence it is sufficient to show $1 + \frac{1}{2n^{3/2}} < \sqrt{\frac{n+1}{n}}$. But $(1 + \frac{1}{n})^{1/2} > 1 + \frac{1}{2n} - \frac{1}{8n^2}$; hence it is sufficient to show that $\frac{1}{2n^{3/2}} < \frac{1}{2n} - \frac{1}{8n^2}$, or equivalently that $1 < \sqrt{n} - \frac{1}{4\sqrt{n}}$, which holds for $n \geq 2$. □

4. PROOF OF THEOREM 1.1

Lemma 4.1. *For some constant A_q, let q be a continuous and strictly monotone increasing function in the interval $[A_q, \infty)$ such that for all $y \geq A_q$ the integral $\int_y^\infty q(x)e^{-x}dx$ is finite. Further, defining*

$$Q(y) = \int_y^\infty q(x)e^{-x}dx - q(y)e^{-y} - 1, \tag{4.1}$$

suppose $Q(A_q) > 0$. *Then,*

$$\lim_{y\to\infty} Q(y) = -1, \tag{4.2}$$

$Q(y)$ *is monotone decreasing, and there exists a unique value* $\beta \in [A_q, \infty)$ *such that* $Q(\beta) = 0$.

Proof. The assumption that the integral in (4.1) is finite and q is increasing implies that $q(y)e^{-y} \to 0$ as $y \to \infty$; thus (4.2) holds. The function Q is differentiable with $dQ(y)/dy = -q'(y)e^{-y} < 0$; thus Q is monotone decreasing. Since $Q(A_q) > 0$, $Q(y)$ is continuous, and negative for all y sufficiently large. Hence the root β exists and is unique in $[A_q, \infty)$. □

Theorem 4.1. *Let* A_q *and* q *be as in Lemma* 4.1. *Then, there exists* n_0 *such that for any* $r \geq n_0$ *and* $\beta_r \in [A_q, \infty)$, *the sequence* β_n *for* $n \geq r$ *is well defined by the recursion*

$$q(\beta_{n+1}) = q(\beta_n)\left(1 - \frac{1}{n}e^{-\beta_n}\right) + \frac{1}{n}\int_{\beta_n}^{\infty} q(x)e^{-x}dx - a_n, \tag{4.3}$$

and satisfies

$$\lim_{n\to\infty} \beta_n = \beta,$$

where β *is the root of* (4.1) *whose existence and uniqueness in* $[A_q, \infty)$ *is guaranteed in Lemma* 4.1.

Proof. First, rewrite (4.3) as

$$q(\beta_{n+1}) - q(\beta_n) = \frac{Q(\beta_n)}{n} + \left(\frac{1}{n} - a_n\right). \tag{4.4}$$

Note that for all $n \geq 1$,

$$0 < \frac{1}{n} - a_n < \frac{1}{2n^2}, \tag{4.5}$$

and that

$$\frac{Q(c)}{n} + \left(\frac{1}{n} - a_n\right)$$

is positive and decreasing in n with limit 0 for all $c \leq \beta$, and is decreasing in n and negative for all n sufficiently large with limit 0 for $c > \beta$.

We show that for any $\underline{\beta}$ and $\bar{\beta}$ with $A_q < \underline{\beta} < \beta < \bar{\beta}$, for all n sufficiently large, β_n is well defined and $\underline{\beta} < \beta_n < \bar{\beta}$; clearly the theorem follows.

Let $A_q < \underline{\beta} < \beta < \bar{\beta}$ be given, and let n_0 be so large that for all $n \geq n_0$

$$\frac{Q(A_q)}{n} + \left(\frac{1}{n} - a_n\right) < q(\bar{\beta}) - q(\beta) \tag{4.6a}$$

$$\left(\frac{1}{n} - a_n\right) < q(\bar{\beta}) - q\left(\frac{\beta + \bar{\beta}}{2}\right) \tag{4.6b}$$

$$\frac{1}{n}Q\left(\frac{\beta+\bar{\beta}}{2}\right)+\left(\frac{1}{n}-a_n\right)<0 \tag{4.6c}$$

$$a_n<q(\beta)-q(\underline{\beta}). \tag{4.6d}$$

We first show that if $A_q<\underline{\beta}<\beta_n<\bar{\beta}$ for $n\geq n_0$, then β_{n+1} is well defined and satisfies $\underline{\beta}<\beta_{n+1}<\bar{\beta}$; thus the sequence β_n remains in the interval $(\underline{\beta},\bar{\beta})$ for all $n\geq n_0$. We show this fact by considering the following cases.

Case A is

$$\underline{\beta}<\beta_n\leq\beta.$$

By (4.4), (4.5), and (4.6a), and the fact that q is increasing and Q is decreasing,

$$\begin{aligned} q(\beta_n) &< q(\beta_n)+\frac{Q(\beta_n)}{n}+\left(\frac{1}{n}-a_n\right)=q(\beta_{n+1})\\ &< q(\beta_n)+\frac{Q(A_q)}{n}+\left(\frac{1}{n}-a_n\right)\\ &< q(\beta_n)+(q(\bar{\beta})-q(\beta))\leq q(\bar{\beta}); \end{aligned} \tag{4.7}$$

thus β_{n+1} exists uniquely by the strict monotonicity of q and satisfies

$$\underline{\beta}<\beta_n<\beta_{n+1}<\bar{\beta}.$$

Case B is

$$\beta<\beta_n<\frac{\beta+\bar{\beta}}{2}.$$

There are two subcases—B1,

$$\frac{Q(\beta_n)}{n}+\left(\frac{1}{n}-a_n\right)>0,$$

which may happen for small n, and B2,

$$\frac{Q(\beta_n)}{n}+\left(\frac{1}{n}-a_n\right)\leq 0. \tag{4.8}$$

In Subcase B1, by (4.6b),

$$\begin{aligned} q(\beta)<q(\beta_n)&<q(\beta_n)+\frac{Q(\beta_n)}{n}+\left(\frac{1}{n}-a_n\right)=q(\beta_{n+1})\\ &<q(\beta_n)+\left(\frac{1}{n}-a_n\right)<q(\beta_n)+\left(q(\bar{\beta})-q\left(\frac{\beta+\bar{\beta}}{2}\right)\right)<q(\bar{\beta}), \end{aligned}$$

so again β_{n+1} is well defined and $\underline{\beta}<\beta_{n+1}<\bar{\beta}$. Subcase B2 can be combined with Case C.

Case C is

$$\frac{\beta+\bar{\beta}}{2} \le \beta_n < \bar{\beta}.$$

In this case, by (4.6a), and in subcase B2 by (4.8), if β_{n+1} exists, it must be smaller than β_n; thus $q(\bar{\beta}) > q(\beta_{n+1})$, but also

$$q(\beta_{n+1}) = q(\beta_n) + \frac{Q(\beta_n)}{n} + \left(\frac{1}{n} - a_n\right) > q(\beta_n) - a_n > q(\beta_n) - (q(\beta) - q(\underline{\beta})) > q(\underline{\beta}), \tag{4.9}$$

where the inequalities are justified by (4.2), (4.6d), and $\beta_n > \beta$, this last of which holds for Case C as well as for Case B, so in particular for subcase B2. Thus again β_{n+1} exists and $\underline{\beta} < \beta_{n+1} < \bar{\beta}$.

It remains to show that for any $r \ge n_0$ and any starting value $\beta_r \in [A_q, \infty) \cap (\underline{\beta}, \bar{\beta})^c$, β_n is well defined and β_n will eventually enter the interval $(\underline{\beta}, \bar{\beta})$. First, suppose $\beta_r \in [A_q, \underline{\beta}]$. Then the sequence will be well defined and start out monotone increasing, and (4.7) and its subsequent inequalities continue to hold as long as $\beta_n \le \beta$, and for all such n one has $\beta_{n+1} < \bar{\beta}$. There are two possibilities: Either (a) for some k the inequality

$$\underline{\beta} < \beta_k < \bar{\beta}$$

holds, in which case we have shown that $\underline{\beta} < \beta_n < \bar{\beta}$ for all $n > k$, or (b) the sequence β_n is monotone increasing throughout with $\lim \beta_n = \beta_0$, which necessarily satisfies $\beta_0 \le \underline{\beta}$. We show that (b) leads to a contradiction. Clearly $Q(\beta_0) > 0$. By (4.4),

$$q(\beta_{n+1}) - q(\beta_n) > \frac{Q(\beta_0)}{n} + \left(\frac{1}{n} - a_n\right);$$

thus for n arbitrarily large and $m > n$,

$$q(\beta_m) - q(\beta_n) > Q(\beta_0) \sum_{k=n}^{m-1} \frac{1}{k} + \sum_{k=n}^{m-1} \left(\frac{1}{k} - a_k\right).$$

Now, the right-hand side tends to infinity as $m \to \infty$; thus the value $q(\beta_m)$ must also tend to infinity, contradicting the fact that $\beta_m \le \underline{\beta}$.

Now consider a starting value β_r for $r \ge n_0$ satisfying $\bar{\beta} \le \beta_r < \infty$. By (4.6c) the sequence will be well defined and decreasing, as long as $\beta_n \ge (\beta + \bar{\beta})/2$, and (4.9) continues to hold; thus $\beta_{n+1} > \underline{\beta}$. Again there are two possibilities. Either (a) for some n we have $\bar{\beta} > \beta_n > \underline{\beta}$, in which case the theorem holds, or (b) the sequence is monotone decreasing for all n, with $\beta_n \ge \bar{\beta}$, and thus the limit $\beta^0 \ge \bar{\beta}$ exists, and clearly satisfies $Q(\beta^0) < 0$. We suppose (b) and show that this leads to a contradiction. By (4.4) and (4.5),

$$q(\beta_{n+1}) - q(\beta_n) < \frac{Q(\beta^0)}{n} + \frac{1}{2n^2};$$

thus for m arbitrarily large,

$$q(\beta_m) - q(\beta_n) < Q(\beta^0) \sum_{k=n}^{m-1} \frac{1}{m} + \frac{1}{2} \sum_{k=n}^{m-1} \frac{1}{k^2}. \tag{4.10}$$

Now, the last summand on the right-hand side of (4.10) converges to a finite limit, while the first term there tends to $-\infty$ as $m \to \infty$. Thus $q(\beta_m)$ must also tend to $-\infty$, contradicting the fact that $q(\beta_m) \geq q(\bar{\beta})$.

Let H be given by (4.1) with q replaced by h of (2.11); the change of variable $x = B + u$ in the integral in this definition of H shows that (2.12) is the equation $H(B) = 0$, and using Lemma 4.1 we conclude that the solution $-\log(e-1)$ is unique. Let $-\infty < A \leq -1$ be some constant, and define

$$\tilde{h}_j(x) = h(x) + \frac{e^{-x}}{\sqrt{j}} \quad \text{for } A \leq x < \infty.$$

Then by Lemmas 3.2 and 3.3, for all $j > j(A) = e^{-A}$ we have

$$h(x) < h_j(x) < \tilde{h}_j(x) \quad \text{for } A \leq x < \infty. \tag{4.11}$$

Also, since there is some $j_0(A) \geq j(A)$ such that for all $j \geq j_0(A)$,

$$\frac{d\tilde{h}_j(x)}{dx} = \frac{e^x}{1+e^x} - \frac{e^{-x}}{\sqrt{j}} > 0,$$

the functions $\tilde{h}_j(x)$, $j \geq j_0(A)$ are strictly increasing in $[A, \infty)$.

Lemma 4.2. *Let $\widetilde{H}_j(x)$ be defined as in* (4.1), *with $q(x)$ replaced by $\tilde{h}_j(x)$. Then, for all $j \geq j_0(A)$ there exists a value $\tilde{\beta}_j \in [A, \infty)$ such that $\widetilde{H}_j(\tilde{\beta}_j) = 0$,*

$$\lim_{j\to\infty} \tilde{\beta}_j = -\log(e-1) \quad \textit{and} \quad \lim_{j\to\infty} \tilde{h}_j(\tilde{\beta}_j) = h(-\log(e-1)) = 1 - \log(e-1). \tag{4.12}$$

Proof. Since $\widetilde{H}_j(x) \to H(x)$ uniformly on $[A, \infty)$, in particular $\lim_{j\to\infty} \widetilde{H}_j(A) = H(A) > H(-\log(e-1)) = 0$. Thus for all $j > j_0(A)$ the value $\tilde{\beta}_j$ exists uniquely in $[A, \infty)$. Now (4.12) follows from the uniform convergence of $\widetilde{H}_j(x)$ and $\tilde{h}_j(x)$ to $H(x)$ and $h(x)$, respectively, on $[A, \infty)$. □

Note (2.5) can be rewritten as

$$W_{n+1} = h_{n+1}(B_{n+1}) = \frac{1}{n} \int_{-\log n}^{\infty} \left[h_n(B_n) \vee h_n(y)\right] e^{-y} dy - a_n, \tag{4.13}$$

whereas (4.3) can be rewritten, with h instead of q (keeping the β_n notation), as

$$h(\beta_{n+1}) = \frac{1}{n} \int_{-\log n}^{\infty} [h(\beta_n) \vee h(y)] e^{-y} dy - a_n. \tag{4.14}$$

Comparing (4.13) and (4.14), we see that the only difference between the two expressions is that in (4.13) the function in the integral depends on n, whereas in (4.14) this function is fixed.

We can now prove our main result.

Proof of Theorem 1.1. We apply Theorem 4.1 to (4.14) for $n \geq n_0$ with starting value $\beta_{n_0} = B_{n_0}$ as in (2.4), where n_0 is the value given by Theorem 4.1 for A and h, after which recursion (4.14) is well defined. For all $j > j_0(A)$ let $r_j = n_0 \vee j$, and for $n \geq r_j$, define the sequence $\tilde{\beta}_{j,n}$ through (4.14) with h replaced by $\tilde{h}_j$, and initial value $\tilde{\beta}_{j,r_j} = B_{r_j}$. Then by (4.11), (4.13), and (4.14), $\beta_n < B_n < \tilde{\beta}_{j,n}$, and thus the inequality

$$h(\beta_n) < h_n(B_n) = W_n < \tilde{h}_j(\tilde{\beta}_{j,n})$$

holds for all $n > r_j$, noting that the right-hand side of (4.14), say, is made larger by replacing h by a larger function. Thus, as $n \to \infty$,

$$1 - \log(e-1) = \lim h(\beta_n) \leq \liminf W_n \leq \limsup W_n \leq \lim \tilde{h}_j(\tilde{\beta}_{j,n}) = \tilde{h}_j(\tilde{\beta}_j). \tag{4.15}$$

Now, by Lemma 4.1, if we let $j \to \infty$, from (4.15),

$$1 - \log(e-1) \leq \liminf W_n \leq \limsup W_n \leq 1 - \log(e-1),$$

from which (1.8) follows, to which the theorem is equivalent. □

Remark 4.1. The limiting one-choice value can be obtained in a similar, but simpler way. Let $\{V_n^1\}$ denote the sequence of one-choice optimal values and let $W_n^1 = V_n^1 - \log n$. Since $V_{n+1}^1 = E[X \vee V_n^1]$ it follows that the $\{W_n^1\}$ sequence satisfies (4.3) with $q(x) = x$ and $\beta_n = W_n^1$. By Theorem 4.1, it therefore follows that $\lim_{n\to\infty} W_n^1 = W^1$ is the solution β of $Q(\beta) = 0$, where

$$Q(y) = \int_y^\infty xe^{-x}ds - ye^{-y} - 1, \quad \text{i.e., } Q(y) = e^{-y} - 1,$$

which implies $W^1 = 0$. This clearly agrees with the more general result of Kennedy and Kertz (1991); see (1.9).

Remark 4.2. A measure of the limiting effectiveness of having a second choice is the value $\lim_{n\to\infty}(V_n^2 - V_n^1)/(EM_n - V_n^1)$. It compares the relative advantage of having two choices over having only one choice, divided by the similar advantage for the "prophet," whose value is EM_n. For the exponential distribution we have

$$\lim_{n\to\infty} \frac{V_n^2 - V_n^1}{EM_n - V_n^1} = \frac{1 - \log(e-1)}{\gamma} = 0.7946\ldots, \tag{4.16}$$

where γ is the Euler constant. For the large subclasses of distributions of Types III and II, treated in Assaf et al. (2004, 2006), the corresponding minimal values over all α-values is (4.16) and 0.7880..., respectively. Thus the minimal saving in all the known cases is near 80%.

Remark 4.3. The model considered here is where observations arrive deterministically, one per time unit. If instead observations were to arrive according to a Poisson process with rate 1, various quantities that are approximate or asymptotic here become exact. For example, it is easy to see that $V(n)$, the optimal value for the one-stop problem on the interval $(0, n]$, satisfies the differential equation

$$V'(n) = \int_0^\infty [(x - V(n)) \vee 0] e^{-x} dx.$$

The equation can be solved explicitly, giving

$$V(n) = \log(n + 1),$$

as compared to the approximate expression (2.9). Subject to solving the corresponding equations that give the optimal two-stop value, this approach may be carried out to yield results such as Theorem 1.1; see Kennedy and Kertz (1990) for use of the Poisson process setting in the one-stop problem.

Added in Proof: After the present work was completed, the paper of Kühne and Rüschendorf (2002) came to our attention. That paper treats the same problem as the one here using Poisson-approximations, basing their results on their earlier detailed paper Kühne and Rüschendorf (2000). Detailed results are given for the domain of attraction $\mathcal{D}(e^{-e^{-x}})$ considered here, and our conjecture, that (1.5) holds for all F in this domain, is proven.

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