Two Choice Stopping: How much is an extra chance worth?

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Two Choice Optimal Stopping

Independent and identically distributed random variables $X_n, \ldots, X_1$ with known distribution are presented one at a time. You can either choose a variable, or pass it up forever. Your goal is to stop on as small a variable as possible. If you are given 2 choices and receive the smaller, what is the optimal strategy, and how small on the average is the variable you stop on?

Your strategy cannot depend on future, unseen values of the sequence; that is, you must use *stopping rules*.

Prophet Value

If you could see the entire sequence, you would receive the 'prophet value'

$$E\left(\min_{1\leq i\leq n} X_i\right) = EX_{(1,n)}.$$

For i.i.d. variables, extremal types theorem says if for some a_n, c_n

$$Z_{(1,n)} = a_n(X_{(1,n)} - c_n) \quad \text{converges in distribution,}$$

then the limit distribution is one of three types.

Extremal Type III

Example: If the X variables have distribution function $\mathcal{U}^\alpha(x) = x^\alpha, x \in [0, 1]$ for some $\alpha > 0$, then $Z_{(1,n)}$ converges to a Type III extreme value distribution:

$$G_\alpha(x) = 1 - e^{(-x^\alpha)}, \quad x \geq 0.$$

Shifting to 0, a necessary and sufficient condition for $F \in \mathcal{D}(G_\alpha)$ is that $F(0) = 0, F(x) > 0$ all $x > 0$, and

$$\lim_{h \to 0} \frac{F(xh)}{F(h)} = x^\alpha, \forall x > 0.$$

Optimal One Choice Stopping

Let V_n^1 be the optimal one stop value. If $X_{n+1} < V_n^1$, then it is better (smaller) than what you can expect from $X_n, \ldots, X_1$, so you should take it; therefore

$$V_{n+1}^1 = E[X_{n+1} \wedge V_n^1],\ n \geq 1, \quad \text{and} \quad V_1^1 = EX_1.$$

If you are guaranteed no worse (not greater) than x, then

$$V_1^1(x) = E[X_1 \wedge x] \quad \text{and}$$

$$V_{n+1}^1(x) = E[X_{n+1} \wedge V_n^1(x)],\ n \geq 1;$$

hence

$$V_n^1(x) = V_1^1(\cdots(V_1^1(x)).$$

Asymptotics of One Choice Stopping

If $F \in \mathcal{D}(G_\alpha)$, then

$$\lim_{n\to\infty} nF(EX_{(1,n)}) = \Gamma(1+1/\alpha)^\alpha \quad \text{Resnick, 1987}$$
$$\lim_{n\to\infty} nF(V_n^1) = (1+1/\alpha) \quad \text{Kennedy and Kertz, 1991}$$

For $F(x) = x^\alpha$,

$$n(V_n^1)^\alpha \to (1+1/\alpha)$$

so

$$V_n^1 \cong n^{-1/\alpha}(1+1/\alpha)^{1/\alpha} = O(n^{-1/\alpha}).$$

One Choice vs. Prophet Value

For $\mathcal{U}[0,1], \alpha = 1$, and $P(X_{(1,n)} > x) = (1-x)^n$, so

$$nEX_{(1,n)} = n\int_0^1 (1-x)^n = \frac{n}{n+1} \to 1 = \Gamma(2),$$

or

$$EX_{(1,n)} \simeq \frac{1}{n}.$$

With one choice $(1 + 1/\alpha) = 2$, so

$$nV_n^1 \to 2 \quad \text{or} \quad V_n^1 \simeq \frac{2}{n} \quad \text{for the optimal strategy.}$$

How much better can one do with two?

$$\lim_{n\to\infty} nEX_{(1,n)} = 1 < ? < 2 = \lim_{n\to\infty} nV_n^1$$

Optimal Two Choice Value

If there are two variables, $n = 2$, take the best one.

$$V_2^2 = E[X_2 \wedge X_1].$$

When deciding to take X_{n+1}, choose between having one choice left and a guarantee of X_{n+1}, or passing the variable and using two choices on what remains.

$$V_{n+1}^2 = E[V_n^1(X_{n+1}) \wedge V_n^2] \quad \text{for } n \geq 2.$$

With one choice, expectation is over same distribution for each n,

$$V_{n+1}^1 = E[X_{n+1} \wedge V_n^1] \quad \text{for } n \geq 2.$$

Optimal Two Choice Policy

Since

$$V_{n+1}^2 \quad = \quad E[V_n^1(X_{n+1}) \wedge V_n^2],$$

if

$$V_n^1(X_{n+1}) < V_n^2$$

take X_{n+1}. Define the indifference value b_n by

$$V_n^2 = V_n^1(b_n);$$

having one choice with a guarantee of b_n is the same as having two choices with no guarantee. Take X_{n+1} iff it is less than b_n.

Two Choice Stopping for $F(x) = x^\alpha$ on $[0,1]$

Recursion $V_{n+1}^2 = E[V_n^1(X_{n+1}) \wedge V_n^2]$ can be written

$$V_{n+1}^2 = \int_0^1 (V_n^1(x) \wedge V_n^2) \alpha x^{\alpha-1} dx,$$

scaling $W_n = n^{1/\alpha} V_n^2$ and $h_n(y) = n^{1/\alpha} V_n^1((y/n)^{1/\alpha})$,

$$\left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1} = \int_0^1 (n^{1/\alpha} V_n^1(x) \wedge W_n) \alpha x^{\alpha-1} dx$$
$$= \frac{1}{n} \int_0^n (h_n(y) \wedge W_n) dy \quad \text{by } y = nx^\alpha.$$

Fundamental Equation

$$\left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1} \quad = \quad \frac{1}{n}\int_0^n (h_n(y) \wedge W_n) dy.$$

Identify limiting function $h_n(y) \to h(y)$, and hope W_n has same limit as Z_n, where

$$\left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} \quad = \quad \frac{1}{n}\int_0^n (h(y) \wedge Z_n) dy.$$

Recursion with fixed function

Theorem 1 *Under smoothness and monotonicity on q, the limit of the recursion*

$$\left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\int_0^n (q(y) \wedge Z_n)dy.$$

exists and equals

$$b = q(d),$$

where d is the unique root of

$$Q(y) = \int_0^y q(x)dx + (1/\alpha - y)q(y).$$

Warmup: Recover One Stop Behavior

For $X \sim x^\alpha$ we have

$$V^1_{n+1} = E[X_{n+1} \wedge V^1_n] = \int_0^1 [x \wedge V^1_n] \alpha x^{\alpha-1} dx.$$

Set $W^1_n = n^{1/\alpha} V^1_n$, change of variable $y = nx^\alpha$,

$$\begin{aligned} \left(\frac{n}{n+1}\right)^{1/\alpha} W^1_{n+1} &= \int_0^1 [n^{1/\alpha} x \wedge W^1_n] \alpha x^{\alpha-1} dx \\ &= \frac{1}{n} \int_0^n [y^{1/\alpha} \wedge W^1_n] dy. \end{aligned}$$

One Stop Behavior

Thus W_n^1 satisfies *fixed* function recursion with

$$q(y) = y^{1/\alpha},$$

Theorem 1 says $W_n^1 \to q(d)$ where d is the unique root of

$$Q(y) = \int_0^y x^{1/\alpha} dx + (1/\alpha - y) y^{1/\alpha} = 0,$$

giving $d = 1 + 1/\alpha$. Hence,

$$\lim_{n\to\infty} W_n^1 = q(d) = (1 + 1/\alpha)^{1/\alpha},$$

or,

$$\lim_{n\to\infty} nF(V_n^1) = (1 + 1/\alpha), \quad \text{as claimed earlier, 2 at } \alpha = 1.$$

Limit function two stops, $X \sim \mathcal{U}[0,1]$

For given large n and x, find k so that

$$V_n^1(x) \simeq V_{n+k}^1(1) = V_n^1(V_k^1(1)) = V_n^1(V_k^1),$$

hence

$$x \simeq V_k^1 \simeq \frac{2}{k} \quad \text{or} \quad k \simeq \frac{2}{x}.$$

Therefore

$$V_n^1(x) \simeq V_{n+\frac{2}{x}}^1(1) = V_{n+\frac{2}{x}}^1 \simeq \frac{2}{n+\frac{2}{x}} = \frac{2x}{nx+2},$$

so with $y = nx$,

$$h_n(y) = nV_n^1(y/n) \simeq n\frac{2y/n}{ny/n+2} = \frac{2y}{y+2} = h(y).$$

Functions h and Q for $\mathcal{U}[0,1]$

$$
\begin{aligned}
h(y) &= \frac{2y}{y+2}, \\
Q(y) &= \int_0^y \frac{2x}{x+2}dx + (1-y)\frac{2y}{y+2},
\end{aligned}
$$

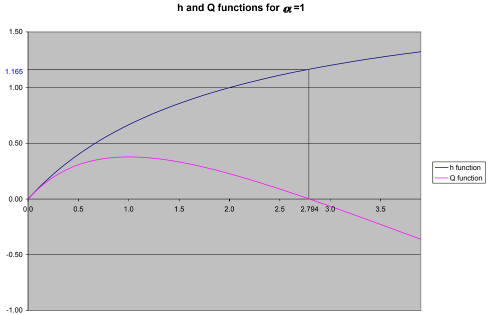
and

$$Q(2.794) = 0 \quad \text{so} \quad h(2.794) = 1.165.$$

For large n, take X_n as first of your two choices if

$$X_n < 2.794/n,$$

and then proceed optimally in the resulting one choice problem.


h and Q functions for α =1
1.50
1.165
1.00
0.50
0.00
-0.50
-1.00
0.0
0.5
1.0
1.5
2.0
2.5
2.794
3.0
3.5
h function
Q function

Limiting $h(x)$ for x^α family

Differential equation heuristic suggests

$$h(y) = y(1 + \frac{\alpha y}{\alpha + 1})^{-1/\alpha},$$

reduces for $\alpha = 1$ to

$$h(y) = \frac{2y}{y + 2}.$$

Closing the Gaps

1. Prove result for limiting value of Z_n sequence generated by fixed function recursion: if above limit, pushed down, if below limit, pushed up.
2. Show $h_n(y) \to h(y)$, with bounds good enough to show that W_n and Z_n have same limit; induction.
3. Extend from $F(x) = x^\alpha$ to large class of $\mathcal{D}(G_\alpha)$; show that only the part of the distribution near zero matters, and bound using the x^α distribution with known limiting behavior.

Extend Result to a Wider Class of F

Write

$$F(x) = x^\alpha L(x).$$

For $F \in \mathcal{D}(G_\alpha)$, we need

$$\lim_{h\to 0} \frac{F(xh)}{F(h)} = x^\alpha \quad \text{or that} \quad \lim_{h\to 0} \frac{L(xh)}{L(h)} = 1.$$

Wide special case is where $\lim_{x\downarrow 0} L(x) = B \in (0, \infty)$.

We take $B = 1$ without loss of generality.

Two ideas used to extend class from canonical x^α

1. The indifference sequence $b_n \downarrow 0$, Therefore only the distribution near 0 matters asymptotically.

2. Use stochastic dominance comparing to x^α type of known behavior:

$$\begin{array}{rrcl} \text{if} & X_i & \leq & Y_i \\ \text{then} & V_n^2(X_n, \ldots, X_1) & \leq & V_n^2(Y_n, \ldots, Y_1). \end{array}$$

Indifference Sequence is Monotone

For any sequence of nonnegative (non necessarily identically distributed) independent random variables $X_n, \ldots, X_1$ with $E[X_2 \wedge X_1] < \infty$ the b_n sequence is monotone non-increasing.

First notice that

$$V_{n+1}^2 \leq E[X_{n+1} \wedge V_n^2],$$

since the two choice problem has smaller value than that achieved by the suboptimal rule of choosing X_{n+1} and forgetting about any second choice if X_{n+1} is less than V_n^2 and retaining two choices otherwise.

Monotonicity

We now have, by $V_n^1(b_n) = V_n^2$,

$$\begin{aligned} V_{n+1}^1(b_{n+1}) &= V_{n+1}^2 \le E[X_{n+1} \wedge V_n^2] \\ &= E[X_{n+1} \wedge V_n^1(b_n)] = V_{n+1}^1(b_n). \end{aligned}$$

The functions $V_n^1(x)$ are monotone increasing, so $b_{n+1} \le b_n$; when options become limited, get less picky.

Therefore $\lim_{n\to\infty} b_n$ exists. Then show limit is zero when $F(x) = x^\alpha L(x)$.

Hence only the distribution near zero matters.

Bounds Using Family x^α

Let $X \sim F(x) = x^\alpha L(x)$ with $L(x) \to 1$. Then

$$X = F^{-1}(U) \quad \text{for } U \sim \mathcal{U}[0,1],$$

and we can construct X as

$$X = U^{1/\alpha} L^*(U^{1/\alpha}),$$

where $\lim_{u \downarrow 0} L^*(u) = 1$. Hence, for any $\epsilon \in (0,1)$ there exists $\delta > 0$ such that

$$1 - \epsilon \le L^*(u^{1/\alpha}) \le 1 + \epsilon \quad \text{for } 0 < u \le \delta,$$

so, where it matters, we have

$$(1-\epsilon)U^{1/\alpha} \le X \le (1+\epsilon)U^{1/\alpha}.$$

Theorem

Let $X_n, \ldots, X_1$ be non-negative integrable i.i.d. random variables with distribution function $F(x) = x^\alpha L(x)$ where $\lim_{x \downarrow 0} L(x) = B \in (0, \infty)$. Then the optimal two choice value V_n^2 satisfies

$$\lim_{n \to \infty} nF(V_n^2) = h^\alpha(d_\alpha)$$

where $d_\alpha > 0$ is the unique solution d to

$$\int_0^d h(y)dy + (1/\alpha - d)h(d) = 0,$$

and $h(y)$ is the function

$$h(y) = \left(\frac{y}{1 + \alpha y/(\alpha + 1)} \right)^{1/\alpha} \quad \text{for } y \geq 0.$$

Table 1: Limiting Values for Various α.

α	d_α	$\lim nF(V_n^1)$	$\lim nF(V_n^2)$	$\lim nF(Em_n)$
0.1	11.9312	11.0000	5.72334	4.52873
0.5	3.8342	3.0000	1.68310	1.41421
1.0	2.7940	2.0000	1.16562	1.00000
2.0	2.2634	1.5000	0.90214	0.78540
5.0	1.9388	1.2000	0.74123	0.65255
10.0	1.8291	1.1000	0.68689	0.60731

Three Directions

1. Consider Type I and II Extremal Classes (partly done).
2. What can one say knowing only that L is slowly varying at 0:
$$L(xh)/L(h) \to 1.$$
3. The k stop problem (some progress)
$$V_{n+1}^k = E[V_n^{k-1}(X_{n+1}) \wedge V_n^k].$$