Addendum to: A Statistical Characterization of Regular Simplices

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The orthogonality $\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} \perp \mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}$, in the proof of the converse near the claim that $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$ is the point closest to $\mathbf{x}$ on the hyperplane $\mathcal{H}$ containing the p points $\mathbf{u} - \{\mathbf{x}\}$, was not sufficiently justified.

Note that the matrix

$$\mathbf{A} = \sigma^{-2}\mathbf{X}'\mathbf{X} \in \mathbf{R}^{n\times n}$$

is symmetric, $\mathbf{A}' = \mathbf{A}$, and idempotent, $\mathbf{A}^2 = \sigma^{-4}\mathbf{X}'\mathbf{X}\mathbf{X}'\mathbf{X} = \sigma^{-4}\mathbf{X}'\mathbf{B}_{\mathbf{u}}\mathbf{X} = \mathbf{A}$. Hence $\mathbf{A}$ is an orthogonal projection, and therefore has rank equal to its trace,

$$\mathrm{rank}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}) = \sigma^{-2}\mathrm{tr}(\mathbf{X}'\mathbf{X}) = \sigma^{-2}\mathrm{tr}(\mathbf{X}\mathbf{X}') = \sigma^{-2}\mathrm{tr}(\mathbf{B}_{\mathbf{u}}) = p,$$

using the cyclic invariance of the trace. With $\mathbf{1}_n \in \mathbf{R}^n$ the vector with all components equal to 1, $\mathbf{A}\mathbf{1}_n = \mathbf{0}$ by virtue of $\bar{\mathbf{x}}_{\mathbf{u}} = 0$. By the rank plus nullity theorem the null space of $\mathbf{A}$ has dimension one, and must therefore equal $\mathrm{span}(\mathbf{1}_n)$, the span of $\mathbf{1}_n$. Hence $\mathbf{A} = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$, as this is the unique orthogonal projection of rank p with null space $\mathrm{span}(\mathbf{1}_n)$.

The inner products $\mathbf{x}'\mathbf{y}$ for all $\mathbf{y} \in \mathbf{u} - \{\mathbf{x}\}$, being off-diagonal elements of $\mathbf{A}$, are equal, yielding $\mathbf{x}'\mathbf{y} = \mathbf{x}'\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, and therefore $\mathbf{x} \perp \mathbf{y} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$. Hence $\mathbf{x} \perp \mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}$, and since $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} = -\mathbf{x}/p$, we have justified $\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} \perp \mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}} = \mathcal{H} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$.

We also note that once the matrix $\mathbf{A}$ has been identified, as its elements are σ^{-2} times the inner products of the vectors in $\mathbf{u}$, the squared interpoint distances between $\mathbf{x}_i \neq \mathbf{x}_j$ in $\mathbf{u}$ can be directly calculated by

$$||\mathbf{x}_i - \mathbf{x}_j||^2 = \mathbf{x}_i'\mathbf{x}_i - 2\mathbf{x}_i'\mathbf{x}_j + \mathbf{x}_j'\mathbf{x}_j = 2\sigma^2\left(\left(1 - \frac{1}{n}\right) + \left(\frac{1}{n}\right)\right) = 2\sigma^2.$$

The authors thank Richard E. Stone for bringing a shortcoming in the original version of this work to their attention.