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6. Problem Set for the Course Mathematical Statistics

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1. Problem: *Uniformly most powerful test for the mean of* $\mathcal{N}(\mu, \sigma^2)$
Suppose that $X_1, \dots, X_n$ are independent and identically $\mathcal{N}(\mu, \sigma^2)$-distributed with known $\sigma > 0$.

(a) Write the continuous density of $X = (X_1, \dots, X_n)$ as a one-parameter exponential family of the form

$$p(x, \mu) = h(x) \exp(\eta(\mu) T_n(x) - B(\mu)), \quad x \in \mathbb{R}^n,\ \mu \in \mathbb{R}, \tag{$*$}$$

where $\mathbb{R} \ni \mu \mapsto \eta(\mu)$ is strictly increasing and

$$T_n(x) := \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} x_i, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

(b) Show that a one-parameter exponential family as in $(*)$ is a monotone likelihood ratio family in T_n, i.e., for all $\mu < \mu'$ in $\mathbb{R}$, the distributions $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(\mu', \sigma^2)$ are different and the likelihood quotient $p(x, \mu')/p(x, \mu)$ is an increasing function of the statistic $T_n(x)$.

(c) Given a level $\alpha \in (0, 1)$, construct a uniformly most powerful test of size α for testing the null hypothesis $\mu \le 0$ versus the alternative $\mu > 0$.

2. Problem: *Testing precision of normally distributed measurements*
Suppose that $X_1, \dots, X_n$ are independent and identically $\mathcal{N}(\mu, \sigma^2)$-distributed measurements of a known standard quantity $\mu \in \mathbb{R}$, using a new measurement instrument. To decide whether it has a higher precision than the old instrument producing values with variance $\sigma_0^2 > 0$, we want to test the null hypothesis $\sigma \ge \sigma_0$ (low precision) versus the alternative $\sigma \in (0, \sigma_0)$ (higher precision). Show that a uniformly most powerful test of level $\alpha \in (0, 1)$ is given by $\varphi = 1_{\{S_n(X) \le x_{\alpha,n}\}}$, where the statistic $S_n(X)$ is given by

$$S_n(X) = \frac{1}{\sigma_0^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

and $x_{\alpha,n}$ denotes the α-quantile of the χ^2-distribution with n degrees of freedom.

Hint: Proceed in a similar way as in the first problem using $T_n = -S_n$.

3. Problem: *A uniformly most powerful goodness-of-fit test*
Let G be a distribution function which has a non-zero density on some interval I of $\mathbb{R}$. For every $\theta > 0$ the function F_θ, defined by $F_\theta(x) := G^{1/\theta}(x)$ for $x \in \mathbb{R}$, is again a continuous distribution function. For a $\theta > 0$, let $X_1, \dots, X_n$ be independent

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and identically F_θ-distributed. Show that a uniformly most powerful test at level $\alpha \in (0,1)$ of the null hypothesis $\theta \leq 1$ versus the alternative $\theta > 1$ is given by $T_n(X) := -2\sum_{i=1}^n \ln G(X_i) \geq x_{1-\alpha}$, where $x_{1-\alpha}$ is the $(1-\alpha)$-quantile of the χ^2-distribution with $2n$ degrees of freedom.

Hints: Proceed as in the first problem. You may use Problem 3 of Set 5.