Scuola Matematica Interuniversitaria
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5. Problem Set for the Course Mathematical Statistics

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1. Problem:
Suppose that $U_1, \dots, U_n$ are idependent and identically distributed according to the uniform distribution $\mathcal{U}(0,\theta)$ with $\theta > 0$ and let $c > 0$. Define $M_n = \max\{U_1, \dots, U_n\}$.

(a) Compute the power function $\beta_n(\theta) = \mathbb{P}_\theta(M_n \geq c)$ and show that it is non-decreasing in θ.

(b) Testing the null hypothesis $\theta \leq \frac{1}{2}$ against $\theta > \frac{1}{2}$, what choice of c would make the test have size (or probability of type I error) exactly 5%?

(c) Draw a rough graph of the power function for the test specified in (b) for $n = 20$.

(d) Determine the minimal n so that the test specified in (b) has power at least 0.98 for $\theta = \frac{3}{4}$?

(e) If $M_{20} = 0.48$, what is the p-value of the test?

2. Problem: *Kolmogorov's test*
Assume that the distribution functions F and F_0 are continuous. Define the Kolmogorov test statistic based on n observations by $D_n = \sup_{x\in\mathbb{R}} |\hat{F}_n(x) - F_0(x)|$, where $\hat{F}_n = \frac{1}{n}\sum_{i=1}^n 1_{[X_i,\infty)}$ denotes the empirical distributionfunction based on independent $X_1, \dots, X_n \sim F$.

(a) Show the following lower bound for the power of the test with critical value c:

$$\mathbb{P}_F(D_n \geq c) \geq \sup_{x\in\mathbb{R}} \mathbb{P}_F(|\hat{F}_n(x) - F_0(x)| \geq c).$$

(b) Show for $F \neq F_0$ and every level $\alpha \in (0,1)$, that the power of Kolmogorov's test tends to 1 as $n \to \infty$.

Hint: You may use the Glivenko–Cantelli theorem. Note that the citical value $c_{\alpha,n}$ depends on the level α and the sample size n.

3. Problem: *Fisher's method for aggregating independent tests*
Suppose that $T_1, \dots, T_r$ are independent tests for a simple null hypothesis, under which each of them has a continuous distribution function. Let $\alpha(T_1), \dots, \alpha(T_r)$ denote the p-values and define the test statistic $\tilde{T} = -2\sum_{j=1}^r \ln \alpha(T_j)$. Show that $\tilde{T}$ under the null hypothesis has a χ^2-distribution with $2r$ degrees of freedom.

Hint: You may use Problem 4 from Set 2.