Problem Set 4

1. Consider replications of the linear model

$$Y = \beta x + \epsilon$$

where $x \in [-1, 1]$ and $\epsilon_1, \ldots, \epsilon_n$ have mean zero and are uncorrelated with common variance σ^2.

   (a) Find the least squares estimate of β and compute its variance.

   (b) How should the observations $x_1, \ldots, x_n$ be placed in order to minmize the variance in part a)? Is the solution unique?

2. For any matrix $A \in \mathbb{R}^{n \times m}$ let

$$\mathcal{R}(A) = \{Ax : x \in \mathbb{R}^m\}$$

denote the range space of A, and $r(A) = \dim(\mathcal{R}(A))$, the rank of A. When $n = m$ let $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$, the sum of the diagonal elements of A.

Let P be an $n \times n$ matrix satisfying $P^T = P, P^2 = P$.

   (a) Prove that $v \in \mathcal{R}(P)$ if and only if $Pv = v$.

   (b) Prove that if λ is an eigenvalue of P then $\lambda \in \{0, 1\}$.

   (c) Prove that $r(P) = \mathrm{tr}(P)$.

   (d) Let $X \in \mathbb{R}^{n \times p}$ with $p < n$ and $r(X) = p$. Prove that

$$P = X(X^T X)^{-1} X^T$$

   is a projection matrix that satisfies $\mathcal{R}(P) = \mathcal{R}(X)$.

3. Let $\mathbf{Z} \in \mathbb{R}^n$ have components $Z_1, \ldots, Z_n$, independent $\mathcal{N}(0, 1)$ random variables.

   (a) Compute the moment generating function of $\mathbf{Z}$,

$$M(s) = E[\exp(s^T Z)], \quad s \in \mathbb{R}^n.$$

(b) Let Σ be a non-negative definite $n \times n$ matrix, diagonalized as $\Sigma = U^T \Lambda U$ for $U^T U = I$ and Λ diagonal, and let $\Sigma^\alpha = U^T \Lambda^\alpha U$ for all α, where the power of the diagonal matrix Λ is taken componentwise. With $\mu \in \mathbb{R}^n$, compute the moment generating function of

$$X = \Sigma^{1/2} Z + \mu$$

We write $X \sim \mathcal{N}_n(\mu, \Sigma)$, the multivariate normal distribution with mean μ and variance matrix Σ.

(c) Prove that any linear transformation of a multivariate normal is again multivariate normal.

(d) For $X \sim \mathcal{N}_n(\mu, \Sigma)$ and n_1 and n_2 positive integers summing to n, write $X = (X_1^T, X_2^T)^T$ for $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$. Prove that X_1 and X_2 are independent if and only if the covariance matrix between X_1 and X_2 is zero, that is, if and only if $\text{Cov}(X_1, X_2) = 0$.

(e) With X decomposed into two subvectors X_1, X_2 as in part d), show that if $\text{Var}(X_2)$ is positive definite, then the conditional distribution of X_1 given X_2 is multivariate normal, and determine the parameters of this conditional distribution. Hint: Consider the covariance between $(X_1 - E[X_1]) + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - E[X_2])$ and $X_2 - E[X_2]$, where $\Sigma_{ij}, 1 \leq i, j \leq 2$ are the block matrices resulting when the $n \times n$ matrix Σ is partitioned according to the dimensions of X_1 and X_2 as $(n_1 + n_2) \times (n_1 + n_2)$, e.g. $\text{Var}(X_2) = \Sigma_{22}$.

4. Let $\mathbf{Z} \sim \mathcal{N}_n(0, \sigma^2 I)$, and let $P \in \mathbb{R}^{n \times n}$ be a projection matrix. Prove that

$$\mathbf{Z}^T P \mathbf{Z} \sim \sigma^2 \chi_k^2 \quad \text{where } k = \text{r}(P), \text{ the rank of } P.$$

You may use the results of exercise 2.