Problem Set 3

1. Let $p(x, \theta), \theta \in \mathbb{R}$ be a parametric family of densities, and consider the two parameter family

$$g(x; \theta_1, \theta_2) = p(x, \theta_1 + \theta_2), (\theta_1, \theta_2) \in \mathbb{R}^2. \tag{1}$$

In the following, we assume the usual regularity of the density that, for instance, allows differentiation under the integeral.

   (a) Compute the score functions $U_1(\theta_1, X)$ and $U_2(\theta_2, X)$ for model (1) and show that they have correlation one, and hence that the information matrix will be singular.

   (b) Show that the parameters θ_1 and θ_2 in model (1) are not identifiable.

2. Compute the variance lower bound provided by the information inequality for the unbiased estimation of the 'signal to noise' parameter

$$q(\mu, \sigma^2) = \mu/\sigma$$

when data is sampled independently from the normal $\mathcal{N}(\mu, \sigma^2)$ distribution with both parameters unknown.

3. (a) Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be non-zero vectors in $\mathbf{R}^n$, and define the projection of $\mathbf{u}_1$ along $\mathbf{u}_2$ to be

$$P_{\mathbf{u}_2}\mathbf{u}_1 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}(\mathbf{u}_1 \cdot \frac{\mathbf{u}_2}{||\mathbf{u}_2||}).$$

   Show that $\mathbf{u}_2 \perp \mathbf{u}_1 - P_{\mathbf{u}_2}\mathbf{u}_1$ and interpret this fact geometrically with a diagram.

   (b) Show that

$$||\mathbf{u}_1 - P_{\mathbf{u}_2}\mathbf{u}_1||^2 = ||\mathbf{u}_1||^2 - \frac{(\mathbf{u}_1 \cdot \mathbf{u}_2)^2}{||\mathbf{u}_2||^2}.$$

   (c) Consider a smooth density $p(\mathbf{x}, \theta), \theta \in \Theta \subset \mathbf{R}^2$. By inverting the information matrix $I = (I_{ij})_{1 \le i,j \le 2}$, show that the Cramer Rao bound lower bound for estimating θ_1 in the presence of θ_2 is the reciprocal of

$$I_{11} - \frac{I_{12}^2}{I_{22}}.$$

(d) For random variables V, W define a dot product

$$V \cdot W = EVW \quad \text{so that} \quad ||V||^2 = EV^2,$$

and orthogonality and projections can be defined as usual. Let $U_1 = U(\mathbf{X}, \theta_1), U_2 = U(\mathbf{X}, \theta_2)$ be score functions associated with θ_1, θ_2 respectively. Show that the lower bound for estimating θ_1 in the presence of θ_2 is the reciprocal of

$$||U_1 - P_{U_2} U_1||^2.$$

4. Consider n independent observations of the linear regression model

$$Y = \beta_1 x_1 + \beta_2 x_2 + \epsilon,$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$, and the vectors $\mathbf{x}_1, \mathbf{x}_2$ are not colinear. Determine the Information matrix for the unknown vector parameter (β_1, β_2), and the effective information for the estimation of β_1 in the presence of β_2.

5. Let $X_1, \ldots, X_m$ be independent samples from the $\mathcal{U}[0, \theta]$ distribution with $\theta > 0$ unknown. Here we consider the asymptotic behavior of $X_{(n)}$, the maximum likelihood estimator. Prove that there exists a sequence of positive constants a_n and a non-trivial distribution on a random variable X such that

$$a_n (X_{(n)} - \theta) \rightarrow_d X,$$

and determine a_n and the distribution of X.