Problem Set 2

1. Prove that the Gamma function $\Gamma(\cdot)$ satisfies $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$ for all $\alpha > 0$

2. Find the mean, variance and moment generating function of the $\Gamma(\alpha, \beta)$ distribution. Using moment generating functions, show that the sum of two independent Gamma variables with the same scale parameter β is Gamma, and find the parameters of the sum.

3. Let X be a continuous random variable with support $[0, \infty)$. Show that

$$P(X > t | X > s) = P(X > t - s)$$

   for all $t \geq s \geq 0$ if and only if $X \sim \mathcal{E}(\lambda)$ for some $\lambda > 0$.

4. Let $Z \sim \mathcal{N}(0, 1)$, and compute and identify the distribution of Z^2. Using moment generating functions, identify the distribution of $Z_1^2 + Z_2^2$ where Z_1, Z_2 are independent standard normals, and show that it is the same as that of $-2 \log U$ where $U \sim \mathcal{U}[0, 1]$.

5. Prove the Cauchy Schwarz inequality

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

   for vectors in $\mathbb{R}^n$, but determine what the minimal requirements are for a space for it to be satisfied. Find a necessary and sufficient condition for the inequality to be an equality.

6. Let $p(\mathbf{x}; \theta), \theta \in \Theta \subset \mathbb{R}$. Assuming sufficient smoothness, let and that differentiation and integration can be interchanged, prove that

$$\text{Var}_\theta(U(\theta, \mathbf{X})) = -E_\theta \left[\partial_\theta U(\theta, \mathbf{X}) \right]$$

   where

$$U(\theta, \mathbf{x}) = \partial_\theta \log p(\mathbf{x}; \theta).$$

7. Find a formula for all the moments $E[Z^n], n \geq 1$ of a standard normal variable Z, and demonstrate that $\text{Var}(Z^2) = 2$.

8. For $\mathbf{X}$ and $\mathbf{Y}$ vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, let

$$\text{Cov}(\mathbf{X}, \mathbf{Y}) = E[(\mathbf{X} - E\mathbf{X})(\mathbf{Y} - E\mathbf{Y})^T],$$

and

$$\text{Var}(\mathbf{X}) = \text{Cov}(\mathbf{X}, \mathbf{X})$$

Prove that

(a)

$$\text{Cov}(A\mathbf{X}, B\mathbf{Y}) = A\text{Cov}(\mathbf{X}, \mathbf{Y})B^T$$

(b)

$$\text{Cov}(\mathbf{X}, \mathbf{Y}) = \text{Cov}(\mathbf{Y}, \mathbf{X})^T$$

(c) That $\text{Var}(\mathbf{X})$ is always non-negative definite, and fails to be positive definite if and only if $\mathbf{X}$ has a linearly dependency, that is, when there exists a vector $\mathbf{a}$ and a constant c such that

$$P(\mathbf{a}^T\mathbf{X} = c) = 1.$$