Problem Set 1

1. Show Benford's law enjoys the following scale invariance property: If the digits $\{1, 2, ..., 9\}$ have the Benford distribution, then multiplying by 2 results in a set of numbers where the leading digit of 1 has the probability as predicted by Benford's law. Show the uniform distribution over this same set of numbers does not have this property. Similarly explore the behavior of the Benford distribution when changing from base 10 to base 3.

2. We say the sequence of distribution functions F_n converges to F in distribution when $\lim_{n\to\infty} F_n(x) = F(x)$ for all points x at which F is continuous. Explain why the restriction to the continuity points of F is needed.

3. Let $b_{n,k}(p) = P(X = k)$ for a Binomial $\mathcal{B}(n, p)$ random variables, and $p_k(\lambda) = P(Y = k)$ for a Poisson $\mathcal{P}(\lambda)$ random variable. Show $\lim_{n\to\infty} b_{n,k}(\lambda/n) = p_k(\lambda)$ for all non-negative integers k.

4. Find the mean, variance and moment generating function of the $P(\lambda)$ distribution. Using moment generating functions, show that the sum of independent Poissons is Poisson, and determine the parameter of the sum.

5. Let X be a random variable with support $\{1, 2, \ldots\}$. Show that

$$P(X > t|X > s) = P(X > t - s)$$

   for all $t \geq s \geq 1$ if and only if $X \sim \mathcal{G}(p)$ for some $p \in (0, 1)$

6. Let $F(\cdot)$ be a strictly increasing continuous distribution function, and $U \sim \mathcal{U}[0, 1]$. Show that $F^{-1}(U)$ has distribution F and that if X has distribution F then $F(X) \sim \mathcal{U}[0, 1]$. Consider relaxing the given assumptions.