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A Statistical Characterization of Regular Simplices

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1. INTRODUCTION. Picture three points at the vertices of an equilateral triangle in two dimensions, or four points at the vertices of a regular tetrahedron in three dimensions. Thought of as scatterings of data they wouldn't seem to reveal strong linear associations between the coordinates. There are no clear axes of elongation in the scatterplots, which would suggest that change in some variable is predictable as a function of the others. In general, such associations are usually indicated by the covariance matrix $\mathbf{S_u}$ of the set of points $\mathbf{u} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ in $\mathbb{R}^p$, which is given by

$$\mathbf{S_u} = \frac{1}{|\mathbf{u}|} \sum_{\mathbf{x} \in \mathbf{u}} (\mathbf{x} - \overline{\mathbf{x}}_\mathbf{u})(\mathbf{x} - \overline{\mathbf{x}}_\mathbf{u})',$$

where

$$\overline{\mathbf{x}}_\mathbf{u} = \frac{1}{|\mathbf{u}|} \sum_{\mathbf{x} \in \mathbf{u}} \mathbf{x}.$$

The off-diagonal entries of $\mathbf{S_u}$, the pairwise covariances, tell us something about dependencies. If the coordinate variables are independent these entries are zero. Though the converse is false, a diagonal covariance matrix roughly says that the coordinates are not mutually linearly predictable from each other. Indeed, for our equilateral triangle in $\mathbb{R}^2$, tetrahedron in $\mathbb{R}^3$, and the generalized configurations in higher dimensions having equal interpoint distances, the covariance matrix turns out to be diagonal. In fact, it's a scalar multiple of the identity. Furthermore, the converse is also true: any $n = p + 1$ points in p dimensions whose covariance matrix is a positive multiple of the identity are equidistant from each other. We formalize this result in the following theorem:

Theorem. *Let* $\mathbf{u} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *be a set of* n *points in* $\mathbb{R}^p$, *with* $n = p + 1 \geq 2$, *and let* σ^2 *be an arbitrary positive number. Then the interpoint distances of* $\mathbf{u}$ *satisfy* $||\mathbf{x}_i - \mathbf{x}_j||^2 = 2\sigma^2\delta_{ij}$ *if and only if* $n\mathbf{S_u} = \sigma^2\mathbf{I}_p$.

In other words, $p + 1$ points in p dimensions lie at the vertices of a regular simplex if and only if their covariance matrix is a multiple of the identity. A proof of this statistical characterization of regular simplices is given in section 2, after some preliminaries.

2. STATISTICAL CHARACTERIZATION OF REGULAR SIMPLICES. The reader is assumed to be familiar with the basic elements of linear algebra in $\mathbb{R}^p$ (linear subspaces, span, linear dependence and independence, basis and dimension), as treated, for example, in the text of Seber [**1**]. For a finite subset $\mathbf{u}$ of $\mathbb{R}^p$ let $\mathcal{V}_{\mathbf{u}} = \mathrm{span}\{\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}} : \mathbf{x} \in \mathbf{u}\}$.

Lemma. *With $n > 1$ let $\mathbf{u}$ be any collection of n points in $\mathbb{R}^p$ with common squared interpoint distance $2\sigma^2 > 0$. Then $\dim(\mathcal{V}_{\mathbf{u}}) = n - 1$, and with $r_{\sigma,n}^2 = \sigma^2(n-1)/n$ and $s_{\sigma,n}^2 = \sigma^2/(n(n-1))$, the following are true for each $\mathbf{x}$ in $\mathbf{u}$:*

$$||\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}}|| = r_{\sigma,n}, \quad ||\bar{\mathbf{x}}_{\mathbf{u}} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}|| = s_{\sigma,n}, \quad \mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} \perp \mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}.$$

Proof. We argue by induction. The three claims are easily verified if $n = 2$. When $n > 2$, for every $\mathbf{x}$ in $\mathbf{u}$ the points of $\mathbf{u} - \{\mathbf{x}\}$ are equidistant from $\mathbf{x}$, and by the induction hypotheses also equidistant from their average $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, albeit at a smaller distance. Hence, the points of $\mathbf{u} - \{\mathbf{x}\}$ lie on the intersection of two spheres with distinct centers, $\mathbf{x}$ and $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, which implies that $\mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}$ is perpendicular to the direction vector of the line

$$L_{\mathbf{u},\mathbf{x}}(\alpha) = \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} + \alpha\left(\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}\right) \quad (\alpha \in \mathbb{R})$$

passing through these centers and that the points of $\mathbf{u} - \{\mathbf{x}\}$ are equidistant from each point of $L_{\mathbf{u},\mathbf{x}}$. In particular, all points of $\mathbf{u} - \{\mathbf{x}\}$ are equidistant from $L_{\mathbf{u},\mathbf{x}}(1/n) = \bar{\mathbf{x}}_{\mathbf{u}}$, hence so are all points of $\mathbf{u}$. Because $\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} \perp \mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}$ but $\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}} \in \mathcal{V}_{\mathbf{u}}$, $\dim(\mathcal{V}_{\mathbf{u}}) = \dim\left(\mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}}\right) + 1$. By orthogonality $||\bar{\mathbf{x}}_{\mathbf{u}} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}||^2 = r_{\sigma,n}^2 - r_{\sigma,n-1}^2$ and does not depend on $\mathbf{x}$. Using the fact that $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, $\bar{\mathbf{x}}_{\mathbf{u}}$, and $\mathbf{x}$ all lie on $L_{\mathbf{u},\mathbf{x}}$ in tandem with orthogonality gives $2\sigma^2 = r_{\sigma,n-1}^2 + (s_{\sigma,n} + r_{\sigma,n})^2$; solving these two equations for $r_{\sigma,n}$ and $s_{\sigma,n}$ finishes the induction. ∎

Proof of the theorem. Let $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, an element of $\mathbb{R}^{p\times n}$. Since $\mathbf{S}_{\mathbf{T}(\mathbf{u})} = \mathbf{S}_{\mathbf{u}}$ for any translation $\mathbf{T}$, we can assume without loss of generality that the members of $\mathbf{u}$ have already been centered by subtraction of their mean, so $\bar{\mathbf{x}}_{\mathbf{u}} = \mathbf{0}$ and in general, letting $\mathbf{B}_{\mathbf{v}} := |\mathbf{v}|\mathbf{S}_{\mathbf{v}}$ for any finite set $\mathbf{v}$ of vectors, we have

$$\mathbf{B}_{\mathbf{u}} = \sum_{\mathbf{x}\in\mathbf{u}} \mathbf{x}\mathbf{x}' = \mathbf{X}\mathbf{X}'. \tag{1}$$

Assuming that the points are equidistant, we infer from (1) and the lemma that

$$\begin{aligned}\mathbf{B}_{\mathbf{u}}\mathbf{x} &= \sum_{\mathbf{y}\in\mathbf{u}-\{\mathbf{x}\}} \mathbf{y}\mathbf{y}'\mathbf{x} + \mathbf{x}\mathbf{x}'\mathbf{x} = (r_{\sigma,n}^2 - \sigma^2) \sum_{\mathbf{y}\in\mathbf{u}-\{\mathbf{x}\}} \mathbf{y} + r_{\sigma,n}^2\mathbf{x} \\ &= (\sigma^2 - r_{\sigma,n}^2)\mathbf{x} + r_{\sigma,n}^2\mathbf{x} = \sigma^2\mathbf{x}\end{aligned}$$

for each $\mathbf{x}$ in $\mathbf{u}$. Hence $\mathbf{B}_{\mathbf{u}}\mathbf{x} = \sigma^2\mathbf{I}_p\mathbf{x}$ on $\mathcal{V}_{\mathbf{u}}$. Since $\dim(\mathcal{V}_{\mathbf{u}}) = p$ by the lemma, $\mathbf{B}_{\mathbf{u}} = \sigma^2\mathbf{I}_p$.

For the converse, assume that $\mathbf{B}_{\mathbf{u}} = \sigma^2\mathbf{I}_p$ and that $n > 2$, the base case being trivial. As any $p - 1$ points in $\mathbb{R}^p$ lie in a hyperplane of dimension $p - 1$, for $\mathbf{x}$ in $\mathbf{u}$ let $\mathcal{H}$ denote such a hyperplane that contains $\mathbf{u} - \{\mathbf{x}\}$, and note by orthogonality that $\bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$ is the point on $\mathcal{H}$ closest to $\mathbf{x}$. Now let $\mathbf{T}$ be the translation $\mathbf{T}\mathbf{y} = \mathbf{y} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, and, with $\{\mathbf{e}_i\}_{1\le i\le p}$ the standard basis for $\mathbb{R}^p$, let $\mathbf{O}$ be the rotation that maps $\mathbf{T}\mathbf{x}$ to βe_p, where

$\beta = ||\mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}||$. That is, $\mathbf{V}(\mathbf{x}) = \beta \mathbf{e}_p$ for $\mathbf{V} = \mathbf{OT}$, and

$$\begin{aligned} \mathbf{B}_{\mathbf{V}(\mathbf{u}-\{\mathbf{x}\})} + \beta^2 \mathbf{e}_p \mathbf{e}_p' &= \mathbf{O}\left(\mathbf{B}_{\mathbf{T}(\mathbf{u}-\{\mathbf{x}\})} + \mathbf{Tx}(\mathbf{Tx})'\right)\mathbf{O}' \\ &= \mathbf{O}\left(\mathbf{B}_{\mathbf{u}-\{\mathbf{x}\}} + \mathbf{Tx}(\mathbf{Tx})'\right)\mathbf{O}' = \mathbf{OB}_{\mathbf{T}(\mathbf{u})}\mathbf{O}' \\ &= \mathbf{OB}_{\mathbf{u}}\mathbf{O}' = \sigma^2 \mathbf{OI}_p \mathbf{O}' = \sigma^2 \mathbf{I}_p. \end{aligned} \tag{2}$$

Since $\mathcal{V}_{\mathbf{u}-\{\mathbf{x}\}} \perp \mathbf{x} - \bar{\mathbf{x}}_{\mathbf{u}-\{\mathbf{x}\}}$, $\mathbf{V}(\mathcal{H}) \subset \mathbb{R}^{p-1} \times \{0\}$, and we can consider the points $\mathbf{V}(\mathbf{u} - \{\mathbf{x}\})$ as lying in $\mathbb{R}^{p-1}$. By (2), the $(p-1) \times (p-1)$ submatrix $[\mathbf{B}_{\mathbf{V}(\mathbf{u}-\{\mathbf{x}\})}]_{1 \le i,j \le p-1}$ equals $\sigma^2 \mathbf{I}_{p-1}$, so applying the induction hypotheses to $\mathbf{V}(\mathbf{u} - \{\mathbf{x}\})$ we conclude that the interpoint distances of $\mathbf{u} - \{\mathbf{x}\}$, unchanged by $\mathbf{V}$, are all $2\sigma^2$. The induction is completed by noting that this is true for each $\mathbf{x}$ in $\mathbf{u}$. ■

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How Terminal Is Terminal Velocity?

Lyle N. Long and Howard Weiss

Several years ago the authors of the present piece wrote a MONTHLY article [**1**] describing the velocity dependence of aerodynamic drag. Recently, a science reporter for the *Guardian* newspaper, who found this reference, solicited our help in writing an article about how a 102-year-old woman from Turin could fall out of a fourth floor window and survive [**2**]. In particular, the reporter wanted to know about the role of terminal velocity in her survival.

An internet search found many webpages dealing with terminal velocity, but none relevant to understanding whether terminal velocity is reached during falls from relatively low heights. We also found an interesting posting from a Coast Guard officer in New York Harbor inquiring about the chance of survival when a person falls from a 230-foot bridge into water and wanting to know how fast the person would hit the water. We also learned about a popular book, *New York Dead* by Stuart Woods [**3**], in which a character, a popular TV newscaster, plummets twelve stories onto a heap of dirt twenty yards away and survives. The book contains a discussion and speculation about how the character survived the fall. One idea is that the character was "saved by terminal velocity." An important part of the plotline of another popular novel, *Angels and Demons* by Dan Brown [**4**], is devoted to a character falling from a helicopter. It raises the question of how a two foot-by-two foot piece of cloth could slow one's speed by 25%. Clearly there are many practical and interesting aspects to studying people's falling through the air.

In this self-contained short addendum to [**1**], we explain why terminal velocity played no role in the falls from the fourth through twelfth story windows. We explicitly