A Berry-Esseen Theorem for the Lightbulb Process

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The Lightbulb Process

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In the lightbulb process of Rao, Rao and Zhang, on days $r = 1, \ldots, n$, out of n lightbulbs, all initially off, exactly r bulbs, selected uniformly and independent of the past, have their status changed from off to on, or vice versa.

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Their histograms indicated that X is asymptotically normal, but left the question open. 'Out of the box' central limit theorems seem to not apply easily to this process.

Mean and Variance

For $s = 1, \ldots, n$, let

$$\lambda_{n,1,s} = 1 - \frac{2s}{n} \quad \text{and} \quad \lambda_{n,2,s} = 1 - \frac{4s}{n} + \frac{4s(s-1)}{n(n-1)},$$

and $\lambda_{n,b,\mathbf{s}} = \prod_{r=1}^{n} \lambda_{n,b,s_r}$. Then with switch pattern $\mathbf{s} = (s_1, \ldots, s_n)$,

$$EX_{\mathbf{s}} = \frac{n}{2}\left(1 - \lambda_{n,1,\mathbf{s}}\right)$$

and

$$\mathrm{Var}(X_{\mathbf{s}}) = \frac{n}{4}(1 - \lambda_{n,2,\mathbf{s}}) + \frac{n^2}{4}(\lambda_{n,2,\mathbf{s}} - \lambda_{n,1,\mathbf{s}}^2).$$

Berry Esseen Theorem for Lightbulb

Theorem 1 *Let X be the number of bulbs on at the terminal time n, an even integer, and $\mu = n/2$ and $\sigma^2 = \mathit{Var}(X)$. Then*

$$\sup_{z \in \mathbb{R}} \left| P\left(\frac{X-\mu}{\sigma} \le z\right) - P(Z \le z) \right| \le \frac{n}{2\sigma^2}\Psi_0 + 1.64\frac{n}{\sigma^3} + \frac{2}{\sigma}$$

where Z is standard normal and

$$\Psi_0 \le \frac{1}{2\sqrt{n}} + \frac{1}{2n} + e^{-n/2} \quad \textit{for } n \ge 6.$$

Yields a bound of order $O(n^{-1/2})$ as $n \to \infty$.

Composition Markov chains of multinomial type

Considered by Zhou and Lange, such chains in general are based on a $d \times d$ Markov transition matrix P which describes the transition of a single particle in a system of n identical particles, a subset of which is selected uniformly to undergo transition at each time step according to P.

Very complete spectral decompositions of such chains may be available; this decomposition becomes essential to the calculation of the Berry Esseen bound for this problem.

Ehrenfest chains, Hoare-Rahman chains, Kimura's model for DNA base-pair substitution.

Size Bias Coupling

Let X be nonnegative with finite mean μ. Recall X^s has the X-sized biased distribution if

$$E[Xf(X)] = \mu E[f(X^s)]$$

for all smooth f.

If $X = \sum_{i=1}^n X_i$ is the sum of exchangeable indicators, then one may form X^s by picking i uniformly and summing the variables having distribution

$$\mathcal{L}(X_1^i, \ldots, X_n^i) = \mathcal{L}(X_1, \ldots, X_n | X_i = 1)$$

Size biased Coupling

With $W = (X-\mu)/\sigma$, and $W^s = (X^s - \mu)/\sigma$,

$$\begin{aligned} & E\left(h(W) - Eh(Z)\right) \\ &= E\left(f'(W) - Wf(W)\right) \\ &= E\left(f'(W) - \frac{\mu}{\sigma}(f(W^s) - f(W))\right) \\ &= E\left(f'(W)(1 - \frac{\mu}{\sigma}(W^s - W))\right. \\ &\qquad \left. - \frac{\mu}{\sigma}\int_0^{W^s - W}(f'(W+t) - f'(W))dt\right). \end{aligned}$$

When monotone $W^s \geq W$.

Concentration Inequality

Size bias version of exchangeable pair concentration inequality of Shao and Su (2005).

Lemma 1 *Let X be a nonnegative random variable with mean μ and variance σ^2, both finite and positive, and let X^s be given on the same space as X, having the X size biased distribution and satisfying $X^s \geq X$. Then with $W = (X - \mu)/\sigma$ and $W^s = (X^s - \mu)/\sigma$, for any $z \in \mathbb{R}$ and $a \geq 0$,*

$$\frac{\mu}{\sigma} E(W^s - W) \mathbf{1}_{\{W^s - W \leq a\}} \mathbf{1}_{\{z \leq W \leq z+a\}} \leq a.$$

Monotone Size Biased Coupled Pair

Theorem 2 *Let X be a nonnegative random variable with mean μ and variance σ^2, both finite and positive, and let X^s be given on the same space as X, with the X size biased distribution, satisfying $X \le X^s \le X + B$ for some $B > 0$. Then with $W = (X - \mu)/\sigma$, we have*

$$\sup_{z \in \mathbb{R}} |P(W \le z) - P(Z \le z)| \le \frac{\mu}{\sigma^2}\Psi + 0.82\frac{\delta^2 \mu}{\sigma} + \delta,$$

where

$$\Psi = \sqrt{\mathit{Var}(E(X^s - X|X))} \quad \textit{and} \quad \delta = B/\sigma.$$

Size biased coupling: Lightbulb

The number of bulbs on at time n is

$$X = \sum_{k=1}^{n} X_k,$$

where X_k is the indicator that bulb k is on. Need to select bulb at random, and turn it on if not already, keeping correct conditional distribution, and without greatly affecting the status of the other bulbs.

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With X_{rk} the switch variable of bulb k at time r, we have

$$X_k = \left(\sum_{r=0}^{n} X_{rk} \right) \bmod 2.$$

Size biased coupling: Lightbulb

Let $n = 2m$ and $\mathbf{X} = \{X_{rk}, r, k = 1, \ldots, n\}$. For given bulb i, if $X_i = 1$ then set $\mathbf{X}^i = \mathbf{X}$.

Otherwise $J^i \sim \mathcal{U}\{j : X_{n/2,j} = 1 - X_{n/2,i}\}$, independent of $\{X_{rk} : r \neq n/2, k = 1, \ldots, n\}$. Let $\mathbf{X}^i$ have components

$$X^i_{rk} = \begin{cases} X_{rk} & r \neq n/2 \\ X_{n/2,k} & r = n/2, k \notin \{i, J^i\} \\ X_{n/2,J^i} & r = n/2, k = i \\ X_{n/2,i} & r = n/2, k = J^i. \end{cases}$$

Size biased coupling: Lightbulb

In words, if $X_i = 0$ then uniformly select a j whose switch variable $X_{n/2,j}$ at stage $n/2$ is opposite to $X_{n/2,i}$, and interchange.

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If I is the uniformly chosen index, and J the index of the bulb with opposite status at stage $n/2$, then

$$X^s - X = 2\mathbf{1}_{\{X_I=0, X_J=0\}}.$$

The coupling is monotone and bounded.

Conditional Variance Calculation, $n = 2m$

With $\mathcal{F}$ the σ-algebra generated by the switch variables $\mathbf{X}$,

$$E\left(X^s - X \,\middle|\, \mathcal{F}\right) = \frac{4}{n^2} \sum_{i \neq j} \mathbf{1}_{\{X_i=0, X_j=0, X_{n/2,i}=0, X_{n/2,j}=1\}}.$$

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The variance calculation requires the computation of (four) joint probabilities such as $g_{2,2,n,(1,\ldots,n),n/2}$,($g_{\alpha,\beta,n,\mathbf{s},l}$)

$$P(X_{i_1} = 0, X_{j_1} = 0, X_{i_2} = 0, X_{j_2} = 0, \\ X_{n/2,i_1} = 0, X_{n/2,j_1} = 1, X_{n/2,i_2} = 0, X_{n/2,j_2} = 1).$$

Conditioning on switches at stage $n/2$ they later become 'initial conditions.'

Spectral Decomposition

For a given subset of b of the n bulbs, the eigenvalues of one, and multiple steps of the chain are given by

$$\lambda_{n,b,s} = \sum_{t=0}^{b} \binom{b}{t} (-2)^t \frac{(s)_t}{(n)_t} \quad \text{and} \quad \lambda_{n,b,\mathbf{s}} = \prod_{r=1}^{k} \lambda_{n,b,s_r},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ denotes the falling factorial, and the empty product is 1.

Spectrum of Composition Markov Chains of Multinomial Type

In the case of the lightbulb chain there are $d = 2$ states and the transition matrix P of a single bulb is given by

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With $b \in \{0, 1, \ldots, n\}$ let $P_{n,b,s}$ be the $2^b \times 2^b$ transition matrix of a subset of size b of the n total lightbulbs when s of the n bulbs are selected uniformly to be switched.
Letting $P_{n,0,s} = 1$ for all n and s, and I_2 the 2×2 identity matrix, for $n \geq 1$ the matrix $P_{n,b,s}$ is given recursively by

$$P_{n,b,s} = \frac{s}{n}\left(P \otimes P_{n-1,b-1,s-1}\right) + \left(1 - \frac{s}{n}\right)\left(I_2 \otimes P_{n-1,b-1,s}\right).$$

Spectral Decomposition

Transition matrices are simultaneously diagonalizable by

$$P_{n,b,s} = \otimes^b T^{-1} \Gamma_{n,b,s} \otimes^b T,$$

where $\Gamma_{n,b,s} = \text{diag}(\lambda_{n,a_1,s}, \ldots, \lambda_{n,a_{2^b},s})$, e.g.,

$$\mathbf{a}_1 = (0,1), \quad \mathbf{a}_2 = (0,1,1,2) \quad \text{and} \quad \mathbf{a}_3 = (0,1,1,2,1,2,2,3),$$

and

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Spectral Decomposition

For calculation of $g_{2,2,n,(1,\ldots,n),n/2}$,

$$P(X_{i_1} = 0, X_{j_1} = 0, X_{i_2} = 0, X_{j_2} = 0,\\ X_{n/2,i_1} = 0, X_{n/2,j_1} = 1, X_{n/2,i_2} = 0, X_{n/2,j_2} = 1),$$

spectral decomposition yields

$$g_{2,2,n,\mathbf{s},l} = \frac{1}{16}(1 - 2\lambda_{n,2,\mathbf{s}_l} + \lambda_{n,4,\mathbf{s}_l})\frac{(s_l)_2(n-s_l)_2}{(n)_4},$$

where

$$\mathbf{s}_l = (s_1, \ldots, s_{l-1}, s_{l+1}, \ldots, s_n).$$

Spectral Decomposition

Now turning to $\lambda_{n,4,\mathbf{s}}$, for $n \geq 4$ consider the fourth degree polynomial

$$f_4(x) = 1 - \frac{8x}{n} + \frac{24(x)_2}{(n)_2} - \frac{32(x)_3}{(n)_3} + \frac{16(x)_4}{(n)_4}, \qquad 0 \leq x \leq n.$$

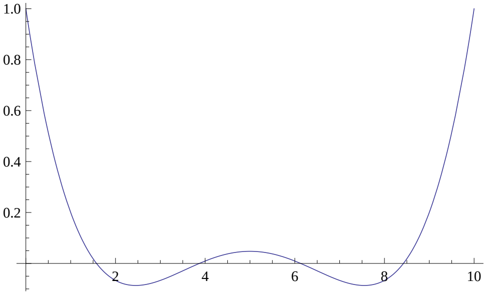
It can be checked that the four roots of $f_4(x)$ are given by

$$x_{1\pm} = \frac{n \pm \sqrt{\sqrt{2}\sqrt{3n^2 - 9n + 8} + 3n - 4}}{2},$$

with $0 < x_{1-} < x_{2-} < x_{2+} < x_{1+} < n$ where

$$x_{2\pm} = \frac{n \pm \sqrt{-\sqrt{2}\sqrt{3n^2 - 9n + 8} + 3n - 4}}{2},$$

$$f_4(x)$$



Need Bounds to compute Conditional Variance

$$|f_4(x)| \leq \begin{cases} \frac{6}{(n-3)^2} & \text{for } x \in [x_{1-}, x_{1+}] \\ f_2^2(x) & \text{for } x \notin [x_{1-}, x_{1+}], x \in [0, n]. \end{cases}$$

where

$$f_2(x) = 1 - \frac{4x}{n} + \frac{4(x)_2}{(n)_2}, \quad 0 \leq x \leq n.$$

$\lambda_{n,4,\mathbf{s}}$

$$|\lambda_{n,4,\mathbf{s}}| = \prod_{s=0}^{\lfloor x_{1-} \rfloor} |\lambda_{n,4,s}| \prod_{s \in \mathbf{u}} |\lambda_{n,4,s}| \prod_{s=\lceil x_{1+} \rceil}^{n} |\lambda_{n,4,s}|$$

Obtain

$$|\lambda_{n,4,\mathbf{s}}| \leq \frac{1}{2} e^{-n} \quad \text{for } n \geq 6,$$

for $s = (1, 2, \ldots, n/2 - 1, n/2 + 1, \ldots, n)$.

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Yields a bound of order $O(n^{-1/2})$ as $n \to \infty$.

Stein's Method for the Lightbulb Process

I hope that the talk was clear and illuminating

To Louis:

Thank you for all your hard work, and Happy Birthday!