Exercise 5

1. Find the maximum likelihood estimate $\widehat{\theta}$ of the location parameter $\theta \in \mathbb{R}$ when observing $X_1, \ldots, X_n$ iid with Cauchy distribution

$$f(x;\theta) = \frac{1}{\pi} \frac{1}{1 + (x-\theta)^2}.$$

2. Let $w_1, \ldots, w_n$ be iid from the mixture distribution

$$f(w;\psi) = \sum_{i=1}^{g} \pi_i f_i(w),$$

where $\psi = (\pi_1, \ldots, \pi_g)$ is a vector of unknown probabilities, summing to one, and $f_1, \ldots, f_g$ are known density functions.

a) Construct a moment estimator of ψ.

b) Describe the E step of the EM algorithm when taking the missing data to be

$$Z_{ij} = \mathbf{1}(\text{observation } j \text{ comes from group } i)$$

3. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be iid with a bivariate normal distribution, and all (five) parameters unknown.

a) Find the MLE of the unknown parameters.

b) With $n = m + m_1 + m_2$, describe the steps of the EM algorithm when m of the pairs are observed completely, m_1 are missing the first coordinate, and m_2 are missing the second coordinate. You may assume the data has been labeled so that the first m indices correspond to the first group, and so forth.

4. Let $Y_1, \ldots, Y_n$ be independent random variables distributed as

$$Y_i \sim \mathcal{P}(\mu_i) \quad \text{where} \quad \mu_i = \sum_{j=1}^{d} p_{ij}\lambda_j,$$

where for each i the vector $(p_{i1}, \ldots, p_{id})$ is of known probabilities summing to no more than one, and $\psi = (\lambda_1, \ldots, \lambda_d)$ is a vector of uknown, nonnegative intensities. Describe the steps of the EM algorithm for the estimation of ψ.