Exercise 4

1. Wald's Lemma: Let $X_1, X_2, \ldots$ be i.i.d. random variables with expectation μ. Let N be a stopping time, that is, a random variables assuming values in $\{0, 1, 2, \ldots,\}$ and such that for all i, events of the type $\{N < i\}$ are independent of any events determined by the variables $X_i, X_{i+1}, \ldots$. Assume $EN < \infty$, or if you want to simplify, make the stronger assumption that $N \leq B$ for some B.Prove

$$E\left(\sum_{i=1}^{N} X_i\right) = EN\mu$$

2. Let $X_1, , X_2, \ldots$ be i.i.d. random variables taking values ± 1 with probability 1/2. Let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ and let N be the first n satisfying $S_n = 1$.

a. Is N a stopping time?

b. Show by conditioning on X_1 that $EN = \infty$.

c. Show that if $EN < \infty$ then Wald's Lemma would imply $S_N = 0$. However, clearly $ES_N = 1$.

2. Let $N \sim$ Geometric(p), that is, $P(n = k) = q^{k-1}p$ for $k = 1, 2, \ldots$. Compute $E(1/N)$ and show that it is stricly larger than p.

3. Suppose you observe a sequence of 10 independent coin tosses with 9 zeros followed by a single 1. It could be generated by 10 coin tosses, or by tossing until the first 1 is observed. Propose other ways that this data could be generated.

4. a. Compute that MLE of the parameter based on a sample $X_1, \ldots, X_n$ of i.i.d. variables with X_i from any distribution you know, such as $\mathcal{P}(\lambda)$ and $\mathcal{N}(\mu, \sigma^2)$. In the latter case the parameter is a vector taking values in a subset of $\mathbb{R}^2$. (For a sample of independent variables it is often more convenient

to maximize the log of the likelihood than the likelihood itself.)

b. For each of the computed MLE's try to compute the bias, and to compare the variance to the Cramer-Rao lower bound.