Exercise set 2

1. Let $\theta(F) = \mathrm{Var}_F(X)$ and let $\widehat{BIAS}$ be the Jackknife estimate of the bias of $\theta(\widehat{F}_n)$. Prove or disprove that

$$E_F[\widehat{BIAS}] = \mathrm{BIAS}$$

where BIAS is the true bias of $\theta(\widehat{F}_n)$. That is, show that in this case the Jackknife estimate of bias is unbiased for the true bias.

2. Let $\theta(F) = E_F(X)$ and show that

$$\widehat{VAR} = \frac{1}{n}S^2$$

where $\widehat{VAR}$ is the Jackknife estimate of the variance of $\theta(\widehat{F}_n)$, and

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

In particular, show that

$$E_F[\widehat{VAR}] = \mathrm{Var}(\theta(\widehat{F}_n)),$$

That is, show in this case that the Jackknife estimate of variance is unbiased for the true variance.

3. Let $\theta(F)$ be the median of the distribution F, and consider a sample $X_1, \ldots, X_{2m}$ from a continuous distribution, and the order statitics of the sample

$$X_{(1)} < X_{(2)} < \cdots < X_{(2m)}.$$

a) Compute $\widehat{BIAS}$, the Jackknife estimate of bias of $\theta(\widehat{F}_n)$.

b) Show that the Jackknife estimate $\widehat{VAR}$ of the variance of $\theta(\widehat{F}_n)$ is given by

$$\widehat{VAR} = \frac{n-1}{4}\left(X_{(m)} - X_{(m+1)}\right)^2$$

In the following parts, assume that F is the Uniform distribution on $[0, 1]$.

c) (*) Prove that

$$n\widehat{VAR} \rightarrow_d \left(\frac{Y}{2}\right)^2,$$

where $\rightarrow_d$ denotes convergence in distribution, and $Y \sim \chi^2_2$, a chi squared distribution on 2 degrees of freedom.

d) (*) We say the Jackknife estimate $\widehat{VAR}$ of variance is consistent for the variance of $\theta(\widehat{F}_n)$ when

$$\frac{\widehat{VAR}}{\mathrm{Var}_F(\theta(\widehat{F}_n))} \rightarrow_p 1.$$

Compute the limit of the ratio above. Is the Jackknife estimate of variance consistent for the median?

e) (**) What if F is a continuous distribution, not necessarily $\mathcal{U}[0, 1]$.