Exercise: let $X_1, \ldots, X_n$ be iid with mean μ and variance σ^2, and set

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad T = \sum_{i=1}^{n}(X_i - \overline{X})^2.$$

1. Show that $T/(n-1)$ is unbiased for σ^2.

   From here on assume the sample follows the normal distribution.

2. Prove that $\text{Var}(T) = 2(n-1)\sigma^4$. Note that under normality many useful properties are available, in particular the sample is multivariate normal, and the Chi-square distributions, which are a special case of the Gamma, arise. Find also the distribution of $T/(n-1)$.

   If you are more ambitious, you can show that $\text{Var}(T) = 2(n-1)\sigma^4$ holds anytime the variables in the sample satisfy certain moment properties, that is, normality is not necessary. To find more general sufficient conditions, you may first assume without loss of generality that $\mu = 0$ (why?). Note which order moments arise in the computation of the variance, and argue that the result holds whenever the moments of the sample agree with the moments of the normal for those orders.

3. Use part 2, even if you did not prove it, to find the MSE of T/k as an estimate of σ^2 for any $k > 0$.

4. Minimize the MSE above over k to find the best estimator of σ^2 the form T/k with respect to MSE.

5. Find the MLE of (μ, σ^2), and also the MLE of σ^2 in the case that μ is known, and the MLE of μ in the case where σ^2 is known. Is it the case here that knowing μ helps to estimate σ^2, but knowing σ^2 does not help to estimate μ?