A Curious Connection Between Optimal Stopping and Branching Processes

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Optimal Stopping

Independent random variables $X_1, \ldots, X_n$ with known distributions are to be presented one at a time. You can either choose a variable, or pass it up forever. Your goal is to stop on a variables with as large a value as possible.

What strategy, which cannot depend on future, so expressed by a stopping time t, maximizes EX_t?

Chow, Robbins, and Siegmund 1971

Optimal Rule

Let V_{i+1}^n be the optimal value for stopping on $X_{i+1}, \ldots, X_n$, with finite expectations.

If $X_i < V_{i+1}^n$, pass up this variable.

If $X_i \geq V_{i+1}^n$, take it.

Getting the better of X_i and V_{i+1} yields that

$$V_i^n = E[X_i \vee V_{i+1}^n]; \quad \text{setting} \quad V_{n+1}^n = -\infty,$$

we have

$$t_n^* = \min\{i : X_i \geq V_{i+1}^n\} \quad \text{satisfies} \quad P(t_n^* \leq n) = 1.$$

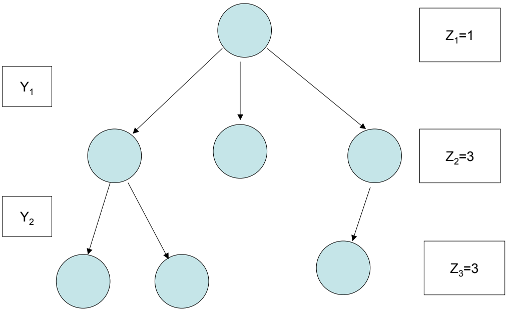
Branching Process

Let the offspring distribution in generation n be that of the integer valued variable Y_n, and the population size Z_n, with $Z_1 = 1$, given by

$$Z_{n+1} = \sum_{i=1}^{Z_n} Y_{ni} \quad \text{for } n \geq 1,$$

where Y_{ni} are independent and distributed as Y_n.

Harris 1963, Jagers 1975



Z_1=1
Y_1
Z_2=3
Y_2
Z_3=3

Generating Function

For a non-negative integer valued random variable $Y \in \{0, 1 \ldots, \}$ with distribution $P(Y = k) = p_k$, consider the generating function

$$g(s) = Es^Y = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \leq s \leq 1,$$

for which

$$g(0) = p_0, \quad g(1) = 1, \quad g \uparrow$$

and g is convex; in fact, derivatives of all order exist and are non-nonegative. Note also that $g'(1) = EY$.

Generating Function Relation

Let

$$g_n(s) = Es^{Y_n} \quad \text{and} \quad g^{(n)}(s) = Es^{Z_n}.$$

Then

$$\begin{aligned} g^{(n+1)}(s) &= Es^{Z_{n+1}} \\ &= E\{E[s^{Z_{n+1}}|Z_n]\} \\ &= E\{E[s^{\sum_{i=1}^{Z_n} Y_{ni}}|Z_n]\} \\ &= E\{g_n(s)^{Z_n}\} \\ &= g^{(n)}(g_n(s)) \\ &= g^{(n-1)}(g_{n-1}(g_n(s))) \\ &= g_1(g_2(\cdots g_n(s))) \end{aligned}$$

Extinction Probability

By

$$g^{(n)}(s) = \sum_{k=0}^{\infty} P(Z_n = k)s^k,$$

we know

$$\tilde{q}_n = P(Z_n = 0) = g^{(n)}(0).$$

Since $Z_n = 0$ implies $Z_{n+1} = 0$ we have

$$0 \le \tilde{q}_1 \le \tilde{q}_2 \le \cdots \quad \text{and} \lim_{n\to\infty} \tilde{q}_n = \tilde{\pi}.$$

We exclude trivial cases by assuming $0 < P(Y = 0) < 1$ and $P(Y = 0) + P(Y = 1) < 1$.

Nuclear Interactions, Polymerase Chain Reaction (PCR)

Extinction Probability: Common Offspring Distribution

If $Y_n =_d Y$ for all n, having generating function g, then $q_n = P(Z_n = 0)$ is given by

$$g^{(n)}(0) = g(g(\cdots g(0))),$$

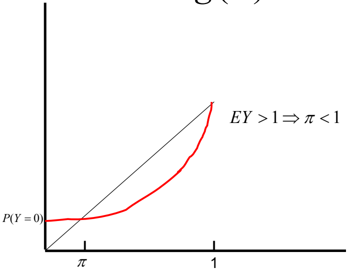
and

$$\lim_{n\to\infty} q_n = \pi$$

is the smallest root of the equation

$$g(s) = s.$$

$$g(\pi) = \pi$$



Behavior Determined by EY

Supercritical case: $EY > 1$ then $\pi < 1$

Critical and Subcritical case: $EY \leq 1$ then $\pi = 1$

Optimal Value

Recall that $V_i^n = E[X_i \vee V_{i+1}^n]$, or

$$V_i^n = h_i(V_{i+1}^n) \quad \text{where} \quad h_i(x) = E[X_i \vee x].$$

When $X_i \geq 0$ we may take $V_{n+1}^n = 0$, so iteration gives

$$V_1^n = h_1(V_2^n) = h_1(h_2(V_3^n)) = \cdots = h_1(h_2(\cdots h_n(0))).$$

Note the similarity to the recursion for extinction in branching!

Homogeneous Case

Branching: Common offspring distribution Y so

$$g_i(s) = g(s).$$

Stopping: iid sequence with distribution X so

$$h_i(s) = h(s).$$

Then

$$q_n = g^{(n)}(0) \quad \text{and} \quad V_1^n = h^{(n)}(0).$$

Do these ever agree?

Example

Branching process with offspring distribution

$$P(Y=0)=\frac{1}{2}=P(Y=2)$$

has generating function

$$g(a)=Ea^Y=\frac{1}{2}a^0+\frac{1}{2}a^2=\frac{1}{2}+\frac{1}{2}a^2.$$

Stopping $X_1,\ldots,X_n$ independent $\mathcal{U}[0,1]$ variables, we have $V_i^n=E[X_i\vee V_{i+1}^n]=h(V_{i+1}^n)$ where

$$h(a) \quad = \quad E[X\vee a]=aP(X\leq a)+\int_a^1 xdx=\frac{1}{2}+\frac{1}{2}a^2.$$

Curious Conclusion

The probability that the branching process with offspring distribution

$$P(Y = 0) = \frac{1}{2} = P(Y = 2)$$

is extinct in generation n equals the value for stopping optimally on the sequence

$$X_1, \ldots, X_n \quad \text{independent } \mathcal{U}[0,1] \text{ variables,}$$

Is this case special?

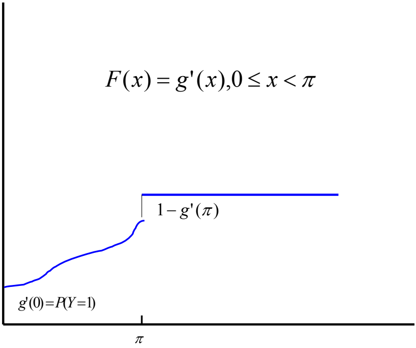
$Y \to X$ Correspondence

Theorem 1 *Let Y be a non-negative integer valued random variable with generating function g, and let π be the smallest root of $g(s) = s$. Then the function $F(x)$ given by*

$$F(x) = \begin{cases} 0 & x < 0 \\ g'(x) & 0 \le x < \pi \\ 1 & \pi \le x, \end{cases}$$

is a distribution function, and for $h(a) = E[X \vee a]$ with $X \sim F$ we have

$$h(a) = g(a) \text{ for } 0 \le a \le \pi.$$



$F(x) = g'(x), 0 \le x < \pi$
$1 - g'(\pi)$
$g'(0) = P(Y = 1)$
π

Identities following the Correspondence

$$h(a) = E[X \vee a] \quad \text{so that} \quad h(0) = EX.$$

But

$$g(0) = P(Y = 0) \quad \text{so that} \quad EX = P(Y = 0).$$

The jump of $F(x) = g'(0)$ at zero is both

$$P(X = 0) = P(Y = 1).$$

Reason $h(s) = g(s), 0 \leq s \leq \pi$

Notice that for $0 \leq a \leq \pi$,

$$\begin{aligned}
h(a) &= E[X \vee a] \\
&= \int_0^\infty P([X \vee a] > x)dx \\
&= \int_0^\pi P([X \vee a] > x)dx \\
&= \int_0^\pi [1 - P([X \vee a] \leq x)]dx \\
&= \pi - \int_a^\pi g'(x)dx \\
&= \pi - g(\pi) + g(a) \\
&= g(a).
\end{aligned}$$

Branching Process → Stopping

Theorem 2 *Let* $\{Z_n\}$ *be a branching process with offspring distribution* $Y_1, Y_2, \ldots$*, and let* $X_1, X_2, \ldots$ *be the corresponding* X *variables. If* $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$*, then*

$$h^{(n)}(a) = g^{(n)}(a) \quad \textit{for } 0 \leq a \leq \pi_n,$$

and in particular

$$V_1^n = \tilde{q}_n.$$

The optimal stopping value equals the extinction probability.

$Y \to X$: Generalized Geometric

$Y \sim \mathcal{GG}(b, c)$ with b, c non-negative and $0 < b + c < 1$,

$$P(Y = 0) = \frac{1 - b - c}{1 - c}, \quad P(Y = k) = bc^{k-1}, \quad k = 1, 2, \ldots$$

Usual Geometric is $\mathcal{GG}(pq, q)$. Generating function

$$g(s) = \frac{\alpha + \beta s}{\gamma + \delta s}$$

and corresponding X distribution

$$F(x) = \begin{cases} 0 & x < 0 \\ b/(1 - cx)^2 & 0 \leq x < \pi \\ 1 & \pi \leq x. \end{cases}$$

Branching Analysis by Stopping

Stopping proof that $\lim P(Z_n = 0) = \pi$, common offspring distribution Y. Let X correspond to Y, and consider stopping on the independent sequence $X_1, X_2, \ldots$ distributed as X. Since $P(X_i \leq \pi) = 1$ we have $V_1^n \leq \pi$. But for all $\epsilon > 0$ we have

$$P(X_i > \pi - \epsilon) = \delta > 0,$$

so the suboptimal rule t_n which stops at the first i for which X_i exceed the threshold $\pi - \epsilon$, and n otherwise, has value

$$\pi \geq V_1^n \geq EX_{t_n} \geq (\pi - \epsilon)(1 - (1 - \delta)^n) \to \pi - \epsilon.$$

Prophet Value: Upper Bounds

A Prophet can pick the maximum:

$$P_1^n = E(\max(X_1, \ldots, X_n)) \uparrow P_1^\infty.$$

Gives upper bounds

$$q_n = V_1^n \leq P_1^n \leq P_1^\infty$$

Prophet Inequalities, and Suboptimal Rules: Lower Bounds

Prophet Inequality: for $X_1, \ldots, X_n$ independent non-negative

$$P_1^n < 2V_1^n \quad \text{and} \quad P_1^\infty < 2V_1^\infty$$

gives lower bounds

$$\frac{P_1^n}{2} < V_1^n = q_n \leq \tilde{\pi}, \quad \text{and} \quad \frac{P_1^\infty}{2} < V_1^\infty = \tilde{\pi}.$$

Any suboptimal Rule t_n also gives lower bounds,

$$EX_{t_n} \leq V_1^n = q_n \leq \tilde{\pi}.$$

Upper Bound: Example

Consider offspring distributions Y_i where for all i, $\pi_i = 1/2$,

$$Y_i = \begin{cases} 0 & p_i/3 \\ 1 & 1 - p_i \\ 2 & 2p_i/3. \end{cases}$$

Corresponding X_i satisfies

$$P(X_i = 0) = P(Y_i = 1) = 1 - p_i,$$

so $X_i \leq X_i^*$ stochastically, where

$$P(X_i^* = 0) = 1 - p_i, \quad P(X_i^* = 1/2) = p_i,$$

and hence

$$P_1^\infty \leq P_1^{\infty *} = \frac{1}{2} P(X_i^* = 1/2 \text{ any } i) = \frac{1}{2}[1 - \prod_{i=1}^{\infty}(1 - p_i)].$$

Lower Bound: Example

We have

$$P(X_i \leq 1/2) = 1, \quad P(X_i = 1/2) = 1 - g'(1/2) = 1 - p_i/3.$$

Considering the suboptimal rule

$$t = \inf\{i : X_i = \frac{1}{2}\}$$

gives the complimentary lower bound

$$\frac{1}{2}[1 - \prod_{i=1}^{\infty}(1 - p_i/3)] \leq \tilde{\pi} \leq \frac{1}{2}[1 - \prod_{i=1}^{\infty}(1 - p_i)].$$

Branching $\rightarrow$ Stopping

For every branching Y there is a stopping X, but not conversely. Nevertheless, taking

$$Y_i \sim \mathcal{GG}(b_i, c_i)$$

which according to Athreya and Ney (1972) is essentially the only non-trivial example where $g^{(n)}(s)$ can be computed explicitly, the corresponding sequence of independent X_i variables, when $b_i = (1 - c_i)^2$, has a permutation invariant optimal stopping value.

Deep Connection, or Coincidence?

From 'An Unexpected Connection Between Branching Processes and Optimal Stopping'

... This correspondence is analytical, and in particular, we are not able to present a probabilistic reason, such as a coupling, which explains it...

Connection or Coincidence: Yes or No?

No: Many mathematical objects are described by composition, and they are not all therefore related.

Yes: Given Y, there is a unique X for which

$$h(s) = E[X \vee s] = Es^Y = g(s), \quad 0 \leq s \leq \pi.$$

And could it be merely coincidence that

$$\tilde{q}_n = V_1^n \quad \text{for all } n = 1, 2, \ldots,$$

and that in both problems there can be termination, in one by stopping, in the other extinction, at any stage $i = 1, \ldots, n$?