INDAM II: Probability, Puzzles and Paradoxes

1. You are in a room containing n people, including yourself. You would like to bet that there are at least two people in the room who share a birthday. How large does n have to be in order for this to be a favorable bet?

   Answer the same question if you want to bet that there is at least one other person in the room who shares your birthday.

2. Before me there are three drawers. Drawer 1 has two silver coins. Drawer 2 has one silver and one gold coin. Drawer 3 has two gold coins. I pick a drawer (uniformly) at random and from that draw I select one of the coins in that drawer, (uniformly) at random. Suppose the coin I select is gold. What is the probability that the second coin in that drawer is also gold?

3. One person in 100,000 has a rare disease. There is test for this disease, but it is not 100% reliable. The probability of having a false positive (where the test says a healthy person has the disease) is 0.03, and the probability of a false negative (when the test says a diseased person is healthy) is 0.01. If someone has a positive test result, what is the probability they have the disease?

4. The Monty Hall Problem. You are in a game show where behind one of three doors is the Grand Prize; behind the remaining two doors are unwanted items, such as goats. First you select one of the doors, say door number 1. Of course, no matter which door you select, there is always going to be at least one goat behind one of two unselected doors. The host now opens one of those doors, say for example door number 2, saying to you: 'It's a good thing you did not select door number 2, since as you can see, your prize in that case would have been this goat.' Lastly, the host gives you the following choice: 'Would you like to stay with door number 1, or would you like to switch to door number 3.' What should you do?

5. There are two sealed envelopes with money on the table. The amount of money in the envelopes can be any positive number, not necessarily an integer, but one envelope contains twice the amount of money in the other envelope. You select one of the envelopes with probability

one half; I get the other one. You open your envelope and find it has X Euro. Now you know that my envelope has either $X/2$ or $2X$ Euro, with equal probability. You find that it is advantageous for you to switch envelopes with me by computing the expected value of what you would have after switching as follows:

$$\frac{1}{2}\left(\frac{X}{2}\right) + \frac{1}{2}\,(2X) = \frac{5}{4}X > X.$$

But I could make the same calculation, and clearly it cannot be to both of our advantages to switch! What is wrong with the calculation?

6. For b any positive integer ($b = 10$ would be a good choice) show that

$$p_d = \log_b\left(1 + \frac{1}{d}\right) \qquad d = 1, \ldots, b-1.$$

is a probability distribution. Does this distribution have any interesting properties?

7. Urn U contains 2280 balls, 1000 white and 1280 black. Of these balls, two are selected at random and put in urn A. In Urn B there is one white and one black ball.

   You are invited to play the following two stage game. In the first stage, you choose either urn A or urn B and pick a ball. If the ball is white, you win \$10,000 and if it is black you win nothing. The ball is returned to the urn from which it is picked. In the second stage you again choose either urn A or urn B from which to randomly draw a single ball. The payoff is the same as for the first draw. In the first stage, from which of the urns do you choose to draw?

8. There are three die on a table, A, B and C. Each of them has six sides, as usual, but the numbers on their faces are as follows: A has the six numbers $\{2, 2, 4, 4, 9, 9\}$, B has $\{1, 1, 6, 6, 8, 8\}$, and C has $\{3, 3, 5, 5, 7, 7\}$. You will pick whichever one you want, and I will select from the two that remain. We both roll, and whoever has the higher number will win 100E. Which die do you pick to roll?

9. Consider two unequal positive numbers x and y, where without loss of generality we assume $x < y$. I am assigned one of the numbers with

probability 1/2 and you are given the other. I want to have the larger of the two numbers, y, and have the option of switching the number I was assigned with yours. Consider the following strategy: First, I will generate a continuous random variable V with support equal to $[0, \infty)$, say an exponential variable with density e^{-v}. If my assigned number is less than V (that is, if my number is 'small'), then I will switch my number with yours, and otherwise I will keep my number. If I use this strategy, what is the probability that I will end up with the number y?

10. A monkey is typing randomly on a typewriter, striking one typewriter key every second. If the typewriter has only keys for H and T, how long does it take, on average, for the letter T to appear? For HTH? For HHH? For a keyboard of 26 letters, how long before ABRACADABRA appears? For a keyboard with a space, for 27 keys, how long until your name appears?

11. You are to pick your favorite three letter sequence from the letters $\{H, T\}$ and then I will pick mine from the 7 which remain. A fair coin is to be tossed and the sequence of heads (H) and tails (T) recorded until one of our two sequences appears. Which sequence do you pick as your favorite, if you would like yours to come up earlier than mine?