

BOUNDS ON THE CONSTANT IN THE MEAN CENTRAL LIMIT THEOREM

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Let X_1, \dots, X_n be independent with zero means, finite variances $\sigma_1^2, \dots, \sigma_n^2$ and finite absolute third moments. Let F_n be the distribution function of $(X_1 + \dots + X_n)/\sigma$, where $\sigma^2 = \sum_{i=1}^n \sigma_i^2$, and Φ that of the standard normal. The L^1 -distance between F_n and Φ then satisfies

$$\|F_n - \Phi\|_1 \leq \frac{1}{\sigma^3} \sum_{i=1}^n E|X_i|^3.$$

In particular, when X_1, \dots, X_n are identically distributed with variance σ^2 , we have

$$\|F_n - \Phi\|_1 \leq \frac{E|X_1|^3}{\sigma^3 \sqrt{n}} \quad \text{for all } n \in \mathbb{N},$$

corresponding to an L^1 -Berry–Esseen constant of 1.

1. Introduction. The classical central limit theorem allows the approximation of the distribution of sums of “comparable” independent real-valued random variables by the normal. As this theorem is an asymptotic, it provides no information as to whether the resulting approximation is useful. For that purpose, one may turn to the Berry–Esseen theorem, the most classical version giving supremum norm bounds between the distribution function of the normalized sum and that of the standard normal. Various authors have also considered Berry–Esseen-type bounds using other metrics and, in particular, bounds in L^p . The case $p = 1$, where the value

$$\|F - G\|_1 = \int_{-\infty}^{\infty} |F(x) - G(x)| dx$$

is used to measure the distance between distribution functions F and G , is of some particular interest and results using this metric are known as mean central limit theorems (see, e.g., [1, 4, 11] and [12]; the latter three of these works consider nonindependent summand variables). One motivation for studying L^1 -bounds is that, combined with one of type L^∞ , bounds on L^p -distance for all $p \in (1, \infty)$ may be obtained by the inequality

$$\|F - G\|_p^p \leq \|F - G\|_\infty^{p-1} \|F - G\|_1.$$

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For $\sigma \in (0, \infty)$, let \mathcal{F}_σ be the collection of distributions with mean zero, variance σ^2 and finite absolute third moment. We prove the following Berry–Esseen-type result for the mean central limit theorem.

THEOREM 1.1. *For $n \in \mathbb{N}$, let X_1, \dots, X_n be independent mean zero random variables with distributions $G_1 \in \mathcal{F}_{\sigma_1}, \dots, G_n \in \mathcal{F}_{\sigma_n}$ and let F_n be the distribution of*

$$W = \frac{1}{\sigma} \sum_{i=1}^n X_i \quad \text{where } \sigma^2 = \sum_{i=1}^n \sigma_i^2.$$

Then

$$\|F_n - \Phi\|_1 \leq \frac{1}{\sigma^3} \sum_{i=1}^n E|X_i|^3.$$

In particular, when X_1, \dots, X_n are identically distributed with distribution $G \in \mathcal{F}_\sigma$,

$$\|F_n - \Phi\|_1 \leq \frac{E|X_1|^3}{\sigma^3 \sqrt{n}} \quad \text{for all } n \in \mathbb{N}.$$

For the case where all variables are identically distributed as X having distribution G , letting

$$(1) \quad c_m = \inf \left\{ C : \frac{\sqrt{n}\sigma^3 \|F_n - \Phi\|_1}{E|X|^3} \leq C \text{ for all } G \in \mathcal{F}_1 \text{ and } n \geq m \right\},$$

the second part of Theorem 1.1 yields the upper bound $c_1 \leq 1$. Regarding lower bounds, we also prove

$$(2) \quad c_1 \geq \frac{2\sqrt{\pi}(2\Phi(1) - 1) - (\sqrt{\pi} + \sqrt{2}) + 2e^{-1/2}\sqrt{2}}{\sqrt{\pi}} = 0.535377\dots$$

Clearly, the elements of the sequence $\{c_m\}_{m \geq 1}$ are nonnegative and decreasing in m , and so have a limit, say c_∞ . Regarding limiting behavior, Esseen [3] showed that

$$\lim_{n \rightarrow \infty} n^{1/2} \|F_n - \Phi\|_1 = A(G)$$

for an explicit constant $A(G)$ depending only on G . Zolotarev [19] provides the representation

$$(3) \quad A(G) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-1/2}^{1/2} \int_{-\infty}^{\infty} \left| \frac{\omega}{2}(1 - x^2) + hu \right| e^{-x^2/2} dx du,$$

where $\omega = |EX^3|/(3\sigma^2)$ and h is the span of the distribution G in the case where G is lattice, and is zero otherwise. Zolotarev obtains

$$\sup_{G \in \mathcal{F}_\sigma} \frac{\sigma^3 A(G)}{E|X|^3} = \frac{1}{2},$$

showing that $c_\infty = 1/2$, hence giving the asymptotic L^1 -Berry–Esseen constant value.

Here, the focus is on nonasymptotic constants and, in particular, on the constant c_1 which gives a bound for all $n \in \mathbb{N}$. Theorem 1.1 is shown using Stein’s method (see [15, 17]), which uses the characterizing equation (5) for the normal, and an associated differential equation to obtain bounds on the normal approximation. More particularly, we employ the zero bias transformation, introduced in [9], and the evaluation of a Stein functional, as in [13]; see, in particular, Proposition 4.1 there. In [9], it was shown that for all X with mean zero and finite nonzero variance σ^2 , there exists a unique distribution for a random variable X^* such that

$$(4) \quad \sigma^2 E f'(X^*) = E[X f(X)]$$

for all absolutely continuous functions f for which these expectations exist. The zero bias transformation, mapping the distribution of X to that of X^* , was motivated by the Stein characterization of the normal distribution [16], which states that Z is normal with mean zero and variance σ^2 if and only if

$$(5) \quad \sigma^2 E f'(Z) = E[Z f(Z)]$$

for all absolutely continuous functions f for which these expectations exist. Hence, the mean zero normal with variance σ^2 is the unique fixed point of the zero bias transformation. How closeness to normality may be measured by the closeness of a distribution to its zero bias transform, and related applications, are the topics of [5, 6] and [7].

As shown in [7] and [9], for a random variable X with $EX = 0$ and $\text{Var}(X) = \sigma^2$, the distribution of X^* is absolutely continuous with density and distribution functions given, respectively, by

$$(6) \quad g^*(x) = \sigma^{-2} E[X \mathbf{1}(X > x)] \quad \text{and} \quad G^*(x) = \sigma^{-2} E[X(X - x) \mathbf{1}(X \leq x)].$$

Theorem 1.1 results by showing that the functional

$$(7) \quad B(G) = \frac{2\sigma^2 \|G^* - G\|_1}{E|X|^3}$$

is bounded by 1 for all X with distribution $G \in \mathcal{F}_\sigma$. As in (3), one may write out a more “explicit” form for $B(G)$ using (6) and expressions for the moments on which $B(G)$ depends, but such expressions appear to be of little value for the purposes of proving Theorem 1.1. In turn, the proof here employs convexity properties of $B(G)$ which depend on the behavior of the zero bias transformation on mixtures.

We also note that the functional $B(G)$ has a different character than $A(G)$; for instance, $A(G)$ is zero for all nonlattice distributions with vanishing third moment, whereas $B(G)$ is zero only for mean zero normal distributions.

Let $\mathcal{L}(X)$ denote the distribution of a random variable X . Since the L^1 -distance scales, that is, since, for all $a \in \mathbb{R}$,

$$(8) \quad \|\mathcal{L}(aX) - \mathcal{L}(aY)\|_1 = |a| \|\mathcal{L}(X) - \mathcal{L}(Y)\|_1,$$

by replacing σ_i^2 by σ_i^2/σ^2 and $\|G_i^* - G_i\|_1$ by $\|G_i^* - G_i\|_1/\sigma$ in equation (16) of Theorem 2.1 of [7], we obtain the following.

PROPOSITION 1.1. *Under the hypotheses of Theorem 1.1, we have*

$$\|F_n - \Phi\|_1 \leq \frac{1}{\sigma^3} \sum_{i=1}^n B(G_i) E|X_i|^3.$$

For \mathcal{F} a collection of nontrivial mean zero distributions with finite absolute third moments, we let

$$B(\mathcal{F}) = \sup_{G \in \mathcal{F}} B(G).$$

Clearly, Theorem 1.1 follows immediately from Proposition 1.1 and the following result.

LEMMA 1.1. *For all $\sigma \in (0, \infty)$,*

$$B(\mathcal{F}_\sigma) = 1.$$

The equality to 1 in Lemma 1.1 improves the upper bound of 3 shown in [7]. Although our interest here is in best universal constants, we note that Proposition 1.1 shows that $B(G)$ is a distribution-specific L^1 -Berry–Esseen constant, in that

$$\|F_n - \Phi\|_1 \leq \frac{B(G) E|X_1|^3}{\sigma^3 \sqrt{n}} \quad \text{for all } n \in \mathbb{N},$$

when X_1, \dots, X_n are identically distributed according to $G \in \mathcal{F}_\sigma$. For instance, $B(G) = 1/3$ when G is a mean zero uniform distribution and $B(G) = 1$ when G is a mean zero two-point distribution; see Corollary 2.1 of [7], and Lemmas 1.2 and 1.3 below.

We close this section with two preliminaries. The first collects some facts shown in [7] and the second demonstrates that to prove Lemma 1.1, it suffices to consider the class of random variables \mathcal{F}_1 . Then, following Hoeffding [10] (see also [13]), in Section 2, we use a continuity property of $B(G)$ to show that its supremum over \mathcal{F}_1 is attained on finitely supported distributions. Exploiting a convexity-type property of the zero bias transformation on mixtures over distributions having equal

variances, we reduce the calculation further to the calculation of the supremum over D_3 , the collection of all mean zero distributions with variance 1 and supported on at most three points. As three-point distributions are, in general, a mixture of two two-point distributions with unequal variances, an additional argument is given in Section 3, where a coupling of an X with distribution $G \in D_3$ to a variable X^* having the X zero bias distribution is constructed, using the optimal L^1 -couplings on the component two-point distributions of which G is the mixture, in order to obtain $B(G) \leq 1$ for all $G \in D_3$. The lower bound (2) on c_1 is calculated in Section 4.

The following simple formula will be of some use. For $a \geq 0, b > 0$ and $l > 0$, we have

$$(9) \quad \int_0^l \left| (a+b)\frac{u}{l} - a \right| du = \frac{l}{2} \frac{a^2 + b^2}{a+b}.$$

LEMMA 1.2. *Let G be the distribution of a nontrivial mean zero random variable X supported on the two points $x < y$. Then X^* is uniformly distributed on $[x, y]$,*

$$EX^2 = -xy, \quad E|X^3| = \frac{-xy(y^2 + x^2)}{y - x}$$

and

$$\|\mathcal{L}(X^*) - \mathcal{L}(X)\|_1 = \frac{1}{2} \frac{y^2 + x^2}{y - x}.$$

In particular, $B(G) = 1$ and

$$B(\mathcal{F}_1) \geq 1.$$

PROOF. Being nontrivial, G has positive variance and, from (6), we see that the density g^* of G^* at u , which is proportional to $E[X\mathbf{1}(X > u)]$, is zero outside $[x, y]$ and constant within it, so $G^*(w) = (w - x)/(y - x)$ for $w \in [x, y]$. That G has mean zero implies that the support points x and y satisfy $x < 0 < y$ and that G gives positive probabilities $y/(y - x)$ and $-x/(y - x)$ to x and y , respectively. The moment identities are immediate.

Making the change of variable $u = w - x$ and applying (9) with $a = y/(y - x)$, $b = -x/(y - x)$ and $l = y - x$ yields

$$\|\mathcal{L}(X^*) - \mathcal{L}(X)\|_1 = \int_x^y \left| \frac{w - x}{y - x} - \frac{y}{y - x} \right| dw = \frac{1}{2} \left(\frac{y^2 + x^2}{y - x} \right),$$

and (7) now gives $B(G) = 1$. \square

LEMMA 1.3. *Let $G \in \mathcal{F}_\sigma$ for some $\sigma \in (0, \infty)$, let X have distribution G and, for $a \neq 0$, let G_a denote the distribution of aX . Then $B(G_a) = B(G)$ and, in particular,*

$$B(\mathcal{F}_\sigma) = B(\mathcal{F}_1) \quad \text{for all } \sigma \in (0, \infty).$$

PROOF. That aX^* has the same distribution as $(aX)^*$ follows from (4). The identities $\sigma_{aX}^2 = a^2\sigma_X^2$, $E|aX|^3 = |a|^3E|X^3|$ and (8) now imply the first claim. The second claim now follows from

$$\{B(G) : G \in \mathcal{F}_\sigma\} = \{B(G) : G \in \mathcal{F}_1\}. \quad \square$$

2. Reduction to three-point distributions. Let (S, Σ) be a measurable space and let $\{m_s\}_{s \in S}$ be a collection of probability measures on \mathbb{R} such that for each Borel subset $A \subset \mathbb{R}$, the function from S to $[0, 1]$ given by

$$s \rightarrow m_s(A)$$

is measurable. When μ is a probability measure on (S, Σ) , the set function given by

$$m_\mu(A) = \int_S m_s(A)\mu(ds)$$

is a probability measure, called the μ mixture of $\{m_s\}_{s \in S}$. With some slight abuse of notation, we let E_μ and E_s denote expectations with respect to m_μ and m_s , and let X_μ and X_s be random variables with distributions m_μ and m_s , respectively. For instance, for all functions f which are integrable with respect to μ , we have

$$E_\mu f(X) = \int E_s f(X)\mu(ds),$$

which we also write as

$$Ef(X_\mu) = \int Ef(X_s)\mu(ds).$$

In particular, if $\{m_s\}_{s \in S}$ is a collection of mean zero distributions with variances $\sigma_s^2 = EX_s^2$ and absolute third moments $\gamma_s = E|X_s^3|$, the mixture distribution m_μ has variance σ_μ^2 and third absolute moment γ_μ given by

$$\sigma_\mu^2 = \int_S \sigma_s^2 d\mu \quad \text{and} \quad \gamma_\mu = \int_S \gamma_s d\mu,$$

where both may be infinite. Note that $\sigma_\mu^2 < \infty$ implies that $\sigma_s^2 < \infty$ μ -almost surely and, therefore, that m_s^* , the m_s zero bias distribution, exists μ -almost surely.

Theorem 2.1 shows that the zero bias distribution of a mixture is a mixture of zero bias distributions, the mixing measure of which is the original measure weighted by the variance and rescaled. Define (arbitrarily, see Remark 2.1) the zero bias distribution of δ_0 , a point mass at zero, to be δ_0 . Write $X =_d Y$ when X and Y have the same distribution.

THEOREM 2.1. *Let $\{m_s, s \in S\}$ be a collection of mean zero distributions on \mathbb{R} and μ a probability measure on S such that the variance σ_μ^2 of the mixture distribution is positive and finite. Then m_μ^* , the m_μ zero bias distribution, exists and is given by the mixture*

$$m_\mu^* = \int m_s^* d\nu \quad \text{where } \frac{d\nu}{d\mu} = \frac{\sigma_s^2}{\sigma_\mu^2}.$$

In particular, $\nu = \mu$ if and only if σ_s^2 is a constant μ a.s.

PROOF. The distribution m_μ^* exists as m_μ has mean zero and finite nonzero variance. Let X_μ^* have the m_μ zero bias distribution and let Y have distribution m_μ^* . For any absolutely continuous function f for which the expectations below exist, we have

$$\begin{aligned} \sigma_\mu^2 E f'(X_\mu^*) &= E X_\mu f(X_\mu) \\ &= \int E X_s f(X_s) d\mu \\ &= \int \sigma_s^2 E f'(X_s^*) d\mu \\ &= \sigma_\mu^2 \int E f'(X_s^*) d\nu \\ &= \sigma_\mu^2 E f'(Y). \end{aligned}$$

Since $E f'(X_\mu^*) = E f'(Y)$ for all such f , we conclude that $X_\mu^* =_d Y$. \square

REMARK 2.1. If $m_s = \delta_0$ for any $s \in S$, then $\sigma_s^2 = 0$ and, therefore,

$$\nu\{s \in S : m_s = \delta_0\} = 0.$$

Hence, the mixture X_μ^* gives zero weight to all corresponding m_s^* , showing that $(\delta_0)^*$ may be defined arbitrarily.

We now recall an equivalent form of the L^1 -distance involving expectations of Lipschitz functions L on \mathbb{R} ,

$$(10) \quad \|F - G\|_1 = \sup_{f \in L} |E f(X) - E f(Y)|$$

where $L = \{f : |f(x) - f(y)| \leq |x - y|\}$,

X and Y having distributions F and G , respectively. With a slight abuse of notation, we may write $B(X)$ in place of $B(G)$ when X has distribution G .

THEOREM 2.2. *If X_μ is the μ mixture of a collection $\{X_s, s \in S\}$ of mean zero, variance 1 random variables satisfying $E|X_\mu^3| < \infty$, then*

$$(11) \quad B(X_\mu) \leq \sup_{s \in S} B(X_s).$$

If \mathcal{C} is a collection of mean zero, variance 1 random variables with finite absolute third moments and $\mathcal{D} \subset \mathcal{C}$ such that every distribution in \mathcal{C} can be represented as a mixture of distributions in \mathcal{D} , then

$$(12) \quad B(\mathcal{C}) = B(\mathcal{D}).$$

PROOF. Since the variances σ_s^2 of X_s are constant, the distribution X_μ^* is the μ mixture of $\{X_s^*, s \in S\}$, by Theorem 2.1. Hence, applying (10), we have

$$(13) \quad \begin{aligned} \|\mathcal{L}(X_\mu^*) - \mathcal{L}(X_\mu)\|_1 &= \sup_{f \in L} |Ef(X_\mu^*) - Ef(X_\mu)| \\ &= \sup_{f \in L} \left| \int_S Ef(X_s^*) d\mu - \int_S Ef(X_s) d\mu \right| \\ &\leq \sup_{f \in L} \int_S |Ef(X_s^*) - Ef(X_s)| d\mu \\ &\leq \int_S \|\mathcal{L}(X_s^*) - \mathcal{L}(X_s)\|_1 d\mu. \end{aligned}$$

Noting that $\text{Var}(X_\mu) = \int_S EX_s^2 d\mu = 1$ and applying (13), we find that

$$\begin{aligned} B(X_\mu) &= \frac{2\|\mathcal{L}(X_\mu^*) - \mathcal{L}(X_\mu)\|_1}{E|X_\mu^3|} \\ &\leq \frac{\int_S 2\|\mathcal{L}(X_s^*) - \mathcal{L}(X_s)\|_1 d\mu}{E|X_\mu^3|} \\ &= \frac{\int_S B(X_s)E|X_s^3| d\mu}{E|X_\mu^3|} \\ &\leq \sup_{s \in S} B(X_s) \frac{\int_S E|X_s^3| d\mu}{E(X_\mu^3)} \\ &= \sup_{s \in S} B(X_s). \end{aligned}$$

Regarding (12), clearly, $B(\mathcal{D}) \leq B(\mathcal{C})$ and the reverse inequality follows from (11). \square

REMARK 2.2. Note that no bound of the type provided by Theorem 2.2 holds, in general, when taking mixtures of variables that have unequal variances. In particular, if $X_s \sim \mathcal{N}(0, \sigma_s^2)$ and σ_s^2 is not constant in s , then X_μ is a mixture of normals with unequal variances, which is not normal. Hence, in this case, $B(X_\mu) > 0$, whereas $B(X_s) = 0$ for all s .

To apply Theorem 2.2 to reduce the computation of $B(\mathcal{F}_1)$ to finitely supported distributions, we apply the following continuity property of the zero bias transformation; see Lemma 5.2 in [8]. We write $X_n \Rightarrow X$ for the convergence of X_n to X in distribution.

LEMMA 2.1. *Let X and $X_n, n = 1, 2, \dots$, be mean zero random variables with finite, nonzero variances. If*

$$X_n \xrightarrow{d} X \quad \text{and} \quad \lim_{n \rightarrow \infty} EX_n^2 = EX^2,$$

then

$$X_n^* \xrightarrow{d} X^*.$$

For a distribution function F , let

$$(14) \quad F^{-1}(w) = \sup\{a : F(a) < w\} \quad \text{for all } w \in (0, 1).$$

If U is uniform on $[0, 1]$, then $F^{-1}(U)$ has distribution function F . If X_n and X have distribution functions F_n and F , respectively, and $X_n \Rightarrow X$, then $F_n^{-1}(U) \rightarrow F^{-1}(U)$ a.s. (see, e.g., Theorem 2.1 of [2], Chapter 2). For distribution functions F and G , we have

$$(15) \quad \|F - G\|_1 = \inf E|X - Y|,$$

where the infimum is over all joint distributions on X, Y which have marginals F and G , respectively, and the variables $F^{-1}(U)$ and $G^{-1}(U)$ achieve the minimal L^1 -coupling, that is,

$$(16) \quad \|F - G\|_1 = E|F^{-1}(U) - G^{-1}(U)|;$$

see [14] for details.

With the use of Lemma 2.1, we are able to prove the following continuity property of the functional $B(X)$.

LEMMA 2.2. *Let X and $X_n, n \in \mathbb{N}$, be mean zero random variables with finite, nonzero absolute third moments. If*

$$(17) \quad X_n \xrightarrow{d} X, \quad \lim_{n \rightarrow \infty} EX_n^2 = EX^2 \quad \text{and} \quad E|X_n^3| \rightarrow E|X^3|,$$

then

$$B(X_n) \rightarrow B(X) \quad \text{as } n \rightarrow \infty.$$

PROOF. By Lemma 2.1, we have $X_n^* \Rightarrow X^*$. Let U be a uniformly distributed variable and set

$$(Y, Y_n, Y^*, Y_n^*) = (F_X^{-1}(U), F_{X_n}^{-1}(U), F_{X^*}^{-1}(U), F_{X_n^*}^{-1}(U)),$$

where F_W denotes the distribution function of W . Then $Y =_d X, Y_n =_d X_n, Y^* =_d X^*$ and $Y_n^* =_d X_n^*$. Furthermore, $Y_n \xrightarrow{a.s.} Y, Y_n^* \xrightarrow{a.s.} Y^*$ and, by (16),

$$\|\mathcal{L}(X_n^*) - \mathcal{L}(X_n)\|_1 = E|Y_n^* - Y_n| \quad \text{and} \quad \|\mathcal{L}(X^*) - \mathcal{L}(X)\| = E|Y^* - Y|.$$

By (4) with $f(x) = x^2 \text{sgn}(x)$, we find, for Y , for example, that

$$E|Y^3| = 2 \text{Var}(Y)E|Y^*|.$$

Hence, as $n \rightarrow \infty$, we have $EY_n^2 = EX_n^2 \rightarrow EX^2 = EY^2$ and

$$E|Y_n^*| = \frac{E|Y_n^3|}{2EY_n^2} = \frac{E|X_n^3|}{2EX_n^2} \rightarrow \frac{E|X^3|}{2EX^2} = \frac{E|Y^3|}{2EY^2} = E|Y^*| \quad \text{as } n \rightarrow \infty.$$

Hence, $\{Y_n\}_{n \in \mathbb{N}}$ and $\{Y_n^*\}_{n \in \mathbb{N}}$ are uniformly integrable, so $\{Y_n^* - Y_n\}_{n \in \mathbb{N}}$ is uniformly integrable. As $Y_n^* - Y_n \xrightarrow{a.s.} Y^* - Y$ as $n \rightarrow \infty$, we have

$$(18) \quad \begin{aligned} \lim_{n \rightarrow \infty} \|\mathcal{L}(X_n^*) - \mathcal{L}(X_n)\|_1 &= \lim_{n \rightarrow \infty} E|Y_n^* - Y_n| = E|Y^* - Y| \\ &= \|\mathcal{L}(X^*) - \mathcal{L}(X)\|. \end{aligned}$$

Combining (18) with the convergence of the variances and the absolute third moments, as provided by (17), the proof is complete. \square

Lemmas 2.3 and 2.4 borrow much from Theorem 2.1 of [10], the latter lemma indeed being implicit. However, the results of [10] cannot be applied directly as $B(G)$ is not expressed as the expectation of $K(X)$ for some K when $\mathcal{L}(X) = G$. For $m \geq 2$, let D_m denote the collection of all mean zero, variance 1 distributions which are supported on at most m points.

LEMMA 2.3.

$$B(\mathcal{F}_1) = B\left(\bigcup_{m \geq 3} D_m\right).$$

PROOF. Letting \mathcal{M} be the collection of distributions in \mathcal{F}_1 which have compact support, we first show that

$$(19) \quad B(\mathcal{F}_1) \leq B(\mathcal{M}).$$

Let $\mathcal{L}(X) \in \mathcal{F}_1$ be given and, for $n \in \mathbb{N}$, set $Y_n = X\mathbf{1}_{|X| \leq n}$. Clearly, $Y_n \xrightarrow{d} X$. As $E|X^3| < \infty$ and $|Y_n^p| \leq |X^p|$ for all $p \geq 0$, by the dominated convergence theorem, we have

$$(20) \quad \begin{aligned} EY_n &\rightarrow EX = 0, \\ EY_n^2 &\rightarrow EX^2 = 1 \quad \text{and} \quad E|Y_n^3| \rightarrow E|X^3| \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Letting

$$(21) \quad X_n = Y_n - EY_n,$$

we have $X_n \Rightarrow X$, by Slutsky's theorem, so, in view of (20), the hypotheses of Lemma 2.2 are satisfied, yielding

$$B(X_n) \rightarrow B(X) \quad \text{as } n \rightarrow \infty, \text{ with } \{X_n\}_{n \in \mathbb{N}} \subset \mathcal{M},$$

showing (19).

Now, consider $\mathcal{L}(X) \in \mathcal{M}$ so that $|X| \leq M$ a.s. for some $M > 0$. For each $n \in \mathbb{N}$, let

$$Y_n = \sum_{k \in \mathbb{Z}} \frac{k}{2^n} \mathbf{1}\left(\frac{k-1}{2^n} < X \leq \frac{k}{2^n}\right).$$

Since $|X| \leq M$ a.s., each Y_n is supported on finitely many points and uniformly bounded. Clearly, $Y_n \rightarrow X$ a.s. and (20) holds by the bounded convergence theorem. Now, defining X_n by (21), the hypotheses of Lemma 2.2 are satisfied, yielding

$$B(X_n) \rightarrow B(X) \quad \text{as } n \rightarrow \infty, \text{ with } \{X_n\}_{n \in \mathbb{N}} \subset \bigcup_{m \geq 3} D_m,$$

showing $B(\mathcal{M}) \leq B(\bigcup_{m \geq 3} D_m)$. Combining this inequality with (19) yields $B(\mathcal{F}_1) \leq B(\bigcup_{m \geq 3} D_m)$ and therefore the lemma, the reverse inequality being obvious. \square

LEMMA 2.4. *Every distribution in $\bigcup_{m \geq 3} D_m$ can be expressed as a finite mixture of D_3 distributions.*

PROOF. The lemma is trivially true for $m = 3$, so consider $m > 3$ and assume that the lemma holds for all integers from 3 to $m - 1$.

The distribution of any $X \in D_m$ is determined by the supporting values $a_1 < \dots < a_m$ and a vector of probabilities $\mathbf{p} = (p_1, \dots, p_m)'$. If any of the components of \mathbf{p} are zero, then $X \in D_k$ for $k < m$ and the induction would be finished, so we assume that all components of \mathbf{p} are strictly positive. As $X \in D_m$, the vector \mathbf{p} must satisfy

$$A\mathbf{p} = \mathbf{c} \quad \text{where } A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_m \\ a_1^2 & a_2^2 & \dots & a_m^2 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $A \in \mathbb{R}^{3 \times m}$ with $m > 3$, $\mathcal{N}(A) \neq \{0\}$, that is, there exists $\mathbf{v} \neq 0$ with

$$(22) \quad A\mathbf{v} = 0.$$

Since $\mathbf{v} \neq 0$ and the equation specified by the first row of A is $\sum_i v_i = 0$, the vector \mathbf{v} contains both positive and negative numbers. Since the vector \mathbf{p} has strictly positive components, the numbers t_1 and t_2 given by

$$t_1 = \inf\left\{t > 0 : \min_i (p_i + tv_i) \geq 0\right\} \quad \text{and} \quad t_2 = \inf\left\{t > 0 : \min_i (p_i - tv_i) \geq 0\right\}$$

are both strictly positive. Note that

$$\mathbf{p}_1 = \mathbf{p} + t_1 \mathbf{v} \quad \text{and} \quad \mathbf{p}_2 = \mathbf{p} - t_2 \mathbf{v}$$

satisfy

$$A\mathbf{p}_1 = A(\mathbf{p} + t_1 \mathbf{v}) = A\mathbf{p} = \mathbf{c} = A\mathbf{p} = A(\mathbf{p} - t_2 \mathbf{v}) = A\mathbf{p}_2,$$

by (22), so that \mathbf{p}_1 and \mathbf{p}_2 are probability vectors since their components are non-negative and sum to one. Additionally, the corresponding distributions have mean zero and variance 1, and in each of these two vectors, at least one component has been set to zero. Hence, we may express the m -point probability vector \mathbf{p} as the mixture

$$\mathbf{p} = \frac{t_2}{t_1 + t_2} \mathbf{p}_1 + \frac{t_1}{t_1 + t_2} \mathbf{p}_2$$

of probability vectors on at most $m - 1$ support points, thus showing X to be the mixture of two distributions in D_{m-1} , completing the induction. \square

The following theorem is an immediate consequence of Theorem 2.2 and Lemmas 2.3 and 2.4.

THEOREM 2.3.

$$B(\mathcal{F}_1) = B(D_3).$$

Hence, we now restrict our attention to D_3 .

3. Bound for D_3 distributions. Lemma 1.3 and Theorem 2.3 imply that $B(\mathcal{F}_\sigma) = B(\mathcal{F}_1) = B(D_3)$. Hence, Lemma 1.1 follows from Theorem 3.1 below, which shows that $B(D_3) = 1$. We prove Theorem 3.1 with the help of the following result.

LEMMA 3.1. *Let $x < y < 0 < z$, and let m_1 and m_0 be the unique mean zero distributions with support $\{x, z\}$ and $\{y, z\}$, respectively, that is,*

$$m_1(\{w\}) = \begin{cases} \frac{z}{z-x}, & \text{if } w = x, \\ \frac{-x}{z-x}, & \text{if } w = z, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad m_0(\{w\}) = \begin{cases} \frac{z}{z-y}, & \text{if } w = y, \\ \frac{-y}{z-y}, & \text{if } w = z, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(23) \quad \|m_1^* - m_0\|_1 \leq \|m_1^* - m_1\|_1.$$

PROOF. Let F_1, F_0 and F_1^* denote the distribution functions of m_1, m_0 and m_1^* , respectively. By Lemma 1.2, m_1^* is uniform over $[x, z]$. There are two cases, depending on the relative magnitudes of $F_1^*(y) = (y - x)/(z - x)$ and $F_0(y) = z/(z - y)$.

We first consider the case

$$(24) \quad F_1^*(y) \leq F_0(y) \quad \text{or, equivalently,} \quad y(x + z) \leq y^2 + z^2.$$

By Lemma 1.2,

$$(25) \quad \begin{aligned} \|m_1^* - m_1\|_1 &= \frac{z^2 + x^2}{2(z - x)} \\ &= \frac{(z^2 + x^2)(z - y)^2}{2(z - x)(z - y)^2} \\ &= \frac{z^4 - 2yz^3 + y^2z^2 + x^2z^2 - 2x^2yz + x^2y^2}{2(z - x)(z - y)^2} \\ &= \frac{(z^4 - 2yz^3 + x^2z^2 - 2x^2yz) + y^2z^2 + x^2y^2}{2(z - x)(z - y)^2}. \end{aligned}$$

Letting $J_1 = [x, y]$ and $J_2 = [y, z]$, we have

$$\|m_1^* - m_0\|_1 = I_1 + I_2 \quad \text{where } I_i = \int_{J_i} |F_1^*(w) - F_0(w)| dw \quad \text{for } i \in \{1, 2\}.$$

Since $F_1^*(w) \geq 0 = F_0(w)$ for all $w \in J_1$,

$$(26) \quad I_1 = \int_x^y \left(\frac{w - x}{z - x} \right) dw = \frac{1}{2} \frac{(y - x)^2}{z - x} = \frac{(y - x)^2(z - y)^2}{2(z - x)(z - y)^2}.$$

Recalling that $F_1^*(y) \leq F_0(y)$, applying (9) with $a = \frac{z}{z - y} - \frac{y - x}{z - x}$, $b = -\frac{y}{z - y}$ and $l = z - y$, after the change of variable $u = w - y$, yields

$$(27) \quad \begin{aligned} I_2 &= \int_y^z \left| \frac{w - x}{z - x} - \frac{z}{z - y} \right| dw \\ &= \left(\frac{z - y}{2} \right) \frac{(z/(z - y) - (y - x)/(z - x))^2 + (y/(z - y))^2}{1 - (y - x)/(z - x)} \\ &= \frac{1}{2}(z - x) \left(\left(\frac{z}{z - y} - \frac{y - x}{z - x} \right)^2 + \left(\frac{y}{z - y} \right)^2 \right) \\ &= \frac{(z(z - x) - (y - x)(z - y))^2 + (y(z - x))^2}{2(z - x)(z - y)^2} \\ &= \frac{(y^2 + z^2)(z - x)^2 - 2z(z - x)(y - x)(z - y) + (y - x)^2(z - y)^2}{2(z - x)(z - y)^2}. \end{aligned}$$

Adding (26) to (27) yields

$$\begin{aligned} & \|m_1^* - m_0\|_1 \\ &= \frac{(y^2 + z^2)(z - x)^2 - 2z(z - x)(y - x)(z - y) + 2(y - x)^2(z - y)^2}{2(z - x)(z - y)^2} \\ &= \frac{(z^4 - 2yz^3 + x^2z^2 - 2x^2yz) + 5y^2z^2 + 3x^2y^2 - 4xy^3}{2(z - x)(z - y)^2} \\ &\quad + \frac{4xy^2z - 4xyz^2 + 2y^4 - 4y^3z}{2(z - x)(z - y)^2}. \end{aligned}$$

Now, subtracting from (25) and simplifying by noting that the terms inside the parentheses in the numerators of these two expressions are equal, we find that

$$\begin{aligned} (28) \quad & \|m_1^* - m_1\|_1 - \|m_1^* - m_0\|_1 \\ &= \frac{-4y^2z^2 - 2x^2y^2 + 4xy^3 - 4xy^2z + 4xyz^2 - 2y^4 + 4y^3z}{2(z - x)(z - y)^2} \\ &= \frac{-y(y - x)(y^2 + 2z^2 - y(x + 2z))}{(z - x)(z - y)^2}. \end{aligned}$$

The denominator in (28) is positive, as is $-y$ and $y - x$. For the remaining term, (24) yields

$$y^2 + 2z^2 - y(x + 2z) = y^2 + 2z^2 - yz - y(x + z) \geq z(z - y) > 0.$$

Hence, (28) is positive, thus proving (23) when $F_1^*(y) \leq F_0(y)$.

When $F_1^*(y) > F_0(y)$, we have $F_1^*(w) \geq F_0(w)$ for all $w \in [x, z]$ as $F_0(w)$ is zero in $[x, y]$ and equals $F_0(y)$ in $[y, z]$, and $F_1^*(w)$ is increasing over $[y, z]$. Hence,

$$\begin{aligned} \|m_1^* - m_0\|_1 &= \int_x^z |F_1^*(w) - F_0(w)| dw = \int_x^z (F_1^*(w) - F_0(w)) dw \\ &= \int_x^z \frac{w - x}{z - x} dw - \int_y^z \frac{z}{z - y} dw = \frac{1}{2} \frac{(z - x)^2}{z - x} - z = \frac{1}{2}(z - x) - z \\ &= -\frac{x + z}{2}. \end{aligned}$$

Now, since $(x + z)(x - z) = x^2 - z^2 \leq z^2 + x^2$ and $z - x > 0$, using Lemma 1.2, we obtain

$$\|m_1^* - m_0\|_1 = -\frac{x + z}{2} \leq \frac{z^2 + x^2}{2(z - x)} = \|m_0^* - m_0\|_1,$$

thus proving inequality (23) when $F_1^*(y) > F_0(y)$ and, therefore, proving the lemma. \square

THEOREM 3.1.

$$B(D_3) = 1.$$

PROOF. Lemma 1.2 shows that $B(X) = 1$ if X is supported on two points, so $B(D_3) \geq 1$ and it only remains to consider X positively supported on three points. We first prove that

$$(29) \quad B(X) \leq 1$$

when $X \in D_3$ is positively supported on the nonzero points x, y, z .

$EX = 0$ implies that $x < 0 < z$. After proving (29), we treat the remaining case, where $y = 0$, by a continuity argument.

Let X be supported on $x < y < z$ with $y \neq 0$. Lemma 1.3 with $a = -1$ implies that $B(-X) = B(X)$, so we may assume, without loss of generality, that $x < y < 0 < z$. Let m_1 and m_0 be the unique mean zero distributions supported on $\{x, z\}$ and $\{y, z\}$, respectively, and let $\mathcal{L}(X_1) = m_1$ and $\mathcal{L}(X_0) = m_0$. As, in general, every mean zero distribution having no atom at zero can be represented as a mixture of mean zero two-point distributions (as in the Skorokhod representation, see [2]), letting

$$(30) \quad \mathcal{L}(X_\alpha) = \alpha m_1 + (1 - \alpha)m_0,$$

we have $\mathcal{L}(X) = \mathcal{L}(X_\alpha)$ for some $\alpha \in [0, 1]$; in fact, for the given X , one may verify that $P(X = x)/P(X_1 = x) \in (0, 1)$ and that (30) holds when α assumes this value. Therefore, to prove (29), it suffices to show that

$$(31) \quad B(X_\alpha) \leq 1 \quad \text{for all } \alpha \in [0, 1].$$

By Lemma 1.2,

$$(32) \quad EX_1^2 = -zx \quad \text{and} \quad EX_0^2 = -zy,$$

and, by (30), the variance of X_α is given by

$$(33) \quad \begin{aligned} EX_\alpha^2 &= \alpha EX_1^2 + (1 - \alpha)EX_0^2 = -(\alpha zx + (1 - \alpha)zy) \\ &= -z(\alpha x + (1 - \alpha)y). \end{aligned}$$

Applying Theorem 2.1 with $S = \{0, 1\}$ and μ being the probability measure putting mass α and $1 - \alpha$ on the points 1 and 0, respectively, in view of (32) and (33), m_α^* , the X_α zero bias distribution, is given by the mixture

$$(34) \quad m_\alpha^* = \beta m_1^* + (1 - \beta)m_0^* \quad \text{where } \beta = \frac{\alpha x}{\alpha x + (1 - \alpha)y}.$$

Since $x < y < 0$, we have

$$\frac{\beta}{1 - \beta} = \frac{\alpha}{1 - \alpha} \frac{x}{y} > \frac{\alpha}{1 - \alpha} \quad \text{and, therefore, } \beta > \alpha.$$

Let F_1, F_0, F_1^* and F_0^* denote the distribution functions of m_1, m_0, m_1^* and m_0^* , respectively. Let U be a standard uniform variable and, with the inverse functions below given by (14), set

$$(Y_1, Y_0, Y_1^*, Y_0^*) = (F_1^{-1}(U), F_0^{-1}(U), (F_1^*)^{-1}(U), (F_0^*)^{-1}(U)).$$

Then $Y_i =_d X_i, Y_i^* =_d X_i^*$ for $i \in \{1, 2\}$ and, by (16), all pairs of the variables Y_1, Y_0, Y_1^*, Y_0^* achieve the L^1 -distance between their respective distributions. Now, let (Y_α, Y_α^*) be defined on the same space with joint distribution given by the mixture

$$\mathcal{L}(Y_\alpha, Y_\alpha^*) = \alpha \mathcal{L}(Y_1, Y_1^*) + (1 - \beta) \mathcal{L}(Y_0, Y_0^*) + (\beta - \alpha) \mathcal{L}(Y_0, Y_1^*).$$

Then (Y_α, Y_α^*) has marginals $Y_\alpha =_d X_\alpha$ and $Y_\alpha^* =_d Y_\alpha^*$, hence, by (15),

$$\begin{aligned} \|m_\alpha^* - m_\alpha\|_1 &\leq \alpha \|m_1^* - m_1\|_1 + (1 - \beta) \|m_0^* - m_0\|_1 \\ (35) \qquad &\qquad \qquad + (\beta - \alpha) \|m_1^* - m_0\|_1. \end{aligned}$$

Lemma 1.2 shows that $G(X_i) = 1$, that is, $E|X_i^3| = 2EX_i^2\|m_i^* - m_i\|_1$ for $i = 1, 2$, so (30) yields

$$E|X_\alpha^3| = 2(\alpha EX_1^2\|m_1^* - m_1\|_1 + (1 - \alpha)EX_0^2\|m_0^* - m_0\|_1)$$

and, by (32), (33) and (34), we now find that

$$\begin{aligned} \frac{E|X_\alpha^3|}{2EX_\alpha^2} &= \frac{\alpha x \|m_1^* - m_1\|_1 + (1 - \alpha)y \|m_0^* - m_0\|_1}{\alpha x + (1 - \alpha)y} \\ (36) \qquad &= \beta \|m_1^* - m_1\|_1 + (1 - \beta) \|m_0^* - m_0\|_1. \end{aligned}$$

Lemma 3.1 shows that the right-hand side and, therefore, the left-hand side of (35) are bounded by (36), that is, that $B(X_\alpha) = 2EX_\alpha^2\|m_\alpha^* - m_\alpha\|_1/E|X_\alpha^3| \leq 1$, completing the proof of (31) and hence of (29).

Finally, we consider the case where the mean zero random variable X is positively supported on $\{x, 0, z\}$ with $x < 0 < z$ and $P(X = 0) = q \in (0, 1)$. For $n \in \mathbb{N}$, let

$$Y_n = X + n^{-1}\mathbf{1}(X = 0) \quad \text{and} \quad X_n = Y_n - EY_n.$$

As $n \rightarrow \infty$, we see that $Y_n \xrightarrow{\text{a.s.}} X$ and $EY_n = q/n \rightarrow 0$ so that $X_n \xrightarrow{\text{a.s.}} X$, and the bounded convergence theorem shows that $\{X_n\}_{n \in \mathbb{N}}$ satisfies the hypothesis of Lemma 2.2. Hence, $B(X_n) \rightarrow B(X)$ as $n \rightarrow \infty$. For all $n \in \mathbb{N}$ such that $1/n < z$, the distribution of X_n is positively supported on the three distinct, nonzero points $x - q/n < (1 - q)/n < z - q/n$, so, by (29), $B(X_n) \leq 1$ for all such n . Therefore, the limit $B(X)$ is also bounded by 1. \square

4. Lower bound. By (1), with $m = 1$ and $\mathcal{L}(X) = G \in \mathcal{F}_1$,

$$\|F_n - \Phi\|_1 \leq \frac{c_1 E|X^3|}{\sqrt{n}} \quad \text{for all } n \in \mathbb{N}$$

and, in particular, for $n = 1$,

$$(37) \quad c_1 \geq \frac{\|F_1 - \Phi\|_1}{E|X^3|} = \frac{\|G - \Phi\|_1}{E|X^3|}.$$

Motivated by Theorem 2.3, that two-point distributions achieve the suprema of $B(G)$, for $p \in (0, 1)$, let

$$X = \frac{\xi - p}{\sqrt{pq}},$$

where ξ is a Bernoulli variable with $P(\xi = 1) = p = 1 - P(\xi = 0)$. The distribution function G_p of X is given by

$$G_p(x) = \begin{cases} 0, & \text{for } x \leq -\sqrt{\frac{p}{q}}, \\ q, & \text{for } -\sqrt{\frac{p}{q}} < x \leq \sqrt{\frac{q}{p}}, \\ 1, & \text{for } \sqrt{\frac{q}{p}} < x, \end{cases}$$

and, therefore, the L^1 -distance between G_p and the standard normal is given by

$$\|G_p - \Phi\|_1 = \int_{-\infty}^{-\sqrt{p/q}} \Phi(x) dx + \int_{-\sqrt{p/q}}^{\sqrt{q/p}} |\Phi(x) - q| dx + \int_{\sqrt{q/p}}^{\infty} |\Phi(x) - 1| dx.$$

As $G_p \in \mathcal{F}_1$ for all $p \in (0, 1)$ and $E|X^3| = (p^2 + q^2)/\sqrt{pq}$, letting

$$\psi(p) = \frac{\sqrt{pq}}{p^2 + q^2} \|G_p - \Phi\|_1 \quad \text{for } p \in (0, 1),$$

inequality (37) gives $c_1 \geq \psi(p)$ for all $p \in (0, 1)$ and $\psi(1/2)$ yields (2).

5. Remarks. This article was submitted on November 18th, 2008. In the article [18], submitted on June 8th, 2009, Ilya Tyurin independently proved Theorem 1.1, also by applying the zero bias method. The current article was posted on arXiv on June 28, 2009; article [18] was posted on December 3rd, 2009. In [18], Theorem 1.1 is used to prove the upper bound 0.4785 on the L^∞ -Berry–Esseen constant.

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