Two Choice Optimal Stopping*†

David Assaf[0], Larry Goldstein and Ester Samuel-Cahn[0]
Hebrew University, University of Southern California and Hebrew University

November 15, 2004

Abstract

Let $X_n, \ldots, X_1$ be i.i.d. random variables with distribution function F. A statistician, knowing F, observes the X values sequentially and is given two chances to choose X's using stopping rules. The statistician's goal is to stop at a value of X as small as possible. Let V_n^2 equal the expectation of the smaller of the two values chosen by the statistician when proceeding optimally. We obtain the asymptotic behavior of the sequence V_n^2 for a large class of F's belonging to the domain of attraction (for the minimum) $\mathcal{D}(G^\alpha)$, where $G^\alpha(x) = [1 - \exp(-x^\alpha)]\mathbf{I}(x \geq 0)$. The results are compared with those for the asymptotic behavior of the classical one choice value sequence V_n^1, as well as with the "prophet value" sequence $V_n^p = E(\min\{X_n, \ldots, X_1\})$.

1 Introduction

Kennedy and Kertz (1990, 1991) study the asymptotic behavior of the value sequence, as $n \to \infty$, when optimally stopping an n long sequence of i.i.d. random variables with common distribution function F, with the objective being to stop on as large a value as possible. They show that the asymptotic behavior of the value sequence depends upon the domain of attraction, for the maximum, to which F belongs.

Recently Assaf and Samuel-Cahn (2000) and Assaf, Goldstein, and Samuel Cahn (2002) have studied optimal stopping problems where the statistician is given several choices, and his return is the expected value of the maximal element chosen. The goals in these works were the derivation of "prophet inequalities."

In the present paper we study the limiting behavior of the value sequence when the statistician, knowing F, is given two choices. It turns out to be more convenient here to take as objective to stop on as small a value as possible, and therefore to take as the statistician's goal the minimization of the expected value upon stopping. In particular, we consider a situation where the statistician would like to choose the smallest possible value from the n i.i.d variables $X_n, \ldots, X_1$ presented sequentially, and, with the luxury of two choices, can

*AMS 2000 subject classifications. Primary 60G40.

†Key words and phrases: multiple choice stopping rules, domains of attraction, prophet value.

[0]This research was supported by the ISRAEL SCIENCE FOUNDATION (grant number 879/01)

take a first choice as a 'fallback' value to use in case that none of the remaining variables are small enough to take as a second choice.

The two choice problem we consider is more difficult by an order of magnitude than the optimal one-choice problem. To be convinced of this, let $V_n^1(x)$ (which we will also denote by $g_n(x)$) and $V_n^2(x)$ be the value of the optimal one and two choice policy respectively, when applied to the i.i.d. sequence $X_n, \ldots, X_1$, when the statistician is already guaranteed the value x. Note that for convenience we are indexing the variables so that the first one observed is X_n and the last is X_1. Then by the dynamic programming principle, for one choice $V_1^1(x) = E[X_1 \wedge x]$ and we have

$$V_{n+1}^1(x) = E[X_{n+1} \wedge V_n^1(x)] \quad \text{for } n \geq 1, \tag{1}$$

whereas with two choices, $V_2^2(x) = E[X_2 \wedge X_1 \wedge x]$ and, for $n \geq 2$,

$$V_{n+1}^2(x) \quad = \quad E[V_n^1(X_{n+1}) \wedge V_n^2(x)]. \tag{2}$$

The first term inside the square brackets in (2) corresponds to choosing the current variable X_{n+1} and being left with only one additional choice among the remaining n observations, while the second term corresponds to passing up the current random variable X_{n+1} and retaining two choices, with the guaranteed bound x, among the remaining n observations.

Comparing (1) and (2) we see that for one choice the expectation computed in (1) is with respect to the random variables X_{n+1} with identical distributions, whereas the distribution of the random variable $V_n^1(X_{n+1})$ in (2) depends on the function V_n^1 which changes with n even though the sequence $X_n, \ldots, X_1$ is identically distributed.

Let

$$x_F = \sup\{x : F(x) < 1\}. \tag{3}$$

When nothing is guaranteed, the value for the one and two stop problems will be denoted V_n^1 and V_n^2 respectively, and satisfy $V_n^1 = V_n^1(x_F)$ and $V_n^2 = V_n^2(x_F)$.

The optimal stopping rules can be specified in the one and two stop cases by the values V_n^1, and the values V_n^2 and functions $V_n^1(x)$, respectively, as follows. For the one stop case, if X_{n+1} is smaller than V_n^1 the variable X_{n+1} should be taken. For the two stop case, if $V_n^1(X_{n+1}) < V_n^2$ then the variable X_{n+1} should be taken as the first choice, and the optimal one stop strategy then used on the remaining n variables when there is a guaranteed upper bound of X_{n+1}. In other words, if X_{m_1} has already been chosen as the first choice, then take $X_m, m < m_1$ as the second choice when $X_m < V_{m-1}^1(X_{m_1})$.

As in the one choice problem, the asymptotic behavior of the value sequence depends on which of the three extreme value classes the distribution function F belongs to. In the present paper, we only consider F which belongs to one of these domains of attraction and take up the study of the remaining two classes in subsequent work. Specifically in this paper, by a suitable shift of the origin, we assume that the distribution function F of the i.i.d. random variables belongs to the domain of attraction (for the minimum) $\mathcal{D}(G^\alpha)$, where $\alpha > 0$ and

$$G^\alpha(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp(-x^\alpha) & x \geq 0, \end{cases} \tag{4}$$

and satisfies $F(0) = 0$ and $F(x) > 0$ for all $x > 0$. (This is the Type III of Leadbetter, Lindgren and Rootzén, 1983, and Type Ψ_α of Resnick, 1987.) A necessary and sufficient condition for $F \in \mathcal{D}(G^\alpha)$ is

$$F(x) = x^\alpha L(x), \quad \text{where } L(x) \text{ is slowly varying at 0, i.e.} \quad \lim_{x\downarrow 0} \frac{L(tx)}{L(x)} = 1 \quad \text{for all } t > 0;$$

a sufficient (and close to necessary) condition is

$$\lim_{x\downarrow 0} \frac{xF'(x)}{F(x)} = \alpha,$$

see e.g. de Haan, 1976, Theorem 4.

Let V_n^p be the expected value of the minimum of n i.i.d. random variables. The results for the maximum (see e.g. Resnick 1987, Chapter 2.1) and the work of Kennedy and Kertz (1991) translate for the minimum as follows: If $F \in \mathcal{D}(G^\alpha)$, then

$$\begin{aligned} \lim_{n\to\infty} nF(V_n^p) &= \Gamma(1+1/\alpha)^\alpha \quad \text{and} \\ \lim_{n\to\infty} nF(V_n^1) &= (1+1/\alpha). \end{aligned} \tag{5}$$

Our main result for a statistician with two choices is as follows.

Theorem 1.1 *Let $X_n, \ldots, X_1$ be non-negative integrable i.i.d. random variables with distribution function*

$$F(x) = x^\alpha L(x) \quad \textit{where } \lim_{x\downarrow 0} L(x) \textit{ exists and equals } \mathcal{L} \in (0,\infty). \tag{6}$$

Then the optimal two choice value V_n^2 satisfies

$$\lim_{n\to\infty} nF(V_n^2) = h^\alpha(b_\alpha) \tag{7}$$

where $b_\alpha > 0$ is the unique solution to

$$\int_0^y h(u)du + (1/\alpha - y)h(y) = 0, \tag{8}$$

and $h(y)$ is the function

$$h(y) = \left(\frac{y}{1+\alpha y/(\alpha+1)}\right)^{1/\alpha} \quad \textit{for } y \geq 0. \tag{9}$$

The value $h(b_\alpha)$ depends only on α but unfortunately, unlike the values (5) cannot be given in closed form in terms of α. A short table of the limiting values (5) and of $h^\alpha(b_\alpha)$ are given in Table 1. The performance improvement in having two choices over having only one is substantial, in that the optimal stopping value becomes much closer to that of the prophet. For example, for a distribution with $\alpha = 1$ such as the uniform, the limiting values (for the minimum) for the statistician with one choice is 2, with two choices it is $1.165\ldots$, while the value for the prophet is 1. More explicitly, with n variables the optimal value for a statistician with one choice is roughly $2/n$, for the prophet it is roughly $1/n$, and for a statistician with two choices it is $1.165\ldots/n$.

The paper is organized as follows. In Section 2 we derive some fundamental identities when F belongs to the family

$$\mathcal{U}^\alpha(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\alpha & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1, \end{cases} \tag{10}$$

for a fixed value of $\alpha > 0$; we also show heuristics which explain the form of the function $h(y)$ of (9). In Section 3 we show that a particular sequence of functions h_n, which determine V_{n+1}^2, converges to h. Section 4 contains some general convergence results. In Section 5 we prove Theorem 1.1 for the special family (10), and some results concerning the finiteness of the limit of the moments of properly scaled randomly selected values. In Section 6 Theorem 1.1 is generalized to a wide class of distributions in $\mathcal{D}(G^\alpha)$. Section 7 contains numerical results presented in Table 1, along with explanations and several additional remarks.

2 The Fundamental Equations and Heuristics

In general, for X with distribution function F, we let

$$g(x) = E[X \wedge x].$$

When $F(0) = 0$, writing $g(x) = x - \int_0^x F(u)du$, we see easily that $g(x)$ is positive and strictly increasing on the interval $(0, x_F)$; hence the same is true for $g_{n+1}(x) = g(g_n(x))$.

In the remainder of this Section we consider $F = \mathcal{U}^\alpha$ as in (10), and in all the following we consider $\alpha > 0$ as fixed, to avoid the necessity of indexing quantities by α. For $\mathcal{U}^\alpha$ we have explicitly on the interval $[0, 1]$

$$g(x) = E[X \wedge x] = x - \frac{x^{\alpha+1}}{\alpha + 1}, \tag{11}$$

and with $g_1(x) = g(x)$,

$$g_{n+1}(x) = g_n(x) - \frac{g_n(x)^{\alpha+1}}{\alpha + 1}, \quad n \ge 1. \tag{12}$$

Since a statistician with two choices does at least as well as one with a single choice

$$g_n(0) = 0 \le V_n^2 \le V_n^1 = g_n(1), \quad n \ge 2.$$

As we are interested in the two choice case, we will henceforth write V_n to denote V_n^2 whenever convenient. Because the function g_n is strictly increasing on $[0, 1]$, there exists a unique number $b_n \in [0, 1]$ satisfying

$$V_n = g_n(b_n). \tag{13}$$

We call b_n the "threshold value" for the following reason; by (2) the statistician at stage $n + 1$ will choose X_{n+1} when $g_n(X_{n+1}) < V_n$, that is, when $X_{n+1} < b_n$.

Since $b_n \in [0, 1]$, $P(X > b_n) = 1 - b_n^\alpha$, and the basic equation (2) becomes

$$V_{n+1} = \int_0^{b_n} g_n(x)\alpha x^{\alpha-1}dx + (1 - b_n^\alpha)V_n, \quad n \ge 2. \tag{14}$$

Letting U_k be independent $\mathcal{U}[0,1]$ variables, $U_k^{1/\alpha}$ has distribution $\mathcal{U}^\alpha$, and hence we may begin recursion (14) at $V_2 = E[U_2^{1/\alpha} \wedge U_1^{1/\alpha}]$. Scaling,

$$W_n = n^{1/\alpha} V_n, \quad B_n = n^{1/\alpha} b_n \tag{15}$$

and

$$g_n(x) = x f_n(n x^\alpha). \tag{16}$$

Since $g_n(x)$ is defined and positive for $0 < x \le 1$, the function $f_n(x)$ is defined and positive for $0 < x \le n$, and setting $f_n(0) = 1$ makes $f_n(x)$ continuous as $x \downarrow 0$, since $g_n'(0) = 1$.

Substituting (16) into (14) and making the change of variable $y = n x^\alpha$ we obtain

$$V_{n+1} = n^{-(1+1/\alpha)} \int_0^{B_n^\alpha} y^{1/\alpha} f_n(y) dy + (1 - b_n^\alpha) V_n, \quad n \ge 2.$$

Multiplying by $n^{1/\alpha}$ and setting

$$h_n(y) = y^{1/\alpha} f_n(y) \tag{17}$$

we have

$$\left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1} = \frac{1}{n} \int_0^{B_n^\alpha} h_n(y) dy + (1 - b_n^\alpha) W_n, \quad n \ge 2. \tag{18}$$

By (13),(15),(16) and (17),

$$W_n = h_n(B_n^\alpha) \tag{19}$$

and we can now write (18) as our fundamental equation

$$W_m = c \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1} = \frac{1}{n} \int_0^n (h_n(y) \wedge W_n)\, dy \quad \text{for } n \ge m, \tag{20}$$

with $m = 2$ and $c = 2^{1/\alpha} E[U_2^{1/\alpha} \wedge U_1^{1/\alpha}]$. Later we allow for arbitrary initial times $m \ge 1$ and any positive starting values c.

The remainder of this Section is devoted to a heuristic argument explaining (7) and (8), the appearance and form of the function h in (9) and of Theorem 1.1. Firstly, $((n+1)/n)^{1/\alpha} = 1 + 1/(\alpha n) + O(n^{-2})$, and if B_n^α and the integral below remain bounded, we have from (18)

$$W_{n+1} - W_n = n^{-1} \int_0^{B_n^\alpha} h_n(y) dy + n^{-1}(1/\alpha - B_n^\alpha) W_n + O(n^{-2}).$$

If $W_n \to d_\alpha$ such that $W_n = d_\alpha + a/n + O(n^{-2})$, then $n(W_{n+1} - W_n) \to 0$, and multiplying by n we have

$$0 = \int_0^{B_n^\alpha} h_n(y) dy + (1/\alpha - B_n^\alpha) W_n + o(1), \tag{21}$$

and if $B_n^\alpha \to b_\alpha$ and $h_n \to h$ as $n \to \infty$, then (21) suggests

$$0 = \int_0^{b_\alpha} h(y) dy + (1/\alpha - b_\alpha) d_\alpha,$$

where from (19) also

$$d_\alpha = h(b_\alpha),$$

which explains (7) and (8) of Theorem 1.1.

By (17), finding the limiting h is equivalent to finding the limiting f, since

$$h(y) = y^{1/\alpha} f(y). \tag{22}$$

Using (12) and (16) and the substitution $y = nx^\alpha$, it follows that

$$f_{n+1}((1+\frac{1}{n})\, y) = f_n(y) - \frac{y}{(\alpha+1)n} f_n(y)^{\alpha+1}. \tag{23}$$

Subtracting $f_n(y)$ from both sides, dividing by y/n and taking limits as $n \to \infty$ indicates that the limiting function f should satisfy the differential equation

$$f'(y) = -f(y)^{\alpha+1}/(\alpha+1) \quad \text{with initial condition } f(0) = 1. \tag{24}$$

Equation (24) has the unique solution

$$f(y) = (1 + \frac{\alpha y}{\alpha+1})^{-1/\alpha}, \tag{25}$$

which together with (22) yields the function h of (9).

3 Preliminary Lemmas

In this Section we continue to consider $F = \mathcal{U}^\alpha$ as in (10). With f_n as in (16) and h_n as in (17), we have the following Lemma.

Lemma 3.1 *The function $f_n(y)$ is strictly decreasing in y for $y \in [0,n]$ and $h_n(y)$ is strictly increasing in y for $y \in [0,n]$.*

Proof: We prove the lemma by induction on n. For $n = 1$ from (11) and (16)

$$f_1(y) = 1 - \frac{y}{\alpha+1},$$

so the result is immediate for f_1, and for h_1 by (17). Now assume the assertions are true for n. We shall show they are true for $n+1$. Note that for $0 \le y \le n$ we have $0 \le y(n+1)/n \le n+1$. Differentiating (23), for $0 < y \le n$,

$$\begin{aligned} \left(\frac{n+1}{n}\right) f'_{n+1}(\frac{n+1}{n}y) &= f'_n(y) - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} - \frac{y}{n} f_n^\alpha(y) f'_n(y) \\ &= f'_n(y)[1 - \frac{y}{n} f_n^\alpha(y)] - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} \\ &< f'_n(y)[1 - \frac{y}{n}] - \frac{1}{(\alpha+1)n} f_n(y)^{\alpha+1} \\ &< 0, \end{aligned}$$

where we have used $f'_n(y) < 0$ and $0 < f_n^\alpha(y) < 1$ for $0 < y \le n$.

From (23) and (17) we have

$$(\frac{n}{n+1})^{1/\alpha} h_{n+1}(\frac{n+1}{n}y) = h_n(y) - \frac{1}{(\alpha+1)n} h_n(y)^{\alpha+1}.$$

Thus for $0 \leq y < n$ we have

$$(\frac{n}{n+1})^{1/\alpha-1}h'_{n+1}(\frac{n+1}{n}y) = h'_n(y)[1-\frac{1}{n}h_n(y)^\alpha] > 0$$

since by the induction hypothesis $h'_n(y) > 0$ and

$$h_n(y)^\alpha < h_n(n)^\alpha = [n^{1/\alpha}f_n(n)]^\alpha < nf_n^\alpha(0) = n. \qquad \blacksquare$$

Let $f(y)$ be given by (25) and define

$$\epsilon_n(y) = f(y) - f_n(y). \tag{26}$$

Lemma 3.2 *With $\epsilon_n(y)$ as in (26),*

$$\epsilon_n(y) > 0 \quad \text{for } 0 < y \leq n. \tag{27}$$

Proof. We use the following two well known inequalities.

$$\text{For } 0 < \alpha \leq 1 \text{ and } x \geq -1, \quad (1+x)^\alpha \leq 1+\alpha x, \tag{28}$$

and

$$\text{for } \alpha \geq 1 \text{ and } x \geq -1, \quad 1+\alpha x \leq (1+x)^\alpha. \tag{29}$$

We prove the lemma by induction. For $n = 1$ we must show that for $0 < y \leq 1$

$$1 - \frac{y}{\alpha+1} < (1+\frac{\alpha y}{\alpha+1})^{-1/\alpha}$$

which is equivalent to

$$(1-\frac{y}{\alpha+1})^\alpha < (1+\frac{\alpha y}{\alpha+1})^{-1}$$

or

$$(1+\frac{\alpha y}{\alpha+1})(1-\frac{y}{\alpha+1})^\alpha < 1. \tag{30}$$

Now for $0 < \alpha \leq 1$ we have by (28) that the left hand side of (30) is less than or equal to

$$(1+\frac{\alpha y}{\alpha+1})(1-\frac{\alpha y}{\alpha+1}) = 1-(\frac{\alpha y}{\alpha+1})^2 < 1.$$

For $\alpha > 1$ the left hand side of (30) is by (29) less than

$$(1+\frac{y}{\alpha+1})^\alpha(1-\frac{y}{\alpha+1})^\alpha = [1-(\frac{y}{\alpha+1})^2]^\alpha < 1.$$

Thus $\epsilon_1(y) > 0$ for $0 < y \leq 1$.

Now suppose $\epsilon_n(y) > 0$ for $0 < y \leq n$. That $\epsilon_{n+1}(y) > 0$ for $0 < y \leq n+1$, is equivalent to

$$f_{n+1}(y) < (1+\frac{\alpha y}{\alpha+1})^{-1/\alpha}.$$

By the induction hypothesis

$$f_n(y) < (1+\frac{\alpha y}{\alpha+1})^{-1/\alpha} \quad \text{for } 0 < y \leq n$$

and thus by (16)

$$g_n(x) < x(1+\frac{\alpha n x^\alpha}{\alpha+1})^{-1/\alpha} \quad \text{for } 0 < x \le 1,$$

and since $g(\cdot)$ is an increasing function, using (12),

$$g_{n+1}(x) < x(1+\frac{\alpha n x^\alpha}{\alpha+1})^{-1/\alpha}[1-\frac{x^\alpha}{\alpha+1}(1+\frac{\alpha n x^\alpha}{\alpha+1})^{-1}]. \tag{31}$$

Thus, again by (16), it suffices to show that the right hand side of (31) is less than

$$x(1+\frac{\alpha(n+1)x^\alpha}{\alpha+1})^{-1/\alpha}, \quad \text{for } 0 < x \le 1.$$

Set $y = x^\alpha/(\alpha+1)$. Then it suffices to show that

$$(1+\alpha n y)^{-1/\alpha}[1-\frac{y}{1+\alpha n y}] < (1+\alpha(n+1)y)^{-1/\alpha} \quad \text{for } 0 < y \le 1,$$

i.e. that

$$[1+\frac{\alpha y}{1+\alpha n y}]^{1/\alpha}[1-\frac{y}{1+\alpha n y}] < 1,$$

which is equivalent to

$$[1+\frac{\alpha y}{1+\alpha n y}][1-\frac{y}{1+\alpha n y}]^\alpha < 1. \tag{32}$$

For $\alpha \le 1$ use (28) to get that the left hand side of (32) is less than or equal to

$$[1+\frac{\alpha y}{1+\alpha n y}][1-\frac{\alpha y}{1+\alpha n y}] = 1-(\frac{\alpha y}{1+\alpha n y})^2 < 1.$$

For $\alpha > 1$ use (29) to get that the left hand side of (32) is less than

$$[1+\frac{y}{1+\alpha n y}]^\alpha[1-\frac{y}{1+\alpha n y}]^\alpha = [1-(\frac{y}{1+\alpha n y})^2]^\alpha < 1.\blacksquare$$

Lemma 3.3 *With $\epsilon_n(y)$ as in (26),*

$$\epsilon_n(y) < \frac{y}{2n} \quad \textit{for } 0 < y \le n. \tag{33}$$

Proof: We prove (33) by induction. For $n = 1$ we must show that

$$(1+\frac{\alpha y}{\alpha+1})^{-1/\alpha} < 1-y(\frac{1}{\alpha+1}-\frac{1}{2}) \quad \text{for } 0 < y \le 1. \tag{34}$$

For $\alpha \ge 1$, equation (34) is obvious, since the left hand side is less than 1 and the right hand side is greater than 1. For $\alpha < 1$ we have, by (29) that

$$(1+\frac{\alpha y}{\alpha+1})^{1/\alpha} \ge 1+\frac{y}{\alpha+1}.$$

Thus to show (34) it suffices to show

$$\frac{1}{1+y/(\alpha+1)} < 1-\frac{y(1-\alpha)}{2(\alpha+1)},$$

i.e. that

$$1 < (1 + \frac{y}{\alpha+1})(1 - \frac{y(1-\alpha)}{2(\alpha+1)}) = 1 + \frac{y}{2} - \frac{y^2(1-\alpha)}{2(\alpha+1)^2}$$

which clearly holds for $0 < y \leq 1$.

Now suppose (33) holds for n. Let $0 < y \leq n+1$, and $p_n = n/(n+1)$. By (23)

$$\begin{aligned}
\epsilon_{n+1}(y) &= f(y) - f_{n+1}(y) \\
&= f(y) - f_n(p_n y) + \frac{p_n y}{(\alpha+1)n} f_n(p_n y)^{\alpha+1} \\
&= (f(y) - f(p_n y)) + (f(p_n y) - f_n(p_n y)) + \frac{y}{(\alpha+1)(n+1)} f_n(p_n y)^{\alpha+1}.
\end{aligned}$$

Thus

$$\epsilon_{n+1}(y) = f(y) - f(p_n y) + \epsilon_n(p_n y) + \frac{y}{(\alpha+1)(n+1)} f_n(p_n y)^{\alpha+1}. \tag{35}$$

Note that

$$f'(y) = -f(y)^{\alpha+1}/(\alpha+1) < 0 \quad \text{for } y > 0 \tag{36}$$

and

$$f''(y) = f(y)^{2\alpha+1}/(\alpha+1) > 0 \quad \text{for } y > 0. \tag{37}$$

Thus if we use the Taylor expansion

$$f(x+\Delta) = f(x) + \Delta f'(x) + \frac{\Delta^2}{2} f''(x + \xi\Delta) \quad \text{for some } 0 < \xi < 1$$

with $x = p_n y$ and $\Delta = y/(n+1)$ so that $x + \Delta = y$, we get, by use of (36) and (37)

$$f(y) - f(p_n y) = -\frac{y}{(\alpha+1)(n+1)} f(p_n y)^{\alpha+1} + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1} \tag{38}$$

where $p_n < \theta < 1$. Substituting (38) into (35) yields

$$\epsilon_{n+1}(y) = \epsilon_n(p_n y) - \frac{y}{(\alpha+1)(n+1)} [f(p_n y)^{\alpha+1} - f_n(p_n y)^{\alpha+1}] + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1}. \tag{39}$$

Since by (27) $f(p_n y) > f_n(p_n y)$ for $0 < y \leq n+1$, we have

$$f(p_n y)^{\alpha+1} - f_n(p_n y)^{\alpha+1} > f(p_n y)^{\alpha}[f(p_n y) - f_n(p_n y)] = f(p_n y)^{\alpha} \epsilon_n(p_n y). \tag{40}$$

Substituting (40) into (39) yields

$$\epsilon_{n+1}(y) < \epsilon_n(p_n y)[1 - \frac{y}{(\alpha+1)(n+1)} f(p_n y)^{\alpha}] + \frac{y^2}{2(\alpha+1)(n+1)^2} f(\theta y)^{2\alpha+1}. \tag{41}$$

It follows from the induction hypothesis that for $0 < y \leq n+1$ (so that $0 < p_n y \leq n$)

$$\epsilon_n(p_n y) < \frac{p_n y}{2n} = \frac{y}{2(n+1)}.$$

Thus (41) yields

$$\begin{aligned}\epsilon_{n+1}(y) &< \frac{y}{2(n+1)}[1-\frac{y}{(\alpha+1)(n+1)}f(p_n y)^\alpha] + \frac{y^2}{2(\alpha+1)(n+1)^2}f(\theta y)^{2\alpha+1} \\ &< \frac{y}{2(n+1)}[1-\frac{yf(p_n y)^\alpha\{1-f(\theta y)^{\alpha+1}\}}{(\alpha+1)(n+1)}] < \frac{y}{2(n+1)},\end{aligned}$$

where we have used the fact that f is decreasing, $\theta > p_n$, and $f < 1$. ■

Corollary 3.1

$$\begin{aligned}f_n(y) \to f(y) &= (1+\frac{\alpha y}{\alpha+1})^{-1/\alpha} \quad \textit{for all } y>0, \textit{ as } n\to\infty \\ h_n(y) \to h(y) &= \left(\frac{y}{1+\alpha y/(\alpha+1)}\right)^{1/\alpha} \quad \textit{for all } y>0, \textit{ as } n\to\infty.\end{aligned}$$

Remark 3.1 *Note that by (16),(17) and (1)*

$$h_n(n) = n^{1/\alpha} g_n(1) = n^{1/\alpha} V_n^1$$

and thus, by (5)

$$\lim_{n\to\infty} h_n(n) = [1+1/\alpha]^{1/\alpha}.$$

On the other hand, we also have

$$\lim_{y\to\infty} h(y) = [1+1/\alpha]^{1/\alpha}$$

Thus, the convergence to h in Corollary 3.1 satisfies

$$\lim_{n\to\infty} h_n(n) = \lim_{y\to\infty}\lim_{n\to\infty} h_n(y).$$

4 Convergence of Recursions

To prove convergence of the sequence W_n determined by the recursion (20), we first study the behavior of a sequence Z_n, whose values are given by the simpler recursion

$$Z_m = c \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\int_0^n (q(y)\wedge Z_n)dy \quad \text{for } n \geq m, \tag{42}$$

where the function in the integral does not depend on n.

For $\alpha > 0$ a fixed value and $q(\cdot)$ a given function, define

$$Q(y) = \int_0^y q(u)du + (1/\alpha - y)q(y). \tag{43}$$

We prove the convergence of Z_n under the following conditions:
(i) $q(0) = 0$
(ii) $q(u)$ for $0 < u < \infty$ is non-decreasing everywhere and strictly increasing and differentiable for $0 < u < A$ where $1/\alpha < A \leq \infty$.
(iii) There exists a unique positive root $b \in (1/\alpha, A)$ to the equation $Q(y) = 0$.

Lemma 4.1 *Under conditions (i) and (ii), the function $Q(\cdot)$ is strictly increasing for $0 < y < 1/\alpha$, strictly decreasing for $1/\alpha < y < A$, and non-increasing for $A < y$. Hence $Q(A) = \lim_{y\uparrow A} Q(y)$ exists and is in $[-\infty, \infty)$, even when $A = \infty$, and (iii) holds if $Q(A) < 0$.*

Proof: For $0 \leq y_1 < y_2 < 1/\alpha$ straightforward calculations yield

$$Q(y_2) - Q(y_1) \geq (q(y_2) - q(y_1))(1/\alpha - y_2),$$

and for $1/\alpha < y_1 < y_2$,

$$Q(y_2) - Q(y_1) \leq (q(y_2) - q(y_1))(1/\alpha - y_1).$$

The claims now follow directly. ■

The main result of this Section is

Theorem 4.1 *Let (i), (ii) and (iii) hold, and let Z_n be given by (42) with $m \geq 1$ any integer and $c \in (0, \infty)$ any constant. Then the limit of Z_n exists and*

$$\lim_{n\to\infty} Z_n = d,$$

where $d = q(b)$ with b the unique root of $Q(y) = 0$.

Lemma 4.2 is the crux of of the proof of Theorem 4.1.

Lemma 4.2 *Assume that (i), (ii) and (iii) hold. Let $m \geq 1$ be any integer and $c \in (0, \infty)$ any constant, and suppose that Z_n for $n \geq m$ is defined by (42). Then for every $\delta \in (0, \min\{q(A) - d, d - q(1/\alpha)\})$ there there exists $\Delta > 0$ and n_0 such that for all $n \geq n_0$,*

$$\text{if } Z_n < d - \delta \text{ then } Z_{n+1} \geq (1 + \Delta/n) Z_n, \tag{44}$$

$$\text{if } Z_n > d + \delta \text{ then } Z_{n+1} \leq (1 - \Delta/n) Z_n, \tag{45}$$

$$\text{if } Z_n < d \text{ then } Z_{n+1} < d, \text{ and} \tag{46}$$

$$\text{if } |Z_n - d| \leq \delta \text{ then } |Z_{n+1} - d| \leq \delta. \tag{47}$$

Proof: We have

$$(1 + \frac{1}{n})^{1/\alpha} = 1 + \frac{1}{\alpha n} + \frac{1}{\alpha}(\frac{1}{\alpha} - 1)\frac{1}{2n^2} + O_\alpha(n^{-3}),$$

and hence for $\gamma > 0$

$$(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n\gamma}) = 1 - \frac{1}{n}(\frac{1}{\gamma} - \frac{1}{\alpha}) + \frac{1}{n^2}\left(\frac{1}{2\alpha}(\frac{1}{\alpha} - 1) - \frac{1}{\alpha\gamma}\right) + O_{\alpha,\gamma}(n^{-3}), \tag{48}$$

where we write $O_\lambda(f_n)$ to indicate a sequence bounded in absolute value by f_n times a constant depending only on λ, a collection of parameters.

Define

$$M(t) = \int_0^{q^{-1}(t)} \left(1 - \frac{q(y)}{t}\right) dy \quad \text{for } 0 \le t < q(A).$$

From (43), $Q(b) = 0$ and $d = q(b)$, we have

$$M(d) = 1/\alpha.$$

It is not hard to see that $M(t)$ is strictly increasing over its range. Hence, setting $\Delta_1 = (1/\alpha - M(d-\delta))/2$ and $\Delta_2 = (M(d+\delta) - 1/\alpha)/2$ we have $\Delta = \min\{\Delta_1, \Delta_2\} > 0$. Now consider the function

$$r_n(t) = \frac{1}{n}\int_0^n \left(\frac{q(y)}{t} \wedge 1\right) dy = 1 - \frac{1}{n}\int_0^{q^{-1}(t)\wedge n} \left(1 - \frac{q(y)}{t}\right) dy.$$

Since $Z_m > 0$ we have $Z_n > 0$ for all $n \ge m$, and now by (42) we have

$$Z_{n+1}/Z_n = \left(\frac{n+1}{n}\right)^{1/\alpha} r_n(Z_n). \tag{49}$$

By definition

$$r_n(t) = 1 - \frac{1}{n}M(t) \quad \text{for } 0 \le t < q(n).$$

To prove (44), assume $Z_n < d - \delta$. Since r_n is decreasing, using (49) and (48), we have for all $n > q^{-1}(d-\delta)$,

$$\begin{aligned}
Z_{n+1} &\ge Z_n(\frac{n+1}{n})^{1/\alpha} r_n(d-\delta) \\
&= Z_n(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n}M(d-\delta)) \\
&= (1 + \frac{1}{n}(\frac{1}{\alpha} - M(d-\delta)) + O_{\alpha,d-\delta}(n^{-2}))Z_n \\
&\ge (1 + \frac{\Delta_1}{n})Z_n \ge (1 + \frac{\Delta}{n})Z_n
\end{aligned}$$

for all n sufficiently large, showing (44).

Next we prove (45). When $Z_n \ge d + \delta$, we have similarly that for $n > q^{-1}(d+\delta)$,

$$\begin{aligned}
Z_{n+1} &\le Z_n(\frac{n+1}{n})^{1/\alpha} r_n(d+\delta) \\
&= Z_n(\frac{n+1}{n})^{1/\alpha}(1 - \frac{1}{n}M(d+\delta)) \\
&= (1 - \frac{1}{n}(M(d+\delta) - \frac{1}{\alpha}) + O_{\alpha,d+\delta}(n^{-2}))Z_n \\
&\le (1 - \frac{\Delta_2}{n})Z_n \le (1 - \frac{\Delta}{n})Z_n
\end{aligned}$$

for all n sufficiently large.

Turning now to (46) and (47), for $Z_n \le d+\delta$, since $d+\delta < q(A)$, β_n is well defined by

$$q(\beta_n) = Z_n.$$

Now by (42) and (43)

$$\left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1} = \frac{1}{n}\left(\int_0^{\beta_n} q(y)dy + (n-\beta_n)q(\beta_n)\right) = \frac{1}{n}Q(\beta_n) + (1-\frac{1}{\alpha n})Z_n;$$

thus

$$Z_{n+1} = \left(1+\frac{1}{n}\right)^{1/\alpha} \frac{1}{n}Q(\beta_n) + R_n Z_n \tag{50}$$

where

$$R_n = (1+\frac{1}{n})^{1/\alpha}(1-\frac{1}{\alpha n}). \tag{51}$$

Consider

$$Q(q^{-1}(u)) = \int_0^{q^{-1}(u)} q(y)dy + (1/\alpha - q^{-1}(u))u.$$

Since $q^{-1}(u)$ is differentiable for $0 < u < q(A)$,

$$\frac{d}{du}Q(q^{-1}(u)) = 1/\alpha - q^{-1}(u).$$

Hence, evaluating $Q(q^{-1}(u))$ by a Taylor expansion around d, and using $Q(b) = Q(q^{-1}(d)) = 0$, we obtain that there exists some ξ_{Z_n} between d and Z_n such that

$$Q(\beta_n) = Q(q^{-1}(Z_n)) = (Z_n - d)(1/\alpha - q^{-1}(\xi_{Z_n})). \tag{52}$$

Subtracting d from both sides of (50) and using (52) we obtain

$$Z_{n+1} - d = \left\{1 - (1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n}) - \frac{1}{\alpha})\right\}(Z_n - d) + [R_n - 1]Z_n. \tag{53}$$

Take n_1 such that for all $n \ge n_1$

$$(1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(d) - 1/\alpha)) < 1.$$

Then for $Z_n < d$ we have $\xi_{Z_n} < d$ and hence $q^{-1}(\xi_{Z_n}) < q^{-1}(d)$, and so

$$0 < \left\{1 - (1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n}) - \frac{1}{\alpha})\right\}.$$

Hence the first term on the right hand side of (53) is strictly negative. Next, there exists $n_2 \ge n_1$ so that for $n \ge n_2$ we have $0 < R_n < 1$, by (51) and (48) with $\gamma = \alpha$. For such n the second term on the right hand side is also negative, and the sum of these two terms is therefore negative. This proves (46).

To consider (47) suppose that $|Z_n - d| \le \delta$. Then $|\xi_{Z_n} - d| \le \delta$, and therefore

$$q^{-1}(d-\delta) \le q^{-1}(\xi_{Z_n}) \le q^{-1}(d+\delta).$$

Hence, for all n sufficiently large so that

$$(1+\frac{1}{n})^{1/\alpha}\frac{1}{n}\left(q^{-1}(d+\delta)-1/\alpha\right)\leq 1,$$

letting $\Delta_3 = q^{-1}(d-\delta)-1/\alpha$, which is strictly positive by choice of $\delta < d-q(1/\alpha)$, we have $q^{-1}(\xi_{Z_n})-1/\alpha \geq \Delta_3$ and therefore

$$0 \leq \left\{1-(1+\frac{1}{n})^{1/\alpha}\frac{1}{n}(q^{-1}(\xi_{Z_n})-\frac{1}{\alpha})\right\} \;\leq\; 1-\frac{\Delta_3}{n}. \tag{54}$$

Further, from (51), again using (48) with $\gamma=\alpha$, there exists K_α such that

$$|R_n-1|\leq \frac{K_\alpha}{n^2}.$$

Then for all n so large that

$$\frac{K_\alpha}{n}(d+\delta)\leq \Delta_3\delta$$

we have, using (53) and (54),

$$\begin{aligned}
|Z_{n+1}-d| &\leq (1-\frac{\Delta_3}{n})|Z_n-d|+|R_n-1|Z_n\\
&\leq (1-\frac{\Delta_3}{n})\delta+\frac{K_\alpha}{n^2}(d+\delta)\\
&\leq \delta.
\end{aligned}$$

This proves (47). ■

Proof of Theorem 4.1: For $\delta\in(0,\min\{q(A)-d, d-q(1/\alpha)\})$, let Δ and n_0 be as in Lemma 4.2.
Case I: $Z_{n_0} > d+\delta$. If $Z_n > d+\delta$ for all $n\geq n_0$ then by (45) we would have

$$Z_{n+1}\leq \prod_{j=n_0}^{n}(1-\frac{\Delta}{j})Z_{n_0}\to 0,$$

a contradiction. Hence for some $n_1\geq n_0$ we have $Z_{n_1}\leq d+\delta$, and we would therefore be in Case II or Case III.

Case II: $Z_{n_1} < d-\delta$ for some $n_1\geq n_0$. If $Z_n < d-\delta$ for all $n\geq n_1$ we would have by (44) that

$$Z_{n+1}\geq \prod_{j=n_1}^{n}(1+\frac{\Delta}{j})Z_{n_1}\to\infty,$$

a contradiction. Hence there exists $n_2\geq n_1$ such that $Z_{n_2}\geq d-\delta$. By (46), $Z_{n_2} < d$, reducing to Case III.

Case III: $|Z_{n_1}-d|\leq\delta$ for some $n_1\geq n_0$. In this case $|Z_n-d|\leq\delta$ for all $n\geq n_1$, by (47). Since δ can be taken arbitrarily small, the Theorem is complete. ■

The following Lemma may be of general interest, and presumably has been noticed independently by others. We will apply it to obtain asymptotic properties of moments in Section 5.

Lemma 4.3 *A. Let $D_n, n \geq n_0$ be a non-negative sequence satisfying*

$$D_{n+1} \leq \vartheta_n D_n + \gamma_n, \quad n \geq n_0, \tag{55}$$

where

$$0 \leq \vartheta_n \leq (1 - \vartheta/n) \quad \text{and} \quad 0 \leq \gamma_n \leq \frac{C}{n}$$

for some $\vartheta > 0$ and $C \geq 0$. Then

$$\limsup_{n\to\infty} D_n < \infty.$$

B. Let $D_{n_0} > 0$ and let

$$D_{n+1} \geq \vartheta_n D_n + \gamma_n, \quad n \geq n_0, \tag{56}$$

where

$$\vartheta_n \geq (1 + \vartheta/n), \quad \text{and} \quad \gamma_n \geq 0.$$

for some $\vartheta > 0$. Then

$$\lim_{n\to\infty} D_n = \infty.$$

Proof: Consider A. If (55) holds, then by induction, for all $n \geq n_0$ and $k \geq 0$,

$$D_{n+k+1} \leq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n + \sum_{j=n}^{n+k} \left(\prod_{l=j+1}^{n+k} \vartheta_l\right) \gamma_j. \tag{57}$$

Using $\vartheta_n \leq (1 - \vartheta/n)$ and $1 - x \leq e^{-x}$ we have

$$\begin{aligned}
\prod_{l=j+1}^{n+k} \vartheta_l &\leq \prod_{l=j+1}^{n+k} e^{-\vartheta/l} \\
&= \exp(-\vartheta \sum_{l=j+1}^{n+k} 1/l) \\
&\leq \exp(-\vartheta(\log(n+k) - \log(j+1))) \\
&= \left(\frac{j+1}{n+k}\right)^{\vartheta}.
\end{aligned}$$

Hence, from (57), for all $k \geq 0$,

$$\begin{aligned}
D_{n+k+1} &\leq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n + \sum_{j=n}^{n+k} \left(\prod_{l=j+1}^{n+k} \vartheta_l\right) \gamma_j \\
&\leq D_n + \sum_{j=n}^{n+k} \left(\frac{j+1}{n+k}\right)^{\vartheta} \frac{C}{j} \\
&\leq D_n + \frac{2^{\vartheta} C}{(n+k)^{\vartheta}} \sum_{j=n}^{n+k} j^{\vartheta-1} \\
&\leq D_n + \frac{2^{\vartheta} C}{\vartheta} \left(\frac{n+k+1}{n+k}\right)^{\vartheta}.
\end{aligned}$$

Letting $k \to \infty$ we see that the D_n sequence is bounded.

To prove B, note that $D_n > 0$ for all $n \geq n_0$ and that for all j sufficiently large

$$\vartheta_j \geq (1 + \vartheta/j) \geq \exp(\vartheta/(2j)),$$

which gives, by (56),

$$D_{n+k+1} \geq \left(\prod_{j=n}^{n+k} \vartheta_j\right) D_n \geq \exp(\frac{\vartheta}{2}\sum_{j=n}^{n+k}\frac{1}{j})D_n \to \infty \quad \text{as } k \to \infty. \blacksquare$$

5 The Family $\mathcal{U}^\alpha$

As in (43), with $h(\cdot)$ defined in (9), let

$$H(y) = \int_0^y h(u)du + (1/\alpha - y)h(y);$$

note that $h(\cdot)$ is strictly increasing for $0 \leq y < \infty$.

Lemma 5.1 *There exists a unique value $b_\alpha > 1/\alpha$ such that $H(b_\alpha) = 0$, and*

$$h^\alpha(b_\alpha) < 1 + \frac{1}{\alpha}. \tag{58}$$

Proof: By Lemma 4.1, $H(y)$ is strictly increasing for $0 < y < 1/\alpha$ and strictly decreasing for $1/\alpha < y < \infty$. Hence a root exists in $(1/\alpha, \infty)$ and is unique if H is ever negative. Since

$$H'(y) = (1/\alpha - y)h'(y),$$

for some constant a

$$H(y) = a + \int_{1/\alpha}^y (1/\alpha - u)h'(u)du. \tag{59}$$

Now, since $h(y)$ converges to a finite positive limit at infinity, and

$$h'(y) = \frac{1}{\alpha}h(y)^{1-\alpha}\frac{1}{(1 + \alpha y/(\alpha+1))^2},$$

we have that $y^2h'(y)$ is bounded away from zero and infinity as $y \to \infty$, and therefore

$$\int_{1/\alpha}^\infty h'(u)du < \infty \quad \text{and} \quad \int_{1/\alpha}^y uh'(u)du \to \infty \quad \text{as } y \to \infty,$$

yielding from (59) that

$$\lim_{y\to\infty} H(y) = -\infty.$$

Inequality (58) follows from $\lim_{y\to\infty} h^\alpha(y) = 1 + 1/\alpha$. $\blacksquare$

For $f(y)$ as given in (25), setting

$$f_j^*(y) = f(y) - y/2j \tag{60}$$

we have

$$\frac{d}{dy}\left(y^{1/\alpha}f_j^*(y)\right) = y^{1/\alpha-1}\left(\frac{f(y)}{\alpha} - \frac{yf(y)^{\alpha+1}}{\alpha+1} - \frac{y}{2j}(1/\alpha+1)\right). \tag{61}$$

Since $yf(y)^\alpha$ is strictly increasing with limit $(\alpha+1)/\alpha$ at infinity, $f(y)/\alpha > yf(y)^{\alpha+1}/(\alpha+1)$ for all $y \geq 0$. Hence, for any fixed $A > b_\alpha$ we have

$$\inf_{0\leq y\leq A}\left(\frac{f(y)}{\alpha} - \frac{yf(y)^{\alpha+1}}{\alpha+1}\right) > 0.$$

It follows that there exists $j_0 = j_0(A)$ such that the derivative in (61) is positive for all $0 < y \leq A$ and all $j > j_0$. For these j, set

$$k_j(y) = \begin{cases} y^{1/\alpha}f_j^*(y) & \text{for } 0 \leq y < A \\ A^{1/\alpha}f_j^*(A) & \text{for } A \leq y < \infty \end{cases} \tag{62}$$

and

$$K_j(y) = \int_0^y k_j(u)du + (1/\alpha - y)k_j(y).$$

Lemma 5.2 *There exists j_1 such that for all $j > j_1$ there are unique roots $b_{j,\alpha}$ to $K_j(y) = 0$ and $b_{j,\alpha} > 1/\alpha$. Setting $d_{j,\alpha} = k_j(b_{j,\alpha})$ we have*

$$b_{j,\alpha} \to b_\alpha \quad \text{and} \quad d_{j,\alpha} \to d_\alpha \quad \text{as } j \to \infty, \text{ where } d_\alpha = h(b_\alpha)\ . \tag{63}$$

Proof: We apply Lemma 4.1. The functions $k_j(\cdot)$ satisfy $k_j(0) = 0$, are non-decreasing everywhere and are strictly increasing and differentiable for $0 < y < A$. Further, $k_j(y)$ converges uniformly to $h(y)$ in $[0, A]$, yielding the uniform convergence of $K_j(y)$ to $H(y)$ in $[0, A]$. Since H is strictly decreasing in $(1/\alpha, \infty)$, it follows that $H(A) < H(b_\alpha) = 0$. Hence, since $K_j(A) \to H(A)$ as $j \to \infty$, for all j sufficiently large $K_j(A) < 0$. For such j Lemma 4.1 now yields the existence of a unique root $b_{j,\alpha} > 1/\alpha$ satisfying $K_j(b_{j,\alpha}) = 0$.

The uniform convergence of K_j to H implies $H(b_{j,\alpha}) \to 0$ as $j \to \infty$, from which the convergence of $b_{j,\alpha}$ to b_α follows. That $d_{j,\alpha}$ converges to d_α follows from the uniform convergence of k_j to h in $[0, A]$. ■

Lemma 5.3 *Let $m \geq 1$ be any integer and $c \in (0, \infty)$ be any constant. For $n > m$ let W_n be determined by the recursion (20) with starting value $W_m = c$, and let*

$$Z_m^+ = c \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{n+1}^+ = \frac{1}{n}\int_0^n \left(h(y) \wedge Z_n^+\right)dy \quad \text{for } n \geq m. \tag{64}$$

With j_1 as in Lemma 5.2, for all $j > j_1$ let $m_j^* = \max\{m, j\}$. Now define sequences $Z_{j,n}^-$ for $n \geq m_j^*$, by

$$Z_{j,m_j^*}^- = W_{m_j^*} \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{1/\alpha} Z_{j,n+1}^- = \frac{1}{n}\int_0^n \left(k_j(y) \wedge Z_{j,n}^-\right)dy \quad \text{for } n \geq m_j^*. \tag{65}$$

Then for all $n \geq m_j^*$,

$$Z_{j,n}^- \leq W_n \leq Z_n^+ \tag{66}$$

and

$$\lim_{n\to\infty} Z_{j,n}^- = d_{j,\alpha} \quad \text{and} \quad \lim_{n\to\infty} Z_n^+ = d_\alpha. \tag{67}$$

Proof: With $j > j_1$ and f_j^* defined in (60), Lemmas 3.3, 3.2 and monotonicity of f_n give

$$f_j^*(y) < f_n(y) < f(y) \quad \text{for all } n \geq j \text{ and } 0 < y \leq n.$$

Therefore, by (62), (17) and (22),

$$k_j(y) < h_n(y) < h(y) \quad \text{for all } n \geq j \text{ and } 0 < y \leq n.$$

Equation (66) now follows by a comparison of (65), (20) and (64), and (67) follows directly from Theorem 4.1. ■

It is convenient to consider the value and scaled value arising from stopping a sequence $U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, X_{m-1}, \ldots, X_1$ of independent variables with a finite initial subsequence from a distribution other than that of $U^{1/\alpha}$. The scaled value sequence for this problem satisfies (20) with $c = m^{1/\alpha} V_m(X_m, \ldots, X_1)$. Note that for any m and c there exists $X_m, \ldots, X_1$ such that $c = m^{1/\alpha} V_m(X_m, \ldots, X_1)$; the simplest construction is obtained by letting $X_j = cm^{-1/\alpha}$ for $1 \leq j \leq m$. Our suppression of the dependence of W_n on m and c is justified by Theorem 5.1, which states that the limiting value of W_n is the same for all such sequences.

Theorem 5.1 *Let $m \geq 2$ be any integer and suppose the variables $U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, \ldots, X_1$ are independent. Let*

$$V_{n,m} = V_n(U_n^{1/\alpha}, \ldots, U_{m+1}^{1/\alpha}, X_m, \ldots, X_1), \tag{68}$$

be the optimal two choice value, and suppose $V_m(X_m, \ldots, X_1) = c \in (0, \infty)$. Then

$$W_n = n^{1/\alpha} V_{n,m} \quad \text{for } n > m,$$

satisfies

$$\lim_{n\to\infty} W_n = h(b_\alpha), \tag{69}$$

where b_α is the unique solution to (8).

In particular, the optimal two stop value V_n for a sequence of i.i.d. variables with distribution function $\mathcal{U}^\alpha(x) = x^\alpha$ for $0 \leq x \leq 1$ and $\alpha > 0$ satisfies

$$\lim_{n\to\infty} n\mathcal{U}^\alpha(V_n) = h^\alpha(b_\alpha); \tag{70}$$

that is, the conclusion of Theorem 1.1 holds for the $\mathcal{U}^\alpha$ family of distributions.

Proof: We apply Lemma 5.3 with the given m and c. Letting $n \to \infty$ in (66) and using (67),

$$d_{j,\alpha} \leq \liminf_{n\to\infty} W_n \leq \limsup_{n\to\infty} W_n \leq d_\alpha \quad \text{for all } j > j_1.$$

Now letting $j \to \infty$ and using (63) gives (69). The W_n values for the i.i.d. sequence with distribution function $\mathcal{U}^\alpha$ are generated by recursion (20) for the particular case $m = 2$ and $c = 2^{1/\alpha} E[U_2^{1/\alpha} \wedge U_1^{1/\alpha}]$, thus proving (70). ■

We conclude this section with some results on the existence of moments for both the one and two-stop problems.

Theorem 5.2 *Let $U_n^{1/\alpha}, \ldots, U_1^{1/\alpha}$ be an i.i.d. sequence with distribution function $\mathcal{U}^\alpha$, a_n a sequence of constants in $[0,1]$ with $a_0 = 1$, and*

$$T_n = \max\{1 \le k \le n : U_k^{1/\alpha} \le a_{k-1}\}. \tag{71}$$

When $A_n = n^{1/\alpha} a_n$ satisfies

$$0 < \underline{\kappa} = \liminf_{n\to\infty} A_n \le \limsup_{n\to\infty} A_n = \overline{\kappa} < \infty$$

we have

$$\limsup_{n\to\infty} E(n^{1/\alpha} U_{T_n}^{1/\alpha})^r < \infty \quad \text{for all } r < \alpha \underline{\kappa}^\alpha. \tag{72}$$

If

$$\overline{\kappa} < \infty,$$

we have

$$\lim_{n\to\infty} E(n^{1/\alpha} U_{T_n}^{1/\alpha})^r = \infty \quad \text{for all } r > \alpha \overline{\kappa}^\alpha. \tag{73}$$

Proof: Let

$$M_n(r) = E(U_{T_n}^{r/\alpha})$$

be the r^{th} moment of the a_k stopped sequence. The sequence $M_n(r)$ satisfies the recursion

$$M_{n+1}(r) \;=\; \int_0^{a_n} x^r \alpha x^{\alpha-1} dx + (1 - a_n^\alpha) M_n(r), \quad n \ge 1.$$

Substituting $y = n x^\alpha$,

$$M_{n+1}(r) \;=\; \frac{1}{n^{1+r/\alpha}} \int_0^{A_n^\alpha} y^{r/\alpha} dy + (1 - a_n^\alpha) M_n(r).$$

Multiplying by $n^{r/\alpha}$, and letting $n^{r/\alpha} M_n(r) = S_n(r)$,

$$\begin{aligned} \left(\frac{n}{n+1}\right)^{r/\alpha} S_{n+1}(r) &= \frac{1}{n}\int_0^{A_n^\alpha} y^{r/\alpha} dy + (1 - \frac{A_n^\alpha}{n}) S_n(r) \\ &= \frac{A_n^{\alpha+r}}{n(1+r/\alpha)} + (1 - \frac{A_n^\alpha}{n}) S_n(r). \end{aligned} \tag{74}$$

To show (72), multiplying (74) by $((n+1)/n)^{r/\alpha}$ and noting that

$$\left(\frac{n+1}{n}\right)^{r/\alpha} = 1 + \frac{r}{\alpha n} + O_{r/\alpha}(n^{-2})$$

by the boundedness of the sequence A_n and $\underline{\kappa}^\alpha > r/\alpha$, we obtain for all n sufficiently large,

$$S_{n+1}(r) \le \frac{2^{r/\alpha} A_n^{\alpha+r}}{n(1+r/\alpha)} + (1 - \frac{(A_n^\alpha - r/\alpha)}{2n}) S_n(r);$$

(72) now follows from Lemma 4.3 A.

To show (73) we note that for all n sufficiently large, using (74),

$$
\begin{aligned}
S_{n+1}(r) &\geq (\frac{n+1}{n})^{r/\alpha}(1-\frac{A_n^\alpha}{n})S_n(r) \\
&= (1+\frac{r}{\alpha n}+O_{r/\alpha}(n^{-2}))(1-\frac{A_n^\alpha}{n})S_n(r) \\
&= \left(1+\frac{(r/\alpha - A_n^\alpha)}{n}+O_{r/\alpha,\overline{\kappa}}(n^{-2})\right)S_n(r) \\
&\geq (1+\frac{(r/\alpha - A_n^\alpha)}{2n})S_n(r).
\end{aligned}
$$

Now, recalling that $r/\alpha > \overline{\kappa}^\alpha$, apply Lemma 4.3 B.

Corollary 5.1 *Let $\mathbf{1}_n^{U^{1/\alpha}}$ and $\mathbf{2}_n^{U^{1/\alpha}}$ be the one and two choice random values obtained from optimally stopping an independent sequence of variables having distribution $\mathcal{U}^\alpha$. In the one choice case,*

$$
\begin{aligned}
\text{if } r < 1+\alpha, &\quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}})^r < \infty, \\
\text{while if } r > 1+\alpha, &\quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}})^r = \infty.
\end{aligned}
\tag{75}
$$

In the two choice case,

$$
\text{if } r < 1+\alpha, \quad \limsup_{n\to\infty} E(n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}})^r < \infty. \tag{76}
$$

Proof: For one choice, apply Theorem 5.2 with $a_n = V_n^1$, and therefore $U_{T_n}^{1/\alpha} = \mathbf{1}_n^{U^{1/\alpha}}$. By (5),

$$
\lim_{n\to\infty} n^{1/\alpha}V_n^1 = \lim_{n\to\infty}(n\mathcal{U}^\alpha(V_n^1))^{1/\alpha} = (1+1/\alpha)^{1/\alpha}.
$$

The one choice results now follow from (72) and (73) of Theorem 5.2 with $\overline{\kappa} = \underline{\kappa} = (1+1/\alpha)^{1/\alpha}$.

For two choices, let T_n be defined as in (71) with b_n, the first choice thresholds given in (13), replacing a_n, and let $B_n = n^{1/\alpha}b_n$. Then as $\mathbf{2}_n^{U^{1/\alpha}} \leq U_{T_n}^{1/\alpha}$, it clearly suffices to show that for $r < 1+\alpha$,

$$
\limsup_{n\to\infty} E(n^{1/\alpha}U_{T_n}^{1/\alpha})^r < \infty.
$$

Reiterating (19), $W_n = h_n(B_n^\alpha)$, and by Theorem 5.1

$$
\lim_{n\to\infty} W_n = d_\alpha = h(b_\alpha).
$$

We show $\lim_{n\to\infty} B_n^\alpha = b_\alpha$. Suppose $\limsup_{n\to\infty} B_n^\alpha = B^\alpha > b_\alpha$. Then there exists $\epsilon > 0$ such that $B^\alpha - \epsilon > b_\alpha$. But then $\limsup_{n\to\infty} h_n(B_n^\alpha) \geq \limsup_{n\to\infty} h_n(B^\alpha-\epsilon) = h(B^\alpha-\epsilon) > h(b_\alpha)$, a contradiction. Similarly if $\liminf_{n\to\infty} B_n^\alpha < b_\alpha$. Thus the limit of B_n exists and

$$
n^{1/\alpha}b_n = B_n \to b_\alpha^{1/\alpha}.
$$

By (72) it suffices to show that $b_\alpha > 1 + 1/\alpha$, which, by Lemmas 4.1 and 5.1, would follow from $H(1 + 1/\alpha) > 0$. Now

$$\begin{aligned} H(1+1/\alpha) &= (1+\alpha)^{1/\alpha}\left[\int_0^{1+1/\alpha}\left(\frac{y}{\alpha+1+\alpha y}\right)^{1/\alpha}dy - \left(\frac{1+1/\alpha}{2(1+\alpha)}\right)^{1/\alpha}\right] \\ &> (1+\alpha)^{1/\alpha}\left[\left(\frac{1}{2(1+\alpha)}\right)^{1/\alpha}\int_0^{1+1/\alpha} y^{1/\alpha}dy - \left(\frac{1+1/\alpha}{2(1+\alpha)}\right)^{1/\alpha}\right] \\ &= 0, \end{aligned}$$

completing the proof. ■

Remark 5.1 *Kennedy and Kertz (1991, Theorem 1.4) obtain the limiting distribution of the scaled optimal one stop random variable* $n^{1/\alpha}\mathbf{1}_n^{U^{1/\alpha}}$. *It is easily checked that this limiting distribution has a finite* r^{th} *moment if and only if* $r < 1 + \alpha$, *which is not surprising, when compared with (75) in Corollary 5.1.*

Remark 5.2 *From the proof that* $b_\alpha > 1 + 1/\alpha$ *in Corollary 5.1, it follows that the limiting thresholds* b_n *for the first choice in the optimal two-choice problem are larger than the corresponding values* V_n^1 *for the optimal one choice problem, for all* $\alpha > 0$. *This is reasonable, as with two choices one 'can afford' to make the first of the two choices in the two stop problem earlier than the only choice in the one stop problem.*

Another interpretation of the inequality $b_n > V_n^1$ *is gained by applying* $V_n^1()$ *to both sides, to obtain* $V_n > V_{2n}^1$ *i.e. one is better off having one choice among* $2n$ *variables than having two choices among* n *variables.*

Remark 5.3 *Whereas it follows from Resnick, (1987, Proposition 2.1) that all scaled moments of the minimum exist, it is of interest to note that no moment with* $r > 1 + \alpha$ *exists for the optimal scaled one-choice value.*

6 Extension to General Distributions

Theorem 5.1 treats the special case where the variables have distribution function $\mathcal{U}^\alpha(x)$ as in (10). At the end of this section we prove Theorem 1.1 for an i.i.d. sequence of random variables in a much wider class.

To prove Theorem 1.1 the two stop problem is considered for $X_n, \ldots, X_1$, non-trivial independent but not necessarily identically distributed random variables. It is direct to see that the dynamic programming equations given in the introduction for an i.i.d. sequence hold under the assumption of independence alone. In particular, the one and two stop value functions $V_n^1(x)$ and $V_n^2(x)$ are again given through (1) and (2) respectively. With nothing guaranteed, we have that $V_n^1 = V_n^1(\infty)$ and $V_n = V_n^2(\infty)$ are the one and two choice optimal stopping values, respectively. However, Lemma 6.1 gives an alternative representation for V_n^1 which reduces to $V_n^1(x_F)$ as given earlier for the i.i.d. case, as well as conditions which guarantee that the 'threshold' indifference sequences are uniquely defined for independent but not necessarily identically distributed sequences.

Lemma 6.1 *Let* $X_n, \ldots, X_1$ *be non-negative independent random variables with distribution functions* $F_n, \ldots, F_1$ *respectively, and* x_F *given in (3). Then for all* $x \geq 0$ *the function* $V_k^1(x)$ *given by (1) satisfies* $0 \leq V_k^1(x) \leq x$ *and is non-decreasing and continuous.*

Letting $v_1 = x_{F_1}$ *and*

$$v_k = v_{k-1} \wedge w_k \quad \text{where } w_k = \inf\{y : V_{k-1}^1(y) \geq x_{F_k}\} \quad \text{for } 2 \leq k \leq n,$$

the function $V_k^1(x)$ *is strictly monotone increasing for* $0 < x < v_k$*, and satisfies* $V_k^1(x) = V_k^1(v_k)$ *for* $x \geq v_k$*; in particular*

$$V_k^1 = V_k^1(v_k).$$

Furthermore, the indifference numbers b_k*,* $2 \leq k \leq n$ *given by the solutions to*

$$V_k = V_k^1(b_k)$$

exist, and are uniquely defined in $[0, v_k]$*.*

Proof: For all $x \geq 0$ the function $V_1^1(x) = E[X_1 \wedge x] = x - \int_0^x F_1(y)dy$ satisfies $0 \leq V_1^1(x) \leq x$ and is non-decreasing and continuous; further, $V_1^1(x)$ is strictly increasing for x in $[0, x_{F_1}]$, and $V_1^1(x) = EX_1$ for $x \geq x_{F_1}$. Now assume for all $x \geq 0$ that $0 \leq V_{k-1}^1(x) \leq x$, and $V_{k-1}^1(x)$ is non-decreasing and continuous. Then $V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] \leq E[x \wedge X_k] \leq x$. For $0 \leq x \leq y$ we have $V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] \leq E[V_{k-1}^1(y) \wedge X_k] = V_k^1(y)$, and $V_k^1(x)$ is continuous for all x by the bounded convergence theorem, using the continuity of $V_{k-1}^1(x)$ and its upper bound of x.

To prove strict monotonicity, assume that $V_{k-1}^1(x)$ is strictly increasing for $0 \leq x \leq v_{k-1}$ and take $0 \leq x < y \leq v_k$. Since $v_k \leq v_{k-1}$ we have $V_{k-1}^1(x) < V_{k-1}^1(y)$, and since $x < w_k$ we have $V_{k-1}^1(x) < x_{F_k}$ and therefore $P(X_k > V_{k-1}^1(x)) > 0$. Hence

$$V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] < E[V_{k-1}^1(y) \wedge X_k] = V_k^1(y).$$

Now, assuming that $V_{k-1}^1(x)$ is constant for $x \geq v_{k-1}$, then for all $x \geq v_{k-1}$,

$$\begin{aligned} V_k^1(x) &= E[V_{k-1}^1(x) \wedge X_k] = E[V_{k-1}^1(v_{k-1}) \wedge X_k] = E[V_{k-1}^1(v_{k-1}) \wedge x_{F_k} \wedge X_k] \\ &= E[V_{k-1}^1(v_{k-1}) \wedge V_{k-1}^1(w_k) \wedge X_k] = E[V_{k-1}^1(v_{k-1} \wedge w_k) \wedge X_k] = V_k^1(v_k). \end{aligned}$$

Similarly, for all $x \geq w_k$,

$$V_k^1(x) = E[V_{k-1}^1(x) \wedge X_k] = E[V_{k-1}^1(x) \wedge V_{k-1}^1(w_k) \wedge X_k] = E[V_{k-1}^1(w_k) \wedge X_k] = V_k^1(w_k),$$

from which it follows that $V_k^1(v_k) = V_k^1(w_k)$ and $V_k^1(x) = V_k^1(v_k)$ for all $x \geq v_k$.

Since

$$0 = V_k^1(0) \leq V_k = V_k^1(b_k) \leq V_k^1 = V_k^1(v_k),$$

and $V_k^1(x)$ is continuous and strictly monotone increasing in $[0, v_k]$, the solution b_k exists uniquely in $[0, v_k]$. ■

In the case where the variables are i.i.d., since $V_{k-1}^1(y) \leq y$ we have $w_k \geq x_F$, and hence $v_k = x_F$, as given in Section 1.

Lemma 6.2 *For any sequence of nonnegative independent random variables* $X_n, \ldots, X_1$ *the sequence* $b_k, 2 \leq k \leq n$ *is monotone non-increasing.*

Proof: We first show that

$$V^2_{k+1} \leq E[X_{k+1} \wedge V^2_k], \quad 2 \leq k \leq n-1.$$

The right hand side is the value obtained by applying, on the sequence $X_{k+1}, \ldots, X_1$, the suboptimal two choice rule where X_{k+1} is chosen as the first and second choice if $X_{k+1} < V^2_k$ (this is the same as taking X_{k+1} as the first choice and not taking any second choice), and when $X_{k+1} \geq V^2_k$ the optimal two choice rule is applied on $X_k, \ldots, X_1$. The inequality reflects that the optimal rule does as well as this, or any other, two choice rule on this sequence. Therefore

$$V^1_{k+1}(b_{k+1}) = V^2_{k+1} \leq E[X_{k+1} \wedge V^2_k] = E[X_{k+1} \wedge V^1_k(b_k)] = V^1_{k+1}(b_k).$$

Since $V^1_{k+1}(x)$ is strictly monotone increasing in the interval $[0, v_{k+1}]$, which contains b_{k+1}, the Lemma is shown. ■

Below we consider stochastic dominance between two random variables, and write $Y \leq_d X$ when $P(Y > t) \leq P(X > t)$ for all t.

Lemma 6.3 *Let $X_n, \ldots, X_1$ and $Y_n, \ldots, Y_1$ be sequences of independent non-negative random variables having two choice value and threshold sequences V^X_j, V^Y_j and $b^X_j, b^Y_j, j = 1, \ldots, n$ respectively. If for some $m \geq 2$,*

$$Y_j \leq_d X_j, \quad j = 1, \ldots, m, \tag{77}$$

and there exists $\tau \geq \max\{b^X_m, b^Y_m\}$ such that

$$\tau \wedge Y_{j+1} \leq_d \tau \wedge X_{j+1} \quad \text{for } m \leq j < n, \tag{78}$$

then

$$V^Y_j \leq V^X_j, \quad \text{for } j = 2, \ldots, n; \tag{79}$$

hence, if the inequalities in (77) and (78) are replaced by equalities, then $V^Y_j = V^X_j, j = 2, 3, \ldots, n$. Finally, V^X_n is unchanged upon replacing any X_{j+1} by $\tau \wedge X_{j+1}, 2 \leq j < n$, for any $\tau \geq b^X_j$.

Proof: Let $V^{X,1}_n(x)$ and $V^{Y,1}_n(x)$ denote the optimal one choice value functions for the X and Y sequences respectively, with guaranteed value x, as in (1). A simple induction using (77) gives $V^{Y,1}_j(x) \leq V^{X,1}_j(x)$ for all x and $1 \leq j \leq m$.

First suppose that (78) holds for some arbitrary τ, and that for some $m \leq j < n$,

$$V^{Y,1}_j(x) \leq V^{X,1}_j(x) \quad \text{for all } x \leq \tau. \tag{80}$$

Then for $x \leq \tau$, using that $V^{X,1}_j(x) \leq x \leq \tau$ and $V^{Y,1}_j(x) \leq x \leq \tau$ by Lemma 6.1, we have

$$Y_{j+1} \wedge V^{Y,1}_j(x) \leq Y_{j+1} \wedge V^{X,1}_j(x) = (Y_{j+1} \wedge \tau) \wedge V^{X,1}_j(x) \leq_d (X_{j+1} \wedge \tau) \wedge V^{X,1}_j(x) = X_{j+1} \wedge V^{X,1}_j(x),$$

giving

$$V^{Y,1}_{j+1}(x) = E[Y_{j+1} \wedge V^{Y,1}_j(x)] \leq E[X_{j+1} \wedge V^{X,1}_j(x)] = V^{X,1}_{j+1}(x), \quad \text{for } x \leq \tau,$$

and thus (80) holds for $1 \le j \le n$.

For $\tau \ge \max\{b_j^X, b_j^Y\}$ and $j = m$, Lemma 6.2 implies this inequality holds for $m \le j < n$, and therefore, for instance,

$$Y_{j+1} \wedge b_j^Y = (Y_{j+1} \wedge \tau) \wedge b_j^Y \le_d (X_{j+1} \wedge \tau) \wedge b_j^Y = X_{j+1} \wedge b_j^Y.$$

Now note that (77) yields $V_j^Y \le V_j^X$ for $1 \le j \le m$, so assuming this inequality for some j, $m \le j < n$, we now have

$$\begin{aligned} V_{j+1}^Y &= E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^Y] = E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^{Y,1}(b_j^Y) \wedge V_j^Y] = E[V_j^{Y,1}(Y_{j+1} \wedge b_j^Y) \wedge V_j^Y] \\ &\le E[V_j^{Y,1}(X_{j+1} \wedge b_j^Y) \wedge V_j^Y] \le E[V_j^{X,1}(X_{j+1} \wedge b_j^Y) \wedge V_j^Y] \\ &= E[V_j^{X,1}(X_{j+1}) \wedge V_j^{X,1}(b_j^Y) \wedge V_j^{Y,1}(b_j^Y)] = E[V_j^{X,1}(X_{j+1}) \wedge V_j^{Y,1}(b_j^Y)] \\ &= E[V_j^{X,1}(X_{j+1}) \wedge V_j^Y] \le E[V_j^{X,1}(X_{j+1}) \wedge V_j^X] \\ &= V_{j+1}^X; \end{aligned}$$

thus (79) holds.

To prove the final claim, let $Y_n, \ldots, Y_1$ be the sequence where any number of variables $X_{j+1}, 2 \le j < n$ have been replaced by $X_{j+1} \wedge \tau$ with $\tau \ge b_j^X$. Note that (77) and (78) hold with equality, and hence so does (80). Clearly $V_2^Y = V_2^X$, so assuming $V_j^Y = V_j^X$ for $2 \le j < n$, we have, taking the non-trivial case of j for which $Y_{j+1} = X_{j+1} \wedge \tau$

$$\begin{aligned} V_{j+1}^Y &= E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^Y] = E[V_j^{Y,1}(X_{j+1} \wedge \tau) \wedge V_j^Y] = E[V_j^{X,1}(X_{j+1}) \wedge V_j^{X,1}(\tau) \wedge V_j^X] \\ &= E[V_j^{X,1}(X_{j+1}) \wedge V_j^{X,1}(\tau) \wedge V_j^{X,1}(b_j^X)] = E[V_j^{X,1}(X_{j+1}) \wedge V_j^{X,1}(b_j^X)] = V_{j+1}^X. \blacksquare \end{aligned}$$

Let now $X_n, \ldots, X_1$ be i.i.d. as X with distribution function F satisfying the hypotheses of Theorem 1.1. Without loss of generality we may assume that the function L in (6) satisfies $\lim_{x \downarrow 0} L(x) = 1$, since if $F_X(x) = x^\alpha L_\mathcal{L}(x)$ with $\lim_{x \downarrow 0} L_\mathcal{L}(x) = \mathcal{L} \in (0, \infty)$, then $Z = \mathcal{L}^{1/\alpha} X$ has distribution function $F_Z(z) = z^\alpha (1/\mathcal{L}) L_\mathcal{L}(\mathcal{L}^{-1/\alpha} z)$ with $\lim_{z \downarrow 0} (1/\mathcal{L}) L_\mathcal{L}(\mathcal{L}^{-1/\alpha} z) = 1$. Since $V_n^Z = \mathcal{L}^{1/\alpha} V_n^X$, we have

$$F_Z(V_n^Z) = F_X(V_n^X),$$

and hence we can assume that X has distribution function F such that

$$F(x) = x^\alpha L(x) \quad \lim_{x \downarrow 0} L(x) = 1. \tag{81}$$

Corollary 6.1 *Let $X_n, \ldots, X_1$ be a sequence of i.i.d. non-negative random variables with $E[X_2 \wedge X_1] < \infty$ and distribution function satisfying (81). Then there exists an i.i.d. sequence $Y_n, \ldots, Y_1$ of bounded non-negative random variables with distribution function satisfying (81) such that $V_n^Y = V_n^X$ for all $n \ge 2$.*

Proof: Assume $x_F = \infty$, else there is nothing to prove. For all $x > 0$ sufficiently small, using the non-degeneracy of the distribution F on $[0, x]$, Jensen's inequality applied to the concave function $\psi(u) = u \wedge x$ yields

$$E[x \wedge X_1] \le x \wedge EX_1, \quad \text{with strict inequality for all } x \text{ sufficiently small.}$$

Thus

$$E[X_2 \wedge X_1 | X_2] \le X_2 \wedge EX_1, \quad \text{with strict inequality having positive probability}$$

and therefore, since $V_1^1(\infty) = EX_1$ (which may be infinite),

$$0 < V_2 = E[X_2 \wedge X_1] < E[X_2 \wedge EX_1] = V_2^1(\infty).$$

Using $x_F = \infty$ and Lemma 6.1, $V_2^1(x)$ is continuous and strictly monotone increasing on $(0, \infty)$, hence the solution b_2 to

$$V_2 = V_2^1(x)$$

exists, is unique, and satisfies $0 < b_2 < \infty$.

For $j = 1, \ldots, n$ and any $K \geq b_2$ let

$$Y_j = \begin{cases} X_j & \text{for } X_j \leq b_2 \\ K & \text{for } X_j > b_2. \end{cases}$$

Using Lemma 6.3 with $m = 2$ and $\tau = b_2$, we see that the two stop values of $X_j, \ldots, X_1$ and of $Y_j, \ldots, Y_3, X_2, X_1$ are the same for $2 \leq j \leq n$, i.e. $V_j^X = V_j^2(X_j, \ldots, X_1) = V_j^2(Y_j, \ldots, Y_3, X_2, X_1)$. Since the distribution of X_j is unbounded, $P(X_j > b_2) > 0$, which guarantees that $K \geq b_2$ can be chosen to yield $E[Y_2 \wedge Y_1] = E[X_2 \wedge X_1]$. But now, with the equality $V_j^2(Y_j, \ldots, Y_1) = V_j^2(Y_j, \ldots, Y_3, X_2, X_1)$ between the optimal two stop values on the sequences indicated now true for $j = 2$ by choice of K, assuming it true for $j \geq 2$ and using the notation as in the proof of Lemma 6.3 yields

$$\begin{aligned} V_{j+1}^Y &= V_{j+1}^2(Y_{j+1}, \ldots, Y_1) = E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^2(Y_j, \ldots, Y_1)] \\ &= E[V_j^{Y,1}(Y_{j+1}) \wedge V_j^2(Y_j, \ldots, Y_3, X_2, X_1)] = V_{j+1}^2(Y_{j+1}, \ldots, Y_3, X_2, X_1) = V_{j+1}^X. \end{aligned}$$

Since $P(X_j \leq x) = P(Y_j \leq x)$ for all $0 \leq x < b_2$, the distribution $P(Y_j \leq x)$ satisfies (81) and the bounded i.i.d. sequence $Y_n, \ldots, Y_1$ has all the claimed properties. ∎

We have the following Lemma.

Lemma 6.4 *Let X have distribution function $F(x) = P(X \leq x)$, and set*

$$F^{-1}(u) = \sup\{x : F(x) < u\} \quad \textit{for } 0 < u < 1.$$

Then

$$F(x) \geq u \quad \textit{if and only if} \quad x \geq F^{-1}(u), \tag{82}$$

and with $U \sim \mathcal{U}(0,1)$ we have

$$X =_d F^{-1}(U). \tag{83}$$

In addition, if

$$F(x) = x^\alpha L_F(x), \quad \textit{for all } x \geq 0, \textit{ with } \lim_{x \downarrow 0} L_F(x) = 1,$$

then there exists a function L^* such that

$$F^{-1}(u) = u^{1/\alpha} L_{F^{-1}}(u) = u^{1/\alpha} L^*(u^{1/\alpha}), \quad \textit{with} \quad \lim_{u \downarrow 0} L^*(u) = 1, \tag{84}$$

so that by (83) and (84),

$$X =_d U^{1/\alpha} L^*(U^{1/\alpha}). \tag{85}$$

Proof: Let $A_u = \{x : F(x) < u\}$. If $F(x) \geq u$ then $x \notin A_u$ and therefore $F^{-1}(u) \leq x$. If $F(x) < u$ then by right continuity there exists $\epsilon > 0$ such that $F(x+\epsilon) < u$. Thus $x + \epsilon \in A_u$, which gives that $F^{-1}(u) \geq x + \epsilon > x$. This demonstrates (82). Now replacing u by a random variable U having the $\mathcal{U}[0,1]$ distribution we obtain (83), by $P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$.

The claim in (84) is equivalent to

$$\lim_{u \downarrow 0} \frac{(F^{-1}(u))^\alpha}{u} = 1. \tag{86}$$

Using that $F(x) = x^\alpha L_F(x)$,

$$F^{-1}(u) = \sup\{x : x^\alpha L_F(x) < u\},$$

and hence, setting $L_\alpha(y) = L_F(y^{1/\alpha})$,

$$(F^{-1}(u))^\alpha = \sup\{x^\alpha : x^\alpha L_F(x) < u\} = \sup\{y : yL_F(y^{1/\alpha}) < u\} = \sup\{y : yL_\alpha(y) < u\}. \tag{87}$$

Note $yL_\alpha(y) = F(y^{1/\alpha})$ is non-decreasing. Let $\epsilon \in (0,1)$ be given. Since $\lim_{y \downarrow 0} L_\alpha(y) = 1$, there exists $\delta > 0$ such that

$$1 - \epsilon < L_\alpha(y) < 1 + \epsilon \quad \text{for all } 0 < y < \delta. \tag{88}$$

Let $0 < u < \delta(1-\epsilon)$. Then if $0 < y < u/(1+\epsilon)$ we have $y < \delta$ and so

$$yL_\alpha(y) < y(1+\epsilon) < u,$$

so

$$\{y : 0 < y < u/(1+\epsilon)\} \subset \{y : yL_\alpha(y) < u\}.$$

Thus

$$u/(1+\epsilon) \leq (F^{-1}(u))^\alpha \quad \text{for all } 0 < u < \delta(1-\epsilon).$$

Now, with $0 < u < \delta(1-\epsilon)$ and any $y \in (u/(1-\epsilon), \delta)$, by (88),

$$u < (1-\epsilon)y < yL_\alpha(y),$$

and it follows by (87) that

$$(F^{-1}(u))^\alpha \leq u/(1-\epsilon).$$

Hence,

$$1/(1+\epsilon) \leq \frac{(F^{-1}(u))^\alpha}{u} \leq 1/(1-\epsilon) \quad \text{for } 0 < u < \delta(1-\epsilon),$$

and (86) is shown. ∎

Lemma 6.5 *Let $\chi_n, n = 1, 2, \ldots$ be a uniformly integrable non-negative sequence of random variables, and $0 \leq L_n \leq L$, L a constant, with $L_n \to_p 1$ as $n \to \infty$. Then*

$$\limsup_{n \to \infty} E\chi_n L_n = \limsup_{n \to \infty} E\chi_n$$

so that in particular, if $\lim_{n \to \infty} E\chi_n$ exists,

$$\limsup_{n \to \infty} E\chi_n L_n = \lim_{n \to \infty} E\chi_n.$$

Proof: Let $\epsilon > 0$ be given. Since χ_n is uniformly integrable, there exists $\delta > 0$ such that

$$E\chi_n \mathbf{1}_A \leq \epsilon \quad \text{whenever} \quad P(A) \leq \delta. \tag{89}$$

Since $L_n \to_p 1$ as $n \to \infty$, there exists n_0 such that for all $n \geq n_0$

$$\Omega_n = \{|L_n - 1| \leq \epsilon\} \quad \text{satisfies} \quad P(\Omega_n) \geq 1 - \delta.$$

Hence, for $n \geq n_0$, using (89) and that $\chi_n \geq 0$, with $A = \Omega_n^c$,

$$(1-\epsilon)E\chi_n \mathbf{1}_{\Omega_n} \leq E\chi_n L_n \leq (1+\epsilon)E\chi_n \mathbf{1}_{\Omega_n} + L\epsilon \leq (1+\epsilon)E\chi_n + L\epsilon$$

and

$$E\chi_n - \epsilon \leq E\chi_n \mathbf{1}_{\Omega_n},$$

so that for $n \geq n_0$ we have

$$(1-\epsilon)(E\chi_n - \epsilon) \leq E\chi_n L_n \leq (1+\epsilon)E\chi_n + L\epsilon.$$

Taking lim sup and recalling $\epsilon > 0$ was arbitrary completes the proof. ■

Lemma 6.6 *Let $X_n, \ldots, X_1$ be an integrable i.i.d. sequence with distribution function $F(x)$ satisfying (81). Let $W_n^X = n^{1/\alpha} V_n^X$ and $W_n^{U^{1/\alpha}} = n^{1/\alpha} V_n^{U^{1/\alpha}}$. Then*

$$\limsup_{n\to\infty} W_n^X \leq \lim_{n\to\infty} W_n^{U^{1/\alpha}}.$$

Proof: Using Lemma 6.4, we construct i.i.d. pairs (U_i, X_i) with $U_i \sim \mathcal{U}, X_i \sim F$, and

$$X_i = U_i^{1/\alpha} L^*(U_i^{1/\alpha}).$$

By Corollary 6.1, without loss of generality we can take the X variables to be bounded, and since $L^*(u) \to 1$ as $u \downarrow 0$, it follows that L^* is bounded.

Let $\mathbf{2}_n^{U^{1/\alpha}}$ and $\mathbf{2}_n^X$ be the optimal random n-variable two-stop value for the $U_n^{1/\alpha}, \ldots, U_1^{1/\alpha}$ and $X_n, \ldots, X_1$ sequences respectively. Since $En^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} = n^{1/\alpha} V_n^{U^{1/\alpha}} = W_n^{U^{1/\alpha}}$ converges (to $h(b_\alpha)$), we have

$$P(\mathbf{2}_n^{U^{1/\alpha}} > \epsilon) = P(n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} > n^{1/\alpha}\epsilon) \leq \frac{W_n^{U^{1/\alpha}}}{n^{1/\alpha}\epsilon} \to 0 \quad \text{as } n \to \infty.$$

Hence $\mathbf{2}_n^{U^{1/\alpha}} \to_p 0$, and therefore $L^*(\mathbf{2}_n^{U^{1/\alpha}}) \to_p 1$. Furthermore, by Corollary 5.1, the collection $n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}}$ has a bounded r^{th} moment for some $r > 1$ and hence is uniformly integrable.

Let $\mathbf{2}_n^{X,U^{1/\alpha}}$ denote the X sequence stopped on the optimal rules for the $U^{1/\alpha}$ sequence. Then $\mathbf{2}_n^{X,U^{1/\alpha}} = \mathbf{2}_n^{U^{1/\alpha}} L^*(\mathbf{2}_n^{U^{1/\alpha}})$, and since these rules may not be optimal for the X sequence we have

$$En^{1/\alpha}\mathbf{2}_n^X \leq En^{1/\alpha}\mathbf{2}_n^{X,U^{1/\alpha}} = En^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}} L^*(\mathbf{2}_n^{U^{1/\alpha}}).$$

Taking limsup and using that $n^{1/\alpha}\mathbf{2}_n^{U^{1/\alpha}}$ is uniformly integrable and L^* is bounded and $L^*(\mathbf{2}_n^{U^{1/\alpha}}) \to_p 1$, the result follows from Lemma 6.5 and the fact that $W_n^{U^{1/\alpha}}$ converges. ■

Lemma 6.7 *Let $X_n, \ldots, X_1$ be i.i.d. random variables with distribution function F satisfying (81). Then the indifference values b_n for X satisfy*

$$\lim_{n\to\infty} b_n = 0.$$

Proof: Let $V_n^1(x) = V_n^1(X_n, \ldots, X_1; x)$ and $V_n^1(X_n, \ldots, X_1)$ denote the optimal one stop value on $X_n, \ldots, X_1$ with and without the guaranteed bound of x, respectively. Note that trivially for $k = 1$ we have that

$$V_k^1(X_k, \ldots, X_1; x) = V_k^1(X_k \wedge x, \ldots, X_1 \wedge x),$$

and assuming it true for some k, $1 \le k < n$ and using $V_k^1(X_k, \ldots, X_1; x) = V_k^1(x) \le x$ gives

$$\begin{aligned} & V_{k+1}^1(X_{k+1}, \ldots, X_1; x) = E[X_{k+1} \wedge V_k^1(X_k, \ldots, X_1; x)] = E[(X_{k+1} \wedge x) \wedge V_k^1(X_k, \ldots, X_1; x)] \\ = \;& E[(X_{k+1} \wedge x) \wedge V_k^1(X_k \wedge x, \ldots, X_1 \wedge x)] = V_{k+1}^1(X_{k+1} \wedge x, \ldots, X_1 \wedge x). \end{aligned}$$

Since b_n is monotone non-increasing by Lemma 6.2, $b_n \downarrow b \ge 0$, and we have

$$V_n^1(X_n \wedge b, \ldots, X_1 \wedge b) = V_n^1(b) \le V_n^1(b_n) = V_n^X. \tag{90}$$

Hence the two choice value V_n^X on $X_n, \ldots, X_1$ is greater (worse) than the optimal one choice value of the sequence of i.i.d. random variables $b \wedge X_n, \ldots, b \wedge X_1$. If $b > 0$, by (5), the limit of the scaled optimal one choice value of this sequence, $W_n^{X \wedge b, 1}$ say, is the same as the limit of $W_n^{X,1}$, the scaled optimal one choice value for $X_n, \ldots, X_1$. But then, using (90) in the first inequality, Lemma 6.6 for the second inequality, Theorem 5.1 for the equality, (58) for the strict inequality and the results of Kennedy and Kertz (1991) for the last two equalities we have

$$\lim_{n\to\infty} W_n^{X,1} \le \limsup_{n\to\infty} W_n^X \le \lim_{n\to\infty} W_n^{U^{1/\alpha}} = h(b_\alpha) < (1 + 1/\alpha)^{1/\alpha} = \lim_{n\to\infty} W_n^{U^{1/\alpha},1} = \lim_{n\to\infty} W_n^{X,1},$$

a contradiction. ∎

Lemma 6.8 *Let (U_i, X_i), $i = n, \ldots, 1$ be independent pairs of random variables with U_i uniform on $[0,1]$ and X_i having distribution function F satisfying (81). Let $V_{n,m}$ be defined as in (68), giving in particular $V_{n,n} = V_n^X$. Then for every $\epsilon \in (0,1)$, there exists m such that*

$$\frac{1}{1+\epsilon} \le \liminf_{n\to\infty} \frac{V_{n,m}}{V_{n,n}} \le \limsup_{n\to\infty} \frac{V_{n,m}}{V_{n,n}} \le \frac{1}{1-\epsilon}. \tag{91}$$

Proof: Using (85) of Lemma 6.4, we can construct the i.i.d. X sequence using an i.i.d. sequence $U^{1/\alpha}$ with distribution $\mathcal{U}^\alpha$ by defining X_i as

$$X_i = U_i^{1/\alpha} L^*(U_i^{1/\alpha}) \quad \text{a.s.} \tag{92}$$

where $\lim_{u\downarrow 0} L^*(u) = 1$. Hence, for the given $\epsilon \in (0,1)$ there exists $\delta > 0$ such that

$$1 - \epsilon \le L^*(u^{1/\alpha}) \le 1 + \epsilon \quad \text{for } 0 < u \le \delta, \tag{93}$$

and so by (92) and (93) we have

$$(1+\epsilon)^{-1}X_i \le U_i^{1/\alpha} \le (1-\epsilon)^{-1}X_i \quad \text{when } U_i \le \delta.$$

By condition (81), F is continuous at 0 and satisfies $F(0)=0$, and therefore there exists $\rho > 0$ with $0 < F(\rho) \le \delta$. But by (82), since

$$U_i \le F(\rho) \quad \text{if and only if} \quad X_i \le \rho,$$

we have

$$\text{if} \quad X_i \le \rho \quad \text{then} \quad U_i \le \delta.$$

Let $\tau = \min\{\delta,\rho\}$, and b_n^X and $b_n^{U^{1/\alpha}}$ be the indifference values for the X and $U^{1/\alpha}$ variables, respectively, which by Lemma 6.7 converge monotonically to zero. Hence there exists m with

$$\max\{b_m^{U^{1/\alpha}}, b_m^X\} \le \tau,$$

and for all $n \ge m$, by Lemma 6.3,

$$\begin{aligned}
& (1+\epsilon)^{-1}V_n(X_n,\ldots,X_1) \\
= \ & (1+\epsilon)^{-1}V_n(X_n\wedge\tau,\ldots,X_{m+1}\wedge\tau,X_m,\ldots,X_1) \\
= \ & V_n((1+\epsilon)^{-1}(X_n\wedge\tau),\ldots,(1+\epsilon)^{-1}(X_{m+1}\wedge\tau),(1+\epsilon)^{-1}X_m,\ldots,(1+\epsilon)^{-1}X_1) \\
\le \ & V_n(U_n^{1/\alpha}\wedge\tau,\ldots,U_{m+1}^{1/\alpha}\wedge\tau,X_m,\ldots,X_1) \\
= \ & V_n(U_n^{1/\alpha},\ldots,U_{m+1}^{1/\alpha},X_m,\ldots,X_1) \\
= \ & V_n(U_n^{1/\alpha}\wedge\tau,\ldots,U_{m+1}^{1/\alpha}\wedge\tau,X_m,\ldots,X_1) \\
\le \ & V_n((1-\epsilon)^{-1}(X_n\wedge\tau),\ldots,(1-\epsilon)^{-1}(X_{m+1}\wedge\tau),(1-\epsilon)^{-1}X_m,\ldots,(1-\epsilon)^{-1}X_1) \\
\le \ & (1-\epsilon)^{-1}V_n(X_n\wedge\tau,\ldots,X_{m+1}\wedge\tau,X_m,\ldots,X_1) \\
= \ & (1-\epsilon)^{-1}V_n(X_n,\ldots,X_1).
\end{aligned}$$

Now dividing by $V_{n,n}$ we see that for all $n \ge m$,

$$\frac{1}{1+\epsilon} \le \frac{V_{n,m}}{V_{n,n}} \le \frac{1}{1-\epsilon},$$

completing the proof. ■

Proof of Theorem 1.1: Clearly, for all $0 \le m \le n$,

$$\frac{V_n(U_n^{1/\alpha},\ldots,U_1^{1/\alpha})}{V_n(X_n,\ldots,X_n)} = \frac{V_{n,0}}{V_{n,n}} = \frac{V_{n,0}}{V_{n,m}}\frac{V_{n,m}}{V_{n,n}}.$$

Given $\epsilon \in (0,1)$, let m be such that (91) holds. But for any fixed m we have by Theorem 5.1 that

$$\lim_{n\to\infty}\frac{V_{n,0}}{V_{n,m}} = 1.$$

Hence by Lemma 6.8,

$$\frac{1}{1+\epsilon} \le \liminf_{n\to\infty}\frac{V_{n,0}}{V_{n,n}} \le \limsup_{n\to\infty}\frac{V_{n,0}}{V_{n,n}} \le \frac{1}{1-\epsilon},$$

and therefore the limit of the ratio exists and equals one. Applying Theorem 5.1 to the sequence $n^{1/\alpha}V_{n,0}$ completes the proof of Theorem 1.1. ■

7 Numerical Results and Additional Remarks

In Table 1, for the $\alpha = 0.1, 0.2, \ldots 1, 2, \ldots 10$ values in column (1), we tabulate the following quantities in the columns indicated:

(2) b_α

(3) $\lim_{n\to\infty} nF(V_n^1) = (1 + 1/\alpha)$

(4) $\lim_{n\to\infty} nF(V_n^2) = h^\alpha(b_\alpha) = d_\alpha^\alpha$ and

(5) $\lim_{n\to\infty} nF(V_n^p) = \Gamma(1 + 1/\alpha)^\alpha$,

for $F(x) = x^\alpha L(x)$ and $\lim_{x\to 0} L(x) = \mathcal{L}$ existing in $(0, \infty)$. In columns (6),(7), and (8), we tablulate the ratios (3)/(4), (4)/(5) and (3)/(5). Note that another natural comparison would be among the values listed raised to the power $1/\alpha$, as this would yield a comparison of the actual limiting values of $V_n^1/V_n^2, V_n^2/E(V_n^p)$ and $V_n^1/E(V_n^p)$ respectively. The reason that Table 1 lists the values in the way it does is to display them in a comparable order of magnitude to make numerical comparisons easier. The final column of Table 1 presents the relative improvement attained by using two stops rather than one, as compared to the reference value of the prophet,

$$\lim_{n\to\infty} (V_n^1 - V_n^2)/(V_n^1 - V_n^p). \tag{94}$$

As evident from the table, the improvement is highly significant for all values of α.

The following asymptotic results can be shown to hold:

(i) For $\alpha \to \infty$,

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(V_n^1) = 1$$

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(V_n^2) = 1 - 1/e$$

$$\lim_{\alpha\to\infty} \lim_{n\to\infty} nF(V_n^p) = e^{-\gamma}$$

where $\gamma = .5772\ldots$ is Euler's constant. The limiting value for the relative improvement (94) given in the last column is

$$[1 - \log(e - 1)]/\gamma = 0.7946\ldots$$

(ii) For $\alpha \to 0$,

The quantities in columns (3), (4) and (5) all tend to infinity, but the ratios in columns (6),(7),(8) and (9) tend to a finite limit, and are respectively

$$\lim_{\alpha\to 0} \lim_{n\to\infty} \frac{nF(V_n^1)}{nF(V_n^2)} = 2$$

$$\lim_{\alpha\to 0} \lim_{n\to\infty} \frac{nF(V_n^2)}{nF(V_n^p)} = e/2 = 1.3591\ldots$$

Table 1: Limiting Values of $nF(V_n^1), nF(V_n^2), nF(V_n^p)$, and their ratios.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
α	b_α	$\lim nF(V_n^1)$	$\lim nF(V_n^2)$	$\lim nF(V_n^p)$	(3)/(4)	(4)/(5)	(3)/(5)	Eq. (94)
0.1	11.9312	11.0000	5.72334	4.52873	1.92195	1.26379	2.42894	.99868
0.2	6.8927	6.0000	3.20772	2.60517	1.87049	1.23129	2.30311	.97131
0.3	5.2004	4.3333	2.36372	1.94980	1.83327	1.21229	2.22245	.93248
0.4	4.3485	3.5000	1.93919	1.61670	1.80488	1.19947	2.16490	.90235
0.5	3.8342	3.0000	1.68310	1.41421	1.78242	1.19013	2.12132	.88102
0.6	3.4896	2.6667	1.51157	1.27776	1.76417	1.18298	2.08699	.86571
0.7	3.2423	2.4286	1.38853	1.17940	1.74902	1.17732	2.05916	.85460
0.8	3.0561	2.2500	1.29590	1.10506	1.73624	1.17270	2.03610	.84614
0.9	2.9107	2.1111	1.22362	1.04684	1.72530	1.16887	2.01665	.83958
1.0	2.7940	2.0000	1.16562	1.00000	1.71583	1.16562	2.00000	.83438
2.0	2.2634	1.5000	0.90214	0.78540	1.66270	1.14864	1.90984	.81217
3.0	2.0839	1.3333	0.81309	0.71207	1.63983	1.14186	1.87245	.80556
4.0	1.9934	1.2500	0.76825	0.67497	1.62707	1.13820	1.85193	.80252
5.0	1.9388	1.2000	0.74123	0.65255	1.61895	1.13590	1.83897	.80078
6.0	1.9023	1.1666	0.72316	0.63753	1.61324	1.13432	1.82994	.79967
7.0	1.8762	1.1429	0.71023	0.62677	1.60914	1.13317	1.82343	.79892
8.0	1.8566	1.1250	0.70052	0.61867	1.60592	1.13230	1.81839	.79831
9.0	1.8412	1.1112	0.69296	0.61236	1.60350	1.13162	1.81455	.79789
10.0	1.8291	1.1000	0.68689	0.60731	1.60147	1.13105	1.81134	.79756

$$\lim_{\alpha\to 0}\lim_{n\to\infty}\frac{nF(V_n^1)}{nF(V_n^p)} = e = 2.7182\ldots$$

The relative improvement (94) given in the last column can be shown to tend to 1.

Remark 7.1 *Though we have proven Theorem 1.1 for the case where $F(x) = x^\alpha L(x)$, $\alpha > 0$ and $L(x)$ having finite positive limit as $x \downarrow 0$, we believe it holds true for all $F \in \mathcal{D}(G^\alpha)$ of (4), that is, whenever $L(x)$ is slowly varying as $x \downarrow 0$.*

Remark 7.2 *The approach in the present paper can easily be applied to obtain the asymptotic behavior of the one-choice value (obtained in Kennedy and Kertz (1991) by a different method), when $F(x) = x^\alpha L(x)$ and $\lim_{x\downarrow 0} L(x) = \mathcal{L} \in (0,\infty)$. First assume that $X \sim \mathcal{U}^\alpha(x)$ as in (10). Then for the one choice value V_n^1, we have*

$$V_{n+1}^1 = E[X \wedge V_n^1] = \alpha\int_0^{V_n^1} x^\alpha dx + (1-(V_n^1)^\alpha)V_n^1.$$

Set $W_n^1 = n^{1/\alpha} V_n^1$, and make the change of variable $y = nx^\alpha$, as in Section 2. Now multiply by $n^{1/\alpha}$ to obtain

$$\left(\frac{n}{n+1}\right)^{1/\alpha} W_{n+1}^1 = \frac{1}{n}\int_0^{(W_n^1)^\alpha} y^{1/\alpha} dy + (1-(V_n^1)^\alpha) W_n^1$$

$$= \frac{1}{n}\int_0^n (W_n^1 \wedge y^{1/\alpha}) dy.$$

Thus W_n^1 satisfies (42) with $q(y) = y^{1/\alpha}$, and now Theorem 4.1 can be applied to yield that $W_n^1 \to q(b)$ where b is the unique root of

$$\int_0^y u^{1/\alpha} du + (1/\alpha - y) y^{1/\alpha} = 0,$$

giving $b = 1 + 1/\alpha$. Hence, $\lim_{n\to\infty} W_n^1 = (1+1/\alpha)^{1/\alpha}$, or, $\lim_{n\to\infty} nF(V_n^1) = (1+1/\alpha)$. The general result for the wider class of distribution functions mentioned now follows in a manner similar to, but simpler than, the calculation for two choices.

Remark 7.3 *A similar approach can also be used to obtain the limiting value for more than 2 choices. For three choices one must first obtain the function $h^{(3)}(y)$ which replaces the function $h^{(2)}(y) = h(y)$ of (8). (Note that by Remark 7.2, $h^{(1)}(y) = y^{1/\alpha}$).*

Remark 7.4 *Our results translate easily to the case where the statistician is given two choices and his goal is to pick as large a value as possible, his payoff being the expectation of the larger of the two values chosen. Denote the optimal two-choice value based on n i.i.d. observations by $\tilde{V}_n^2$. Then for $X \sim F(x)$, where $x_F < \infty$, and*

$$F_X(x) = 1 - (x_F - x)^\alpha L(x_F - x)$$

where $L(\cdot)$ satisfies $lim_{y\downarrow 0} L(y) = \mathcal{L}$ and $0 < \mathcal{L} < \infty$, we have

$$\lim_{n\to\infty} n[1 - F(\tilde{V}_n^2)] = h^\alpha(b_\alpha).$$

8 Final Remarks

The last two authors are very saddened to announce that our invaluable colleague and friend David Assaf passed away most suddenly on December 23^{rd} 2003 as this work was nearing completion. On that very day, in a last email from Prof. Assaf to us regarding the final touches on this manuscript, he wrote that he had some ideas and 'I will say more on this in a few days.' We regret on many levels that this work can now only remain more or less in its current form, without the benefit of those further comments, now forever lost, which would have certainly greatly improved the work.

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