Nonlinear dynamics of the Schrödinger equation with periodic boundary conditions

Emanuele Haus

In this course I will present some ideas and results concerning the dynamics of semilinear Schrödinger equations with periodic boundary conditions, of the form

$$i\partial_t u(t,x) = -\Delta_x u(t,x) \pm |u|^{2p} u(t,x), \qquad x \in \mathbb{T}^d, \quad d = 1,2$$

where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ and p is a positive integer.

I will mainly focus on the existence of two types of solutions:

1. Solutions with a recurrent behavior, exhibiting a periodic or quasi-periodic exchange of energy between several Fourier modes, as studied in [2, 7, 8].
2. Unstable solutions that undergo a large growth of Sobolev norms, like the ones constructed in [1, 3–6].

These two types of solutions have a common feature: even if their amplitude is small, they display a genuinely nonlinear behavior, deviating significantly from the dynamics of the free Schrödinger equation.

References

[1] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1):39–113, 2010.

[2] B. Grébert and L. Thomann. Resonant dynamics for the quintic nonlinear Schrödinger equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(3):455–477, 2012.

[3] M. Guardia, Z. Hani, E. Haus, A. Maspero, and M. Procesi. Strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation. *ArXiv e-prints*, October 2018.

[4] M. Guardia, E. Haus, and M. Procesi. Growth of Sobolev norms for the analytic NLS on $\mathbb{T}^2$. *Adv. Math.*, 301:615–692, 2016.

[5] M. Guardia and V. Kaloshin. Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation. *J. Eur. Math. Soc. (JEMS)*, 17(1):71–149, 2015.

[6] E. Haus and M. Procesi. Growth of Sobolev norms for the quintic NLS on T^2. *Anal. PDE*, 8(4):883–922, 2015.

[7] E. Haus and M. Procesi. KAM for beating solutions of the quintic NLS. *Comm. Math. Phys.*, 354(3):1101–1132, 2017.

[8] E. Haus and L. Thomann. Dynamics on resonant clusters for the quintic non linear Schrödinger equation. *Dyn. Partial Differ. Equ.*, 10(2):157–169, 2013.