

Price Calculation for Power Exponential Jump-diffusion Models – A Hermite-series approach.

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ABSTRACT. In this paper we study convolution formulae for the independent sum of a normal random variable and several power exponential distributed random variables. This problem is motivated by the numerical simulation for pricing financial derivatives (such as options) when the underlying assets follow a jump-diffusion model in which the logarithm of the jump sizes are assumed to be within the class of power exponential distributions. When the “*kurtosis parameter*” (denoted by β) of the power exponential distribution equals 1 and $\frac{1}{2}$, the power exponential distribution becomes a standard normal and a double exponential distribution, respectively. Therefore our model contains those of Merton [M] and Kuo-Wang [K-W] as special cases. We propose a closed form convolution formula, represented in terms of infinite series expanded using either Hermite polynomials or parabolic cylindrical functions, depending on the value of kurtosis parameter β . We also analyze the convergence of such series, and perform the numerical experiments to illustrate these formulae.

1. Introduction

In this paper we study series forms of convolution formulae for a normal random variable and several i.i.d. power exponential random variables. To be more precise, we represent the convolution formulae of such independent sums as infinite series of either Hermite polynomials or parabolic cylindrical functions. Compared to the usual integral form of convolution formula, the series form has obvious an advantage in numerical simulation because of its explicit nature. This advantage will become more significant when the number of random variables involved increases, as we shall see in this paper.

The study of such convolution formula was motivated by the numerical simulation and/or calculation of asset prices and their derivatives when the dynamics of the asset prices are assumed to follow jump diffusion models. Recall that the jump

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diffusion model was first considered by Merton (1976) [M], in which the logarithm of the jump sizes are assumed to have normal distributions. It is by now well-understood that the Black-Scholes-Merton model does not incorporate the asymmetric leptokurtic natures (heavy tail and skew to the left) of the return distribution and the so-called “volatility smile”. On the other hand, it is also understood that while the normal distribution is often used by “default” for any random quantity with unknown statistics, the statistical analysis based on the normal distribution often lacks robustness. In a recent work Kuo and Wang (2001) [K-W] considered jump diffusion models assuming that the logarithm of the jump sizes have the “double exponential distributions”. In that paper the authors noted that by replacing the normal distribution by the double exponential, one captures several fundamental features such as skewness, heavy tails, and kurtosis, as it is observed from the empirical investigations. In that paper the authors also emphasize that one of the main reasons that they choose double exponential (instead of t -distribution, another obvious robust class) is the analytical tractability. In fact, using the double exponential distribution the authors were able to derive some closed-form formulae for pricing various types of options, including some path-dependent ones, such as look-back options and barrier options.

At this point we should note that both the normal distribution and double exponential distribution are special cases of the family of *power exponential distributions*, established by Subbotin in 1923 [S] as an extension of the normal distribution (see also Box-Tiao (1965) [B-T], Hogg (1974) [Ho], Rahman, et al. (1995)[R], Agrò (1995) [A], Gokhale-Rahman (1996)[Go-R], and Gómez-Gómez-Villegas-Marín (1998) [G-G-M], for properties and applications of such distributions). The main purpose of this work is to see whether one can develop a class of jump diffusion models based upon the general power exponential distributions, and compare all the outcomes so as to choose one to “best fit” the empirical data. In fact, it would be rather interesting if one can actually prove or disprove that the double exponential distribution is “optimal” at least among the family of the power exponential distributions to match the fundamental features of Skewness, heavy tail, and kurtosis.

As it turns out, an analysis similar to that of [K-W] is possible for power exponential distributions. The main difficulty, however, lies in the explicit formula for the convolution of one normal random variable and a (finite) sequence of i.i.d. power exponential distributions, which leads to this paper. We should note that finding the series form of convolution formula is not only useful for the closed-form solution, but also important in numerical simulation. In fact, it is expected that it is much more efficient than the Monte Carlo simulation for the traditional integral form convolution formulae, especially when the number of power exponential random variables increases, since the main ingredient of the infinite series, the Hermite polynomials or parabolic cylindrical functions, can be computed off line.

We would like to point out that our series representation of the convolution formulae depends on the kurtosis parameter (“ β ”) of the power exponential distributions. We show that if $\beta > 1$, then the series can be expressed in terms of Hermite polynomials, but if $\beta < 1$ the series would be better expressed in terms of *parabolic cylinder functions* (PCF). Since an integer-indexed PCF can be related to a Hermite polynomial explicitly, we shall call both of them “*Hermite-series*” for simplicity. It is interesting to note that the double exponential distribution actually

corresponds to the case when $\beta = 1/2$, for which we would use PCF series which seems to be much more complicated than that of [K-W]. We show that such a series can be reduced to the formula of [K-W] rather easily, with a little help from the properties of PCF's.

This paper is organized as follows. In section 2 we give all the definitions and basic properties of power exponential distributions, Hermite polynomials, and PCF's. In section 3 we describe the pricing problems for jump diffusion models involving power exponential distributions, and introduce our "convolution problems" $NP(1)$ and $NP(n)$, etc. In section 4 we give the series solutions to the problem $NP(1)$, and discuss their convergence. In section 5 we study two special cases when a power distribution is degenerated to a normal or a double exponential ($\beta = 1$ and $\frac{1}{2}$, respectively). Finally in section 6 we give a "formal" series solution for $NP(n)$, and in section 7 we show the numerical results.

2. Preliminaries

In this section we review some important facts regarding one dimensional power exponential distribution, Hermite Polynomials, and Cylinder Parabolic functions, which will play fundamental roles in the rest of the paper.

A. Power exponential distribution

A random variable X is said to have *Power exponential distribution* if its probability density function is given by:

$$(2.1) \quad f(x) = \frac{1}{\phi \Gamma\left(1 + \frac{1}{2\beta}\right) 2^{1 + \frac{1}{2\beta}}} e^{-\frac{1}{2} \left|\frac{x-\mu}{\phi}\right|^{2\beta}}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$, $\phi \in (0, \infty)$, and $\beta \in (0, \infty)$ are traditionally called the *location parameter*, *scale parameter*, and *kurtosis parameter*, respectively. We shall denote $X \sim P(\beta, \mu, \phi)$ to indicate that X has a power exponential distribution with parameters (β, μ, ϕ) .

We note that the power exponential distribution belongs to the family of symmetrical distributions. Two special cases of such distributions are well-understood:

- (i) $\beta = 1$: clearly $P(1, \mu, \phi) = N(\mu, \phi^2)$, the normal distribution;
- (ii) $\beta = \frac{1}{2}$: in this case we have the so-called *double-exponential* distribution.

In general, the parameter β indicates the disparity of a power exponential from a normal distribution. When β decreases, the "tail" of the distribution function gets "heavier". Therefore any $P(\beta, \mu, \phi)$ -distribution with $\beta < 1$ will have a heavier tail than a normal distribution.

B. Hermite polynomials

The family of Hermite polynomials plays an important role in determining the orthogonal basis for an L^2 -Gaussian space. Let $n \in \mathbb{N}$, the n -th *Hermite polynomial*, denoted by $H_n(x)$, is defined by

$$(2.2) \quad H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2),$$

and $H_0(x) \equiv 1$. The family of Hermite polynomials comes from the coefficients of the Taylor expansion (in variable t) of the function $G(x, t) \triangleq \exp(2xt - t^2)$. In fact,

$$(2.3) \quad G(x, t) = \exp\{2xt - t^2\} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}, \quad (t, x) \in \mathbb{R}^2.$$

Furthermore, using the identities

$$\begin{cases} \frac{\partial G}{\partial x} = 2tG, \\ \frac{\partial^2 G}{\partial x^2} - 2x \frac{\partial G}{\partial x} + 2t \frac{\partial G}{\partial t} = 0. \end{cases}$$

one can easily derive the recursive relation on $\{H_n(\cdot)\}$:

$$(2.4) \quad \begin{cases} H'_n(x) = 2nH_{n-1}(x), & n = 1, 2, 3, \dots, \\ H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0, & n = 0, 1, 2, \dots, \\ H_n(x) - 2xH_{n-1}(x) + 2(n-1)H_{n-2}(x) = 0 & n = 2, 3, 4, \dots, \end{cases}$$

with the “initial values”: $H_0 = 1$ and $H_1 = 2x$.

We list some of the properties that will be useful in the future (cf. e.g., Gradshteyn-Ryzhik [**Gr-R**]):

- For each $n \in \mathbb{N}$, one has

$$(2.5) \quad \begin{cases} H_{2n}(0) = (-1)^n 2^n (2n-1)!!, \\ H'_{2n}(0) = 0, \end{cases}$$

and

$$(2.6) \quad \begin{cases} H_{2n+1}(0) = 0, \\ H'_{2n+1}(0) = (-1)^n \frac{2^{2n+1}!}{n!}. \end{cases}$$

- The Hermite polynomials $H_n(x)$ are “orthogonal” in the weighted $L^2(\mathbb{R})$:

$$(2.7) \quad \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0, & n \neq m, \\ \sqrt{\pi} 2^n n!, & n = m. \end{cases}$$

- For any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, it holds that

$$(2.8) \quad |H_{2n}(x)| \leq 2^n (2n-1)!! e^{\frac{x^2}{2}}.$$

C. Parabolic Cylinder Functions

Parabolic Cylinder Functions (PCF) have been used in many fields such as Dirichlet problems in parabolic cylinder coordinates (half-integral order) and statistical thermodynamics, crystallography or lattice field theory (integral order). We are interested in them because of their relation to Hermite polynomials. We refer to [**Gr-R**], [**Ha**], and [**T**] for more detailed information on such functions.

By “Parabolic Cylinder Functions” we mean the solutions of the following second order ordinary differential equations with parameter p :

$$(2.9) \quad y'' + \left(p + \frac{1}{2} - \frac{1}{4}x^2\right) y = 0.$$

Let $p \in \mathbb{R}$, and we consider PCF’s, denoted by D_p , that have closed-form formulae. Such closed-form presentation will facilitate the numerical experiment

tremendously. To write down the explicit formula let us introduce some auxiliary functions. Let $\Gamma(\alpha)$ be the usual Γ -function, and denote

$$(a)_n \triangleq \Gamma(a+n)/\Gamma(a), \quad n = 0, 1, 2, \dots$$

The so-called *confluent hypergeometric function* is defined by

$${}_1F_1(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}.$$

Next, we define the following pair of auxiliary functions:

$$\begin{aligned} y_1(a, z) &= e^{-(1/4)z^2} {}_1F_1\left(-\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}; -\frac{1}{2}z^2\right) = e^{(1/4)z^2} F_1\left(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}; -\frac{1}{2}z^2\right), \\ y_2(a, z) &= ze^{-(1/4)z^2} {}_1F_1\left(\frac{1}{2}a + \frac{3}{4}, \frac{3}{2}; \frac{1}{2}z^2\right) = ze^{(1/4)z^2} {}_1F_1\left(-\frac{1}{2}a + \frac{3}{4}, \frac{3}{2}; -\frac{1}{2}z^2\right). \end{aligned}$$

The Parabolic Cylinder Functions D_p can then be written as follows (see, e.g., Temme [T]):

$$(2.10) \quad D_p(z) = \sqrt{\pi} 2^{\frac{p}{2} + \frac{1}{4}} \left\{ \frac{2^{-\frac{1}{4}} y_1(-p + \frac{1}{2}), z}{\Gamma(\frac{1}{2} - \frac{p}{2})} - \frac{2^{\frac{1}{4}} y_2(-p + \frac{1}{2}), z}{\Gamma(-\frac{p}{2})} \right\}.$$

It is fairly easy to check that the following recursive relations hold for the functions D_p and D_{-p} :

$$(2.11) \quad \begin{cases} D_{p+1}(x) = xD_p(x) - pD_{p-1}(x), & p \in \mathbb{R} \\ \frac{d}{dx} D_p(x) = -\frac{x}{2} D_p(x) + pD_{p-1}(x), & p \in \mathbb{R} \\ \frac{d}{dx} D_p(x) = \frac{x}{2} D_p(x) - D_{p+1}(x). & p \in \mathbb{R}. \end{cases}$$

An important case, which is of particular interest to us, is the case when $p = n$, a natural number. In this case the PCFs have the following relations with the Hermite polynomials:

$$D_n(z) = -2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right).$$

Finally, we note that the parabolic cylinder function $D_p(z)$ has the following integral form for $p < 0$:

$$(2.12) \quad D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^{\infty} t^{-p-1} e^{-tz - \frac{t^2}{2}} dt.$$

3. Problem Formulation

In this section we study the jump diffusion models that motivated the convolution formulae that we are interested in. We should note that our framework is almost parallel to the one proposed by Kou-Wang [K-W], except for the assumption on the logarithm of the jump sizes. Let $(\Omega, \mathcal{F}, \mathbf{P}; \{\mathcal{F}_t\})$ be a complete filtered probability space on which is defined a Brownian motion W and a compound Poisson process J , both adapted to the filtration $\{\mathcal{F}_t\}$. More precisely, we assume that the process J takes the following form:

$$(3.1) \quad J_t = \sum_{j=1}^{N_t} (V_j - 1), \quad t \geq 0,$$

where $N = \{N_t\}$ is a standard Poisson process with rate λ , and $\{V_j\}$ is a sequence of i.i.d. nonnegative random variables. We assume that

- (i) for each j , $X_j = \log(V_j)$ has a power exponential distribution with density given in (2.1);
- (ii) the processes W , N , and X_j 's are independent;
- (iii) $\mathcal{F}_t = \sigma\{W_s, J_s : 0 \leq s \leq t\}$, $t \geq 0$, augmented under \mathbf{P} so that it satisfies the *usual hypotheses* (cf. e.g., [P]).

In our jump diffusion model we assume that all the economics have a finite horizon $[0, T]$, and the price of our underlying risky asset is given by the following stochastic differential equation (SDE):

$$(3.2) \quad \frac{dS_t}{S_t} = \mu dt + \sigma dW_t + dJ_t = \mu dt + \sigma dW_t + d \left(\sum_{i=1}^{N_t} (V_i - 1) \right).$$

We assume that the drift μ and the volatility σ are constants. It is then well-known (see, e.g., Protter [P]) that the solution to the SDE (3.2) is given by the Doléans-Dade stochastic exponential

$$(3.3) \quad S_t = S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\} \prod_{i=1}^{N_t} V_i.$$

Consequently, the return process $Z_t = \log(S_t/S_0)$ is given by

$$(3.4) \quad Z_t = \left\{ \mu - \frac{1}{2} \sigma^2 \right\} t + \sigma W_t + \sum_{i=1}^{N_t} V_i, \quad Z_0 = 0.$$

In light of the results of Kuo-Wang [K-W], we now consider the discretized version of (3.2)–(3.4), following the well-known Euler method.

Let $\pi : t_0 = 0 \leq t_1, \dots, t_n = T$ be any partition of $[0, T]$. Denote $\Delta_i = t_{i+1} - t_i$, and $|\pi| \triangleq \max_{0 \leq i \leq n-1} \{\Delta_i\}$, the mesh size of the partition. In what follows, for any process ξ , we denote $\Delta_i \xi = \xi_{t_{i+1}} - \xi_{t_i}$, $i = 0, 1, \dots, n-1$. Consider the following discretized version of (3.2): $S_0^\pi = S_0$, and for $i = 0, 1, \dots, n-1$, define

$$(3.5) \quad S_{t_{i+1}}^\pi = S_{t_i}^\pi + S_{t_i}^\pi \{ \mu \Delta_i + \sigma [\Delta_i W] + \Delta_i J \}.$$

Noting the definition of the process J (see (3.1)), we derive easily that

$$(3.6) \quad \frac{\Delta_i S^\pi}{S_{t_i}^\pi} = \mu \Delta_i + \sigma [\Delta_i W] + \left\{ \sum_{j=N_{t_i}+1}^{N_{t_{i+1}}} (V_j - 1) \right\}.$$

Recall that $X = \log V$ (or $V = e^X$), we can approximate V by $1 + X$ so as to rewrite (3.6) as

$$(3.7) \quad \frac{\Delta_i S^\pi}{S_{t_i}^\pi} = \mu \Delta_i + \sigma [\Delta_i W] + \left\{ \sum_{j=N_{t_i}+1}^{N_{t_{i+1}}} X_j \right\}.$$

for $i = 0, \dots, n-1$. In what follows we shall consider only the equi-distant discretization, that is $t_i = \frac{iT}{n}$, $i = 0, 1, \dots, n$, so that $|\pi| = \Delta_i = \frac{T}{n}$, for all i .

Note that for each $1 \leq i \leq n$ the random variable $\Delta_i W \sim N(0, |\pi|)$, we can write $\Delta_i W = \sqrt{|\pi|} Z_i$, where Z_i 's are i.i.d. $N(0, 1)$ -random variables. The equation

(3.7) then becomes

$$(3.8) \quad \frac{\Delta_i S^\pi}{S_{t_i}^\pi} = \mu \Delta_i + \sigma \sqrt{|\pi|} Z_i + \left\{ \sum_{j=N_{t_i}+1}^{N_{t_i+1}} X_j \right\}.$$

As it was shown in [K-W], for Δ_i small enough we have

$$\sum_{j=N_{t_i}+1}^{N_{t_i+1}} X_j = \begin{cases} X_{N_{t_i}+\Delta_i} & \text{w.p. } \lambda \Delta_i, \\ 0 & \text{w.p. } 1 - \lambda \Delta_i. \end{cases}$$

In other words, if $\delta \triangleq |\pi|$ is sufficiently small, the return can be approximated in distribution by

$$(3.9) \quad \frac{\Delta S_t}{S_t} = \mu \delta + \sigma Z \sqrt{\delta} + B \cdot X$$

where B is a Bernoulli random variable with $\mathbf{P}(B = 1) = \lambda \delta$ and $\mathbf{P}(B = 0) = 1 - \lambda \delta$, and $Z \sim N(0, 1)$. Note that

$$\begin{aligned} \mathbf{P}\{\sigma \sqrt{\delta} Z + BX \leq x\} &= \mathbf{P}\{\sigma \sqrt{\delta} Z + X \leq x\} \mathbf{P}\{B = 1\} + \mathbf{P}\{\sigma \sqrt{\delta} Z \leq x\} \mathbf{P}\{B = 0\} \\ &= \mathbf{P}\{\sigma \sqrt{\delta} Z + X \leq x\} \lambda \delta + \mathbf{P}\{\sigma \sqrt{\delta} Z \leq x\} (1 - \lambda \delta). \end{aligned}$$

The problem is thus reduced to calculate the distribution of the random variable $\sigma \sqrt{\delta} Z + X$, an independent sum of a normal random variable and a power exponential random variable. In what follows we refer the problem of calculating the distribution of such a sum as *Problem NP(1)*, with “1” meaning that there is only one power exponentially random variable involved.

Let us now look at the option pricing problem. Following the idea of Merton [M] and/or Duffie [D], we shall start with the *risk-neutral* measure \mathbf{P}^* , under which the return process is rewritten as

$$(3.10) \quad \frac{dS_t}{S_t} = (r - \lambda \mathbf{E}(V - 1)) dt + \sigma dW_t + dJ_t = (\mu - \lambda \alpha) dt + \sigma dW_t + dJ_t,$$

where $\alpha = \mathbf{E}(e^X) - 1$, and r is the shot rate of the riskless asset. Note that the unique solution to the SDE (3.10) is given by

$$(3.11) \quad \begin{aligned} S_t &= S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 - \lambda \alpha \right) t + \sigma W_t \right\} \prod_{i=1}^{N_t} V_i \\ &= S_0 \exp \left\{ \left(r - \frac{1}{2} \sigma^2 - \lambda \alpha \right) t + \sigma \sqrt{t} Z + \sum_{i=1}^{N_t} X_i \right\}, \end{aligned}$$

where $Z \sim N(0, 1)$. For an option of the form $g(S_T)$ the hedging price at time 0 is given by

$$\begin{aligned} V_0 &= \mathbf{E}^* \left\{ e^{-\mu T} g \left(S_0 \exp \left\{ \left(r - \frac{\sigma^2}{2} - \lambda \alpha \right) T + \sigma \sqrt{T} Z \right\} \prod_{i=1}^{N_t} V_i \right) \right\} \\ &= \sum_{n=0}^{\infty} \mathbf{E}^* \left\{ e^{-rT} g \left(S_0 e^{(r - \frac{\sigma^2}{2} - \lambda \alpha) T + \sigma \sqrt{T} Z} \prod_{i=1}^{N_t} V_i \right) \middle| N_T = n \right\} \mathbf{P}\{N_T = n\} \\ &= e^{-rT} \sum_{m=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^m}{m!} \mathbf{E}^* \left\{ g \left(S_0 e^{-\lambda \alpha T} e^{(r - \frac{\sigma^2}{2}) T + \sigma \sqrt{T} Z + \sum_{i=1}^m X_i} \right) \right\}. \end{aligned}$$

Clearly, the calculation of the right hand side above would require the knowledge of the distribution of the random variable $\sigma\sqrt{T}Z + \sum_{i=1}^m X_i$, $m = 1, 2, \dots$, which are essentially the independent sums of one normal random variable and m independent power exponentially distributed random variables. In what follows we refer to such a problem as *Problem NP(m)*, $m = 2, 3, \dots$.

We remark that the solution to the Problem NP(m) is nothing but the $m + 1$ -fold convolution of a normal random variable and k independent power exponential random variables. Our main purpose is to find the convolution formulae in terms of either Hermite polynomials or parabolic cylinder functions so as to numerically simulate the underlying prices and calculate option prices. We should also note such series solution may not even converge in general. For this reason, we borrow the notation of the well-known Taylor series expansion. For example, we shall denote in general a Hermite polynomial expansion as

$$f(z) \cong \sum_{n=0}^{\infty} \alpha_n(z) H_n(\beta_n z)$$

or

$$f(z) \cong \sum_{n=0}^{\infty} \alpha_n(z) [D_{\beta_n}(z) + D_{\beta_n}(-z)],$$

and change the sign “ \cong ” to “ $=$ ” after we verify the convergence of the series. The essence here, however, is that with the explicit form of the series expansion, one can always use the partial sum to approximate the density function, even without actually proving the convergence(!).

To end this section we take a closer look at the quantity $\alpha \triangleq \mathbf{E}\{e^X\}$ which plays an important role in the formula (3.10). The following series expansion for α gives an idea for what we are trying to do in the rest of the paper.

PROPOSITION 3.1. *Suppose that $X \sim P(\beta, 0, \phi)$. Then for all $\beta \geq 1/2$ and $\phi < 1/2$ the following formula follows*

$$(3.12) \quad \mathbf{E}(e^X) = \frac{C(\phi, \beta)}{\beta} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!} 2^{\frac{2n+1}{2\beta}} \phi^{2n+1},$$

where

$$(3.13) \quad C(\phi, \beta) = \frac{1}{\phi \Gamma\left(1 + \frac{1}{2\beta}\right) 2^{1 + \frac{1}{2\beta}}}.$$

PROOF. Suppose $X \sim P(\beta, 0, \phi)$. Then its density function $f(x) = f_X(x)$ is given by (2.1) with normalizing constant $C(\phi, \beta)$. Assume that $\beta \geq 1/2$, then first using the Taylor expansion for e^x and then formally applying Fubini's theorem we have

$$\mathbf{E}(e^X) = \int_{\mathbb{R}} e^x f_X(x) dx \cong C(\phi, \beta) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} x^n e^{-\frac{1}{2\phi^{2\beta}} |x|^{2\beta}} dx.$$

Note that the integrals on the right hand side above are all absolutely convergent, and it holds that

$$\int_{\mathbb{R}} x^n e^{-\frac{1}{2\phi^{2\beta}} |x|^{2\beta}} dt = \begin{cases} 2 \int_0^{\infty} x^n e^{-\frac{1}{2\phi^{2\beta}} x^{2\beta}} dx = \Gamma\left(\frac{n+1}{2\beta}\right) \frac{1}{\beta} \phi^{n+1} 2^{\frac{n+1}{2\beta}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus we have

$$\mathbf{E}(e^X) \cong \frac{C(\phi, \beta)}{\beta} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!} 2^{\frac{2n+1}{2\beta}} \phi^{2n+1}.$$

To show that the equality actually holds in the above (whence (3.12)), first recall an important limit (see, e.g., [Gr-R])

$$(3.14) \quad \lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} z^{-a} = 1.$$

Let us define $b_n(\beta) \triangleq \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!} 2^{\frac{2n+1}{2\beta}} \phi^{2n+1}$. Then, applying the ratio test we have, for any n ,

$$(3.15) \quad \left| \frac{b_{n+1}(\beta)}{b_n(\beta)} \right| = \frac{2^{\frac{1}{\beta}} \phi^2}{(2n+2)(2n+1)} \frac{\Gamma\left(\frac{2n+3}{2\beta}\right)}{\Gamma\left(\frac{2n+1}{2\beta}\right)}.$$

Thus for $\beta > 1/2$, if we let $z = \frac{n}{\beta} + \frac{1}{2\beta}$ and $a = \frac{1}{\beta}$ in (3.14), then by (3.15) we obtain that

$$(3.16) \quad \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}(\beta)}{b_n(\beta)} \right| = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{\beta}} \phi^2 \left(\frac{n}{\beta} + \frac{1}{2\beta}\right)^{\frac{1}{\beta}}}{(2n+2)(2n+1)} \left\{ \left(\frac{n}{\beta} + \frac{1}{2\beta}\right)^{-\frac{1}{\beta}} \frac{\Gamma\left(\frac{2n+3}{2\beta}\right)}{\Gamma\left(\frac{2n+1}{2\beta}\right)} \right\} = 0.$$

Hence $\sum b_n(\beta) < \infty$, and the series in (3.12) converges for all $\beta > 1/2$.

Finally, we note that when $\beta = 1/2$, the limit in (3.16) is actually equal to $(2\phi)^2$, thus the series (3.12) converges for $\phi < 1/2$. \square

We remark that when $\beta = 1/2$ the series (3.12) is equal to $\frac{4\phi}{(1-(2\phi)^2)}$, and it coincides with the result in [K-W] for the case of “double exponential”. Since the double exponential distribution has the “heaviest tail” among all the power exponential distributions with $\beta \geq 1/2$, this result essentially shows that the double exponential distribution is more or less the farthest one can go in such jump diffusion models, whenever $\alpha = E[e^X] < \infty$ is satisfied.

4. Solution of Problem NP(1)

We now turn to the (Hermite) series solution to Problem NP(1). Let us assume that $X \sim P(\beta, \mu_1, \phi)$ and $Y \sim N(\mu_2, \sigma^2)$. That is, the density functions of X and Y are, respectively,

$$f_X(x) = \frac{1}{\phi \Gamma\left(1 + \frac{1}{2\beta}\right)} 2^{1 + \frac{1}{2\beta}} e^{-\frac{1}{2} \left| \frac{x - \mu_1}{\phi} \right|^{2\beta}} = C(\phi, \beta) e^{-\frac{1}{2} \left| \frac{x - \mu_1}{\phi} \right|^{2\beta}},$$

where $C(\phi, \beta)$ is given by (3.13), and

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y - \mu_2)^2}{2\sigma^2}}.$$

To simplify notation we assume from now on $\mu_1 = \mu_2 = 0$, the general case can be argued in the same way.

Recall that the complete (Hermite-series) solution of NP(1) consists of two parts: a closed form series representation and its convergence analysis. Since the convergence analysis is usually lengthy, and the series representation is sometimes

already sufficient for practical purposes, we thus present the results separately. We first give a formal Hermite-expansion theorem without studying the convergence.

THEOREM 4.1. *Suppose that $X \sim P(\beta, 0, \phi)$ and $Y \sim N(0, \sigma^2)$, and that X and Y are independent. Then the density function of $X + Y$ has the following series representation:*

(i) for $\beta > 1$,

$$(4.1) \quad f_{X+Y}(z) \cong \frac{C(\phi, \beta)}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!(2\sigma^2)^n} \phi^{2n+1} 2^{\frac{2n+1}{2\beta}} H_{2n}\left(\frac{z}{\sqrt{2\sigma}}\right);$$

(ii) for $0 < \beta < 1$,

$$(4.2) \quad f_{X+Y}(z) \cong \frac{C(\phi, \beta)}{\sqrt{2\pi}} e^{-\frac{z^2}{4\sigma^2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2\beta n + 1) \sigma^{2\beta n}}{n! 2^n \phi^{2\beta n}} \cdot \left[D_{-(2\beta n+1)}\left(\frac{z}{\sigma}\right) + D_{-(2\beta n+1)}\left(-\frac{z}{\sigma}\right) \right],$$

where H_n 's and D_μ 's are the Hermite polynomials and the parabolic cylinder functions, respectively.

PROOF. First assume $\beta > 1$. From (2.3) it is easily seen that

$$e^{\frac{1}{2\sigma^2}(-t^2+2zt)} = \sum_{n=0}^{\infty} \frac{t^n}{n!(\sqrt{2\sigma})^n} H_n\left(\frac{z}{\sqrt{2\sigma}}\right),$$

where H_n 's are the Hermite polynomials. Formally applying the Fubini theorem to the convolution formula we have

$$\begin{aligned} f_{X+Y}(z) &= \int_{\mathbb{R}} f_X(x) f_Y(z-x) dx \\ &\cong \frac{C(\phi, \beta)}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{H_n\left(\frac{z}{\sqrt{2\sigma}}\right)}{n!(\sqrt{2\sigma})^n} \int_{\mathbb{R}} t^n e^{-\frac{1}{2\phi^{2\beta}}|t|^{2\beta}} dt. \end{aligned}$$

We note that the integrals inside the summation on the right hand side above are obviously absolutely convergent, and they can be calculated explicitly as

$$\int_{\mathbb{R}} t^n e^{-\frac{1}{2\phi^{2\beta}}|t|^{2\beta}} dt = \begin{cases} 2 \int_0^{\infty} t^n e^{-\frac{1}{2\phi^{2\beta}}t^{2\beta}} dt = \Gamma\left(\frac{n+1}{2\beta}\right) \frac{1}{\beta} \phi^{n+1} 2^{\frac{n+1}{2\beta}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Consequently we have

$$f_{X+Y}(z) \cong \frac{C(\phi, \beta)}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!(2\sigma^2)^n} \phi^{2n+1} 2^{\frac{2n+1}{2\beta}} H_{2n}\left(\frac{z}{\sqrt{2\sigma}}\right),$$

proving (4.1).

Now let us assume $0 < \beta < 1$. As we will see in the next theorem, the usual technique used to prove the convergence of the series representation for $\beta > 1$ does not work for this case. Thus a different approach is in order. In this case let us first use the Taylor expansion for the exponential function

$$e^{-\frac{1}{2\phi^{2\beta}}|t|^{2\beta}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n \phi^{2\beta n}} |t|^{2\beta n}.$$

Formally applying Fubini's theorem again in the convolution formula we obtain that

$$(4.3) \quad f_{X+Y}(z) \cong \frac{C(\phi, \beta)}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n \phi^{2\beta n}} \int_{\mathbb{R}} |t|^{2\beta n} e^{-\frac{1}{2\sigma^2}(t^2-2tz)} dt.$$

Note that the integrals inside the summation on the right hand side are still absolutely convergent. In fact, we have the following closed form formula (see, for example, [Gr-R]): for $\mu > 0$, $\nu > 0$, and $x \in \mathbb{R}$,

$$(4.4) \quad I(x; \mu, \nu) \triangleq \int_0^{\infty} t^{\nu-1} e^{-\mu t^2 - xt} dt = (2\mu)^{-\frac{\nu}{2}} \Gamma(\nu) e^{\frac{x^2}{8\mu}} D_{-\nu} \left(\frac{x}{\sqrt{2\mu}} \right).$$

Thus the integral on the right hand side of (4.3) should read

$$(4.5) \quad \begin{aligned} & \int_{\mathbb{R}} |t|^{2\beta n} e^{-\frac{1}{2\sigma^2}(t^2-2tz)} dt \\ &= I\left(-\frac{z}{\sigma^2}; \frac{1}{2\sigma^2}, 2\beta n + 1\right) + I\left(\frac{z}{\sigma^2}; \frac{1}{2\sigma^2}, 2\beta n + 1\right) \\ &= \sigma^{2\beta n+1} \Gamma(2\beta n + 1) e^{\frac{z^2}{4\sigma^2}} \left[D_{-(2\beta n+1)} \left(-\frac{z}{\sigma} \right) + D_{-(2\beta n+1)} \left(\frac{z}{\sigma} \right) \right]. \end{aligned}$$

Plugging (4.5) into (4.3) we derive (4.2). \square

We now try to replace the sign “ \cong ” by “=” in (4.1) and (4.2). The following theorem explains why we need to use different representations for the cases $\beta > 1$ and $\beta < 1$.

THEOREM 4.2. *Suppose that X and Y are random variables as defined in Theorem 4.1. Then, the following convergence results hold:*

(i) *if $\beta > 1$ and $\phi \leq \sigma$, then the Hermite series (4.1) converges absolutely and uniformly in $z \in \mathbb{R}$;*

(ii) *if $0 < \beta < 1$ and $\sigma \leq \phi$, then the Hermite series (4.2) converges absolutely and uniformly in $z \in \mathbb{R}$;*

Consequently, in all cases above the “ \cong ” signs in both (4.1) and (4.2) can be replaced by equalities.

PROOF. (i) Assume $\beta > 1$ and $\phi \leq \sigma$. Denote the right hand side of (4.1) by $I_1(z)$. Then we have

$$(4.6) \quad \begin{aligned} |I_1(z)| &\leq \frac{C(\phi, \beta)}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!2^n} \left(\frac{\phi}{\sigma}\right)^{2n+1} 2^{\frac{2n+1}{2\beta}} \left| H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right) \right| \\ &\leq \frac{C(\phi, \beta)}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!2^n} 2^{\frac{2n+1}{2\beta}} \left| H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right) \right|. \end{aligned}$$

Thus it suffices to show the convergence of the series

$$(4.7) \quad \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!2^n} 2^{\frac{2n+1}{2\beta}} \left| H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right) \right| e^{-\frac{z^2}{2\sigma^2}}.$$

Note that the basic estimate of Hermite polynomial (2.8) implies that for all $z \in \mathbb{R}$,

$$\begin{aligned} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!2^n} 2^{\frac{2n+1}{2\beta}} \left| H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right) \right| e^{-\frac{z^2}{2\sigma^2}} &\leq \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!} 2^{\frac{2n+1}{2\beta}} (2n-1)!! e^{-\frac{z^2}{4\sigma^2}} \\ &\leq \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)}{(2n)!} 2^{\frac{2n+1}{2\beta}} (2n-1)!! \triangleq a_n(\beta). \end{aligned}$$

But note that

$$(4.8) \quad \left| \frac{a_{n+1}(\beta)}{a_n(\beta)} \right| = \frac{2^{\frac{1}{\beta}}}{(2n+2)} \frac{\Gamma\left(\frac{2n+3}{2\beta}\right)}{\Gamma\left(\frac{2n+1}{2\beta}\right)}.$$

Since $\beta > 1$, if we set $z = \frac{n}{\beta} + \frac{1}{2\beta}$ and $a = \frac{1}{\beta}$ in the identity (3.14), then from (4.8) we see that

$$(4.9) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(\beta)}{a_n(\beta)} \right| = \lim_{n \rightarrow \infty} \frac{2^{\frac{1}{\beta}} \left(\frac{n}{\beta} + \frac{1}{2\beta}\right)^{\frac{1}{\beta}}}{(2n+2)} \left\{ \left(\frac{n}{\beta} + \frac{1}{2\beta}\right)^{-\frac{1}{\beta}} \frac{\Gamma\left(\frac{2n+3}{2\beta}\right)}{\Gamma\left(\frac{2n+1}{2\beta}\right)} \right\} = 0.$$

Hence $\sum a_n(\beta) < \infty$, and the series in (4.7) converges absolutely and uniformly for all z , proving part (i).

(ii) $0 < \beta < 1$. Again we write the right hand side of (4.2) as $I_2(z)$, using the assumption $\sigma \leq \phi$ we then have

$$(4.10) \quad |I_2(z)| \leq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(2\beta n + 1) [|D_{-(2\beta n+1)}(z)| + |D_{-(2\beta n+1)}(-z)|] \\ \triangleq I_2^1(z) + I_2^2(z),$$

where

$$I_2^1(z) \triangleq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(2\beta n + 1) |D_{-(2\beta n+1)}(z)| \\ I_2^2(z) \triangleq \sum_{n=0}^{\infty} \frac{1}{2^n n!} \Gamma(2\beta n + 1) |D_{-(2\beta n+1)}(-z)|.$$

It then suffices to show that both $I_2^1(\cdot)$ and $I_2^2(\cdot)$ converge uniformly in z . It is readily seen that we need only check the uniform convergence of $I_2^1(z)$ for $z > 0$, thanks to the symmetry.

To this end, let $p = -(2\beta n + 1)$. By (2.12) we have

$$D_{-(2\beta n+1)}(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(2\beta n + 1)} \int_0^{\infty} t^{2\beta n} e^{-tz - \frac{t^2}{2}} dt.$$

Putting this into the right side of I_2^1 we can easily check that, for $z > 0$,

$$\begin{aligned} I_2^1(z) &\leq \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-\frac{z^2}{4}} \int_0^{\infty} t^{2\beta n} e^{-tz - \frac{t^2}{2}} dt \\ &\leq \sum_{n=0}^{\infty} \frac{1}{2^n n!} e^{-z^2/4} \int_0^{\infty} t^{2\beta n} e^{-t^2/2} dt \leq \sum_{n=0}^{\infty} \frac{2^{\frac{2\beta n-1}{2}} \Gamma\left(\frac{2\beta n+1}{2}\right)}{2^n n!}. \end{aligned}$$

Again, let us define

$$a_n(\beta) \triangleq \frac{2^{\frac{2\beta n-1}{2}} \Gamma\left(\frac{2\beta n+1}{2}\right)}{2^n n!}.$$

Then, applying (3.14) again (with $z = \frac{2\beta n+1}{2}$ and $a = \beta$) we get that, for $\beta < 1$,

$$\begin{aligned} (4.11) \quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(\beta)}{a_n(\beta)} \right| &= \lim_{n \rightarrow \infty} \frac{2^{\beta-1} \Gamma\left(\frac{2\beta n+1+2\beta}{2}\right)}{n+1 \Gamma\left(\frac{2\beta n+1}{2}\right)} \\ &\leq \lim_{n \rightarrow \infty} \frac{(\beta n + \frac{1}{2})^\beta}{(n+1)} \left\{ (\beta n + \frac{1}{2})^{-\beta} \frac{\Gamma\left(\frac{2\beta n+1+2\beta}{2}\right)}{\Gamma\left(\frac{2\beta n+1}{2}\right)} \right\} = 0. \end{aligned}$$

Thus, by the ratio test we see that $\sum a_n(\beta) < \infty$, and hence $I_2^1(z)$ converges absolutely and uniformly for $z > 0$. The proof is now complete. \square

5. Two special cases ($\beta = 1$ and $\beta = \frac{1}{2}$)

Observe that in Theorems 4.1 and 4.2 we did not discuss the case $\beta = 1$. In fact in this case neither argument works, since the limits in both ratio tests equal to 1(!). We shall nevertheless prove that in this case the Hermite series (4.1) still converges, and it is in fact the unique solution to the Cauchy problem of a second order ODE. Further, since the case $\beta = 1/2$ corresponds to the double exponential distribution, for which an explicit formula is given by Kuo-Wang [K-W], we shall prove in this section that our Hermite series expansion (4.2) produces exactly the same thing, although starting from a seemingly different formula.

5.1. The normal case ($\beta = 1$). We first note that when $\beta = 1$, $P(1, 0, \phi) = N(0, \phi^2)$, thus the solution to problem NP(1) is nothing but a normal distribution. Furthermore, since both X and Y are normal random variables in this case, in the convolution formula the role of σ and ϕ are interchangeable. Hence we might as well assume in this section that $\phi = \sigma = 1$. That is, $X \sim P(1, 0, 1) = N(0, 1)$ and $Y \sim N(0, 1)$, hence the independent sum $X + Y \sim N(0, 2)$. In other words, we have

$$(5.1) \quad f_{X+Y}(z) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}}.$$

On the other hand, setting $\beta = 1$, $\phi = 1$ in (4.1) we have

$$(5.2) \quad f_{X+Y}(z) \cong \frac{C(1, 1)}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{n=0}^{\infty} \frac{H_{2n}\left(\frac{z}{\sqrt{2}}\right)}{(2n)! 2^n} \Gamma\left(\frac{2n+1}{2}\right) 2^{\frac{2n+1}{2}}.$$

Since, by (3.13), we have

$$C(1, 1) = \frac{1}{\Gamma\left(1 + \frac{1}{2}\right) 2^{1+\frac{1}{2}}} = \frac{1}{\sqrt{2\pi}},$$

we have the following result.

THEOREM 5.1. *Suppose that X and Y are two independent $N(0, 1)$ -random variables. Then it holds that*

$$(5.3) \quad f_{X+Y}(z) = \frac{1}{2\sqrt{\pi}} e^{-\frac{z^2}{4}} = \frac{C(1, 1)}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{n=0}^{\infty} \frac{H_{2n}(\frac{z}{\sqrt{2}})}{(2n)!2^n} \Gamma\left(\frac{2n+1}{2}\right) 2^{\frac{2n+1}{2}}$$

$$(5.4) \quad = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \sum_{n=0}^{\infty} \frac{H_{2n}(\frac{z}{\sqrt{2}})}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right).$$

Furthermore, the series

$$\psi(u) = \sum_{n=0}^{\infty} \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) = \sqrt{\frac{\pi}{2}} e^{\frac{u^2}{2}}$$

is the unique solution of the following second order homogeneous ODE:

$$(5.5) \quad \begin{cases} \psi''(u) - u\psi'(u) - \psi(u) = 0, \\ \psi(0) = \sqrt{\frac{\pi}{2}}, \quad \psi'(0) = 0. \end{cases}$$

PROOF. First, letting $u = \frac{z}{\sqrt{2}}$, we see that it suffices to show the following identity:

$$(5.6) \quad \sum_{n=0}^{\infty} \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) = \sqrt{\frac{\pi}{2}} e^{\frac{u^2}{2}}.$$

We first analyze the particular case $u = 0$. By (2.5) we have

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!! \quad \text{and} \quad \Gamma\left(\frac{2n+1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!.$$

Therefore (5.6) is equivalent to

$$(5.7) \quad \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!! (2n-1)!!}{(2n)!} = \frac{1}{\sqrt{2}}.$$

To prove (5.7) let us make the following elementary observations. First, consider the Taylor expansion:

$$(5.8) \quad \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!n!} x^n, \quad -\frac{1}{4} \leq x < \frac{1}{4}.$$

Now, setting $x = \frac{-1}{4}$ in (5.8) and noting that $(2n)! = (2n-1)!!(2n)!! = (2n-1)!!2^n n!$, we obtain that

$$\frac{1}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2^n n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! ((2n-1)!!)^2}{((2n)!)^2}.$$

The identity (5.7) thus follows.

In the general case we shall make use of the properties of the Hermite polynomials given in (2.4), and the stability result of viscosity solutions (see, e.g., Fleming and Soner [F-S]). For any integer N , define

$$\psi_N(u) \triangleq \sum_{n=0}^N \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) = \sqrt{\pi} + \sum_{n=1}^N \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right).$$

Differentiating both sides above and using the recursive relation (2.4) we obtain that

$$(5.9) \quad \psi'_N(u) = \sum_{n=1}^N \frac{H'_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) = \sum_{n=1}^N 4n \frac{H_{2n-1}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right).$$

Differentiating (5.9) again and applying (2.4) we obtain

$$(5.10) \quad \begin{aligned} \psi''_N(u) &= \sum_{n=1}^N \frac{4n2(2n-1)H_{2n-2}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \\ &= \sum_{n=1}^N \frac{4n2(2n-1)(-H_{2n}(u) + 2uH_{2n-1}(u))}{(2n)!2(2n-1)} \Gamma\left(\frac{2n+1}{2}\right) \\ &= -\sum_{n=1}^N \frac{4nH_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) + 2u\psi'_N(u). \end{aligned}$$

On the other hand, it is easily seen that

$$(5.11) \quad \begin{aligned} \psi''_N(u) &= \sum_{n=1}^N \frac{4n2(2n-1)H_{2n-2}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \\ &= \sum_{n=1}^N \frac{2(2n-1)H_{2n-2}(u)}{(2n-2)!} \Gamma\left(\frac{2n-1}{2}\right) \\ &= 2 \sum_{n=1}^N \frac{H_{2n-2}(u)}{(2n-2)!} \Gamma\left(\frac{2n-1}{2}\right) + \sum_{n=1}^N \frac{4(n-1)H_{2n-2}(u)}{(2n-2)!} \Gamma\left(\frac{2n-1}{2}\right) \\ &= 2[\psi_N(u) - \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)] + \sum_{n=1}^{N-1} \frac{4nH_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right). \end{aligned}$$

Adding (5.10) and (5.11) and dividing both sides by 2 we obtain that

$$(5.12) \quad \psi''_N(u) = \psi_N(u) + u\psi'_N(u) - (2n+1) \frac{H_{2n-2}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right).$$

This amounts to saying that $\psi_N(\cdot)$ is a (classical) solution to the (non-homogeneous) ODE (5.12) for each N . Since

$$\alpha_N(u) \triangleq -(2n+1) \frac{H_{2n-2}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right) \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

uniformly in u on compacts, and $\psi_N(\cdot)$ converges to $\psi(u) \triangleq \sum_{n=0}^{\infty} \frac{H_{2n}(u)}{(2n)!} \Gamma\left(\frac{2n+1}{2}\right)$, uniformly near $u = 0$, we conclude from the stability of the viscosity solution that ψ is a viscosity solution to the ODE (5.5), at least near $u = 0$. But the uniqueness of the viscosity solution would then imply that the ψ must coincide with $\sqrt{\frac{\pi}{2}} e^{\frac{u^2}{2}}$, the unique solution of (5.5), whenever the series converges. The result then follows from some standard arguments using the extension of the solution for ODEs and analytic functions. \square

5.2. The double exponential case ($\beta=1/2$). In the case $\beta = 1/2$ the power exponential distribution $P(\frac{1}{2}, 0, \phi)$ becomes the “double exponential”, that is, the density function is given by

$$f_X(x) = \frac{1}{2\eta} e^{-\frac{|x|}{\eta}},$$

where $\eta = 2\phi$. Such a case was studied by Kuo-Wang [K-W]. In particular, in [K-W] it is proved that in this case the density function of $X + Y$ is given by the following formula

$$(5.13) \quad f_{X+Y}(t) = \frac{1}{\eta} e^{\sigma^2/(2\eta^2)} \left\{ \frac{1}{2} e^{-t/\eta} \Phi\left(\frac{t\eta - \sigma^2}{\sigma\eta}\right) + \frac{1}{2} e^{t/\eta} \Phi\left(-\frac{t\eta + \sigma^2}{\sigma\eta}\right) \right\}.$$

In what follows we show that our Hermite series representation (4.2) gives exactly the same formula. To simplify numerical calculations, we shall consider only the case when $\phi = 1/4$ and $\sigma = 1$. The general cases can be obtained by a simple change of variables. We have the following result.

THEOREM 5.2. *Suppose that $\beta = \frac{1}{2}$, $\phi = \frac{1}{4}$, and $\sigma = 1$. Then the Hermite series (4.2) takes the following form:*

$$(5.14) \quad \begin{aligned} & \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-(n+1)}(z) + D_{-(n+1)}(-z)] \\ & = e^2 (e^{-2z} \Phi(z-2) + e^{2z} \Phi(-z-2)), \end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Furthermore, if we define

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-(n+1)}(z) + D_{-(n+1)}(-z)],$$

then φ is the unique solution to the following second order ODE:

$$(5.15) \quad \begin{cases} \varphi''(z) = 4\varphi(z) - \frac{4}{\sqrt{2\pi}} e^{-z^2/2} \\ \varphi(0) = 2e^2 \Phi(-2), \quad \varphi'(0) = 0. \end{cases}$$

PROOF. First, setting $\beta = \frac{1}{2}$, $\phi = \frac{1}{4}$, and $\sigma = 1$ in (4.2), and noting that $C(\frac{1}{4}, \frac{1}{2}) = 1$ and $\Gamma(n+1) = n!$ we have

$$(5.16) \quad \begin{aligned} f_{X+Y}(z) &= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} \Gamma(n+1) [D_{-(n+1)}(z) + D_{-(n+1)}(-z)] \\ &= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-(n+1)}(z) + D_{-(n+1)}(-z)]. \end{aligned}$$

On the other hand, setting $\beta = \frac{1}{2}$, $\phi = \frac{1}{4}$, and $\sigma = 1$ in (5.13), we have

$$(5.17) \quad f_{X+Y}(z) = e^2 (e^{-2z} \Phi(z-2) + e^{2z} \Phi(-z-2)).$$

Thus the first conclusion follows from Theorem 4.2 and the result of [K-W].

We now prove that the function φ , defined as the right hand side of (5.16), satisfies the ODE (5.15). Again, we first analyze the case $z = 0$. In this case we note that

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n 2^{n+1} D_{-(n+1)}(0) = 2e^2 \Phi(-2).$$

In the general case we follow the same argument as in the previous theorem to show that the function

$$\varphi(u) \triangleq \frac{1}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-(n+1)}(u) + D_{-(n+1)}(-u)]$$

is at least the unique solution to the ODE (5.15). To see this, first note that by taking partial sum if necessary, we can (formally) differentiate the function φ to get

$$\begin{aligned} \varphi'(u) &= \frac{-u}{2} \varphi(u) + \frac{1}{\sqrt{2\pi}} e^{-u^2/4} \left(\sum_{n=0}^{\infty} (-1)^n 2^n \frac{u}{2} [D_{-(n+1)}(u) + D_{-(n+1)}(-u)] \right. \\ &\quad \left. - \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-n}(u) - D_{-n}(-u)] \right) \\ (5.18) \quad &= -\frac{1}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n [D_{-n}(u) - D_{-n}(-u)] \\ &= -2\varphi(u) + \frac{4}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n D_{-(n+1)}(u), \end{aligned}$$

and by using the recursive relation (2.11) to get

$$\begin{aligned} (5.19) \quad \varphi''(u) &= -2\varphi'(u) + \frac{4}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n D'_{-(n+1)}(u) \\ &\quad - \frac{2u}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n D_{-(n+1)}(u) \\ &= -2\varphi'(u) - \frac{4}{\sqrt{2\pi}} e^{-u^2/4} \sum_{n=0}^{\infty} (-1)^n 2^n D_{-n}(u) \\ (5.20) \quad &= -2\varphi'(u) - \frac{4}{\sqrt{2\pi}} e^{-u^2/4} \left(e^{-u^2/4} - 2 \sum_{n=0}^{\infty} (-1)^n 2^n D_{-(n+1)}(u) \right) \\ &= -2\varphi'(u) - \frac{4}{\sqrt{2\pi}} e^{-u^2/2} + e^{-u^2/4} \frac{8}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n 2^n D_{-(n+1)}(u). \end{aligned}$$

Combining (5.18) and (5.19) and following the stability arguments if needed, one shows that the function φ is at least a viscosity solution to the ODE (5.15). We can then repeat the same arguments as those in the previous theorem to conclude that φ must coincide with the unique analytic solution of (5.15), completing the proof. \square

6. Convolution formulae for the NP(m) cases

In the last part of this paper we shall give the series presentations of the solution to the Problem NP(m). As we will see, such a presentation will turn out to be quite complicated in its form, although the idea is rather straightforward. Consequently, we will pay more attention to the actually computational aspect of the convolution formulae, rather than the detailed convergence analysis. We shall

verify the convergence by performing the actual numerical simulation, and compare the results.

To begin with, let X be a normal random variable, and let Y_1, \dots, Y_m be m *i.i.d.* random variables with a same power exponential distribution. We will study the convolution formula for the random variable $Z = X + \sum_{i=1}^m Y_i$. We still consider the cases of $\beta > 1$ and $\beta < 1$ separately again.

Case 1. $\beta > 1$. First assume $m = 2$. Then applying Theorem 4.1 we have

$$\begin{aligned}
(6.1) \quad f_{X+Y_1+Y_2}(z) &= \int_{-\infty}^{\infty} f_{X+Y_1}(z-t)f_{Y_2}(t)dt \\
&\cong \frac{C^2(\phi, \beta)}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{|z-t|^2}{2\sigma^2}} \sum_{n=0}^{\infty} \frac{H_{2n}\left(\frac{z-t}{\sqrt{2}\sigma}\right)\Gamma\left(\frac{2n+1}{2\beta}\right)\phi^{2n+1}2^{\frac{2n+1}{2\beta}} e^{-\frac{1}{2}\left|\frac{t}{\phi}\right|^{2\beta}}}{(2n)!(2\sigma^2)^n} dt \\
&= \frac{C^2(\phi, \beta)}{\sqrt{\pi}\sigma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)\phi^{2n+1}2^{\frac{2n+1}{2\beta}}}{(2n)!(2\sigma^2)^n} \int_{-\infty}^{\infty} H_{2n}\left(\frac{z-t}{\sqrt{2}\sigma}\right) e^{-\frac{|z-t|^2}{2\sigma^2}} e^{-\frac{1}{2}\left|\frac{t}{\phi}\right|^{2\beta}} dt.
\end{aligned}$$

Note that

$$(6.2) \quad H_{2n}\left(\frac{z-t}{\sqrt{2}\sigma}\right) = \sum_{j=0}^n 2^{2n-j} \left(\frac{z-t}{\sqrt{2}\sigma}\right)^{2n-2j} \binom{2n}{2j} (2j-1)!!$$

and

$$e^{-\frac{|z-t|^2}{2\sigma^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k (z-t)^{2k}}{(2\sigma^2)^k},$$

we get that

$$\begin{aligned}
f_{X+Y_1+Y_2}(z) &= \frac{C^2(\phi, \beta)}{\sqrt{\pi}\sigma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)\phi^{2n+1}2^{\frac{2n+1}{2\beta}}}{(2n)!} \\
&\quad \times \sum_{k=0}^{\infty} \sum_{j=0}^n (-1)^k \frac{\binom{2n}{2j} (2j-1)!!}{2^k \sigma^{2(2n+k-j)}} \int_{-\infty}^{\infty} (z-t)^{2(k+n-j)} e^{-\frac{1}{2}\left|\frac{t}{\phi}\right|^{2\beta}} dt \\
&= \frac{C^2(\phi, \beta)}{\sqrt{\pi}\sigma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right)\phi^{2n+1}2^{\frac{2n+1}{2\beta}}}{(2n)!} \sum_{k=0}^{\infty} \sum_{j=0}^n (-1)^k \frac{\binom{2n}{2j} (2j-1)!!}{2^k \sigma^{2(2n+k-j)}} \\
&\quad \cdot \sum_{i=0}^{2(k+n-j)} (-1)^i \binom{2(k+n-j)}{i} z^{2(k+n-j)-i} \int_{-\infty}^{\infty} t^i e^{-\frac{1}{2}\left|\frac{t}{\phi}\right|^{2\beta}} dt.
\end{aligned}$$

It is fairly easy to calculate the last integral in the above as

$$(6.3) \quad \int_{-\infty}^{\infty} t^i e^{-\frac{1}{2}\left|\frac{t}{\phi}\right|^{2\beta}} dt = [1 + (-1)^i] \frac{1}{2\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{i+1}{2\beta}} \Gamma\left(\frac{i+1}{2\beta}\right).$$

We conclude that

$$f_{X+Y_1+Y_2}(z) = \frac{C^2(\phi, \beta)}{\sqrt{\pi}\sigma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right) \phi^{2n+1} 2^{\frac{2n+1}{2\beta}}}{(2n)!} \sum_{k=0}^{\infty} \sum_{j=0}^n (-1)^k \frac{\binom{2n}{2j} (2j-1)!!}{2^k \sigma^{2(2n+k-j)}} \\ \cdot \sum_{i=0}^{k+n-j} \binom{2(k+n-j)}{2i} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i+1}{2\beta}} \Gamma\left(\frac{2i+1}{2\beta}\right) z^{2(k+n-j-i)}.$$

For the general n we can repeat this procedure to obtain the following formula.

$$f_{X+Y_1+\dots+Y_n}(z) \\ = \frac{C^{n-1}(\phi, \beta)}{\sqrt{\pi}\sigma} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2\beta}\right) \phi^{2n+1} 2^{\frac{2n+1}{2\beta}}}{(2n)!} \sum_{k=0}^{\infty} \sum_{j=0}^n (-1)^k \frac{\binom{2n}{2j} (2j-1)!!}{2^k \sigma^{2(2n+k-j)}} \\ \cdot \sum_{i_3=0}^{k+n-j} \binom{2(k+n-j)}{2i_3} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_3+1}{2\beta}} \Gamma\left(\frac{2i_3+1}{2\beta}\right) \\ \cdot \sum_{i_4=0}^{k+n-j-i_3} \binom{2(k+n-j-i_3)}{2i_4} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_4+1}{2\beta}} \Gamma\left(\frac{2i_4+1}{2\beta}\right) \\ \dots \sum_{i_n=0}^{k+n-j-i_3-i_4-\dots-i_{n-1}} \binom{2(k+n-j-i_3-i_4-\dots-i_{n-1})}{2i_n} \\ \times \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_n+1}{2\beta}} \Gamma\left(\frac{2i_n+1}{2\beta}\right) z^{2(k+n-j-i_3-i_4-\dots-i_n)}$$

Case 2. $\beta < 1$. Again we first consider the case $m = 2$. In this case we use the expansion in terms of PCF's to get

$$f_{X+Y_1+Y_2}(z) = \int_{-\infty}^{\infty} f_{X+Y_1}(t) f_{Y_2}(z-t) dt \\ = \frac{C^2(\phi, \beta)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2} \left| \frac{t}{\phi} \right|^{2\beta}} e^{\frac{-(t-z)^2}{4\sigma^2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \left(\frac{\sigma}{\phi} \right)^{2\beta n} \Gamma(2\beta n + 1) \\ \times \left[D_{-(2\beta n+1)} \left(\frac{(t-z)}{\sigma} \right) + D_{-(2\beta n+1)} \left(\frac{-(t-z)}{\sigma} \right) \right].$$

(6.4)

Now recall from §2-C that

$$\left[D_{-(2\beta n+1)} \left(\frac{t-z}{\sigma} \right) + D_{-(2\beta n+1)} \left(\frac{-(t-z)}{\sigma} \right) \right] \\ = \sqrt{2\pi} 2^{-(\beta n + \frac{1}{4})} \left[\frac{2^{-\frac{1}{4}} \left(y_1(2\beta n + \frac{1}{2}, \frac{t-z}{\sigma}) + y_1(2\beta n + \frac{1}{2}, \frac{-t+z}{\sigma}) \right)}{\Gamma(\beta n + 1)} \right. \\ \left. - \frac{2^{\frac{1}{4}} \left(y_2(2\beta n + \frac{1}{2}, \frac{t-z}{\sigma}) - y_1(2\beta n + \frac{1}{2}, \frac{-t+z}{\sigma}) \right)}{\Gamma(\beta n + \frac{1}{2})} \right],$$

where

$$\begin{aligned} y_1(a, u) + y_1(a, -u) &= 2e_1^{u^2/4} F_1(a/2 + 1/4, 1/2; u^2/2), \\ y_2(a, u) + y_2(a, -u) &= u2e_1^{u^2/4} F_1(-a/2 + 3/4, 3/2; -u^2/2), \end{aligned}$$

we obtain that

$$\begin{aligned} & f_{X+Y_1+Y_2}(z) \\ &= C^2(\phi, \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n 2^{\beta n}} \left(\frac{\sigma}{\phi}\right)^{2\beta n} \Gamma(2\beta n + 1) \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{\Gamma(\beta n + \frac{1}{2} + k)}{\Gamma(\beta n + 1)\Gamma(\beta n + \frac{1}{2})(2k-1)!!k!} - \frac{4(-1)^k \Gamma(-\beta n + \frac{1}{2} + k)}{\Gamma(\beta n + \frac{1}{2})\Gamma(-\beta n + \frac{1}{2})(2k+1)!!k!} \\ &\quad \cdot \frac{1}{\sigma^{2k}} \int_{-\infty}^{\infty} (t-z)^{2k} e^{-\frac{t}{\phi}} \left|\frac{t}{\phi}\right|^{2\beta} dt. \end{aligned}$$

Since $(t-z)^{2k} = \sum_{\ell=0}^{2k} \binom{2k}{\ell} (-1)^\ell z^{2k-\ell} t^\ell$, we have

$$\begin{aligned} & f_{X+Y_1+Y_2}(z) \\ &= C^2(\phi, \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n 2^{\beta n}} \left(\frac{\sigma}{\phi}\right)^{2\beta n} \Gamma(2\beta n + 1) \cdot \\ &\quad \cdot \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(\beta n + \frac{1}{2} + k)}{\Gamma(\beta n + 1)(2k-1)!!} - \frac{4(-1)^k \Gamma(-\beta n + \frac{1}{2} + k)}{\Gamma(-\beta n + \frac{1}{2})(2k+1)!!} \right\} \frac{1}{\Gamma(\beta n + \frac{1}{2})k!} \\ &\quad \cdot \frac{1}{\sigma^{2k}} \sum_{\ell=0}^{2k} \binom{2k}{\ell} (-1)^\ell z^{2k-\ell} \int_{-\infty}^{\infty} t^\ell e^{-\frac{t}{\phi}} \left|\frac{t}{\phi}\right|^{2\beta} dt. \end{aligned}$$

Now using (6.3) we finally obtain that

$$\begin{aligned} & f_{X+Y_1+Y_2}(z) \\ &= C^2(\phi, \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n 2^{\beta n}} \left(\frac{\sigma}{\phi}\right)^{2\beta n} \Gamma(2\beta n + 1) \\ &\quad \cdot \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(\beta n + 1/2 + k)}{(2k-1)!!} - \frac{4(-1)^k \Gamma(-\beta n + 1/2 + k)}{\Gamma(-\beta n + 1/2)(2k+1)!!} \right\} \frac{1}{\Gamma(\beta n + 1/2)k!} \\ &\quad \cdot \frac{1}{\sigma^{2k}} \sum_{\ell=0}^k \binom{2k}{2\ell} z^{2(k-\ell)} \frac{2}{2\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{k+1}{2\beta}} \Gamma\left(\frac{\ell+1}{2\beta}\right). \end{aligned}$$

Finally, for general n we have the following formula: denoting $s(k, \ell) = i_k + \dots + i_\ell$,

$$\begin{aligned} & f_{X+Y_1+Y_2}(z) \\ &\cong C^{n-1}(\phi, \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n 2^{\beta n}} \left(\frac{\sigma}{\phi}\right)^{2\beta n} \Gamma(2\beta n + 1) \\ &\quad \cdot \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(\beta n + \frac{1}{2} + k)}{\Gamma(\beta n + 1)(2k-1)!!} - \frac{4(-1)^k \Gamma(-\beta n + \frac{1}{2} + k)}{\Gamma(-\beta n + \frac{1}{2})(2k+1)!!} \right\} \frac{1}{\Gamma(\beta n + \frac{1}{2})k!} \end{aligned}$$

$$\begin{aligned}
 & \cdot \frac{1}{\sigma^{2k}} \sum_{\ell=0}^k \binom{2k}{2\ell} z^{2(k-\ell)} \frac{2}{2\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{\ell+1}{2\beta}} \Gamma\left(\frac{\ell+1}{2\beta}\right) \\
 & \cdot \sum_{i_3=0}^{k-\ell} \binom{2(k-\ell)}{2i_3} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_3+1}{2\beta}} \Gamma\left(\frac{2i_3+1}{2\beta}\right) \\
 & \cdot \sum_{i_4=0}^{k-\ell-i_3} \binom{2(k+\ell-j-i_3)}{2i_4} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_4+1}{2\beta}} \Gamma\left(\frac{2i_4+1}{2\beta}\right) \\
 & \dots \sum_{i_n=0}^{k-\ell-S(3,n-1)} \binom{2(k+n-j-s(3,n-1))}{2i_n} \frac{1}{\beta} \left[\frac{1}{2\phi^{2\beta}} \right]^{\frac{2i_n+1}{2\beta}} \Gamma\left(\frac{2i_n+1}{2\beta}\right) \\
 & \cdot z^{2(k-\ell-s(3,n))}.
 \end{aligned}$$

7. Numerical Illustrations

In this section we illustrate our results by numerical experiments. We will be interested in the following three cases: 1) NP(1) with $\beta = 1$; 2) NP(1) with $\beta \neq 1$; and 3) NP(2) with all β .

7.1. NP(1) with $\beta=1$. This is a special case worth mentioning. On the one hand this is the case the convergence analysis of Theorem 4.2 does not apply. But on the other hand, in this case one actually has a convolution of two standard Normal random variables, thus the NP(1) sum is simply a $N(0, 2)$ random variable. We nevertheless did an experiment just to see how efficient (or inefficient) the Hermite expansion could be in this case. We would note that this numerical computation is only for theoretical purposes.

In Figure 1 the defaulting parameters are $\mu_1 = \mu_2 = 0$ and $\sigma = \phi = 1$. We see that while the Hermite series (5.2) actually does converge, it is extremely slow. For example, the result is far from satisfactory when $n = 200$; and it only becomes more acceptable when $n = 500$.

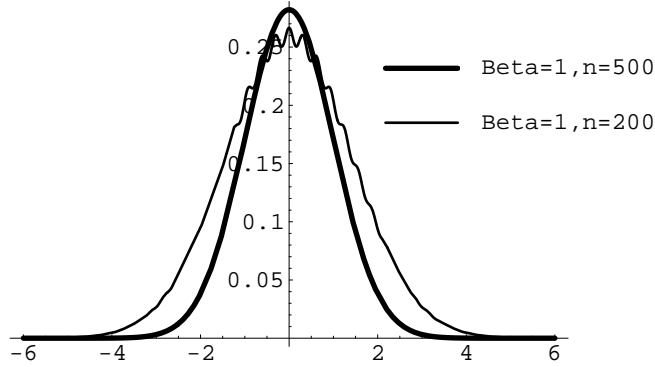


FIGURE 1. NP(1) with $\beta = 1$ ($\sim N(0, 2)$)

7.2. NP(1), $\beta \neq 1$. In this case we have the convolution of a normal random variable and a Power exponential. We would like to see two things: the speed of convergence and the shape of “tails” for different values of β . We fix the default parameters $\mu_1 = \mu_2 = 0$ and $\sigma = \phi = 1$, but let β vary. In Figure 2 we combine the graphs of those with $\beta = 3/10, 1/2, 1, 2,$ and $5/2$. In all the cases (except for $\beta = 1$) we find that $n = 100$ is already sufficient for the satisfactory results. As we can see that the smaller the β value is, the heavier the tail becomes. The example for $\beta = 3/10$ shows that there might be cases in which a power exponential distribution can be more efficient than double exponential ($\beta = 1/2$), if the heavier tails are desired. In general, all power exponentials with $\beta < 1$ may be of independent interest in robustness studies. Among other things, they could be useful in modeling the errors in *Regression Analysis* and *Time Series*, for example.

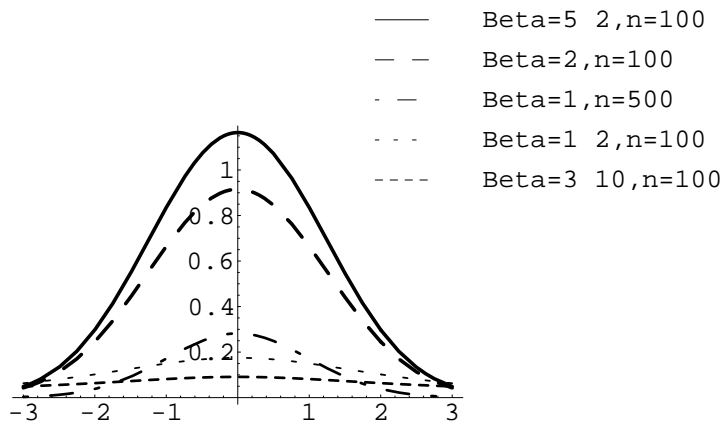
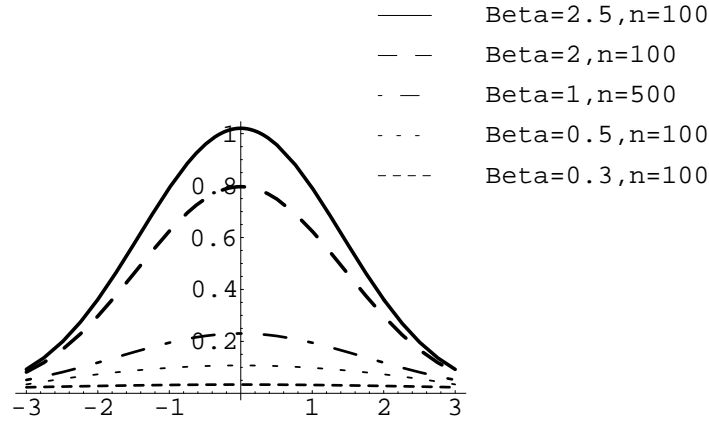


FIGURE 2. NP(1) for different values of β

7.3. NP(2) with all β 's. In this experiment we expect to see the same features of case (2), and we would also like to see the difference in tail shape when m , the number of power exponentials, increases. In Figure 3 we show the graphs of convolution of one $N(0, 1)$ random variable and two Power exponentials with $\mu_1 = \mu_2 = 0$, $\sigma = \phi = 1$. It is a little surprising that although the closed form expression of the solution to NP(2) is much more complicated as we saw in the previous section, the speed of convergence is almost no worse than the case of NP(1). In fact, $n = 100$ is again sufficient for a satisfactory result. One should also note that the solutions of NP(2) have even heavier tails than those of NP(1) with the same β values. We believe that “heaviness” of the tails increases as m increases, but we did not perform further numerical experiment as it goes beyond our original purpose of this paper.

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FIGURE 3. $NP(2)$ for different values of β

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