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Correlated intensity, counter party risks, and dependent mortalities

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ABSTRACT

In this paper we use an intensity-based framework to analyze and compute the correlated default probabilities, both in finance and actuarial sciences, following the idea of "change of measure" initiated by Collin-Dufresne et al. (2004). Our method is based on a representation theorem for joint survival probability among an arbitrary number of defaults, which works particularly effectively for certain types of correlated default models, including the counter-party risk models of Jarrow and Yu (2001) and related problems such as the phenomenon of "flight to quality". The results are also useful in studying the recently observed dependent mortality for married couples involving spousal bereavement. In particular we study in details a problem of pricing Universal Variable Life (UVL) insurance products. The explicit formulae for the joint-life status and last-survivor status (or equivalently, the probability distribution of first-to-default and last-to-default in a multi-firm setting) enable us to derive the explicit solution to the indifference pricing formula without using any advanced results in partial differential equations.

1. Introduction

Determining and computing the joint distributions of correlated default times in a multi-firm model has been a long standing problem in the credit risk theory. Due to the lack of information on the possible correlation among the parties involved, it is essentially impossible to have a universally applicable method that could precisely describe and quantify the possible correlation among the collection of defaults. As a consequence the price of the credit derivatives, in which the correlation among the defaults is known to present but cannot be analytically specified, are in general hard to be accurately evaluated.

There have been several existing methods dealing with the correlated defaults. Most notably are the *copula* method (cf. e.g., Schonbucher and Schubert, 2001), and the intensity-based, contagion models (cf. e.g., Collin-Dufresne et al., 2003, 2004; Jarrow and Yu, 2001; Yu, 2003; Yu, 2007), as well as the idea of "frailties" (cf. e.g., Schonbucher (2003), Gourieroux and Gagliardini (2003), and Duffie et al. (2006)). Since the copula method is based on a heavily ad hoc assumption on the joint distribution, and the frailty models involve an *unobservable* factor process, there seem to be limitations when the tractability and identifiability are the main concerns. On the other hand, the accurate pricing of financial products that contain a portfolio of correlated default risks has become a ubiquitous priority in today's financial industry. As a consequence, an effective method for identifying the joint distribution (or joint survival probability) becomes fundamentally important in the study of correlated default. We are therefore interested in developing methods that could lead to computable, or even explicit formulae of the joint distributions and/or joint survival probabilities in the general multi-firm situations.

The intensity-based (or the reduced form) contagion models have been studied quite extensively in recent years. Roughly speaking, one assumes that there exists a certain explicit structure among the default intensities of a group of interdependent firms, and the default of one firm could directly affect the default of its counter-parties, and/or even trigger a cascade of defaults in the group. (cf. e.g., Jarrow and Yu, 2001; Yu, 2003; Yu, 2007). While it is arguable whether the contagion model is more justified than the other methods for the correlated defaults, the fact that most of these works provide explicit solutions, at least in theory, makes the method rather attractive. It should be noted, however, that obtaining the explicit solution for counter-party risk is by no means trivial. The "looping" structure of the intensities, and the complexity of resulting system (of algebraic or differential equations) often becomes the main obstacle for deriving the analytical solution, even in the simplest two-firm case (see, e.g., Jarrow and Yu, 2001). In a recent work Collin-Dufresne et al. (2004) proposed a method of valuation of defaultable securities using a "change of measure" technique, so that the new measure only concentrates on the paths that do not default before the given maturity. It turns out that such a method has an obvious advantage. That is, it "ignores" the events where the default happens before the maturity, so that the so-called "no-jump" assumptions in

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many previous studied remains true under the new measure (cf. e.g., Collin-Dufresne et al., 2003, 2004).

The main purpose of this paper is two-fold. First, we try to extend the idea of Collin-Dufresne et al. (2004) to a multi-firm setting, and prove a general representation theorem. While such a representation does not actually reduce the degree of difficulty in computing the joint distribution, it nevertheless offers an opportunity for simplifying the computation, especially when the intensities are assumed to take a certain form. It turns out that many well-known counter-party intensities in the literature, especially those emphasize defaults before maturity, can be naturally simplified under the new probability measure. Thus the analytical expressions of the joint distribution can be derived even in the situations where such expressions are otherwise considered impossible.

The second purpose of this paper is to establish some connections between the correlated defaults and some actuarial problems involving correlated multiple lives. A particularly interesting class of problems includes those involving bereaved spouses. In fact, there has been a variety of research on excess mortality in the literatures of demography, epidemiology and psychology. Particular of interest, where the framework of counterpart risk is potentially useful, is the so-called *bereaved partner*. For example, it is noted that there has been a tendency of high excess mortality among the bereaved spouses/partners from accidental, violent, and alcohol-related causes. Particulary, a short duration of bereavement has greater excess mortality than a long one (cf. e.g., Martikainen and Valkonen (1996), Valkonen et al. (2004) and Hu and Goldman (1990) for correlations of excess mortalities among various groups with different ages and marital status). Using the counterparty risk framework we are able to find the joint density of the bereaved couple where the marginal mortality follow the Gompertz law. This, together with an indifference pricing method, enables us to find the explicit expression of the price of a *Universal Variable Life* (UVL) insurance product without using any advanced theory of partial differential equations (cf. Ma and Yu, 2006).

The rest of the paper is organized as follows. In Section 2 we formulate the problem and provide the necessary preparation for the default structure. In Section 3 we prove a representation theorem for the joint survival probability. In Section 4 we use it to study the counterparty risk models, and in Section 5 we apply the result to the dependent mortality models. In Section 6 we study the case of *Flight-to-Quality*, and finally in Section 7 we apply our results to the UVL insurance pricing problems.

2. Problem formulation

Throughout this paper we assume that all the uncertainty comes from a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and we assume that the probability $\mathbb{P}$ is a *risk neutral measure* for a defaultable bond market, namely under $\mathbb{P}$ the discounted underlying bond prices are martingales. We also assume that such a probability measure $\mathbb{P}$ is actually unique, which amounts to saying that the bond market is complete. We refer to Jarrow and Yu (2001) for more details on such a probabilistic set-up.

Similar to Jarrow and Yu (2001) we assume that there exists an $\mathbb{R}^d$-value background (or factor) process X_t which represents the exogenous state variables in the economy. Also, we shall consider a group of I firms in the market. These firms operate independently and each has a possibility of default. We denote τ^i to be the default time of the i-th firm, and assume that there exists a certain correlation among $\tau^1 \cdots \tau^I$ that is to be specified. Denoting the *default process* with respect to τ^i by $N_t^i \overset{\triangle}{=} \mathbf{1}_{\{\tau^i \le t\}}$, we define the filtration generated by factor process and the default processes by:

$$\mathcal{F}_t \overset{\triangle}{=} \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^I, \tag{2.1}$$

where $\mathcal{F}_t^X \overset{\triangle}{=} \sigma(X_s, 0 \le s \le t)$ and $\mathcal{F}_t^i = \sigma(N_s^i, 0 \le s \le t)$ represent the filtrations generated by X_t and N_t^i's, respectively. We also define the filtrations $\mathcal{H}_t^i$, $i = 1, \ldots, I$, by

$$\mathcal{H}_t^i = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^{i-1} \vee \mathcal{F}_t^{i+1} \vee \cdots \vee \mathcal{F}_t^I, \tag{2.2}$$

which represent the information generated by the state variables and the default processes of all but the i-th firm. Thus we have

$$\mathcal{F}_t = \mathcal{H}_t^i \vee \mathcal{F}_t^i.$$

There have been several essentially equivalent ways to define the *hazard process*. Throughout this paper we shall follow the definition by Bielecki et al. (2006) and/or Jeanblanc and Rutkowski (2001): namely we start from the the conditional survival probability $S_t^i = \mathbb{P}\{\tau^i > t | \mathcal{H}_t^i\}$, for $t \in \mathbb{R}_+$. Then it is clear that S^i is a nonnegative, bounded $\mathcal{H}^i$-supermartingale. We shall therefore consider the right-continuous modification of S^i.

Definition 2.1. Assumed that $S_t^i > 0$, $t \ge 0$, $\mathbb{P}$-a.s. The $\mathcal{H}^i$-hazard process of τ^i is defined by $H_t^i \overset{\triangle}{=} -\ln(S_t^i)$, $t \ge 0$. Equivalently, $S_t^i = e^{-H_t^i}$, for all $t \ge 0$, $\mathbb{P}$-a.s.

We note that the assumption that $S_t^i > 0$ amounts to saying that τ^i cannot be a $\mathcal{H}^i$-stopping time. On the other hand, if we assume further that H^i is absolutely continuous and increasing, that is, we assume that there exists a nonnegative $\mathcal{H}_t^i$-adapted stochastic process λ_t^i such that $H_t^i = \int_0^t \lambda_s^i \, ds$, $t \ge 0$, then we have the following expression:

$$S_t^i = \mathbb{P}\{\tau^i > t | \mathcal{H}_t^i\} = \exp\left\{ -\int_0^t \lambda_s^i ds \right\}.$$

The process λ^i is called the *intensity process* of the default time τ^i, and it obviously holds that $\lambda_t^i dt = -dS_t^i / S_t^i$, $t \ge 0$, as it is often seen in the literature (cf., e.g., Lando, 2004).

Our main task is to evaluate the conditional expectation given the σ-field $\{\mathcal{F}_t\}$. The following lemma is frequently cited, often without proof. We provide a sketch of the proof for ready reference.

Lemma 2.2. *For any $\mathcal{F}$-measurable random variable Z we have, for any $t \ge 0$,*

$$\mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} Z | \mathcal{H}_t^i\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} | \mathcal{H}_t^i\}}. \tag{2.3}$$

Proof. The proof is along the lines of that in, e.g., Jeanblanc and Rutkowski (2001). We first define, for each $t \ge 0$,

$$\mathcal{F}_t^* \overset{\triangle}{=} \{A \in \mathcal{F} | \exists B \in \mathcal{H}_t^i,\ A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\}\}.$$

Then clearly $\mathcal{F}_t^*$ is a sub-σ-field of $\mathcal{F}$. We claim that it contains $\mathcal{F}_t$, for $t \ge 0$. Indeed, note that $\mathcal{F}_t = \mathcal{H}_t^i \vee \mathcal{F}_t^i = \sigma\{\mathcal{H}_t^i, \{\tau^i \le s\}, s \le t\}$, it suffice to show that if either $A \in \mathcal{H}_t^i$ or $A = \{\tau^i \le s\}$ for some $s \le t$, then there exists $B \in \mathcal{H}_t^i$ such that $A \cap \{\tau^i > t\} = B \cap \{\tau^i > t\}$. But in the former case we can choose $B = A$, and in in the latter we simply set $B = \emptyset$, proving the claim.

A simple application of Monotone Class Theorem then leads to that, for any $\mathcal{F}$-measurable random variable Z and $t \ge 0$, one can find some $\mathcal{H}_t^i$-measurable random variable X such that

$$\mathbb{E}\{\mathbf{1}_{\{\tau^i > t\}} Z | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} \mathbb{E}\{Z | \mathcal{F}_t\} = \mathbf{1}_{\{\tau^i > t\}} X.$$

Taking conditional expectation $\mathbb{E}\{\cdot\, | \mathcal{H}_t^i\}$ on both sides above and using the measurability of X we can solve X and obtain (2.3) immediately. □

A direct consequence of Lemma 2.2 we now derive the conditional survival probabilities with respect to the filtration $\mathcal{F}_t$. To this end we first note that

$$\mathbb{P}\{\tau^i > T|\mathcal{F}_t\} = \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>T\}}|\mathcal{F}_t\}.$$

Thus, applying Lemma 2.2 we have

$$\mathbb{P}\{\tau^i > T|\mathcal{F}_t\} = \mathbf{1}_{\{\tau^i>t\}}\frac{\mathbb{E}[\mathbf{1}_{\{\tau^i>T\}}|\mathcal{H}^i_t\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathcal{H}^i_t\}}. \tag{2.4}$$

Now, note that

$$\begin{aligned}\mathbb{E}\{\mathbf{1}_{\{\tau^i>T\}}|\mathcal{H}^i_t\} &= \mathbb{E}\left\{\mathbb{P}\{\tau^i > T|\mathcal{H}^i_T\}|\mathcal{H}^i_t\right\} = \mathbb{E}\left\{\exp\left(-\int_0^T \lambda^i_s ds\right)\middle|\mathcal{H}^i_t\right\}\\ &= \exp\left(-\int_0^t \lambda^i_s ds\right)\mathbb{E}\left\{\exp\left(-\int_t^T \lambda^i_s ds\right)\middle|\mathcal{H}^i_t\right\}.\end{aligned}$$

Since $\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathcal{H}^i_t\} = \exp(-\int_0^t \lambda^i_s ds)$ by definition, (2.4) becomes

$$\mathbb{P}\{\tau^i > T|\mathcal{F}_t\} = \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}\left\{\exp\left(-\int_t^T \lambda^i_s ds\right)\middle|\mathcal{H}^i_t\right\}.$$

Finally, we point out an important fact regarding the hazard process H^i. That is, it can be related to the nondecreasing default processes $N^i_t = \mathbf{1}_{\{\tau^i\le t\}}$ as its compensator. More precisely,

$$M^i_t \triangleq N^i_t - H^i_{t\wedge\tau^i} = \mathbf{1}_{\{\tau^i\le t\}} - \int_0^t \mathbf{1}_{\{\tau^i>s\}}\lambda^i_s ds, \quad t \ge 0, \tag{2.5}$$

are $\{\mathcal{F}_t\}$-martingales, for $i = 1, \dots, I$ (cf., e.g., Bremaud, 1981). Consequently we often say that $\mathbf{1}_{\{\tau^i\le t\}}$ admits an $\mathcal{F}_t$-intensity λ^i_t, and (2.5) is often referred to as *martingale characterization of intensity* λ^i.

We conclude this section by making the following *Standing Assumptions* of the intensity processes $\{\lambda^i_t\}_{i=1}^I$.

(H1) λ^i_t satisfy the following condition:

$$\mathbb{E}\left\{\exp\left(2\int_0^t \sum_{i=1}^I \lambda^i_s ds\right)\right\} < \infty, \quad \forall t < \infty.$$

(H2) For each i, $\mathbb{P}\{\tau^i > 0\} = 1$. Furthermore, there are no simultaneous defaults among the I firms. In other words, it holds that $\mathbb{P}\{\tau^i \ne \tau^j\} = 1$, whenever $i \ne j$.

3. Representation of joint survival probability

In this section we give the main representation result for the joint survival probability

$$\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, \dots, \tau^I > t_I\}, \quad (t_1, \dots, t_I) \in \mathbb{R}^I_+.$$

We follow the idea of *change of measure* proposed in Collin-Dufresne et al. (2004). To be more precise, we define, for $i = 1, \dots, I$, $\Gamma^i_t \triangleq \exp\{\int_0^t \lambda^i_s ds\}$, and

$$Z^i_t \triangleq \mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t = \mathbf{1}_{\{\tau^i>t\}}\exp\left\{\int_0^t \lambda^i_s ds\right\}. \tag{3.1}$$

Clearly, Γ^i's are nonnegative, $\{\mathcal{H}^i_t\}$-adapted process satisfying $\Gamma^i_0 = 1$, for all i. We shall use the processes Z^i's to construct a family of probability measures that will facilitate the computation of the joint default probability. The following result is crucial.

Proposition 3.1. *Assume* (H1) *and* (H2). *Then, for* $k = 1, \dots, I$, *the processes*

$$\prod_{i=1}^k Z^i_t \triangleq \prod_{i=1}^k \mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t, \quad t \ge 0 \tag{3.2}$$

are all $\{\mathcal{F}_t\}$*-martingales.*

Proof. We first show that Z^i_t's are martingales for $i = 1, \dots, I$. Note that for $t \ge s$ we have

$$\mathbb{E}\{Z^i_t|\mathcal{F}_s\} = \mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{F}_s\} = \mathbf{1}_{\{\tau^i>s\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{F}_s\}. \tag{3.3}$$

Applying Lemma 2.2 we see that

$$\begin{aligned}&\mathbf{1}_{\{\tau^i>s\}}\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{F}_s\}\\ &\quad= \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{H}^i_t\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{H}^i_s\}} = \mathbf{1}_{\{\tau^i>s\}}\frac{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{H}^i_s\}}{\exp\left\{-\int_0^s \lambda^i_s ds\right\}}\\ &\quad= \mathbf{1}_{\{\tau^i>s\}}\Gamma^i_s\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{H}^i_s\} = Z^i_s\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{H}^i_s\}.\end{aligned}$$

Note that $\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}\Gamma^i_t|\mathcal{H}^i_s\} = \mathbb{E}\{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathcal{H}^i_t\}\Gamma^i_t|\mathcal{H}^i_s\} = 1$. Thus, we derive from (3.3) that $\mathbb{E}\{Z^i_t|\mathcal{F}_s\} = Z^i_s$. Namely Z^i is an $\{\mathcal{F}_t\}$-martingale.

We now prove (3.2) by induction. Since we have just proved the case $k = 1$, we shall assume that $\prod_{i=1}^k Z^i_t$ is a martingale, and prove that $\prod_{i=1}^{k+1} Z^i_t$ is a martingale as well. To simplify notation, let us set $\prod_{i=1}^k Z^i_t \triangleq \tilde{Z}^k_t$ so that $\prod_{i=1}^{k+1} Z^i_t = \tilde{Z}^k_t Z^{k+1}_t$. Applying Itô's formula we have

$$\tilde{Z}^k_t Z^{k+1}_t = \int_{0+}^t \tilde{Z}^k_{s-}dZ^{k+1}_s + \int_{0+}^t Z^{k+1}_{s-}d\tilde{Z}^k_s + [\tilde{Z}^k, Z^{k+1}]_t. \tag{3.4}$$

Note that $\tilde{Z}^k$ and Z^{k+1} are both of finite variation paths, and the inductional hypothesis and (H1) imply that both $\tilde{Z}^k$ and Z^{k+1} are quadratic pure jump martingales (see Protter, 2004). Therefore we have

$$[\tilde{Z}^k, Z^{k+1}]_t = \tilde{Z}^k_0 Z^{k+1}_0 + \sum_{0<s\le t} \Delta\tilde{Z}^k_s \Delta Z^{k+1}_s.$$

Furthermore, the assumption (H2) ensures that $\sum_{0<s\le t} \Delta\tilde{Z}^k_s \Delta Z^{k+1}_s = 0$. Thus (3.4) becomes

$$\tilde{Z}^k_t Z^{k+1}_t = \int_{0+}^t \tilde{Z}^k_{s-}dZ^{k+1}_s + \int_{0+}^t Z^{k+1}_{s-}d\tilde{Z}^k_s + \tilde{Z}^k_0 Z^{k+1}_0.$$

In other words, $\tilde{Z}^k_t Z^{k+1}_t$ is a local martingale. But (H1) further guarantees that $\tilde{Z}^k_t Z^{k+1}_t$ is a true martingale, proving the proposition. □

Since Z^i's are martingales with $Z^i_0 = 1$, we can define probability measures $\mathbb{P}^i$'s that are absolutely continuous with respect to $\mathbb{P}$ by

$$\left.\frac{d\mathbb{P}^i}{d\mathbb{P}}\right|_{\mathcal{F}_T} \triangleq Z^i_T = \mathbf{1}_{\{\tau^i>T\}}\exp\left\{\int_0^T \lambda^i_s ds\right\}, \quad i = 1, \dots, I. \tag{3.5}$$

Similarly, for each $k = 2, \dots, I$ we can define

$$\left.\frac{d\mathbb{P}^{1,\dots,k}}{d\mathbb{P}}\right|_{\mathcal{F}_T} \triangleq \tilde{Z}^k_T = \prod_{i=1}^k Z^i_T.$$

Then, for any $\mathbb{P}^{1,\dots,k}$-integrable random variable X and $k = 1, \dots, I$ it holds that

$$\mathbb{E}\{Z^1_T Z^2_T \dots Z^k_T X\} = \mathbb{E}^{\mathbb{P}^{1,\dots,k}}\{X\} \triangleq \mathbb{E}^{1,\dots,k}\{X\}.$$

Furthermore, using the martingale property of $\tilde{Z}^k_t$ we see that for each k and any $A \in \mathcal{F}_t$, it holds that

$$\begin{aligned}&\mathbb{E}\{\mathbf{1}_A\tilde{Z}^k_t\mathbb{E}^{1,\dots,k}\{X|\mathcal{F}_t\}\}\\ &\quad= \mathbb{E}\{\mathbf{1}_A\mathbb{E}\{\tilde{Z}^k_T|\mathcal{F}_t\}\mathbb{E}^{1,\dots,k}\{X|\mathcal{F}_t\}\} = \mathbb{E}\{\mathbf{1}_A\tilde{Z}^k_T\mathbb{E}^{1,\dots,k}\{X|\mathcal{F}_t\}\}\\ &\quad= \mathbb{E}^{1,\dots,k}\{\mathbf{1}_A\mathbb{E}^{1,\dots,k}\{X|\mathcal{F}_t\}\} = \mathbb{E}^{1,\dots,k}\{1_A X\} = \mathbb{E}\{1_A\tilde{Z}^k_T X\}\\ &\quad= \mathbb{E}\{\mathbf{1}_A\mathbb{E}\{\tilde{Z}^k_T X|\mathcal{F}_t\}\}.\end{aligned}$$

This easily leads to the following identity:

$$\mathbb{E}\{Z_T^1 Z_T^2 \dots Z_T^k X|\mathcal{F}_t\} = Z_t^1 Z_t^2 \dots Z_t^k \mathbb{E}^{1,\dots,k}\{X|\mathcal{F}_t\},\quad \mathbb{P}-a.s., \tag{3.6}$$

for any $0 \le t \le T$ and $k = 1, \dots, I$.

We are now ready to study the representation of joint survival probability $P\{\tau^1 > t_1, \tau^2 > t_2, \dots, \tau^I > t_I\}$. We begin by considering the two-firm case. Assume that $t_1 \le t_2$, we can apply (3.6) to get

$$\begin{aligned}
\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2\} &= \mathbb{E}\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbf{1}_{\{\tau^2>t_2\}}\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbb{E}\left\{Z_{t_2}^2 \exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}Z_{t_1}^2\mathbb{E}^{\mathbb{P}^2}\left\{\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}\left\{Z_{t_1}^1 Z_{t_1}^2\mathbb{E}^{\mathbb{P}^2}\left\{\exp\left\{-\int_0^{t_1}\lambda_s^1 ds\right\}\right.\right.\\
&\quad\times\left.\left.\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}^{1,2}\left\{\mathbb{E}^{\mathbb{P}^2}\left\{\exp\left\{-\int_0^{t_1}\lambda_s^1 ds\right\}\right.\right.\\
&\quad\times\left.\left.\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}.
\end{aligned}\tag{3.7}$$

In particular, if $t_1 = t_2 = t$, then we have

$$\mathbb{P}\{\tau^1 > t, \tau^2 > t\} = \mathbb{E}^{1,2}\left\{\exp\left\{\int_0^t(\lambda_s^1+\lambda_s^2)ds\right\}\right\}.$$

We can easily iterate the above procedure to obtain the general representation of joint survival probability of I firms, which we summarize in the following theorem.

Theorem 3.2. *Assume* (H1) *and* (H2). *Then,*

(i) *For any* $0 \le t_1 \le t_2 \le \dots \le t_I < \infty$, *it holds that*

$$\begin{aligned}
&\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, \dots, \tau^I > t_I\}\\
&= \mathbb{E}^{1,\dots,I}\left\{\mathbb{E}^{2,\dots,I}\left\{\cdots\left\{\mathbb{E}^{\mathbb{P}^I}\left\{\exp\left\{-\int_0^{t_1}\lambda_s^1 ds\right\}\right.\right.\right.\right.\\
&\quad\times\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\cdots\exp\left\{-\int_0^{t_I}\lambda_s^I ds\right\}\middle|\mathcal{F}_{t_{I-1}}\right\}\\
&\quad\times\left.\left.\middle|\mathcal{F}_{t_{I-2}}\right\}\cdots\middle|\mathcal{F}_{t_1}\right\}\right\};
\end{aligned}\tag{3.8}$$

(ii) *For any* $t \ge 0$ *it holds that*

$$\mathbb{P}\{\tau^1 > t, \tau^2 > t, \dots, \tau^I > t\} = \mathbb{E}^{1,\dots,I}\left\{\exp\left\{-\int_0^t(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\right\};\tag{3.9}$$

(iii) *For any* $0 \le t \le T$, *it holds that*

$$\begin{aligned}
&\mathbb{P}\{\tau^1 > T, \tau^2 > T, \dots, \tau^I > T|\mathcal{F}_t\}\\
&= \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}}\cdots\mathbf{1}_{\{\tau^I>t\}}\mathbb{E}^{1,\dots,I}\\
&\quad\times\left\{\exp\left\{-\int_t^T(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\middle|\mathcal{F}_t\right\}.
\end{aligned}\tag{3.10}$$

Proof. (i) Again, we prove the assertion by induction. The case when $k = 2$ is proved in (3.7). We shall prove the induction step by looking at case $k = 3$, since the argument is completely the same for general k.

Similar to (3.7) we have

$$\begin{aligned}
&\mathbb{P}\{\tau^1 > t_1, \tau^2 > t_2, \tau^3 > t_3\} = \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbf{1}_{\{\tau^2>t_2\}}\mathbf{1}_{\{\tau^1>t_3\}}\right\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbf{1}_{\{\tau^2>t_2\}}\mathbb{E}\left\{Z_{t_3}^3\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\middle|\mathcal{F}_{t_2}\right\}\right\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbb{E}\left\{\mathbf{1}_{\{\tau^2>t_2\}}Z_{t_2}^3\mathbb{E}^{\mathbb{P}^3}\left\{\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\right.\right.\right.\\
&\quad\times\left.\left.\left.\middle|\mathcal{F}_{t_2}\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}\mathbb{E}\left\{Z_{t_2}^2 Z_{t_2}^3\mathbb{E}^{\mathbb{P}^3}\left\{\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\right.\right.\right.\\
&\quad\times\left.\left.\left.\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\middle|\mathcal{F}_{t_2}\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}\left\{\mathbf{1}_{\{\tau^1>t_1\}}Z_{t_1}^2 Z_{t_1}^3\mathbb{E}^{2,3}\left\{\mathbb{E}^{\mathbb{P}_3}\left\{\exp\left\{\int_0^{t_2}\lambda_s^2 ds\right\}\right.\right.\right.\\
&\quad\times\left.\left.\left.\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\middle|\mathcal{F}_{t_2}\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}\left\{Z_{t_1}^1 Z_{t_1}^2 Z_{t_1}^3\mathbb{E}^{2,3}\left\{\mathbb{E}^{\mathbb{P}^3}\left\{\exp\left\{-\int_0^{t_1}\lambda_s^1 ds\right\}\right.\right.\right.\\
&\quad\times\left.\left.\left.\exp\left\{-\int_0^{t_2}\lambda_s^2 ds\right\}\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\middle|\mathcal{F}_{t_2}\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}\\
&= \mathbb{E}^{1,2,3}\left\{\mathbb{E}^{2,3}\left\{\mathbb{E}^{\mathbb{P}^3}\left\{\exp\left\{-\int_0^{t_1}\lambda_s^1 ds\right\}\exp\left\{\int_0^{t_2}\lambda_s^2 ds\right\}\right.\right.\right.\\
&\quad\times\left.\left.\left.\exp\left\{-\int_0^{t_3}\lambda_s^3 ds\right\}\middle|\mathcal{F}_{t_2}\right\}\middle|\mathcal{F}_{t_1}\right\}\right\}.
\end{aligned}$$

By the induction hypothesis, one can complete the proof.

(ii) From part (i) we see that, if $t_1 = t_2 = \dots = t_I = t$, then

$$\begin{aligned}
&\mathbb{P}\{\tau^1 > t, \tau^2 > t, \dots, \tau^I > t\}\\
&= \mathbb{E}^{1,2,\dots,I}\left\{\mathbb{E}^{2,3,\dots,I}\left\{\cdots\left\{\mathbb{E}^{\mathbb{P}^I}\{\exp\{\right.\right.\right.\\
&\quad-\left.\left.\left.\int_0^t(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\middle|\mathcal{F}_t\right\}\middle|\mathcal{F}_t\right\}\cdots\middle|\mathcal{F}_t\right\}\right\}.
\end{aligned}$$

Since $\exp\{-\int_0^t(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\}$ is $\mathcal{F}_t$-measurable, by moving $\exp\{-\int_0^t(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\}$ outside the conditional expectations, we see that all the conditional expectations will disappear, proving (3.9).

(iii) It follows from (3.6) and (3.8) that

$$\begin{aligned}
&\mathbb{P}\{\tau^1 > T, \tau^2 > T, \dots, \tau^I > T|\mathcal{F}_t\}\\
&= \mathbb{E}\left\{Z_T^1 Z_T^2\dots Z_T^I\exp\left\{-\int_0^T(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\middle|\mathcal{F}_t\right\}\\
&= Z_t^1 Z_t^2\dots Z_t^I\mathbb{E}^{1,\dots,I}\left\{\exp\left\{-\int_0^T(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}}\cdots\mathbf{1}_{\{\tau^I>t\}}\mathbb{E}^{1,\dots,I}\\
&\quad\times\left\{\exp\left\{-\int_t^T(\lambda_s^1+\lambda_s^2+\dots+\lambda_s^I)ds\right\}\middle|\mathcal{F}_t\right\}.
\end{aligned}$$

Note that $Z_t^i = \mathbf{1}_{\{\tau^i>t\}}\exp\{\int_0^t\lambda_s^i ds\}$. The third equality follows from moving $\exp\{\int_0^t\lambda_s^i ds\}$ into the conditional expectation. □

4. Counter-party risk models

In Jarrow and Yu (2001) the problem of counter-party risk was investigated under the assumption that default correlation takes the following form:

$$\lambda_t^A = a_0 + \mathbf{1}_{\{\tau^B \le t\}} a_1, \qquad \lambda_t^B = b_0 + \mathbf{1}_{\{\tau^A \le t\}} b_1. \tag{4.1}$$

where A and B represent two firms, and a_i and b_i, $i = 0, 1$ are constants. In this section we shall extend this looping default structure to the more general setting.

4.1. Two firm case

We first consider the case when $I = 2$. In light of (4.1), we consider the following general model:

$$\begin{cases} \lambda_t^A = a_0(t) + \mathbf{1}_{\{\tau^B \le t\}} a_1(t - \tau^B), \\ \lambda_t^B = b_0(t) + \mathbf{1}_{\{\tau^A \le t\}} b_1(t - \tau^A), \end{cases} \tag{4.2}$$

where a_0, a_1, b_0, and b_1 are deterministic functions. We shall need the following extra assumptions:

(H3) (i) a_0 and b_0 are positive functions;
(ii) a_1 and b_1 are either positive and decreasing or negative and increasing, such that

$$\lim_{t\to\infty} a_1(t) = 0 \qquad \lim_{t\to\infty} b_1(t) = 0. \tag{4.3}$$

(ii) Both λ_t^A and λ_t^B are positive processes.

We remark that (4.2) and (H3) amount to saying that one firm's intensity will have a jump when the other firm defaults, but the impact of the default will gradually vanish as time passes. Moreover, the positivity of a_1 or b_1 means that the two companies have a certain partnership, whereas the negativity indicates that they are competitors. It should also be noted that the assumption (ii) in (H3) is not needed for the proof of Proposition 4.1.

We now apply the result in the previous section to derive a relatively closed form of the joint survival probability.

Proposition 4.1. *Assume* (H1)–(H3). *Then the joint survival probability* $\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\}$ *is given by*

$$\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} = \begin{cases} c(t_1,t_2)\left(\int_{t_1}^{t_2} a_0(x) e^{-\int_x^{t_2} b_1(s-x)ds - \int_{t_1}^x a_0(s)ds} dx + \int_{t_2}^{\infty} a_0(x) e^{-\int_{t_1}^x a_0(s)ds} dx\right) & t_1 \le t_2; \\ c(t_1,t_2)\left(\int_{t_2}^{t_1} b_0(x) e^{-\int_x^{t_1} a_1(s-x)ds - \int_{t_2}^x b_0(s)ds} dx + \int_{t_1}^{\infty} b_0(x) e^{-\int_{t_2}^x b_0(s)ds} dx\right) & t_1 > t_2 \end{cases}$$

where $c(t_1, t_2) = \exp\{-\int_0^{t_1} a_0(s)ds - \int_0^{t_2} b_0(s)ds\}$.

Proof. We shall check only $t_1 \le t_2$. Recall the process $\Gamma_t \triangleq \exp\{\int_0^t \lambda_s ds\}$, $\lambda = \lambda^A, \lambda^B$, respectively. Applying the change of measure, we have

$$\begin{aligned} \mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} &= \mathbb{E}\left[\mathbf{1}_{\{\tau^A > t_1\}}\mathbf{1}_{\{\tau^B > t_2\}}\Gamma_{t_2}^B \exp\left\{-\int_0^{t_2}\lambda_s^B ds\right\}\right] \\ &= \mathbb{E}^B\left[\mathbf{1}_{\{\tau^A > t_1\}} \exp\left(-\int_0^{t_2}\left(b_0(s) + \mathbf{1}_{\{\tau^A \le s\}} b_1(s-\tau^A)\right)ds\right)\right] \\ &= e^{-\int_0^{t_2} b_0(s)ds}\mathbb{E}^B\left[\mathbf{1}_{\{\tau^A > t_1\}}\exp\left\{-\int_{\tau^A}^{t_2} b_1(s-\tau^A)ds\right\}\right] \\ &= e^{-\int_0^{t_2} b_0(s)ds}\left\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x)ds}\mathbb{P}^B\{\tau^A \in dx\} + \int_{t_2}^{\infty}\mathbb{P}^B\{\tau^A \in dx\}\right\} \\ &= e^{-\int_0^{t_2} b_0(s)ds}\left\{\int_{t_1}^{t_2} e^{-\int_x^{t_2} b_1(s-x)ds} a_0(x) e^{-\int_0^x a_0(s)ds}dx + \int_{t_2}^{\infty} a_0(x)e^{-\int_0^x a_0(s)ds}dx\right\} \\ &= c(t_1,t_2)\left\{\int_{t_1}^{t_2} a_0(x)e^{-\int_x^{t_2} b_1(s-x)ds - \int_{t_1}^x a_0(s)ds}dx + \int_{t_2}^{\infty} a_0(x)e^{-\int_{t_1}^x a_0(s)ds}dx\right\}. \end{aligned}$$

In the second last equality above, we used the fact that $\lambda_s^A = a_0(s)$ under $\mathbb{P}^{A,B}$, and consequently $\mathbb{P}^B(\tau^A > x) = e^{-\int_0^x a_0(s)ds}$. The case for $t_2 < t_1$ can be argued in a similar way. □

We remark that once the joint survival probability is obtained, the marginal survival probabilities can be derived by simply setting $t_1 = 0$ or $t_2 = 0$, respectively. Furthermore, the relation

$$\mathbb{P}\{\tau^A \le t_1, \tau^B \le t_2\} = 1 - \mathbb{P}\{\tau^A > t_1\} - \mathbb{P}\{\tau^B > t_2\} + \mathbb{P}\{\tau^A > t_1, \tau^B > t_2\}$$

gives the joint default probability.

We now give two specific examples where the closed-form formulae can be obtained.

Example 4.2. Assume that $a_1(t), b_1(t) \sim (ce^t + 1)^{-1}$, where $c > 0$ is such that (4.3) holds. More precisely, we assume that

$$\begin{cases} \lambda_t^A = a_0 + \mathbf{1}_{\{\tau^B \le t\}} \dfrac{n}{a_1 e^{(t-\tau^B)} + 1} \\ \lambda_t^B = b_0 + \mathbf{1}_{\{\tau^A \le t\}} \dfrac{m}{b_1 e^{(t-\tau^A)} + 1}, \end{cases}$$

where m and n are positive integers. We note that when $a_1 = b_1 = 0$, then one can allow m and n to take any values in $\mathbb{R}$, as long as both λ_t^A and λ_t^B are positive. This then leads to the two firm counterparty-risk model of Jarrow and Yu (2001).

Applying Proposition 4.1 we see that for $t_1 \le t_2$, it holds that

$$\begin{aligned} \mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} &= c(t_1,t_2)\left\{\int_{t_1}^{t_2} a_0(x)e^{-\int_x^{t_2} b_1(s-x)ds - \int_{t_1}^x a_0(s)ds}dx + \int_{t_2}^{\infty} a_0(x)e^{-\int_{t_1}^x a_0(s)ds}dx\right\} \\ &= e^{-a_0t_1 - b_0t_2}\left\{\int_{t_1}^{t_2} a_0 e^{m\{\ln(\frac{b_1 + e^{-(t_2-x)}}{b_1+1})\} - a_0(x-t_1)}dx + \int_{t_2}^{\infty} a_0 e^{-a_0(x-t_1)}dx\right\} \\ &= \frac{a_0 e^{-b_0t_2}}{(b_1+1)^m}\int_{t_1}^{t_2}\{b_1 + e^{-(t_2-x)}\}^m e^{-a_0x}dx + a_0e^{-b_0t_2}\int_{t_2}^{\infty} e^{-a_0x}dx \\ &= \frac{a_0 e^{-b_0t_2}}{(b_1+1)^m}\int_{t_1}^{t_2}\sum_{k=0}^m \binom{m}{k} b_1^{m-k} \times e^{-kt_2 - (a_0-k)x}dx + e^{-(a_0+b_0)t_2}dx \\ &= \frac{a_0 e^{-b_0t_2}}{(b_1+1)^m}\sum_{k=0}^m \binom{m}{k}\frac{b_1^{m-k}e^{-kt_2}}{a_0-k}[e^{-(a_0-k)t_1} - e^{-(a_0-k)t_2}] \\ &\quad + e^{-(a_0+b_0)t_2}. \end{aligned} \tag{4.4}$$

Arguing the case for $t_2 < t_1$ similarly we then obtain the formula for the joint survival probability:

$$\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} = \begin{cases} \dfrac{a_0}{(b_1+1)^m}\displaystyle\sum_{k=0}^{m}\binom{m}{k}\dfrac{b_1^{m-k}\mathrm{e}^{-(b_0+k)t_2}}{a_0-k} \\ \quad\times[\mathrm{e}^{-(a_0-k)t_1}-\mathrm{e}^{-(a_0-k)t_2}]+\mathrm{e}^{-(a_0+b_0)t_2} & t_1\le t_2; \\ \dfrac{b_0}{(a_1+1)^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}\dfrac{a_1^{n-k}\mathrm{e}^{-(a_0+k)t_1}}{b_0-k} \\ \quad\times[\mathrm{e}^{-(b_0-k)t_2}-\mathrm{e}^{-(b_0-k)t_1}]+\mathrm{e}^{-(a_0+b_0)t_1} & t_1>t_2. \end{cases} \tag{4.5}$$

From (4.5) we can easily derive other statistics of the default times τ^A and τ^B. For example, the joint density of the survival probability and the marginal distributions of τ^A and τ^B are given by, respectively:

$$f(t_1,t_2) = \begin{cases} \dfrac{a_0}{(b_1+1)^m}\displaystyle\sum_{k=0}^{m}\binom{m}{k}(b_0+k)b_1^{m-k} \\ \quad\times\mathrm{e}^{-(a_0-k)t_1-(b_0+k)t_2} & t_1\le t_2; \\ \dfrac{b_0}{(a_1+1)^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}(a_0+k)a_1^{n-k} \\ \quad\times\mathrm{e}^{-(b_0-k)t_2-(a_0+k)t_1} & t_1>t_2, \end{cases}$$

and

$$\begin{aligned} F_A(t_1) &= 1-\frac{b_0}{(a_1+1)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{a_1^{n-k}}{b_0-k} \\ &\quad\times[\mathrm{e}^{-(a_0+k)t_1}-\mathrm{e}^{-(a_0+b_0)t_1}]-\mathrm{e}^{-(a_0+b_0)t_1}, \\ F_B(t_2) &= 1-\frac{a_0}{(b_1+1)^m}\sum_{k=0}^{m}\binom{m}{k}\frac{b_1^{m-k}}{a_0-k} \\ &\quad\times[\mathrm{e}^{-(b_0+k)t_2}-\mathrm{e}^{-(a_0+b_0)t_2}]-\mathrm{e}^{-(a_0+b_0)t_2}. \quad\square \end{aligned}$$

Example 4.3. Assume that $a_1, b_1 \sim (t+c)^{-1}$, where $c > 0$ is a constant. More precisely, we assume that

$$\lambda_t^A = a_0 + 1_{\{\tau^B\le t\}}\frac{-n}{(t-\tau^B)+a_1}$$
$$\lambda_t^B = b_0 + 1_{\{\tau^A\le t\}}\frac{-m}{(t-\tau^A)+b_1},$$

where m and n are positive integers such that λ_t^A and λ_t^B are positive.

In this case we can calculate the joint survival probability via Proposition 4.1 to get:

$$\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} = \begin{cases} \displaystyle\sum_{k=0}^{m}\frac{(-1)^{k+1}{}_m\mathrm{P}_k}{a_0^k b_1^m}[b_1^{m-k}\mathrm{e}^{-(a_0+b_0)t_2}-(t_2-t_1+b_1)^{m-k} \\ \quad\times\mathrm{e}^{-a_0t_1-b_0t_2}]+\mathrm{e}^{-(a_0+b_0)t_2} & t_1\le t_2; \\ \displaystyle\sum_{k=0}^{n}\frac{(-1)^{k+1}{}_n\mathrm{P}_k}{b_0^k a_1^n}[a_1^{n-k}\mathrm{e}^{-(a_0+b_0)t_1}-(t_1-t_2+a_1)^{n-k} \\ \quad\times\mathrm{e}^{-a_0t_1-b_0t_2}]+\mathrm{e}^{-(a_0+b_0)t_1} & t_1>t_2. \end{cases}$$

The joint density and the marginal distributions of default times τ^A and τ^B are given by, respectively:

$$f(t_1,t_2) = \begin{cases} \displaystyle\sum_{k=0}^{m}\frac{(-1)^{k+1}{}_m\mathrm{P}_k}{a_0^k b_1^m}[(m-k)(m-k-1)+(a_0-b_0) \\ \quad\times(m-k)(t_2-t_1+b_1)-a_0b_0(t_2-t_1+b_1)^2] \\ \quad\times(t_2-t_1+b_1)^{m-k-2}\mathrm{e}^{-a_0t_1-b_0t_2} & t_1<t_2; \\ \displaystyle\sum_{k=0}^{n}\frac{(-1)^{k+1}{}_n\mathrm{P}_k}{b_0^k a_1^n}[(n-k)(n-k-1)+(b_0-a_0) \\ \quad\times(n-k)(t_2-t_1+a_1)-a_0b_0(t_2-t_1+b_1)^2] \\ \quad\times(t_2-t_1+a_1)^{n-k-2}\mathrm{e}^{-a_0t_1-b_0t_2} & t_1>t_2. \end{cases}$$

and

$$\begin{aligned} F_A(t_1) &= 1-\sum_{k=0}^{n}\frac{(-1)^{k+1}{}_n\mathrm{P}_k}{b_0^k a_1^n}[a_1^{n-k}\mathrm{e}^{-(a_0+b_0)t_1} \\ &\quad-(t_1+a_1)^{n-k}\mathrm{e}^{-a_0t_1}]-\mathrm{e}^{-(a_0+b_0)t_1}; \\ F_B(t_2) &= 1-\sum_{k=0}^{m}\frac{(-1)^{k+1}{}_m\mathrm{P}_k}{a_0^k b_1^m}[b_1^{m-k}\mathrm{e}^{-(a_0+b_0)t_2} \\ &\quad-(t_2+b_1)^{m-k}\mathrm{e}^{-b_0t_2}]-\mathrm{e}^{-(a_0+b_0)t_2}. \quad\square \end{aligned}$$

4.2. Multiple firm case

We now extend the formulae of the joint default probability and density function to the more general multiple firm case. To be more precise, let us assume that $I > 2$, and that the default intensities are given by

$$\lambda_t^i = a_{i,0}(t) + \sum_{j=1}^{I} a_{i,j}(t-\tau^j)\mathbf{1}_{\{\tau^j\le t\}}, \quad i=1,\ldots,I, \tag{4.6}$$

where $a_{i,i} = 0$ and $a_{i,j}$'s are deterministic functions satisfying (H3). We try to find an internal relationship between the given default intensities and their joint density function. To this end we first introduce the following intensity dynamics:

$$\lambda_t^{i,k} = a_{i,0}(t) + \sum_{j=1}^{k} a_{i,j}(t-\tilde\tau^j)\mathbf{1}_{\{\tilde\tau^j\le t\}}, \quad i=1,\ldots,I, \tag{4.7}$$

where $1 \le k \le I$. We note that $\lambda_t^{i,k}$'s do not jump at defaults by $\tilde\tau^j$ for $j \ge k+1$. In fact, the random variables can be split into two groups: $\{\tilde\tau^1, \tilde\tau^2, \ldots, \tilde\tau^k\}$ and $\{\tilde\tau^{k+1}, \tilde\tau^{k+2}, \ldots, \tilde\tau^I\}$. The first group has an impact on the second group, but not vice versa. In case of $k = 1$, for example $\tilde\tau^1$ has a default intensity of $a_{1,0}(t)$ so that $\mathbb{P}\{\tilde\tau^1 < t\} = 1 - \mathrm{e}^{-\int_0^t a_{1,0}(s)\mathrm{d}s}$, which does not depend on $\tilde\tau^j$ for $j \ne 1$. However, $\tilde\tau^j$'s rely on $\tilde\tau^1$ because $a_{1,0}(t)$ contributes to $\mathbb{P}\{\tilde\tau^j < t\}$ as can be seen in the two-firm case. In particular, $\lambda_t^{i,k}$'s are identical to λ_t^i's when $k = I$.

For $1 \le m \le I$, let us denote $f_m^k(t_1, t_2, \ldots, t_m)$ to be the joint density function of the default times $\tilde\tau^1, \tilde\tau^2, \ldots, \tilde\tau^m$ with the intensity dynamics in (4.7). For instance, $f_1^1(t_1)$ will be the density function of $\tilde\tau^1$ with the intensity $\lambda_t^{1,1} = a_{1,0}(t)$. Thus, $f_1^1(t_1) = a_{1,0}(t_1)\mathrm{e}^{-\int_0^{t_1} a_{1,0}(s)\mathrm{d}s}$. It can be easily seen that $f_2^2(t_1, t_2)$ becomes the joint density function of $\tilde\tau_1$ and $\tilde\tau_2$ with the intensity dynamics:

$$\lambda_t^{1,2} = a_{1,0}(t) + a_{1,2}(t-\tilde\tau^2)\mathbf{1}_{\{\tilde\tau^2\le t\}};$$
$$\lambda_t^{2,2} = a_{2,0}(t) + a_{2,1}(t-\tilde\tau^1)\mathbf{1}_{\{\tilde\tau^1\le t\}}.$$

Our goal is here to find $f_I^I(t_1, t_2, \ldots, t_I)$ representing the joint density function of $\tau^1, \tau^2, \ldots, \tau^I$ with the default intensities in (4.6). We shall obtain a relation between f_k^k and f_{k+1}^{k+1}. For notational convenience f_k^k will be denoted by f_k unless the notation becomes confusing.

Proposition 4.4. *Assume that $\{\lambda_t^{i,k}\}_{i=1}^I$ are given by (4.7), and assume that (H3) is in force. Then the following recursive relation holds:*

$$f_{k+1}(t_1, t_2, \ldots, t_{k+1}) = \left\{\sum_{j=0}^{k} a_{k+1,j}(t_{k+1}-t_j)\right\} \times \exp\left\{-\sum_{j=0}^{k}\int_{t_j}^{t_{k+1}} a_{k+1,j}(s-t_j)\mathrm{d}s\right\} f_k(t_1, t_2, \ldots, t_k), \tag{4.8}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_{k+1}$.

Proof. We first assume that $t_1 < t_2 < \cdots < t_{k+1}$. Given $\{\lambda_t^{i,k+1}\}_{i=1}^I$ by (4.7), we denote the joint distribution of $\tilde\tau^1, \ldots, \tilde\tau^{k+1}$ by $F_{k+1}(t_1, \ldots, t_{k+1}) \stackrel{\triangle}{=} \mathbb{P}\{\tilde\tau^1 \le t_1, \ldots, \tilde\tau^{k+1} \le t_{k+1}\}$. Note that

$$F_{k+1}(t_1, \ldots, t_{k+1}) = \mathbb{P}\{\tilde\tau^1 \le t_1, \tilde\tau^2 \le t_2, \ldots, \tilde\tau^k \le t_k\} - \mathbb{E}\{\mathbf{1}_{\{\tilde\tau^1 \le t_1\}}\mathbf{1}_{\{\tilde\tau^2 \le t_2\}} \cdots \mathbf{1}_{\{\tilde\tau^k \le t_k\}}\mathbf{1}_{\{\tilde\tau^{k+1} > t_{k+1}\}}\}.$$

Now we apply the change of measure as in the previous sections to get

$$\begin{aligned}
&\mathbb{E}\{\mathbf{1}_{\{\tilde\tau^1 \le t_1\}}\mathbf{1}_{\{\tilde\tau^2 \le t_2\}} \cdots \mathbf{1}_{\{\tilde\tau^k \le t_k\}}\mathbf{1}_{\{\tilde\tau^{k+1} > t_{k+1}\}}\} \\
&= \mathbb{E}^{\mathbb{P}^{k+1}}\left\{\mathbf{1}_{\{\tilde\tau^1 \le t_1\}}\mathbf{1}_{\{\tilde\tau^2 \le t_2\}} \cdots \mathbf{1}_{\{\tilde\tau^k \le t_k\}} \exp\left(-\int_0^{t_{k+1}} \lambda_s^{k+1,k+1} \mathrm{d}s\right)\right\} \\
&= \mathbb{E}^{\mathbb{P}^{k+1}}\left\{\mathbf{1}_{\{\tilde\tau^1 \le t_1\}}\mathbf{1}_{\{\tilde\tau^2 \le t_2\}} \cdots \mathbf{1}_{\{\tilde\tau^k \le t_k\}} \times \exp\left\{-\sum_{j=0}^k \int_{\tilde\tau^j}^{t_{k+1}} a_{k+1,j}(s - \tilde\tau^j)\mathrm{d}s\right\}\right\} \\
&= \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_k} \exp\left\{-\sum_{j=0}^k \int_{u^j}^{t_{k+1}} a_{k+1,j}(s-u^j)\mathrm{d}s\right\} \\
&\quad \times f_k(u^1, u^2, \ldots, u^m)\mathrm{d}u^1\mathrm{d}u^2 \ldots \mathrm{d}u^m,
\end{aligned}$$

where $\tilde\tau^0 \equiv 0$ and the last equality follows from the fact that under the measure $\mathbb{P}^{k+1}\lambda_t^{i,k+1} = \lambda_t^{i,k}$ for $t \le t_{k+1}$ and $i = 1, \ldots, k$. Denote $G_{k+1}(t_1, \ldots, t_{k+1}) \stackrel{\triangle}{=} \mathbb{P}\{\tilde\tau^1 \le t_1, \ldots, \tilde\tau^k \le t_k, \tilde\tau^{k+1} > t_{k+1}\}$. Then, it holds that

$$\frac{\partial^k G(t_1, t_2, \ldots, t_{k+1})}{\partial t_1 \partial t_2 \ldots \partial t_k} = \exp\left\{-\sum_{j=0}^k \int_{t_j}^{t_{k+1}} a_{k+1,j}(s-t_j)ds\right\} f_k(t_1, t_2, \ldots, t_k). \tag{4.9}$$

Since we have

$$\frac{\partial^{k+1}F(t_1, t_2, \ldots, t_{k+1})}{\partial t_1 \partial t_2 \ldots \partial t_{k+1}} = -\frac{\partial^{k+1}G(t_1, t_2, \ldots, t_{k+1})}{\partial t_1 \partial t_2 \ldots \partial t_{k+1}},$$

$-f_{k+1}(t_1, t_2, \ldots, t_{k+1})$ is equal to the derivative of (4.9) with respect to t_{k+1}. $\square$

We now consider the general case. In light of the two-firm case, we shall extend the domain of joint density function from the simplex $D \stackrel{\triangle}{=} \{(t_1, t_2, \ldots, t_I) \in \mathbb{R}_+^I : t_1 < t_2 < \cdots < t_I\}$ to the whole $\mathbb{R}_+^I$. To this end, we partition the space $\mathbb{R}_+^I$ as follows. Let σ be a permutation of $\{1, \ldots, I\}$ (we shall simply denote $\sigma(x)$ by (x) if the context is clear), and denote $\mathscr{P}(I)$ to be all such permutations. Clearly $|\mathscr{P}(I)| = I!$. For each $\sigma \in \mathscr{P}(I)$, we denote the corresponding rearrangement of $(t_1, \ldots, t_I)$ by $(t_{(1)}, \ldots, t_{(I)})$, and denote

$$D^{(\sigma)} \stackrel{\triangle}{=} \{(t_1, t_2, \ldots, t_I) \in \mathbb{R}_+^I : t_{(1)} < t_{(2)} < \cdots < t_{(I)}\}, \quad \sigma \in \mathscr{P}(I).$$

Clearly, the space $\mathbb{R}_+^I$ is partitioned into $I!$ disjoint regions $D^{(\sigma)}$, $\sigma \in \mathscr{P}(I)$. We assume that all τ^i's are continuous random variables so that the probability $\mathbb{P}\{\tau^i = \tau^j\} = 0$, if $j \ne j$. Therefore we need only consider the case when none of the two times of $t_1, t_2, \ldots, t_I$ is equal. We now extend the joint density function f_I to the whole space, and denote it by $g_I(t_1, t_2, \ldots, t_I)$, $(t_1, t_2, \ldots, t_I) \in \mathbb{R}_+^I$.

For any $\sigma \in \mathscr{P}(I)$, we permute the default times accordingly, and denote them by $(\tau^{(1)}, \ldots, \tau^{(I)})$. We then define the corresponding intensities, denoted by $(\lambda_t^{(1)}, \ldots, \lambda_t^{(I)})$, as follows

$$\lambda_t^{(i)} = a_{(i),0}(t) + \sum_{j=1}^I a_{(i),(j)}(t - \tau^{(j)})\mathbf{1}_{\{\tau^{(j)} \le t\}}, \quad i = 1, \ldots, I, \tag{4.10}$$

where (i) and (j) are the images of i and j under the permutation σ, respectively. Now, for each $\sigma \in \mathscr{P}(I)$ we can apply Proposition 4.4 on the region $D^{(\sigma)}$, with $(\lambda_t^1, \ldots, \lambda_t^I)$ being replaced by $(\lambda_t^{(1)}, \ldots, \lambda_t^{(I)})$, to obtain the joint density function on $D^{(\sigma)}$, which we denote by $f_I^{(\sigma)}$. We can then define

$$g_I(t_1, t_2, \ldots, t_I) = f_I^{(\sigma)}(t_{(1)}, t_{(2)}, \ldots, t_{(I)}), \quad \text{if } (t_1, t_2, \ldots, t_I) \in D^{(\sigma)}, \sigma \in \mathscr{P}(I). \tag{4.11}$$

It is now easy to check that g_I is indeed the joint density function of $\tau^1, \ldots, \tau^I$. In other words, we have proved the following result for the joint density function on the whole space.

Proposition 4.5. *Assume* (H1)–(H3). *The joint distribution of $\tau_1, \tau_2, \ldots, \tau_I$ can be expressed as*

$$\begin{aligned}
&\mathbb{P}\{\tau^1 \le t_1, \tau^2 \le t_2, \ldots, \tau^I \le t_I\} \\
&= \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_I} g_I(u_1, u_2, \ldots, u_I)\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_I \\
&= \sum_{\sigma\in\mathscr{P}(I)} \int\cdots\int_{D^{(\sigma)}\cap R_I} f_I^{(\sigma)}(u_{(1)}, u_{(2)}, \ldots, u_{(I)}) \times \mathrm{d}u_{(1)}\mathrm{d}u_{(2)}\cdots\mathrm{d}u_{(I)},
\end{aligned}$$

where g_I and $f_I^{(\sigma)}$'s are defined by (4.10)–(4.11), *$(u_{(1)}, u_{(2)}, \ldots, u_{(I)})$ is the reordering of $(u_1, u_2, \ldots, u_I)$ under the permutation $\sigma \in \mathscr{P}(I)$, and $R_I \stackrel{\triangle}{=} [0, t_1] \times \cdots \times [0, t_I]$.* $\square$

To conclude this section we look at a special case where the joint density function could be written down more explicitly. We assume that

$$a_{i,j}(t) = \frac{a_{i,j}}{c_{i,j}\mathrm{e}^t + 1}, \quad t \ge 0,\ i = 1, \ldots, I,\ j = 0, \ldots, I.$$

We assume further that $a_{i,i} = 0$, and $c_{i,0} = 0$ so that $a_{i,0}(t) = a_{i,0}$. Then by Proposition 4.4, we see that for any $1 \le k \le I$, it holds that

$$f_{k+1}(t_1, t_2, \ldots, t_{k+1}) = \left\{\sum_{j=0}^k \frac{a_{k+1,j}}{c_{k+1,j}\mathrm{e}^{(t_{k+1}-t_j)} + 1}\right\} \times \prod_{j=0}^k \left\{\frac{c_{k+1,j} + \mathrm{e}^{-(t_{k+1}-t_j)}}{c_{k+1,j} + 1}\right\}^{a_{k+1,j}} f_k(t_1, t_2, \ldots, t_k),$$

and $f_1(t_1) = a_{1,0}\mathrm{e}^{-a_{1,0}t_1}$. In particular, if $c_{i,j} \equiv 0$ for all i and j, then $a_{ij}(t) = a_{ij}$ so that we have a more explicit form of the joint density function:

$$f_I(t_1, t_2, \ldots, t_I) = \prod_{i=1}^I \left\{\left(\sum_{j=0}^{i-1} a_{i,j}\right) \mathrm{e}^{-\left(\sum_{k=0}^{I-i} a_{I-i+1,k} - \sum_{k=2}^{i} a_{I-k+2,I-i+1}\right)t_{I-i+1}}\right\}$$

where $t_1 < t_2 < \cdots < t_I$. The general version of joint density function is then given by

$$f_I^{(\sigma)}(t_{(1)}, t_{(2)}, \ldots, t_{(I)}) = \prod_{i=1}^I \left\{\left(\sum_{j=0}^{i-1} a_{(i),(j)}\right) \mathrm{e}^{-\left(\sum_{k=0}^{I-i} a_{(I-i+1),(k)} - \sum_{k=2}^{i} a_{(I-k+2),(I-i+1)}\right)t_{(I-i+1)}}\right\}$$

where $f_I^{(\sigma)}$ is defined on $D^{(\sigma)} = \{(t_1, t_2, \ldots, t_I) \in \mathbb{R}_+^I : t_{(1)} < t_{(2)} < \cdots < t_{(I)}\}$ for each $\sigma \in \mathscr{P}(I)$.

5. Dependent mortality models

In this section we turn our attention to the potential application of the correlated intensity model to problems in actuarial science. One such problem is to price insurance products involving multiple lives. The most commonly seen examples would be the pension plan where the multiple life status plays a decisive role. Although the study of more general models is possible, we would like to focus on the case for married couples, because some recently discovered empirical evidence (cf. e.g., Hu and Goldman (1990), Martikainen and Valkonen (1996), and Valkonen et al. (2004), to mention a few) indicates that the structure of the correlated intensities could fit well in these situations. We should note that the results in this section could also be applied to any two-firm models. In these cases they will be the natural extensions of the results in Jarrow and Yu (2001).

We begin by describing the so-called baseline mortality of a certain group of lives. We denote T_{x_i}, $i = 1, \dots, I$, to be the future life time random variables, which represents the remaining lifetimes of these individuals with current age x_i's, respectively. The *force of mortality* of each life, denoted by λ_{x_i}, $i = 1, \dots, I$, is defined by

$$\mathbb{P}\{T_{x_i} > t\} = \exp\left\{-\int_0^t \lambda_{x_i}(s)\mathrm{d}s\right\}, \quad i = 1, \dots, I.$$

It is clear that if we compare the future life time random variable to the default time, then the force of mortality should be the same as default intensity.

5.1. Dependent mortality of married couple

We now assume that $I = 2$, and the two lives under consideration are actually a married couple. Moreover, for technical convenience in what follows we assume that the baseline mortality of both individuals follows the Gompertz's law (cf. Bower et al., 1997). That is, we assume that

$$\lambda_{x_1}(t) = h_1 \mathrm{e}^{g_1(x_1+t)}, \qquad \lambda_{x_2}(t) = h_2 \mathrm{e}^{g_2(x_2+t)},$$

where $h_i > 0$ and $g_i > 0$ are the Gompertz parameters, for $i = 1, 2$.

Our main assumption for actual mortalities of the married couple, denoted by μ_{x_i}, $i = 1, 2$, is that they are correlated in the following way: of the mortality intensity correlation of x_1 and x_2 given as

$$\begin{cases} \mu_t^{x_1} = \lambda_{x_1}(t) + \mathbf{1}_{\{T_{x_2} \le t\}}\gamma_{x_1}(t - T_{x_2}) \\ \mu_t^{x_2} = \lambda_{x_2}(t) + \mathbf{1}_{\{T_{x_1} \le t\}}\gamma_{x_2}(t - T_{x_1}) \end{cases} \tag{5.1}$$

where γ_{x_i}, $i = 1, 2$, are deterministic functions such that the force of mortality $\mu_t^{x_i}$'s are positive for all $t > 0$.

We first give a easy result on the joint survival probability. We note that such a result seems to be new in the actuarial context.

Proposition 5.1. *Suppose that the forces of mortalities of the married couple are give by (5.1), and assume further that the functions γ_{x_i}, $i = 1, 2$, take the following form:*

$$\gamma_{x_1}(t) = \frac{n_1}{r_1 \mathrm{e}^t + 1}, \qquad \gamma_{x_2}(t) = \frac{n_2}{r_2 \mathrm{e}^t + 1},$$

where r_1, r_2 are positive constant and n_1, n_2 are positive integers. Then the joint survival probability of T_{x_1} and T_{x_2} is given by

$$\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\}$$

$$= \begin{cases} \dfrac{c(t_1,t_2)}{(r_2+1)^{n_2}} \displaystyle\sum_{k=0}^{n_2} \binom{n_2}{k} \frac{h_1}{g_1} r_2^{n_2-k} \mathrm{e}^{-k(t_2+x_1)+\frac{h_1}{g_1}\mathrm{e}^{g_1(x_1+t_1)}} \\ \quad \times \left(\Delta_k^1\left(\dfrac{\lambda_{x_1}(t_2)}{h_1}\right) - \Delta_k^1\left(\dfrac{\lambda_{x_1}(t_1)}{h_1}\right)\right) + c(t_2,t_2) \quad t_1 \le t_2; \\ \dfrac{c(t_1,t_2)}{(r_1+1)^{n_1}} \displaystyle\sum_{k=0}^{n_1} \binom{n_1}{k} \frac{h_2}{g_2} r_1^{n_1-k} \mathrm{e}^{-k(t_1+x_2)+\frac{h_2}{g_2}\mathrm{e}^{g_2(x_2+t_2)}} \\ \quad \times \left(\Delta_k^2\left(\dfrac{\lambda_{x_2}(t_1)}{h_2}\right) - \Delta_k^2\left(\dfrac{\lambda_{x_2}(t_2)}{h_2}\right)\right) + c(t_1,t_1) \quad t_1 > t_2, \end{cases} \tag{5.2}$$

where $\Delta_k^i(t) = \int_0^t y^{\frac{k}{g_i}} \mathrm{e}^{-\frac{h_i}{g_i}y}\mathrm{d}y$ for $i = 1, 2$ and

$$c(t_1,t_2) = \exp\left\{-\frac{h_1}{g_1}[\mathrm{e}^{g_1(x_1+t_1)} - \mathrm{e}^{g_1x_1}] - \frac{h_2}{g_2}[\mathrm{e}^{g_2(x_2+t_2)} - \mathrm{e}^{g_2x_2}]\right\}.$$

Proof. The proof is a direct application of the two-firm case in the counter-party risk model. Indeed, applying Proposition 4.1, or more precisely, applying (4.4) with $\tau^A = T_{x_1}$ and $\tau^B = T_{x_2}$ we have, for $t_1 \le t_2$,

$$\begin{aligned}
&\mathbb{P}\{T_{x_1} > t_1, T_{x_2} > t_2\} \\
&= c(t_1,t_2)\left(\int_{t_1}^{t_2} \lambda_{x_1}(x)\,\mathrm{e}^{-\int_x^{t_2}\gamma_{x_2}(s-x)\mathrm{d}s - \int_{t_1}^x \lambda_{x_1}(s)\mathrm{d}s}\mathrm{d}x + \int_{t_2}^{\infty} \lambda_{x_1}(x)\mathrm{e}^{-\int_{t_1}^x \lambda_{x_1}(s)\mathrm{d}s}\mathrm{d}x\right) \\
&= c(t_1,t_2)\int_{t_1}^{t_2}\left[\frac{r_2 + \mathrm{e}^{-(t_2-x)}}{r_2+1}\right]^{n_2} \lambda_{x_1}(x)\mathrm{e}^{-\int_{t_1}^x \lambda_{x_1}(s)\mathrm{d}s}\mathrm{d}x + c(t_1,t_2)\left[-\mathrm{e}^{-\int_{t_1}^x \lambda_{x_1}(s)\mathrm{d}s}\right]_{t_2}^{\infty} \\
&= \frac{c(t_1,t_2)}{(r_2+1)^{n_2}}\int_{t_1}^{t_2}[r_2 + \mathrm{e}^{-(t_2-x)}]^{n_2} h_1 \\
&\quad \times \mathrm{e}^{g_1(x_1+x) - \frac{h_1}{g_1}[\mathrm{e}^{g_1(x_1+x)} - \mathrm{e}^{g_1(x_1+t_1)}]}\mathrm{d}x + c(t_1,t_2)\mathrm{e}^{-\int_{t_1}^{t_2}\lambda_{x_1}(s)\mathrm{d}s} \\
&= \frac{c(t_1,t_2)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k} h_1 r_2^{n_2-k}\mathrm{e}^{-kt_2+g_1x_1+\frac{h_1}{g_1}\mathrm{e}^{g_1(x_1+t_1)}} \\
&\quad \times \int_{t_1}^{t_2} \mathrm{e}^{(k+g_1)x - \frac{h_1}{g_1}\mathrm{e}^{g_1(x_1+x)}}\mathrm{d}x + c(t_2,t_2) \\
&= \frac{c(t_1,t_2)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{h_1}{g_1} r_2^{n_2-k}\mathrm{e}^{-k(t_2+x_1)+\frac{h_1}{g_1}\mathrm{e}^{g_1(x_1+t_1)}} \\
&\quad \times \int_{\mathrm{e}^{g_1(x_1+t_1)}}^{\mathrm{e}^{g_1(x_1+t_2)}} y^{\frac{k}{g_1}}\mathrm{e}^{-\frac{h_1}{g_1}y}\mathrm{d}y + c(t_2,t_2) \\
&= \frac{c(t_1,t_2)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{h_1}{g_1} r_2^{n_2-k}\mathrm{e}^{-k(t_2+x_1)+\frac{h_1}{g_1}\mathrm{e}^{g_1(x_1+t_1)}} \\
&\quad \times \left(\Delta_k^1(\mathrm{e}^{g_1(x_1+t_2)}) - \Delta_k^1(\mathrm{e}^{g_1(x_1+t_1)})\right) + c(t_2,t_2)
\end{aligned}$$

where $\Delta_k^1(t) = \int_0^t y^{\frac{k}{g_1}}\mathrm{e}^{-\frac{h_1}{g_1}y}\mathrm{d}y$; and

$$\begin{aligned}
c(t_1,t_2) &= \exp\left\{-\int_0^{t_1}\lambda_{x_1}(s)\mathrm{d}s - \int_0^{t_2}\lambda_{x_2}(s)\mathrm{d}s\right\} \\
&= \exp\left\{-\frac{h_1}{g_1}[\mathrm{e}^{g_1(x_1+t_1)} - \mathrm{e}^{g_1x_1}] - \frac{h_2}{g_2}[\mathrm{e}^{g_2(x_2+t_2)} - \mathrm{e}^{g_2x_2}]\right\},
\end{aligned}$$

proving (5.2) for the case $t_1 < t_2$. Similarly, for $t_2 \le t_1$ we have

$$\begin{aligned}
\mathbb{P}\{\tau^A > t_1, \tau^B > t_2\} &= \frac{c(t_1,t_2)}{(r_1+1)^{n_1}}\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{h_2}{g_2} \\
&\quad \times r_1^{n_1-k}\mathrm{e}^{-k(t_1+x_2)+\frac{h_2}{g_2}\mathrm{e}^{g_2(x_2+t_2)}}\left(\Delta_k^2(\mathrm{e}^{g_2(x_2+t_1)}) - \Delta_k^2(\mathrm{e}^{g_2(x_2+t_2)})\right)
\end{aligned}$$

$$+c(t_1,t_2),$$

where $\Delta_k^2(t) = \int_0^t y^{\frac{k}{g_2}} \mathrm{e}^{-\frac{h_2}{g_2}y}\mathrm{d}y$. Note that when $t_1 = t_2 = t$, we have $\mathbb{P}\{\tau^A > t, \tau^B > t\} = c(t,t)$. This completes the proof. □

Having obtained the joint probability at hand, we now turn our attention to the two important elements in the multi-life models: the *joint-life* status and the *last survivor* status. We recall that if T_{x_1}, $T_{x_2}, \ldots, T_{x_n}$ be n future life time random variables, then their joint-life status and last-survivor status are given by, respectively:

$$T_m = T_{x_1,\ldots,x_n} \triangleq \min\{T_{x_1}, T_{x_2}, \ldots, T_{x_n}\},$$

$$T_M = T_{\overline{x_1,\ldots,x_n}} \triangleq \max\{T_{x_1}, T_{x_2}, \cdots, T_{x_n}\}.$$

Clearly, the joint-life status and the last survivor status are, respectively, the *smallest* and the *largest order statistics* of the n lives. In the default risk literature, T_m is often called the first-to-default time, and T_M the last-to-default time, for obvious reasons.

The dependent mortality of the married couple could be considered as a special two firm case, which could be treated more specifically. We first observe the following relationships among T_{x_1}, T_{x_2}, T_m, and T_M:

$$T_M + T_m = T_{x_1} + T_{x_2}, \qquad T_M T_m = T_{x_1} T_{x_2}.$$

Furthermore, note that

$$\{T_{x_1} \le t\} \cap \{T_{x_2} \le t\} = \{T_M \le t\},$$
$$\{T_{x_1} \le t\} \cup \{T_{x_2} \le t\} = \{T_m \le t\}.$$

Thus, denoting $F_{T_{x_i}}$, $i = 1, 2$, F_m and F_M to be the distribution function of T_m, T_M, T_{x_1}, and T_{x_2}, respectively, we have

$$F_M(t) + F_m(t) = F_{T_{x_1}}(t) + F_{T_{x_2}}(t), \quad t \ge 0.$$

Consequently, given that we know the marginal distributions F_{x_i}, $i = 1, 2$, we can derive F_M if we can calculate F_m, and vice versa. For example, assume that similar to the counterparty risks studied in the previous sections, the force of mortalities of the couple, denoted by μ_{x_1} and μ_{x_2}, respectively, satisfy the following relations:

$$\begin{cases} \mu_{x_1}(t) = a_0 + \mathbf{1}_{\{T_{x_2}\le t\}} \dfrac{n}{a_1 \mathrm{e}^{(t-T_{x_2})}+1}; \\ \mu_{x_2}(t) = b_0 + \mathbf{1}_{\{T_{x_1}\le t\}} \dfrac{m}{b_1 \mathrm{e}^{(t-T_{x_1})}+1}. \end{cases}$$

Then, borrowing the formula (5.5) to be proved in the next subsection, we see that

$$F_m(t) = 1 - \mathbb{P}\{T_m > t\} = 1 - \mathrm{e}^{-(a_0+b_0)t}.$$

Consequently we obtain

$$\begin{aligned} F_M(t) &= F_{T_{x_1}}(t) + F_{T_{x_2}}(t) - F_m(t) \\ &= 1 - \frac{b_0}{(a_1+1)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{a_1^{n-k}}{b_0-k}[\mathrm{e}^{-(a_0+k)t} \\ &\quad - \mathrm{e}^{-(a_0+b_0)t}] - \mathrm{e}^{-(a_0+b_0)t} + 1 - \frac{a_0}{(b_1+1)^m} \\ &\quad \times \sum_{k=0}^{m}\binom{m}{k}\frac{b_1^{m-k}}{a_0-k}[\mathrm{e}^{-(b_0+k)t} - \mathrm{e}^{-(a_0+b_0)t}] \\ &\quad - \mathrm{e}^{-(a_0+b_0)t} - 1 + \mathrm{e}^{-(a_0+b_0)t}. \end{aligned} \tag{5.3}$$

Finally, taking the derivative we can obtain the density of T_M:

$$\begin{aligned} f_M(t) &= \frac{b_0}{(a_1+1)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{a_1^{n-k}}{b_0-k}[(a_0+k)\mathrm{e}^{-(a_0+k)t} \\ &\quad -(a_0+b_0)\mathrm{e}^{-(a_0+b_0)t}] + \frac{a_0}{(b_1+1)^m}\sum_{k=0}^{m}\binom{m}{k}\frac{b_1^{m-k}}{a_0-k} \\ &\quad \times [(b_0+k)\mathrm{e}^{-(b_0+k)t} - (a_0+b_0)\mathrm{e}^{-(a_0+b_0)t}] \\ &\quad + (a_0+b_0)\mathrm{e}^{-(a_0+b_0)t}. \end{aligned} \tag{5.4}$$

We note that the formulae (5.3) and (5.4) are also valid for the last-to-default time (or the second-to-default time) in a two-firm counterparty risk models.

5.2. Joint-life status in multi-life model

We now turn our attention to the joint-life status for multiple life models. Since this case will naturally correspond to the first-to-default time for a multiple-firm counterparty risk model, we shall return to the language in Section 4 to make the arguments more coherent, and thus our results will be more general. The joint-life status of the married couple will simply be the special case with $I = 2$. It should be worth noting that the first-to-default time is usually the starting point of the chain of events in a mutually dependent economy, and it is often used as a main factor to characterize the default dependence, for instance, for a group of high quality credits (cf., e.g., Yu, 2007).

We begin by assuming that there are I firms in an economy (or I lives), and their default intensities (or force of mortalities) have the following relationship:

$$\begin{aligned} \lambda_t^i &= a_0^i(t) + \sum_{k\ne i} a_k^i(t)\mathbf{1}_{\{\tau^k\le t\}} \\ &= a_0^i(t) + \sum_{k\ne i} a_k^i(t) N_s^i, \quad i = 1,\ldots,I, \end{aligned}$$

where a_0^i's and a_k^i's are $\{\mathcal{F}_t^X\}$-adapted and $\{\mathcal{H}_t^i\}$-adapted processes, respectively, and both are positive processes so that the intensities λ_t^i are positive processes satisfying (H1). The survival probability of the first-to-default time, defined by $\tau_m \triangleq \min\{\tau^1, \tau^2, \ldots, \tau^I\}$, can be expressed as

$$\mathbb{P}\{\tau_m > t\} = \mathbb{P}\{\tau^1 > t, \tau^2 > t, \ldots, \tau^I > t\}.$$

Applying (3.9) we obtain that

$$\begin{aligned} \mathbb{P}\{\tau_m > t\} &= \mathbb{P}\{\tau^1 > t, \tau^2 > t, \ldots, \tau^I > t\} \\ &= \mathbb{E}^{1,2,\ldots,I}\left\{\exp\left\{-\int_0^t (\lambda_s^1 + \lambda_s^2 + \cdots + \lambda_s^I)\mathrm{d}s\right\}\right\} \\ &= \mathbb{E}^{1,2,\ldots,I}\left\{\exp\left\{-\int_0^t [a_0^1(s) + a_0^2(s) + \cdots + a_0^I(s)]\mathrm{d}s\right\}\right\}. \end{aligned}$$

The third equality in the above follows from the fact that $\mathbf{1}_{\{\tau^i\le s\}} = 0$ for $s \le t$ under the probability measure $\mathbb{P}^{1,2,\ldots,I}$.

In the special case when a_0^i's are all deterministic, the calculation becomes even simpler:

$$\mathbb{P}\{\tau_m > t\} = \exp\left\{-\int_0^t [a_0^1(s) + a_0^2(s) + \cdots + a_0^I(s)]\mathrm{d}s\right\}. \tag{5.5}$$

Thus the density function of the first-to-default time τ_m will simply be

$$\begin{aligned} f_m(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{\tau_m > t\} \\ &= \left(\sum_{i=1}^{I} a_0^i(t)\right)\exp\left\{-\int_0^t \left[\sum_{i=1}^{I} a_0^i(s)\right]\mathrm{d}s\right\}. \end{aligned}$$

It is worth noting that in this simple case, the distribution of the first-to-default time τ_m is not affected by the default processes N^i's.

Next, we consider the *conditional* survival probability of τ_m, given $\mathcal{F}_t$. Applying (3.10) we have

$$\begin{aligned} \mathbb{P}\{\tau_m > T|\mathcal{F}_t\} &= \mathbb{P}\{\tau^1 > T, \tau^2 > T, \ldots, \tau^I > T|\mathcal{F}_t\} \\ &= \mathbf{1}_{\{\tau_t^1>t\}}\mathbf{1}_{\{\tau_t^2>t\}}\cdots\mathbf{1}_{\{\tau_t^I>t\}}\mathbb{E}^{1,2,\ldots,I}\left\{\exp\left\{-\int_t^T [\lambda_s^1\right.\right. \end{aligned}$$

$$+\lambda_s^2+\cdots+\lambda_s^I]\mathrm{d}s\Big\}\Big|\mathcal{F}_t\Big\}$$
$$=\mathbf{1}_{\{\tau_m>t\}}\mathbb{E}^{1,2,\ldots,I}\left\{\exp\left\{-\int_t^T[a_0^1(s)\right.\right.$$
$$\left.\left.+\ a_0^2(s)+\cdots+a_0^I(s)]\mathrm{d}s\right\}\Big|\mathcal{F}_t\right\}.$$

Again, the third equality holds because $\mathbf{1}_{\{\tau^i\le s\}}=0$ for $t\le s\le T$ under the probability measure $\mathbb{P}^{1,2,\ldots,I}$. Furthermore, in the special cases where a_0^i's are all deterministic, we have

$$\mathbb{P}\{\tau_m>T|\mathcal{F}_t\}$$
$$=\mathbf{1}_{\{\tau_m>t\}}\exp\left\{-\int_t^T[a_0^1(s)+a_0^2(s)+\cdots+a_0^I(s)]\mathrm{d}s\right\}.$$

In the rest of the section we apply Theorem 3.2 to the aforementioned special case where the first-to-default time is the main reason of correlation among the defaults. Such a case was proposed in Yu (2007), while considering a population of high quality credits with relatively infrequent defaults. We note that in Yu (2007) the following type of intensities were considered:

$$\lambda_t^i\overset{\triangle}{=}(a_1+a_2\mathbf{1}_{\{\tau_m\le t\}})\mathbf{1}_{\{\tau^i>t\}},\quad t\ge0,\ i=1,\ldots,n,$$

where $\tau_m=\min\{\tau^1,\tau^2,\ldots,\tau^I\}$. Let us extend the model slightly by allowing each firm to have its own coefficients a_1 and a_2. That is, in what follows we assume that

$$\lambda_t^i\mathbf{1}_{\{\tau^i>t\}}=(a_i+b_i\mathbf{1}_{\{\tau_m\le t\}})\mathbf{1}_{\{\tau^i>t\}},\quad i=1,\ldots,I.$$

We observe that, if we denote $\tau_m^{-i}=\min\{\tau^1,\ldots,\tau^{i-1},\tau^{i+1},\ldots,\tau^I\}$ to be the first-to-default time among all but firm i, then it is readily seen that

$$\lambda_t^i=a_i+b_i\mathbf{1}_{\{\tau_m^{-i}\le t\}}.$$

We assume that a_i and a_i+b_i are all positive so that the $\lambda_t^i>0$ for all $t\ge0$, and all i.

To obtain the marginal distribution of τ^i, we claim that

$$\mathbb{P}\{\tau^i>t\}=\frac{(A-a_i)\mathrm{e}^{-(a_i+b_i)t}-b_i\mathrm{e}^{-At}}{A-a_i-b_i},$$

where $A=\sum_{k=1}^I a_k$.

To see this, we first apply Theorem 3.2-(ii) to get

$$\mathbb{P}\{\tau^i>t\}=\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_0^t\lambda_s^i\mathrm{d}s\right\}\right\}$$
$$=\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_0^t(a_i+b_i\mathbf{1}_{\{\tau_m^{-i}\le t\}})\mathrm{d}s\right\}\right\}$$
$$=\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{(-a_it-b_i-(t-\tau_m^{-i})\mathbf{1}_{\{\tau_m^{-i}\le t\}})\right\}\right\}$$
$$=\int_0^t\mathrm{e}^{-a_it-b_i(t-x)}\mathbb{P}^i(\tau_m^{-i}\in\mathrm{d}x)+\int_t^\infty\mathrm{e}^{-a_it}\mathbb{P}^i(\tau_m^{-i}\in\mathrm{d}x)$$
$$=(A-a_i)\mathrm{e}^{-(a_i+b_i)t}\int_0^t\mathrm{e}^{[b_i-(A-a_i)]x}\mathrm{d}x+(A-a_i)\mathrm{e}^{-a_it}$$
$$\times\int_t^\infty\mathrm{e}^{-(A-a_i)x}\mathrm{d}x$$
$$=(A-a_i)\mathrm{e}^{-(a_i+b_i)t}\frac{\mathrm{e}^{[b_i-(A-a_i)]t}-1}{b_i-(A-a_i)}+(A-a_i)\mathrm{e}^{-a_it}\frac{\mathrm{e}^{-(A-a_i)t}}{A-a_i}$$
$$=\frac{A-a_i}{b_i-(A-a_i)}[\mathrm{e}^{-At}-\mathrm{e}^{-(a_i+b_i)t}]+\mathrm{e}^{-At}$$
$$=\frac{(A-a_i)\mathrm{e}^{-(a_i+b_i)t}-b_i\mathrm{e}^{-At}}{A-a_i-b_i}.$$

In the above, the fifth equality follows from $\mathbb{P}^i(\tau_m^{-i}\in\mathrm{d}x)=(A-a_i)\mathrm{e}^{-(A-a_i)x}\mathrm{d}x$ since $\mathbb{E}^{\mathbb{P}^i}[\mathbf{1}_{\{\tau_m^{-i}>x\}}]=\mathrm{e}^{-(A-a_i)x}$.

Clearly, the marginal density function of τ^i can be obtained by simply differentiating the expression above:

$$f_i(t)=\frac{(a_i+b_i)(A-a_i)\mathrm{e}^{-(a_i+b_i)t}-b_iA\mathrm{e}^{-At}}{A-a_i-b_i}.$$

In the special case where $a_i\equiv r$, we have

$$\mathbb{P}\{\tau^i>t\}=\frac{(I-1)r\mathrm{e}^{-(r+b_i)t}-b_i\mathrm{e}^{-Irt}}{(I-1)r-b_i}.\tag{5.6}$$

One can regard r as a constant interest rate so that the intensity of a firm is influenced by an interest rate and an impact from a first-to-default time among I firms except the firm. The marginal density function of τ^i is also given by

$$f_i(t)=\frac{(I-1)r(r+b_i)\mathrm{e}^{-(r+b_i)t}-Irb_i\mathrm{e}^{-Irt}}{(I-1)r-b_i}.$$

It is easy to see that as $I\to\infty$, $\mathbb{P}(\tau^i>t)$ in (5.6) converges to $\mathrm{e}^{-(r+b_i)t}$. This means that enlarging the number of firms causes the intensity of τ^i to become $r+b_i$. Yu (2007) states that as I increases the first default occurs sooner so that τ^i is equal to zero and the intensity of τ^i is equal to $r+b_i$ almost surely in the limit of $I\to\infty$.

6. Flight to quality

The term "flight to quality" refers to the phenomenon that investors move their capital away from riskier investments to the safest possible investment vehicles, e.g., treasury bonds. There is ample evidence of such actions in the current economic downturn. In this section we apply the results in the previous sections to give some quantitative description of the impact of such a flight on the term structure of interest rates in the bond market, and consequently the prices of treasury bonds. Our discussion will be based on the model suggested by Collin-Dufresne et al. (2003) and Collin-Dufresne et al. (2004), with modifications using the results established in this paper.

In Collin-Dufresne et al. (2004) the flight to quality was described by assuming that the risk-neutral dynamics of the risk-free interest rate process is given by

$$r_t=r_0+J\mathbf{1}_{\{\tau\le t\}},\quad t\ge0,\tag{6.1}$$

where τ is an exogenous default time, $r_0\ge0$, and $J\ge-r_0$. Roughly speaking, the model suggests that the effect of a certain default (or any event) is that the investors move into (or out of) the bond market which triggers a sudden decrease ($J>0$) or increase ($J<0$) of the interest rate. The requirement that $r_0+J\ge0$ is obvious. To guarantee that the default time is exogenous, it is commonly assumed that τ is a totally inaccessible stopping time, with a constant intensity λ.

We now look a little closer to this model. Let us assume that there are actually I firms in the market, whose performance are closely watched by the investors. We denote their default times by τ^i, $i=1,\ldots,I$, with intensities λ^i, respectively. It should be reasonable to assume that it is not the default of any single firm that triggers flight to quality, but rather the collective default of a group (may assume all) of these firms that do. In other words, we modify the definition of the interest rate process (6.1) as follows:

$$r_t=r_0+J\mathbf{1}_{\{\tau_M\le t\}},\quad t\ge0,\tag{6.2}$$

where $\tau_M\overset{\triangle}{=}\max\{\tau^1,\ldots,\tau^I\}$ is the last-to-default time. Our purpose is then to derive the pricing formula for the defaultable zero-coupon bonds.

We first recall the process $N_t^i \triangleq \mathbf{1}_{\{\tau^i\le t\}}$, $i=1,\ldots,I$. It is readily seen that (6.2) can be written as $r_t = r_0 + J\prod_{i=1}^I N_t^i = r(N_t^1, N_t^2, \ldots, N_t^I)$. Now recall the general setup in Section 2, in what follows we assume that the default-free interest rate takes the following form:

$$r_t = r^0(X_t) + J\prod_{i=1}^I N_t^i = r(X_t, N_t^1, N_t^2, \ldots, N_t^I),$$

where r^0 is a deterministic function, and thus r is an $\{\mathcal{F}_t\}$-adapted process (see (2.1)).

The prices of a default-free zero-coupon bond and the defaultable zero-coupon bonds issued by firm i are thus given by, respectively,

$$p(t,T) = \mathbb{E}\left\{\exp\left(-\int_t^T r_s\mathrm{d}s\right)\middle|\mathcal{F}_t\right\},$$

and

$$v^i(t,T) = \mathbb{E}\left\{\exp\left(-\int_t^T r_s\mathrm{d}s\right)\mathbf{1}_{\{\tau^i>T\}} + \exp\left(-\int_t^T r_s\mathrm{d}s\right)\delta^i\mathbf{1}_{\{\tau^i\le T\}}\middle|\mathcal{F}_t\right\}, \quad (6.3)$$

where δ^i's are the *recovery rates* of firm i, $i=1,\ldots,I$. Now if we recall the definition (2.2) for the filtrations $\{\mathcal{H}_t^i\}$, $i=1,\ldots,I$ and denote $r_s^{-i} = r(X_s, N_s^1, \ldots, N_s^{i-1}, 0, N_s^{i+1}\ldots, N_s^I)$, then

$$r_t\,\mathbf{1}_{\{\tau^i>T\}} = r(X_t, N_t^1, \ldots, N_t^{i-1}, 0, N_t^{i+1}\ldots, N_t^I)\mathbf{1}_{\{\tau^i>T\}}, \quad t\le T$$

and one can easily derive the following:

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(-\int_t^T r_s\mathrm{d}s\right)\mathbf{1}_{\{\tau^i>T\}}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau^i>t\}}\frac{\mathbb{E}\left\{\exp\left(-\int_t^T r_s\mathrm{d}s\right)\mathbf{1}_{\{\tau^i>T\}}|\mathcal{H}_t^i\right\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathcal{H}_t^i\}}\\
&= \mathbf{1}_{\{\tau^i>t\}}\frac{\mathbb{E}\left\{\exp\left(-\int_t^T r_s^{-i}\mathrm{d}s\right)\mathbb{E}\left\{\mathbf{1}_{\{\tau^i>T\}}|\mathcal{H}_T^i\right\}\middle|\mathcal{H}_t^i\right\}}{\mathbb{E}\{\mathbf{1}_{\{\tau^i>t\}}|\mathcal{H}_t^i\}}\\
&= \mathbf{1}_{\{\tau^i>t\}}\frac{\mathbb{E}\left\{\exp\left(-\int_t^T r_s^{-i}\mathrm{d}s\right)\exp\left(-\int_0^T \lambda_s^i\mathrm{d}s\right)\middle|\mathcal{H}_t^i\right\}}{\exp\left(-\int_0^t \lambda_s^i\mathrm{d}s\right)}\\
&= \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}\left\{\exp\left(-\int_t^T (r_s^{-i}+\lambda_s^i)\mathrm{d}s\right)\middle|\mathcal{H}_t^i\right\}. \quad (6.4)
\end{aligned}$$

In the above, the second equality is due to the fact that $\exp(-\int_t^T r_s^{-i}\mathrm{d}s)$ is $\mathcal{H}_T^i$-measurable. Consequently, combining (6.3) and (6.4) we see that the prices of zero-coupon bonds with maturity T issued by firm i at time t are given by

$$v^i(t,T) = \delta^i p(t,T) + (1-\delta^i)\mathbf{1}_{\{\tau^i>t\}} \times\mathbb{E}\left\{\exp\left(-\int_t^T (r_s^{-i}+\lambda_s^i)\mathrm{d}s\right)\middle|\mathcal{H}_t^i\right\}, \quad i=1,\ldots,I. \quad (6.5)$$

Using the representation theorem developed in Section 3 we can write the pricing formulae (6.5) in a different way. Indeed, if we define $Z_t^i = \mathbf{1}_{\{\tau^i>t\}}\exp\{\int_0^t \lambda_s^i\mathrm{d}s\}$ and $\mathrm{d}\mathbb{P}^i = Z_T^i\mathrm{d}\mathbb{P}$ (recall (3.1) and (3.5)), then we have

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left\{-\int_t^T r_s\mathrm{d}s\right\}\mathbf{1}_{\{\tau^i>T\}}\middle|\mathcal{F}_t\right\}\\
&= \mathbb{E}\left\{Z_T^i\exp\left\{-\int_t^T r_s\mathrm{d}s\right\}\exp\left\{-\int_0^T \lambda_s^i\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}\\
&= Z_t^i\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T r_s\mathrm{d}s\right\}\exp\left\{-\int_0^T \lambda_s^i\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T (r_s+\lambda_s^i)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}. \quad (6.6)
\end{aligned}$$

We remark that under $\mathbb{P}^i$ one has

$$\begin{aligned}
&\mathbf{1}_{\{\tau^i>t\}}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T (r_s+\lambda_s^i)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T (r_s^{-i}+\lambda_s^i)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}.
\end{aligned}$$

Thus the price formula (6.5) can be written as

$$v^i(t,T) = \delta^i p(t,T) + (1-\delta^i)\mathbf{1}_{\{\tau^i>t\}} \times\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T (r_s+\lambda_s^i)\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}, \quad i=1,\ldots,I.$$

In the rest of the section we specify the parameters to calculate the prices of the defaultable zero coupon bond in the case of flight to quality. For simplicity we shall consider the two-firm case ($I=2$). Assume that the default intensities of the two firms A and B are given by

$$\lambda_t^1 = a_0 + \mathbf{1}_{\{\tau^2\le t\}}\frac{n}{a_1\mathrm{e}^{(t-\tau^2)}+1},$$
$$\lambda_t^2 = b_0 + \mathbf{1}_{\{\tau^1\le t\}}\frac{m}{b_1\mathrm{e}^{(t-\tau^1)}+1};$$

and we assume that the interest rate of the defaultable bond is given

$$r_t = r_0 + J1_{\{\tau_M\le t\}} = r_0 + JN_t^1N_t^2, \quad t\ge 0,$$

where $\tau_M = \max\{\tau^1, \tau^2\}$, and r_0 is a constant.

We begin by looking at the default-free zero-coupon bond price $p(t,T)$. We first note that in this it holds that

$$\begin{aligned}
p(t,T) &= \mathbb{E}\left\{\exp\left\{-\int_t^T r_s\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau_M\le t\}}\mathrm{e}^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau_M>t\}}\\
&\quad\times\mathbb{E}\left\{\exp\left\{-\int_t^T (r_0+J\mathbf{1}_{\{\tau_M\le s\}})\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau_M\le t\}}\mathrm{e}^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau_M>t\}}\\
&\quad\times\mathbb{E}\left\{\exp\left\{-r_0(T-t)-J(T-\tau_M)\mathbf{1}_{\{t<\tau_M\le T\}}\right\}\middle|\mathcal{F}_t\right\}\\
&= \mathbf{1}_{\{\tau_M\le t\}}\mathrm{e}^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau_M>t\}}\\
&\quad\times\int_t^T \mathrm{e}^{-r_0(T-t)-J(T-u)}\mathbb{P}\{\tau_M\in\mathrm{d}u|\mathcal{F}_t\}\\
&\quad+\mathbf{1}_{\{\tau_M>t\}}\int_T^\infty \mathrm{e}^{-r_0(T-t)}\mathbb{P}\{\tau_M\in\mathrm{d}u|\mathcal{F}_t\}. \quad (6.7)
\end{aligned}$$

Here $\mathbb{P}(\tau_M>\mathrm{d}u|\mathcal{F}_t)$ denotes the condition distribution of τ_M given $\mathcal{F}_t$. Note that $\mathbf{1}_{\{\tau_M>u\}} = \mathbf{1}_{\{\tau^1>u\}} + \mathbf{1}_{\{\tau^2>u\}} - \mathbf{1}_{\{\tau^1>u\}}\mathbf{1}_{\{\tau^2>u\}}$, we have

$$\begin{aligned}
\mathbb{P}\{\tau_M>u|\mathcal{F}_t\} &= \mathbb{E}\{\mathbf{1}_{\{\tau^1>u\}}|\mathcal{F}_t\} + \mathbb{E}\{\mathbf{1}_{\{\tau^2>u\}}|\mathcal{F}_t\}\\
&\quad-\mathbb{E}\{\mathbf{1}_{\{\tau^1>u\}}\mathbf{1}_{\{\tau^2>x\}}|\mathcal{F}_t\}. \quad (6.8)
\end{aligned}$$

Applying Theorem 3.2-(iii) and using the arguments in Example 4.2 we can derive

$$\begin{aligned}
\mathbb{P}\{\tau^1>u|\mathcal{F}_t\} &= \mathbb{E}\{\mathbf{1}_{\{\tau^1>u\}}|\mathcal{F}_t\}\\
&= \mathbf{1}_{\{\tau^1>t\}}\mathbb{E}^{\mathbb{P}^1}\left\{\exp\left\{-\int_t^u \lambda_s^1\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\}
\end{aligned}$$

$$= \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2\le t\}} e^{-a_0(u-t)} \left[\frac{a_1+e^{-(u-\tau^2)}}{a_1+e^{-(t-\tau^2)}}\right]^n + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{b_0}{(a_1+1)^n}\sum_{k=0}^n \binom{n}{k}\frac{a_1^{n-k}}{b_0-k} \times [e^{-(a_0+k)(u-t)} - e^{-(a_0+b_0)(u-t)}] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} e^{-(a_0+b_0)(u-t)}. \tag{6.9}$$

Here we used the fact that for any fixed $u \ge t$, the random variable $\mathbf{1}_{\{\tau^2\le t\}} e^{-a_0(u-t)}[\frac{a_1+e^{-(u-\tau^2)}}{a_1+e^{-(t-\tau^2)}}]^n$ is $\mathcal{F}_t$-measurable (hence the conditional expectation with respect to $\mathcal{F}_t$ disappears). Similarly, we can calculate $\mathbb{P}\{\tau^2 > u|\mathcal{F}_t\}$ correspondingly. Moreover, note that $\lambda_s^1 = a_0$ and $\lambda_s^2 = b_0$ for $s \le u \le T$ under the measure $P^{1,2}$, we have

$$\mathbb{E}[\mathbf{1}_{\{\tau^1>u\}}\mathbf{1}_{\{\tau^2>u\}}|\mathcal{F}_t] = \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}}\mathbb{E}^{\mathbb{P}^{1,2}}\left\{\exp\left\{-\int_t^u(\lambda_s^1+\lambda_s^2)ds\right\}\right\} = \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} e^{-(a_0+b_0)(u-t)}. \tag{6.10}$$

Combining (6.8)–(6.10) we obtain

$$\mathbb{P}\{\tau_M > u|\mathcal{F}_t\} = \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2\le t\}} e^{-a_0(u-t)} \left[\frac{a_1+e^{-(u-\tau^2)}}{a_1+e^{-(t-\tau^2)}}\right]^n + \mathbf{1}_{\{\tau^2>t\}}\mathbf{1}_{\{\tau^1\le t\}} e^{-b_0(u-t)} \left[\frac{b_1+e^{-(u-\tau^1)}}{b_1+e^{-(t-\tau^1)}}\right]^m + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{b_0}{(a_1+1)^n}\sum_{k=0}^n \binom{n}{k}\frac{a_1^{n-k}}{b_0-k} \times [e^{-(a_0+k)(u-t)} - e^{-(a_0+b_0)(u-t)}] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{a_0}{(b_1+1)^m}\sum_{k=0}^m \binom{m}{k}\frac{b_1^{m-k}}{a_0-k} \times [e^{-(b_0+k)(u-t)} - e^{-(a_0+b_0)(u-t)}] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} e^{-(a_0+b_0)(u-t)}. \tag{6.11}$$

Differentiating (6.11) with respect to u we obtain $-\mathbb{P}\{\tau_M \in du|\mathcal{F}_t\}$. Plugging this into (6.7), and after some straightforward but tedious integration, we obtain the price of the default-free zero-coupon bond:

$$p(t,T) = \mathbf{1}_{\{\tau_M\le t\}} e^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau_M\ge t\}} e^{-(r_0+J)(T-t)} + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2\le t\}} \frac{J}{[a_1+e^{-(t-\tau^2)}]^n}\sum_{k=0}^n\binom{n}{k} \times \frac{a_1^{n-k}e^{-k(t-\tau^2)}}{J-a_0-k}[e^{-(r_0+a_0+k)(T-t)} - e^{-(r_0+J)(T-t)}] + \mathbf{1}_{\{\tau^1\le t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{J}{[b_1+e^{-(t-\tau^1)}]^m}\sum_{k=0}^m\binom{m}{k} \times \frac{b_1^{m-k}e^{-k(t-\tau^1)}}{J-b_0-k}[e^{-(r_0+b_0+k)(T-t)} - e^{-(r_0+J)(T-t)}] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{Jb_0}{(a_1+1)^n}\sum_{k=0}^n\binom{n}{k}\frac{a_1^{n-k}}{b_0-k} \times \left[\frac{e^{-(r_0+a_0+k)(T-t)} - e^{-(r_0+J)(T-t)}}{J-a_0-k} - \frac{e^{-(r_0+a_0+b_0)(T-t)} - e^{-(r_0+J)(T-t)}}{J-a_0-b_0}\right] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{Ja_0}{(b_1+1)^m}\sum_{k=0}^m\binom{m}{k}\frac{b_1^{m-k}}{a_0-k} \times \left[\frac{e^{-(r_0+b_0+k)(T-t)} - e^{-(r_0+J)(T-t)}}{J-b_0-k} - \frac{e^{-(r_0+a_0+b_0)(T-t)} - e^{-(r_0+J)(T-t)}}{J-a_0-b_0}\right] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} \frac{J}{J-a_0-b_0} \times [e^{-(r_0+a_0+b_0)(T-t)} - e^{-(r_0+J)(T-t)}]. \tag{6.12}$$

We now consider the zero-recovery zero-coupon bonds issued by the two firms. In light of (6.6), and note that $r_s = r_0$ for $t \le s \le T$ since $\mathbf{1}_{\{\tau_M\le s\}} = 0$ on the sets $\{\tau^1 > T\}$ and $\{\tau^2 > T\}$, we have

$$\mathbb{E}\left\{\exp\left\{-\int_t^T r_s ds\right\}\mathbf{1}_{\{\tau^i>T\}}\Big|\mathcal{F}_t\right\} = \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{\int_t^T (r_s+\lambda_s^i)ds\right\}\Big|\mathcal{F}_t\right\} = \mathbf{1}_{\{\tau^i>t\}} e^{-r_0(T-t)}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T \lambda_s^i ds\right\}\Big|\mathcal{F}_t\right\}.$$

Now similar to (6.9) we can compute $\mathbf{1}_{\{\tau^1>t\}}\mathbb{E}^{\mathbb{P}^1}\{\exp\{-\int_t^T\lambda_s^1 ds\}|\mathcal{F}_t\}$ and obtain that

$$\mathbf{1}_{\{\tau^1>t\}}\mathbb{E}^{\mathbb{P}^1}\left\{\exp\left\{-\int_t^T (r_s+\lambda_s^1)ds\Big|\mathcal{F}_t\right\}\right\} = \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2\le t\}} e^{-(r_0+a_0)(T-t)}\left[\frac{a_1+e^{-(T-\tau^2)}}{a_1+e^{-(t-\tau^2)}}\right]^n + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}}\frac{b_0}{(a_1+1)^n}\sum_{k=0}^n\binom{n}{k}\frac{a_1^{n-k}}{b_0-k} \times [e^{-(r_0+a_0+k)(T-t)} - e^{-(r_0+a_0+b_0)(T-t)}] + \mathbf{1}_{\{\tau^1>t\}}\mathbf{1}_{\{\tau^2>t\}} e^{-(r_0+a_0+b_0)(T-t)}. \tag{6.13}$$

The price of the zero-recovery defaultable bond issued by firm 2 can be derived similarly. We summarize our discussion into the following main result of this section.

Theorem 6.1. *Assume* (H1)–(H2). *Then the prices of the zero-coupon bonds by each firm are given by*

$$v^i(t,T) = \delta^i p(t,T) + (1-\delta^i)P^i(t,T), \quad i = 1, 2,$$

where $\delta^i \in [0, 1]$, $i = 1, 2$ are the recovery rates, $p(t,T)$ is the price of the default free zero-coupon bond, given by (6.12), and

$$P^i(t,T) \triangleq \mathbf{1}_{\{\tau^i>t\}}\mathbb{E}^{\mathbb{P}^i}\left\{\exp\left\{-\int_t^T (r_s+\lambda_s^i)ds\Big|\mathcal{F}_t\right\}\right\}$$

is the price of the zero-recovery, defaultable zero-coupon bond issued by firm i, and it is given by (6.13). □

7. Pricing of UVL insurance involving married couples

In this section we try to apply the results in dependent mortality, or more generally the representation theorem, to the problem of pricing Universal Variable Life (UVL, for short) Insurance or annuity, especially when the benefit is payable based on the multiple survivorship of the insured, for instance, the married couples.

The UVL Insurance problem, as a special type of equity-linked insurance problem, could be treated via the so-called *Principle of Equivalent Utility*, and eventually become a problem of indifference pricing. We refer to Young and Zariphopoulou (2002), Young

(2003), Ma and Yu (2006), and Ludkovski and Young (2008), for different visions on such problems. Our discussion will follow the formulation of Ma and Yu (2006) closely, with necessary modifications to fit the dependent life models. To our best knowledge, such a problem has not been explored in the literature.

We begin by recalling briefly the formulation of UVL insurance problem and the results regarding indifference pricing. Assume that an investor is accessible to two assets in a liquid market, one riskless and one risky, denote their prices by X^0 and X^1, respectively. We assume that the dynamics of X^0 and X^1 follow the stochastic differential equations: for $t \geq 0$,

$$\begin{cases} dX_t^0 = rX_t^0 dt, & X_0^0 = x^0 \\ dX_t^1 = \mu X_t^1 dt + \sigma X_t^1 dB_t, & X_0^1 = x^1 \end{cases}$$

where $\{B_t : t \geq 0\}$ is a Brownian motion. Let us denote by π_t the amount of money invested in the risky asset X^1, and by W_t the total wealth of the investor, at time t. At each time t, the amount $\pi_t^0 = W_t - \pi_t$ will be invested in the riskless asset. A *self-financing* portfolio is defined by those π such that the corresponding wealth process $\{W_t\}$ satisfies the following dynamics:

$$dW_t = rW_t dt + (\mu - r)\pi_t dt + \sigma \pi_t dB_t, \quad t \geq 0.$$

For the technical clarity we shall define $\mathscr{A}$ to be the set of all *admissible* strategies $\pi = \{\pi_t : t \geq 0\}$ that are $\{\mathcal{F}_t^X\}$-adapted and $\mathbb{E}\int_0^T |\pi_t|^2 dt < \infty$. We often denote the solution $W = W^\pi$ if the control π needs to be emphasized.

To be consistent with Section 3 we now specify the filtrations. Assume that there are two (possibly dependent) lives with current ages x_1 and x_2, with future time random variables T_{x_i}, $i = 1, 2$, respectively. Denote $N_t^i = \mathbf{1}_{\{T_{x_i} \leq t\}}$, $i = 1, 2$. Then, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is defined by

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2, \quad t \geq 0,$$

where $\mathcal{F}_t^X = \sigma\{X_s, 0 \leq s \leq t\}$ and $\mathcal{F}_t^i = \sigma\{N_s^i, 0 \leq s \leq t\}$, $t \geq 0$, $i = 1, 2$. Assuming that $\sigma > 0$, then it is easy to see that $\mathcal{F}_t^X = \mathcal{F}_t^B$, $t \in [0, T]$.

An UVL Insurance plan is as follows. Assume that the investor is also the seller of a death benefit, defined as a lump-sum (e.g., \$1) payable at a terminal time T, contingent on the survivorship of, say, a married couple. In what follows we denote $K = \{K_t : t \geq 0\}$ to be a generic benefit process, and in particular we denote JLI to be the benefit of \$1 payable to a joint-life status, whereas SLI to be one corresponding to a last-survivor status. More precisely,

$$JLI_t = \mathbf{1}_{\{T_{x_1 x_2} \leq t\}}, \qquad SLI_t = \mathbf{1}_{\{T_{\overline{x_1 x_2}} \leq t\}}, \quad t \geq 0.$$

The seller is obliged to pay K_T at time T, whenever the death happens before T. The question is then how to price this insurance contract, and we shall concentrate on the cases when $K = JLI$ or $K = SLI$.

To introduce the Principle of Equivalent Utility, let u be a given utility function. For simplicity let us assume that u is an exponential utility function:

$$u(w) = -\frac{1}{\alpha} e^{-\alpha w}, \quad w \in \mathbb{R}. \tag{7.1}$$

The expected utility of the total wealth of the seller/investor is therefore

$$J(t, w; \pi) \triangleq \mathbb{E}_{t,w}\{u(W_T^\pi - K_T)\}. \tag{7.2}$$

Here $\mathbb{E}_{t,w}\{\cdot\}$ denotes the conditional expectation $\mathbb{E}\{\cdot | W_t = w, N_t^1 = N_t^2 = 0\}$. If $K_T = JLI_T$, then $K_T = 1$ unless both spouses are alive at T. If $K_T = SLI_T$, then $K_T = 1$ if both spouses are deceased by T. Note that if $K_T = 0$, then

$$J(t, w; \pi) = \mathbb{E}_{t,w}\{u(W_T^\pi)\} \stackrel{\triangle}{=} J^0(t, w; \pi), \quad \pi \in \mathscr{A}.$$

Thus the cost functional is the same as one in a standard utility maximization problem in finance. Let us now distinguish these two cases by considering two optimization problems:

$$V(t, w) = \sup_{\pi \in \mathscr{A}} J^0(t, w; \pi);$$
$$U(t, w) = \sup_{\pi \in \mathscr{A}} J(t, w; \pi).$$

Assume now that the seller will charge the policy K_T at time $t = 0$, as a lump sum $p > 0$. Namely, one adds to the initial wealth w by p. It is not hard to see, the fair (selling) price at each time t, denoted by p_t^* should be

$$p_t^* = \inf\{p : V(t, w) \leq U(t, w + p), \forall w\}.$$

It can be shown (cf. Ma and Yu, 2006) that the fair price p_t^* satisfies

$$V(t, w) = U(t, w + p_t^*), \quad \forall w, \ t^* \geq 0. \tag{7.3}$$

It has been observed (cf. Ma and Yu (2006) and Young and Zariphopoulou (2002)) that in many cases the valued functions U and V are related by $U(t, w) = V(t, w)\Phi(t, w)$, and Φ is often more tractable than the value functions themselves. This often leads quickly to an explicit solution. For example, in the case when u is an exponential utility, then one can show that

$$V(t, w) = -\frac{1}{\alpha} \exp\left(-\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\right)$$

by solving the corresponding HJB equation, and $\Phi(t, w) = \varphi(t)$, which can be derived by solving an ordinary differential equation (see Young and Zariphopoulou (2002) and Ma and Yu (2006) for its variation).

Extending the idea of "separation of variable", in what follows we shall argue that $U(t, w) = V(t, w)\Phi(t, w)$ still holds in the dependent married couple case, and try to derive $\Phi(t, w)$ in a pure probabilistic way, using the formulae for the conditional joint survival probabilities. Indeed, since W_T and K_T are independent, by assuming (7.1) we can check that

$$J(t, w; \pi) = J^0(t, w; \pi)\mathbb{E}_{t,w}\{e^{\alpha K_T}\}, \quad (t, w) \in [0, T] \times \mathbb{R}_+;$$

and therefore

$$U(t, w) = V(t, w)\mathbb{E}_{t,w}\{e^{\alpha K_T}\}.$$

Note that we condition on survival of the policyholders regarding $J(t, w; \pi)$. Recall (7.2), we see that this implies that $\mathbb{E}_{t,w}\{e^{\alpha K_T}\}$ is equivalent to $\mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t^1 \vee \mathcal{F}_t^2\}$, subject to $N_t^i = 0$ for $i = 1, 2$. Furthermore, note that $e^{\alpha K_T}$ and $\mathcal{F}_t^X$ are independent, it is easy to check that $\mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t\} = \mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t^1 \vee \mathcal{F}_t^2\}$. For notational convenience in the sequel we shall use the filtration $\mathcal{F}_t$ instead of $\mathcal{F}_t^1 \vee \mathcal{F}_t^2$.

Let us first observe the case when $K_T = JLI_T$. Then, we have

$$\begin{aligned} \mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t\} &= \mathbb{E}\{\mathbf{1}_{\{T_{x_1 x_2} > T\}} + e^\alpha \mathbf{1}_{\{T_{x_1 x_2} \leq T\}} | \mathcal{F}_t\} \\ &= e^\alpha + (1 - e^\alpha)\mathbb{E}\{\mathbf{1}_{\{T_{x_1 x_2} > T\}} | \mathcal{F}_t\} \\ &= e^\alpha + (1 - e^\alpha)\mathbb{E}\{\mathbf{1}_{\{T_{x_1} > T\}}\mathbf{1}_{\{T_{x_2} > T\}} | \mathcal{F}_t\}. \end{aligned} \tag{7.4}$$

In the case when $K_T = SLI_T$, we have

$$\begin{aligned} \mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t\} &= \mathbb{E}\{\mathbf{1}_{\{T_{\overline{x_1 x_2}} > T\}} + e^\alpha \mathbf{1}_{\{T_{\overline{x_1 x_2}} \leq T\}} | \mathcal{F}_t\} \\ &= e^\alpha + (1 - e^\alpha)\mathbb{E}\{\mathbf{1}_{\{T_{\overline{x_1 x_2}} > T\}} | \mathcal{F}_t\} \\ &= e^\alpha + (1 - e^\alpha)\mathbb{E}\{\mathbf{1}_{\{T_{x_1} > T\}} + \mathbf{1}_{\{T_{x_2} > T\}} - \mathbf{1}_{\{T_{x_1} > T\}}\mathbf{1}_{\{T_{x_2} > T\}} | \mathcal{F}_t\}. \end{aligned} \tag{7.5}$$

Thus, $\mathbb{E}\{e^{\alpha K_T} | \mathcal{F}_t\}$ will be completely analyzed if one examines the conditional expectations: $\mathbb{E}\{\mathbf{1}_{\{T_{x_1} > T\}} | \mathcal{F}_t\}$, $\mathbb{E}\{\mathbf{1}_{\{T_{x_2} > T\}} | \mathcal{F}_t\}$ and $\mathbb{E}\{\mathbf{1}_{\{T_{x_1} > T\}}\mathbf{1}_{\{T_{x_2} > T\}} | \mathcal{F}_t\}$.

Now following the ideas of the representation theorem, let us define

$$Z_t^i \triangleq \mathbf{1}_{\{T_{x_i}>t\}} \exp\left(\int_0^t \mu^{x_i}(s)\mathrm{d}s\right), \quad i=1,2,$$

and define

$$\left.\frac{\mathrm{d}P^i}{\mathrm{d}P}\right|_{\mathcal{F}_T} \triangleq Z_T^i, \quad i=1,2; \qquad \left.\frac{\mathrm{d}P^{1,2}}{\mathrm{d}P}\right|_{\mathcal{F}_T} \triangleq Z_T^1 Z_T^2.$$

Assume that the force of mortalities of the married couple are similar to those in Section 5:

$$\mu_{x_i}(t) = \lambda_{x_i}(t) + \mathbf{1}_{\{T_{x_j}\le t\}}\theta_{x_i}(t - T_{x_j}), \quad i,j=1,2;\ i\ne j$$

where

$$\lambda_{x_i} \triangleq h_i e^{g_i(x_i+t)}; \qquad \theta_{x_i}(t-T_{x_j}) \triangleq \frac{n_i}{r_i e^{(t-T_{x_j})}+1},$$
$$i,j=1,2;\ i\ne j.$$

To simplify notations let us recall the process Γ in Section 3. In the present context we define:

$$\begin{cases} \Gamma_{x_i}^t(s) \triangleq \exp\left\{-\int_s^t \mu_{x_i}(r)\mathrm{d}r\right\}; \\ \Lambda_{x_i}^t(s) \triangleq \exp\left\{-\int_s^t \lambda_{x_i}(r)\mathrm{d}r\right\} \\ \quad = e^{-\frac{h_i}{g_i}[e^{g_i(x_i+t)} - e^{g_i(x_i+s)}]}, & s\le t, i,j=1,2, i\ne j. \\ \Theta_{x_i,x_j}^t(s) \triangleq \exp\left\{-\int_s^t \theta_{x_i}(r-T_{x_j})\mathrm{d}r\right\} \\ \quad = \left[\dfrac{r_i + e^{-(t-T_{x_j})}}{r_i + e^{-(s-T_{x_j})}}\right]^{n_i}, \end{cases}$$

Our main result of this section is the following theorem.

Theorem 7.1. *Assume* (H2). *Then, the indifference (selling) price of problem (7.3) is given by*

$$p_t^* = \frac{1}{\alpha}e^{-r(T-t)}\log \mathbb{E}_{t,w}[e^{\alpha K_T}].$$

Proof. Applying (3.6) we have

$$\begin{aligned}
&\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}|\mathcal{F}_t\} \\
&\quad = \mathbb{E}[Z_T^1\Gamma_{x_1}^T(0)|\mathcal{F}_t] \\
&\quad = Z_t^1\mathbb{E}^{\mathbb{P}^1}\{\Gamma_{x_1}^T(0)|\mathcal{F}_t\} = \mathbf{1}_{\{T_{x_1}>t\}}\mathbb{E}^{\mathbb{P}^1}\{\Gamma_{x_1}^T(t)|\mathcal{F}_t\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\Lambda_{x_1}^T(t)\mathbb{E}^{\mathbb{P}^1}\left\{\mathbf{1}_{\{T_{x_2}\le t\}}\Theta_{x_1,x_2}^T(t) + \mathbf{1}_{\{T_{x_2}>t\}}\Theta_{x_1,x_2}^T(T_{x_2})|\mathcal{F}_t\right\}.
\end{aligned}$$

Since $\Theta_{x_1,x_2}^T(t)$ is $\mathcal{F}_t$-measurable, the first term becomes

$$\begin{aligned}
&\mathbf{1}_{\{T_{x_1}>t\}}\Lambda_{x_1}^T(t)\mathbb{E}^{\mathbb{P}^1}\{\mathbf{1}_{\{T_{x_2}\le t\}}\Theta_{x_1,x_2}^T(t)|\mathcal{F}_t\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}\le t\}}\Lambda_{x_1}^T(t)\Theta_{x_1,x_2}^T(t).
\end{aligned}$$

The second term becomes

$$\begin{aligned}
&\mathbf{1}_{\{T_{x_1}>t\}}\Lambda_{x_1}^T(t)\mathbb{E}^{\mathbb{P}^1}\left\{\mathbf{1}_{\{T_{x_2}>t\}}\Theta_{x_1,x_2}^T(T_{x_2})|\mathcal{F}_t\right\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\Lambda_{x_1}^T(t)\mathbb{E}^{\mathbb{P}^1}\left\{\mathbf{1}_{\{t<T_{x_2}\le T\}}\Theta_{x_1,x_2}^T(T_{x_2}) + \mathbf{1}_{\{T_{x_2}>T\}}\middle|\mathcal{F}_t\right\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\Lambda_{x_1}^T(t)\left\{\int_t^T\left[\frac{r_1+e^{-(T-u)}}{r_1+1}\right]^{n_1}\right. \\
&\qquad \left.\times\ \mathbb{P}^1(T_{x_2}\in \mathrm{d}u|\mathcal{F}_t) + \mathbb{P}^1(T_{x_2}>T|\mathcal{F}_t)\right\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\Lambda_{x_1}^T(t)\left\{\int_t^T\left[\frac{r_1+e^{-(T-u)}}{r_1+1}\right]^{n_1}\right. \\
&\qquad \left.\times\ \lambda_{x_2}(u)\Lambda_{x_2}^u(t)\mathrm{d}u + \Lambda_{x_2}^T(t)\right\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\frac{\Lambda_{x_1}^T(t)}{(r_1+1)^{n_1}}\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{h_2}{g_2}r_1^{n_1-k} \\
&\qquad \times e^{-k(T+x_2)+\frac{h_2}{g_2}e^{g_2(x_2+t)}}\left(\Delta_k^2(e^{g_2(x_2+T)}) - \Delta_k^2(e^{g_2(x_2+t)})\right) \\
&\qquad + \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\Lambda_{x_1}^T(t)\Lambda_{x_2}^T(t),
\end{aligned}$$

where $\Delta_k^2(t) = \int_0^t y^{\frac{k}{g_2}} e^{-\frac{h_2}{g_2}y}\mathrm{d}y$. In the above, the third equality is due to the fact that

$$\mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\mathbb{P}^1(T_{x_2}>u|\mathcal{F}_t) = \mathbf{1}_{\{T_{x_1}>t\}}1_{\{T_{x_2}>t\}}e^{-\int_t^u\lambda_{x_2}(s)\mathrm{d}s}$$

and the last equality follows from simple integration as in Proposition 5.1. Therefore, we have

$$\begin{aligned}
&\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}|\mathcal{F}_t\} \\
&\quad = (1-N_t^1)N_t^2\Lambda_{x_1}^T(t)\Theta_{x_1,x_2}^T(t) + (1-N_t^1)(1-N_t^2)\Lambda_{x_1}^T(t)\Lambda_{x_2}^T(t) \\
&\qquad + \prod_{i=1}^2(1-N_t^i)\frac{\Lambda_{x_1}^T(t)}{(r_1+1)^{n_1}}\sum_{k=0}^{n_1}\binom{n_1}{k}\frac{h_2}{g_2}r_1^{n_1-k}e^{-k(T+x_2)+\frac{\lambda_{x_2}(t)}{g_2}} \\
&\qquad \times\left(\Delta_k^2\left(\frac{\lambda_{x_2}(T)}{h_2}\right) - \Delta_k^2\left(\frac{\lambda_{x_2}(t)}{h_2}\right)\right).
\end{aligned} \tag{7.6}$$

Similar to (7.6) we have

$$\begin{aligned}
&\mathbb{E}\{\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\} \\
&\quad = (1-N_t^2)N_t^1\Lambda_{x_2}^T(t)\Theta_{x_2,x_1}^T(t) + (1-N_t^1)(1-N_t^2)\Lambda_{x_1}^T(t)\Lambda_{x_2}^T(t) \\
&\qquad + \prod_{i=1}^2(1-N_t^i)\frac{\Lambda_{x_2}^T(t)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k}\frac{h_1}{g_1}r_2^{n_2-k}e^{-k(T+x_1)+\frac{\lambda_{x_1}(t)}{g_1}} \\
&\qquad \times\left(\Delta_k^1\left(\frac{\lambda_{x_1}(T)}{h_1}\right) - \Delta_k^1\left(\frac{\lambda_{x_1}(t)}{h_1}\right)\right).
\end{aligned} \tag{7.7}$$

Applying Theorem 3.2 we have

$$\begin{aligned}
&\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\mathbb{E}^{P^{1,2}}\left\{\exp\left\{-\int_t^T(\mu_s^{x_1}+\mu_s^{x_2})\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\} \\
&\quad = \mathbf{1}_{\{T_{x_1}>t\}}\mathbf{1}_{\{T_{x_2}>t\}}\mathbb{E}^{\mathbb{P}^{1,2}}\left\{\exp\left\{-\int_t^T(\lambda_{x_1}(s)+\lambda_{x_2}(s))\mathrm{d}s\right\}\middle|\mathcal{F}_t\right\} \\
&\quad = (1-N_t^1)(1-N_t^2)\Lambda_{x_1}^T(t)\Lambda_{x_2}^T(t)
\end{aligned} \tag{7.8}$$

where the second equality follows from the fact that $\mu_s^{x_i} = \lambda_{x_i}(s)$ for $s\le T$ under the measure $\mathbb{P}^{1,2}$.

Finally, we can derive $\mathbb{E}_{t,w}\{e^{\alpha K_T}\}$ via $\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\}|_{N_t^1=N_t^2=0}$. In the case when $K_T = JLI_T$ recall that from (7.4) $\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\} = e^\alpha + (1-e^\alpha)\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\}$. In (7.8) we can immediately obtain

$$\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\}|_{N_t^1=N_t^2=0} = \Lambda_{x_1}^T(t)\Lambda_{x_2}^T(t).$$

Therefore we have

$$\mathbb{E}_{t,w}\{e^{\alpha K_T}\} = e^{\alpha} + (1 - e^{\alpha})\Lambda^T_{x_1}(t)\Lambda^T_{x_2}(t).$$

In the case when $K_T = SLI_T$ recall that from (7.5) $\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\} = e^{\alpha} + (1 - e^{\alpha})\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}} + \mathbf{1}_{\{T_{x_2}>T\}} - \mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\}$. Applying $\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\}|_{N^1_t=N^2_t=0}$ in (7.6)–(7.8) we have

$$\begin{aligned}\mathbb{E}_{t,w}\{e^{\alpha K_T}\} &= e^{\alpha} + (1 - e^{\alpha})\frac{\Lambda^T_{x_1}(t)}{(r_1+1)^{n_1}}\sum_{k=0}^{n_1}\binom{n_1}{k} \\ &\times \frac{h_2}{g_2} r_1^{n_1-k} e^{-k(T+x_2)+\frac{h_2}{g_2}e^{g_2(x_2+t)}}\left(\Delta^2_k(e^{g_2(x_2+T)}) - \Delta^2_k(e^{g_2(x_2+t)})\right) \\ &+(1 - e^{\alpha})\frac{\Lambda^T_{x_2}(t)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k} \\ &\times \frac{h_1}{g_1} r_2^{n_2-k} e^{-k(T+x_1)+\frac{h_1}{g_1}e^{g_1(x_1+t)}}\left(\Delta^1_k(e^{g_1(x_1+T)}) - \Delta^1_k(e^{g_1(x_1+t)})\right) \\ &+ (1 - e^{\alpha})\Lambda^T_{x_1}(t)\Lambda^T_{x_2}(t).\end{aligned}$$

We have calculated $\mathbb{E}_{t,w}\{e^{\alpha K_T}\}$ so that $U(t, w) = V(t, w)\mathbb{E}_{t,w}\{e^{\alpha K_T}\}$ is known. To find the fair price p^*_t satisfying $V(t, w) = U(t, w+p^*_t)$, we plug in $w + p^*_t$ and have

$$\begin{aligned}U(t, w + p^*_t) &= V(t, w + p^*_t)\mathbb{E}_{t,w}\{e^{\alpha K_T}\} \\ &= \exp(-\alpha p^*_t e^{r(T-t)})V(t, w)\mathbb{E}_{t,w}\{e^{\alpha K_T}\}.\end{aligned}$$

Therefore, the fair price is finally given by

$$p^*_t = \frac{1}{\alpha}e^{-r(T-t)}\log \mathbb{E}_{t,w}\{e^{\alpha K_T}\}. \quad \square$$

To end this section we consider the so-called *survival benefit*, using both joint-life status and last-survivor status of the married couple. Denote Y_t, $t \geq 0$, to be the generic benefit process, and $SBF_t = 1_{\{T_{x_1x_2}\geq t\}}$, $SBS_t = 1_{\{T_{\overline{x_1x_2}}\geq t\}}$. Similar to the life insurance case, the cost functional is given by

$$J(t, w; \pi) \stackrel{\triangle}{=} \mathbb{E}_{t,w}\{u(W^{\pi}_T - Y_T)\}.$$

Again, since W_T and Y_T are independent, we have

$$U(t, w) = V(t, w)\mathbb{E}_{t,w}\{e^{\alpha Y_T}\}.$$

In the case where $Y_T = SBF_T$, we have

$$\begin{aligned}\mathbb{E}\{e^{\alpha Y_T}|\mathcal{F}_t\} &= \mathbb{E}\{e^{\alpha}\mathbf{1}_{\{T_{x_1x_2}>T\}} + \mathbf{1}_{\{T_{x_1x_2}\leq T\}}|\mathcal{F}_t\} \\ &= 1 + (e^{\alpha} - 1)\mathbb{E}\{\mathbf{1}_{\{T_{x_1x_2}>T\}}|\mathcal{F}_t\} \\ &= 1 + (e^{\alpha} - 1)\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\}.\end{aligned}$$

Applying (7.8) we obtain

$$\mathbb{E}_{t,w}\{e^{\alpha Y_T}\} = 1 + (e^{\alpha} - 1)\Lambda^T_{x_1}(t)\Lambda^T_{x_2}(t).$$

In the case when $Y_T = SBS_T$, we have

$$\begin{aligned}\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\} &= \mathbb{E}\{e^{\alpha}\mathbf{1}_{\{T_{\overline{x_1x_2}}>T\}} + \mathbf{1}_{\{T_{\overline{x_1x_2}}\leq T\}}|\mathcal{F}_t\} \\ &= 1 + (e^{\alpha} - 1)\mathbb{E}\{\mathbf{1}_{\{T_{\overline{x_1x_2}}>T\}}|\mathcal{F}_t\} \\ &= 1 + (e^{\alpha} - 1)\mathbb{E}\{\mathbf{1}_{\{T_{x_1}>T\}} + \mathbf{1}_{\{T_{x_2}>T\}} \\ &\quad - \mathbf{1}_{\{T_{x_1}>T\}}\mathbf{1}_{\{T_{x_2}>T\}}|\mathcal{F}_t\}.\end{aligned}$$

Again applying $\mathbb{E}\{e^{\alpha K_T}|\mathcal{F}_t\}|_{N^1_t=N^2_t=0}$ in (7.6)–(7.8) we have

$$\begin{aligned}\mathbb{E}_{t,w}\{e^{\alpha K_T}\} &= 1 + (e^{\alpha} - 1)\frac{\Lambda^T_{x_1}(t)}{(r_1+1)^{n_1}}\sum_{k=0}^{n_1}\binom{n_1}{k} \\ &\times \frac{h_2}{g_2} r_1^{n_1-k} e^{-k(T+x_2)+\frac{h_2}{g_2}e^{g_2(x_2+t)}}\left(\Delta^2_k(e^{g_2(x_2+T)}) - \Delta^2_k(e^{g_2(x_2+t)})\right) \\ &+(e^{\alpha} - 1)\frac{\Lambda^T_{x_2}(t)}{(r_2+1)^{n_2}}\sum_{k=0}^{n_2}\binom{n_2}{k} \\ &\times \frac{h_1}{g_1} r_2^{n_2-k} e^{-k(T+x_1)+\frac{h_1}{g_1}e^{g_1(x_1+t)}}\left(\Delta^1_k(e^{g_1(x_1+T)}) - \Delta^1_k(e^{g_1(x_1+t)})\right) \\ &+(e^{\alpha} - 1)\Lambda^T_{x_1}(t)\Lambda^T_{x_2}(t).\end{aligned}$$

Finally, the fair price of survival benefit is similarly given by

$$p^*_t = \frac{1}{\alpha}e^{-r(T-t)}\log \mathbb{E}_{t,w}\{e^{\alpha Y_T}\}.$$

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References

Bielecki, T.R., Jeanblanc, M., Rutkowski, M., 2006. Hedging of credit derivatives in models with totally unexpected default. In: Stochastic Processes and Applications to Mathematical Finance. World Sci. Publ., Hackensack, NJ, pp. 35–100.
Bowers, N., Gerber, H., Hickman, J., Jones, D., Nesbitt, C., 1997. Actuarial Mathematics. The Society of Actuaries, Shaunburg, Illnois, USA.
Bremaud, P., 1981. Point Processes and Queues: Martingale Dynamics. Springer-Verlag, New York.
Collin-Dufresne, P., Goldstein, R., Helwege, J., 2003. Is credit event risk priced? Modeling Contagion via the Updating of Beliefs. Working Paper. Carnegie-Mellon University, Washington University, and Ohio State University.
Collin-Dufresne, P., Goldstein, R., Hugonnier, J., 2004. A general formula for valuing defaultable securities. Econometrica 72 (5), 1377–1407.
Duffie, D., Eckner, A., Horel, G., Saita, L., 2006. Frailty correlated default. Working Paper.
Gourieroux, C., Gagliardini, P., 2003. Spread term structure and default correlation. Working Paper.
Hu, Y., Goldman, N., 1990. Mortality differentials by marital status: an international comparison. Demography 27 (2), 233–250.
Jarrow, R., Yu, F., 2001. Counterparty risk and the pricing of defaultable securities. The Journal of Finance 56 (5), 1765–1799.
Jeanblanc, M., Rutkowski, M., 2001. Default risk and hazard process. Working paper, Université d'Evry Val d'Essonne and Warsaw University of Technology.
Lando, D., 2004. Credit Risk Modeling: Theory and Applications. Princeton University Press, Princeton, New Jersey.
Ludkovski, M., Young, V., 2008. Indifference pricing of pure endowments and life annuities under stochastic hazard and interest rates. Insurance: Mathematics and Economics 42 (1), 14–30.
Ma, J., Yu, Y., 2006. Principle of equivalent utility and universal variable life insurance pricing. Scandinavian Actuarial Journal 6, 311–337.
Martikainen, P., Valkonen, T., 1996. Mortality after the death of a spouse: rates and causes of death in a large Finnish cohort. American Journal of Public Health 86, 1087–1093.
Protter, P., 2004. Stochastic Integration and Differential Equations, Second edition. Springer Verlag, Heidelberg.
Schonbucher, P.J., 2003. Information-driven default contagion. Working Paper, ETH Zurich.
Schonbucher, P.J., Schubert, D., 2001. Copula-dependent default risk in intensity models. Working Paper, Department of Statistics, University of Bonn.
Valkonen, T., Martikainen, P., Blomgren, J., 2004. Increasing excess mortality among non-married elderly people in developed countries. Demographic Research 2, 305–330.
Young, V., Zariphopoulou, T., 2002. Pricing dynamic insurance risks using the principle of equivalent utility. Scandinavian Actuarial Journal 4, 246–279.
Young, V., 2003. Equity-indexed life insurance: pricing and reserving using the principle of equivalent utility. North American Actuarial Journal 17 (1), 68–86.
Yu, F., 2003. Correlated defaults in reduced-form models, Working Paper, University of California, Irvine.
Yu, F., 2007. Correlated defaults in intensity-based models. Mathematical Finance 17 (2), 155–173.