Scandinavian Actuarial Journal

ISSN: 0346-1238 (Print) 1651-2030 (Online) Journal homepage: http://www.tandfonline.com/loi/sact20

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To cite this article: Jin Ma & Yuhua Yu (2006) Principle of equivalent utility and universal variable life insurance pricing, Scandinavian Actuarial Journal, 2006:6, 311-337, DOI: 10.1080/03461230600986128

To link to this article: http://dx.doi.org/10.1080/03461230600986128

Published online: 16 Feb 2007.

Scandinavian Actuarial Journal, 2006, 6, 311–337

Taylor & Francis
Taylor & Francis Group

Original Article

Principle of equivalent utility and universal variable life insurance pricing

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In this paper we study the pricing problem for a class of universal variable life (UVL) insurance products, using the idea of *principle of equivalent utility.* As the main features of UVL products we allow the (death) benefit to depend on certain indices or assets that are not necessarily tradable (e.g., pension plans), and we also consider the "multiple decrement" cases in which various status of the insured are allowed and the benefit varies in accordance with the status. Following the general theory of *indifference pricing*, we formulate the pricing problem as stochastic control problems, and derive the corresponding HJB equations for the value functions. In the case of exponential utilities, we show that the prices can be expressed explicitly in terms of the global, bounded solutions of a class of semilinear parabolic PDEs with exponential growth. In the case of general insurance models where multiple decrements and random time benefit payments are all allowed, we show that the price should be determined by the solutions to a system of HJB equations, each component corresponds to the value function of an optimization problem with the particular status of the insurer.

Keywords: Principle of equivalent utility; value functions; HJB equations; indifference pricing

1. Introduction

In recent years, especially during the bullish equity market and the low interest rate environment of the 1990s, the sale of *universal variable life insurance* (UVL) has grown substantially. Introduced in the 1950s in Holland, the *variable life* (VL) insurance initiated the novel idea that would allow the insured to have a low-risk cash account and link the death benefit to the returns of that account. The UVL insurance is a version of VL insurance with further flexibility on the premiums. These special features endow the traditional life insurance with investment growth potential, and thus turn it into a more attractive financial product.

This new trend of "securitizing insurance products" (such as UVL) has brought up many interesting theoretical problems in mathematics, economics and actuarial science. For example, as the death benefit in a UVL contract depends on the policy account whose value is affected by many factors including the investment performance and the withdrawal/addition activities of the insured, the pricing of such a product becomes more complicated than that of both traditional life insurance products and the usual contingent claims in finance theory. In this paper we try to apply the so-called *principle of equivalent utility* to study such a pricing

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Scandinavian Actuarial Journal
ISSN 0346-1238 print/ISSN 1651-2030 online
http://www.tandf.no/saj
DOI: 10.1080/03461230600986128

problem. We note that such a principle can be thought of as a special case of the *indifference pricing*, first introduced by Hodges and Neuberger (1989), which has become one of the important topics in the credit risk theory. The idea of pricing life insurance products via the principle of equivalent utility was explored recently by Young, Zariphopoulou and others in a sequence of papers (see, e.g., Jaimungal, 2004; Jaimungal & Nayak, 2005; Jaimungal & Young, 2005; Young, 2003; Young & Zariphopoulou, 2002, 2004), mostly in the case of single decrement (i.e., death is the only insurance risk). The general framework of the indifference pricing problem as well as the dynamic utility optimization problem have also been studied quite extensively, from various angles. We refer to Cvitanic *et al.* (2001), Delbaen *et al.* (2002), Frittelli (2000), Musiela and Zariphopoulou (2004a,b), Owen (2002), Rouge and El Karoui (2000) and Bielecki *et al.* (2004), to mention a few. Nevertheless, our UVL model has some features that do not seem to have been covered by the existing results so far.

There are two main features in the UVL insurance models considered in this paper. First, we allow the death benefits to depend on certain indices or benchmark assets that could be non-tradable in the given market (e.g., a retirement fund). Thus, the arbitrage price at a given time for such assets cannot be determined *a priori* in order to offset the possible insurance risk, as was done in, say, Young & Zariphopoulou (2002). We will follow more general strategy without using the knowledge of the arbitrage price of the death benefit at any time t, and we show that such a strategy will actually produce the same solution if the benchmark asset is tradable. We should mention that the indifference pricing with non-tradable assets was considered recently by Musiela and Zariphopoulou (2004a,b), but our problem seems to be more general. Another feature of our general insurance model is that it allows multiple decrements, an important building block in insurance theory, and that benefits are payable at the moment of decrements. In such a model benefit payments will depend on the cause of termination of the status (such as disability, withdrawal, death and retirement), therefore it is a mixture of continuous dynamics with a discrete Markov chain. To our best knowledge, the indifference pricing for such a problem under general utility functions is novel. We should remark here that due to the length of this paper, we do not pursue a complete solution for the general insurance model. Instead, we provide a somewhat ad hoc result: we derive the HJB equation, assuming that the value function is $C^{1,2}$. We should note that even such a result is not trivial, without detailed assumptions on the utility function. Of course, many questions remain, and we hope to be able to address them in our future publications.

An important special case that has been studied extensively is when the utility function is exponential. We provide a detailed analysis for such a case as well, assuming the general benefit functions which could depend on the non-tradable assets. It turns out that in this case the method of "separation of variables" (see, e.g., Young, 2003; Young & Zariphopoulou, 2002) still works well, but the result will eventually depend on the solvability of a semilinear (reaction-diffusion) partial differential equation with exponential growths on the reaction term. While such an equation is expected to have a finite time blow-up in general, we show that in our case the global, bounded solution exists. The closed form solutions of the pricing problems then follow.

The rest of the paper is organized as follows. In section 2 we formulate the problems and set up the mathematical bases. In section 3 we study the simple UVL model, in which the death benefits are payable at the end of a fixed term. Two types of the strategies, subject to whether the death benefits depend on the non-tradable assets, will be studied separately. In section 4 we treat the special case where the utility function is exponential. In section 5 we study the general insurance models involving multiple decrements.

2. Problem formulation

Throughout this paper we assume that all the randomness come from a common complete probability space $(\Omega, \mathcal{F}, P)$. We assume that the probability space is rich enough to carry a $d+1$-dimensional Brownian motion $(B, \tilde{B}) = \{(B_t, \tilde{B}_t): t \geq 0\}$, which will be thought of as the source of the randomness of a financial market where all the investments under consideration will be made. For notational clarity we shall denote $\mathbf{F}^B = \{\mathcal{F}_t^B: t \geq 0\}$, $\mathbf{F}^{\tilde{B}} = \{\mathcal{F}_t^{\tilde{B}}: t \geq 0\}$, to be the natural filtrations generated by B and $\tilde{B}$, respectively, with usual P-augmentation so that they satisfy the *usual hypotheses* (cf. e.g., Protter, 1990). Throughout this paper we denote $|\cdot|$ to be the norm of a generic Euclidean space, and $\langle \cdot, \cdot \rangle$ to be its inner product. Further, for any real-valued function $\Phi(t, x, y, \ldots)$, where x and y could be vectors, we denote Φ_t, $\Phi_x, \ldots$, etc., to be the partial derivatives (gradients) with respect to the corresponding variables (vectors). The higher-order derivatives are denoted similarly when there is no danger of confusion.

2.1. The life model

We first give a detailed mathematical description of the life models on which all our future discussion will be based. We refer to the ubiquitous reference Bowers *et al.* (1997) for most of the ideas and notations below.

2.1.1. Simple life model. In this case we assume that the only uncertainty comes from a *future life-time random variable*, denoted by $T(x)$, where x is the current age of the insured. More precisely, $T(x)$ is the time to death from the present time, and we assume that it is independent of the Brownian motion $(B, \tilde{B})$. The random variable $T(x)$ is characterized by the "survival function" $G_x(t) \triangleq P\{T(x) > t\}$, $t \geq 0$. Further, define $X_t \triangleq 1_{\{T(x) \leq t\}}$, $t \geq 0$, and let $\mathbf{F}^X = \{\mathcal{F}_t^X\}_{t \geq 0}$ with $\mathcal{F}_t^X \triangleq \sigma\{X_s, 0 \leq s \leq t\}$. We define a new filtration

$$\mathbf{F} = \mathbf{F}^B \otimes \mathbf{F}^{\tilde{B}} \otimes \mathbf{F}^X = \{\mathcal{F}_t^B \vee \mathcal{F}_t^{\tilde{B}} \vee \mathcal{F}_t^X : t \geq 0\}.$$

Clearly, under such a setting B is still an $\mathbf{F}$-Brownian motion, and the random time $T(x)$ becomes an $\mathbf{F}$-stopping time.

Next, following the conventional actuarial terms we define the "survival probability" of the life (x) by ${}_tp_x = P\{T(x) > t\}$, and denote ${}_tq_x = 1 - {}_tp_x$. The *force of mortality* is then defined by

$$\lambda_x(t) = \lim_{h \to 0} \frac{{}_hq_{x+t}}{h}. \tag{2.1}$$

2.1.2. General life model. A more general model of life will be considered in the last part of this paper, which will be referred to as a "general life insurance model" (see, e.g., Norberg, 1992). In that model we shall allow "multiple decrement", and that benefit will be payable at a random time, such as "the moment of death", instead of a fixed terminal time T. Moreover, the payments may depend on the different status as well as the transitions between them. Typical examples of status include, but are not limited to: short/long term disabilities, withdrawal, retirement, and of course, death.

To describe such a model we need to modify the "state process" $\{X_t\}_{t\geq 0}$ defined in the simple life model. Suppose that the policy starts at time 0 for a person aged x, we now assume that X is a Markov chain with a finite state space $\{0, 1,\ldots,m\}$, representing the numerical code of the status at time t. We specify $i=1$ to be a (absorbing) "cemetary state", representing "death", and $X_0=0$ to be the initial state.

Now denote $I_t^i = 1_{\{X_t=i\}}$, and define the counting process:

$$N_t^{ij} \triangleq \#\{\text{transitions of } X \text{ from state } i \text{ to } j \text{ during } [0,\ t]\}. \tag{2.2}$$

Also, for each t we define a stopping time $\tau_t=\inf\{s\geq t{:}X_s\neq X_t\}$. Namely, τ_t is the first transition time of X after t. We then define, for $i=0,\ldots,m$,

$$\tau_t^i = \begin{cases} \tau_t, & \text{if } X_{\tau_t}=i \\ \infty, & \text{otherwise} \end{cases} \tag{2.3}$$

Using these stopping times we define the following conditional probabilities:

$${}_t\bar{p}_s^i \triangleq P\{\tau_s > t | X_s = i\}; \quad {}_t\bar{q}_s^{ij} \triangleq P\{\tau_s^j = \tau_s \leq t | X_s = i\}, \quad s \leq t, \quad i,j\in\{0,\ldots,m\}. \tag{2.4}$$

Clearly, by definition of state $i=1$ we have ${}_t\bar{p}_s^1 = 1$; ${}_t\bar{q}_s^{1j}=0$, for all $j\neq 1$; and

$${}_t\bar{p}_s^i + \sum_{j\neq i} {}_t\bar{q}_s^{ij} = 1, \qquad \forall i=0,1,\cdots,m, \quad 0\leq s<t. \tag{2.5}$$

Similar to (2.1), we define the "force of decrement of status i due to cause j" as

$$\bar{\lambda}_t^{ij} \triangleq \lim_{h\to 0} \frac{{}_{t+h}\bar{q}_t^{ij}}{h}, \qquad i,j=0,1,\ldots,m. \tag{2.6}$$

We should note that, being a Markov chain, process X has its transition probability ${}_tq_s^{ij} = P\{X_t=j|X_s=i\}$ and the corresponding transition intensity

$$\lambda_t^{ij} \triangleq \lim_{h\downarrow 0} \frac{{}_{t+h}q_t^{ij}}{h}, \qquad i\neq j. \tag{2.7}$$

The following results are not surprising; we prove them for completeness.

Lemma 2.1. *Let $\{\bar{\lambda}_\cdot^{ij}\}_{i,j=0}^m$ be the force of decrements defined by (2.6), and $\{\lambda_\cdot^{ij}\}_{i,j=0}^m$ the transition density of X defined by (2.7). Then*

i. ${}_t\bar{q}_s^{ij} = \int_s^t {}_t\bar{p}_s^i \bar{\lambda}_r^{ij} dr$;

ii. $\bar{\lambda}_t^{ij} = \lambda_t^{ij}$, for all $t\geq 0$, $i, j=0, 1,\ldots,m$;

iii. $\lim_{t\downarrow s} \dfrac{1 - {}_t\bar{p}_s^i}{t-s} = \Sigma_{j\neq i}\ \lambda_s^{ij}$, for all $s\geq 0$.

Proof. (i) For fixed $s \le t$, and i, j, let us denote the following sets:

$$A = \{\exists\, \tau \in (t, t+h], X_\tau = j, X_u = i, \forall u \in [t, \tau)\},$$
$$B = \{X_u = i, \forall u \in [s, t]\}, \quad C = \{X_s = i\}.$$

Then, it is readily seen that $B \subseteq C$, $P\{B|C\} = {}_t\bar{p}^i_s$, and

$$P\{A \cap B|C\} = P\{t < \tau^j_s \le t+h | X(s) = i\} = {}_{t+h}\bar{q}^{ij}_s - {}_t\bar{q}^{ij}_s.$$

Further, by the Markovian property of X one can also check that $P\{A|B\} = {}_{t+h}\bar{q}^{ij}_t$. Consequently,

$$\bar{\lambda}^{ij}_t = \lim_{h\to 0}\frac{1}{h}\,{}_{t+h}\bar{q}^{ij}_t = \lim_{h\to 0}\frac{1}{h}P\{A|B\} = \lim_{h\to 0}\frac{P\{A\cap B|C\}}{hP\{B|C\}} \quad = \frac{1}{{}_t\bar{p}^i_s}\lim_{h\to 0}\frac{1}{h}({}_{t+h}\bar{q}^{ij}_s - {}_t\bar{q}^{ij}_s) = \frac{1}{{}_t\bar{p}^i_s}\frac{\mathrm{d}}{\mathrm{d}t}\,{}_t\bar{q}^{ij}_s.$$

This proves (i).

(ii) As $\{\tau^k_s = r,\ X_s = i\} = \{X_s = i,\ X_r = k,\ X_u = i, \forall u \in [s,\ r)\}$, for $i,\ k > 0$, and $r \ge s$, using the Markov property one shows that $P\{X_t = j | \tau^k_s = r,\ X_s = i\} = {}_t q^{kj}_r$, for $s < r < t$. Next, recalling that $P\{\tau^k_s \le r | X_s = i\} = {}_r\bar{q}^{ik}_s$, and applying (i), we have

$${}_tq^{ij}_s = P\{X_t = j|X_s = i\} = \sum_{k\ne i}\int_s^t P\{X_t = j|\tau^k_s = r, X_s = i\}d[{}_r\bar{q}^{ik}_s] = \int_s^t \sum_{k\ne i} {}_t q^{kj}_r\, {}_r\bar{p}^i_s \bar{\lambda}^{ik}_r dr.$$

Finally, using the facts that ${}_tq^{kj}_t = 1_{\{k=j\}}$ and ${}_t\bar{p}^i_t = 1$, we obtain that

$$\lambda^{ij}_t = \frac{d}{dh}{}_{t+h}q^{ij}_t\bigg|_{h=0} = \bigg\{\sum_{k\ne i} {}_{t+h}\bar{p}^i_t \bar{\lambda}^{ik}_{t+h}\, {}_{t+h}q^{kj}_t + \int_t^{t+h}\sum_{k\ne i} {}_r\bar{p}^i_s \bar{\lambda}^{ik}_r \frac{d}{dh}[{}_{t+h}q^{kj}_r]dr\bigg\}\bigg|_{h=0} = \bar{\lambda}^{ij}_t, \qquad t \ge 0.$$

(iii) This follows from (2.5), (i) and (ii). □

Finally, we remark that if $m = 1$, then there are only two states: life or death. In this case the state process X becomes the one in the simple life model, and the general life model obviously reduces to the simple life model with $\tau^1_0 = T(x)$, though slightly different notations from the simple life model are used here:

$${}_t\bar{p}^0_s = {}_{t-s}\,p_{x+s}, \quad {}_t\,q^{01}_s = {}_{t-s}\,q_{x+s}.$$

Furthermore, with a slight abuse of notation, we shall use the same notation $\mathbf{F}$ for the filtration $\mathbf{F}^X \otimes \mathbf{F}^B \otimes \mathbf{F}^{\tilde{B}}$ in all cases, as long as the context is clear.

2.2. The death benefits

A fundamental part of the UVL is that the death benefit is linked to an investment opportunity. We formulate this fact by assuming that the death benefit at any time t is given by

$$b_t = g(t, S^1_\cdot, \ldots, S^d_\cdot, Z_\cdot) = g(t, S_\cdot, Z_\cdot), \tag{2.8}$$

where $g{:}[0,\infty) \times C([0,\infty); \mathbb{R}^{d+1}) \mapsto (0,\infty)$ is some functional that is progressively measurable, and S^i_t, $i = 1,\ldots,d$ and Z are the prices of $d+1$-risky assets at time t, and

S_t^0 is the value of a riskless asset at time t. We assume that the risky assets S are liquid in a given market, but Z is a non-tradable asset.

To be more specific, let us use the following general set-up for the market: we assume that the dynamics of the prices $S^1,\ldots,S^d$, Z and S^0 are described by the following stochastic differential equations (SDEs): for $t \geq 0$,

$$\begin{cases} dS_t^0 = r_t S_t^0 dt, & S_0^0 = s^0; \\ dS_t^i = S_t^i\{\mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dB_t^j\}, & S_0^i = s^i, \ \ i = 1, \ldots, d; \\ dZ_t = Z_t\{\mu_t^Z dt + \langle \sigma_t^Z, dB_t\rangle + \tilde{\sigma}_t d\tilde{B}_t\}, & Z_0 = z. \end{cases} \tag{2.9}$$

where $\tilde{B}$ is another Brownian motion, independent of B. We shall assume that the investment can be made in the market $S = (S^0, S^1, \ldots, S^d)$, and we denote π_t^i, $i = 1,\ldots,d$ to be the amount of money invested in the i-th stock. Let W_t be the total investment income (wealth) at time t; we then assume that all the rest of the money $\pi_t^0 = W_t - \Sigma_{i=1}^d \pi_t^i$ is put into the money market. Assuming that that the portfolio π is "self-financing", it is known (cf. e.g., Karatzas & Shreve, 1988) that the wealth process W satisfies the following SDE: denoting $\tilde{1} \triangleq (1, \ldots, 1)^T$,

$$dW_t = r_t W_t dt + \langle \pi_t, \mu_t - r_t \tilde{1}\rangle dt + \langle \pi_t, \sigma_t dB_t\rangle. \tag{2.10}$$

Given the nature of a UVL insurance, one can often require that the death benefit is no less than a guaranteed return. Simple examples of such a case are

i. $g(t, S_\cdot, Z_\cdot) = S_t^i \vee S_0^i, t \geq 0$, for some i;
ii. $g(t, S_\cdot, Z_\cdot) = Z_t \vee Z_0$, $t \geq 0$;
iii. $g(t, S_\cdot, Z_\cdot) = Z_t \vee e^{\bar{r}t} z$, $t \geq 0$.

In case (iii), the asset Z can be thought of as a retirement fund, and $\bar{r}$ could then be a certain growth rate which can be chosen simply as the market interest rate or any contractually prescribed rate. Such a form of benefit covers a wide range of rate of return guarantees; we refer to Milevsky & Posner (2001) and Miltersen & Persson (2006), etc., for the case of one period models. Obviously, one can design various products using the combinations of S, Z, and some guaranteed returns so as to make the insurance product more attractive, especially in a bullish stock market.

In the rest of the paper we shall denote $\mathcal{A}$ to be the set of all (portfolio) processes $\pi = (\pi_t : t \geq 0\}$ that are $\mathbf{F}^B \otimes \mathbf{F}^{\tilde{B}}$-adapted, and that

$$E\int_0^T |\pi_t|^2 dt < \infty. \tag{2.11}$$

We will call $\mathcal{A}$ the set of "admissible strategies", as usual.

2.3. Principle of equivalent utility

We are now ready to formulate the main problem of the paper. Let u be a given utility function, that is, it is a non-decreasing, concave function, and we assume that it is smooth for technical simplicity. We shall first consider the contract such that the death benefit is payable at the end of a prescribed terminal time T, with the benefit function being of the form $g(T, S_T, Z_T)X_T$. Let us further assume that the insurance company measures its performance by the simple rule of "expected utility", that is, by the following "cost functional":

$$J(t, w, s, z; \pi) \triangleq E_{t,w,s,z}\{u(W_T^\pi - g(T, S_T, Z_T)X_T)\}, \tag{2.12}$$

where W^π is the solution to (2.10) with given portfolio π, and $E_{t,w,s,z}\{\cdot\}$ denotes the conditional expectation $E\{\cdot|W_t = w, S_t = s, Z_t = z\}$, for $t \geq 0$, $w \in \mathbb{R}$, $s \in \mathbb{R}_+^d$ and $z \geq 0$. Of course, if the benefit is not paid (namely, either the insurance was not sold or the death does not occur before time T, namely $T(x) > T$, P-a.s.), then $b_T = g(T, S_T, Z_T)X_T = 0$, P-a.s. In this case the cost functional becomes

$$J^0(t, w; \pi) \triangleq E_{t,w}\{u(W_T^\pi)\}, \tag{2.13}$$

and the optimization problem is reduced to a standard utility maximization problem in finance. Another special case is when $g = g(S_T)$, namely the death benefit does not involve any non-tradable asset. In this case we denote the cost functional to be

$$\widehat{J}(t, w, s; \pi) \triangleq E_{t,w,s}\{u(W_T^\pi - g(T, S_T)X_T)\}. \tag{2.14}$$

We remark that the cases when $g \equiv 1$ or $X_t \equiv 1$ were studied in Young & Zariphopoulou (2002). But with the combination of the benefit payment function g and the status process, we will be able to treat some more general cases.

Let us now consider the "value functions" of the following optimization problems:

$$V^0(t, w) = \sup_{\pi \in \mathcal{A}} J^0(t, w; \pi); \tag{2.15}$$

$$V(t, w, s) = \sup_{\pi \in \mathcal{A}} \hat{J}(t, w, s; \pi); \tag{2.16}$$

$$U(t, w, s, z) = \sup_{\pi \in \mathcal{A}} J(t, w, s, z; \pi). \tag{2.17}$$

The *principle of equivalent utility* can be described as follows.

For any given benefit function $g: \mathbb{R}^{d+1} \mapsto \mathbb{R}_+$ and any given initial state $(S_0, Z_0, W_0) = (s, z, w)$, we define the "fair price" at time t of a UVL insurance with death benefit $g(S_T, Z_T)$ payable at time $T > t$ to be a lump-sum $p^* \geq 0$ such that

$$p^* = \inf\{p: V^0(t, w) \leq U(t, w + p, s, z), \quad \forall (t, w, s, z)\}. \tag{2.18}$$

The definition of the fair price can be understood in the a slightly different way. For any given initial state (s, z), let us call a price $p \geq 0$ "(s, z)-acceptable" if

$$V^0(t, w) \leq U(t, w + p, s, z), \qquad \forall (t, w) \in [0, T] \times \mathbb{R}. \tag{2.19}$$

Obviously, a reasonable insurer would always look for an "acceptable" price to sell an insurance product, as it gives him/her incentive (in terms of a higher expected utility in this case) than doing nothing, assuming that he/she is confident to perform optimally in investment. On the other hand, if we denote $\mathcal{P}_{s,z}$ to be the set of all (s, z)-acceptable premiums, then (2.9) states that $p^* = \inf \mathcal{P}_{s,z}$, a rather standard way of defining a fair price (compare, e.g., to the fair/hedging price of a contingent claim). The following lemma links the fair price to the principle of equivalent utility.

LEMMA 2.2. *Suppose that for given* (t, s, z) *the mapping* $w \mapsto U(t, \cdot, s, z)$ *is continuous and that* $\mathcal{P}_{s,z} \neq \emptyset$. *Then the fair price* $p* = p^*(s, z)$ *defined by (2.18) satisfies*

$$V^0(t, w) = U(t, w + p^*, s, z), \qquad \forall (t, w). \tag{2.20}$$

Proof. First note that $g(S_T, Z_T)X_T \geq 0$, a.s. The monotonicity of u then implies that $u(W_T^\pi - g(S_T, Z_T)Y_T) \leq u(W_T^\pi)$, and thus

$$U(t, w, s, z) \leq V^0(t, w). \tag{2.21}$$

On the other hand, as the wealth process W follows a linear SDE, a simple application of comparison theorem leads to the following: for given portfolio $\pi \in \mathcal{A}$, the terminal wealth W_T^π increases as the initial endowment W_0^π increases. Therefore, $U(t, w+p, s, z)$ is increasing in p. Now, using the definition of $\mathcal{P}_{s,z}$ and its non-emptiness, as well as the continuity of U, one can check that

$$V^0(t, w) \leq \inf_{p \in \mathcal{P}_{s,z}} U(t, w + p, s, z) = U(t, w + p^*, s, z). \tag{2.22}$$

Combining (2.22) and (2.21), and using the continuity of U again we can find $0 \leq p^{**} \leq p^*$ such that

$$V^0(t, w) = U(t, w + p^{**}, s, z),$$

thanks to the Mean Value Theorem. But by definition of $\mathcal{P}_{s,z}$, we must have $p^{**} \in \mathcal{P}_{s,z}$, and thus $p^* = p^{**}$ by the definition of p^*. This proves (2.20). □

REMARK 2.3. Equation (2.20) is exactly the "*principle of equivalent utility*" for determining the price p^*, first initiated in Hodges and Neuberger (1989). Lemma 2 only provides a slightly different perspective.

REMARK 2.4. The assumptions of Lemma 2 are extremely mild. In fact, the continuity of the value function U in w is almost always true. The only technical requirement is the non-emptiness of the set $P_{s,z}$. But this assumption can be easily removed if the utility function u satisfies $\lim_{w\to\infty} u(w) = +\infty$, or in the case of exponential utility. Indeed, in the former case, by virtue of monotone convergence, we have

$$\begin{aligned}
&\lim_{w\to\infty} U(t, w + p, s, z) \\
&\geq \lim_{p\to\infty} E_{t,w+p,s,z}\left\{ u\left((w + p)e^{\int_t^T r_s ds} - g(S_T, Z_T)X_T \right) \right\} \quad (\pi \equiv 0) \\
&= E_{t,s,z}\left\{ \lim_{p\to\infty} u\left((w + p)e^{\int_t^T r_s ds} - g(S_T, Z_T)X_T \right) \right\} \\
&= E_{t,s,z}\left\{ \lim_{p\to\infty} u(w) \right\} = \infty.
\end{aligned}$$

Therefore $\mathcal{P}_{s,z}$ is always non-empty. In the case of exponential utility, say, $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$, we can still show that $\mathcal{P}_{s,z}$ is non-empty. Indeed, it was shown in Young and Zariphopoulou (2002) that, in such a case, $V^0(t, w)$ is strictly negative, therefore

$$V^0(t,w) < 0 = \lim_{w\to\infty} u(w) \le \lim_{p\to\infty} U(t, w+p, s, z),$$

and (2.22) follows.

To conclude this section we give the *Standing Assumptions* for the rest of the paper:

(H1) All the market parameters μ, σ and r are deterministic, continuous functions of t. Furthermore, the volatility matrix σ is "non-degenerate" in the sense that there exists a $c_0 > 0$, such that

$$\xi^T \sigma_t \sigma_t^T \xi \ge c_0 |\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, t \in [0, T].$$

(H2) The death benefit function g:$[0, T] \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}_+$ is bounded, and continuously differentiable, with bounded derivatives.

3. The simple UVL models

In this and the next section we consider the simplest UVL insurance models in which the only uncertainty for the termination (or decrement) is death. We shall first derive the corresponding HJB equations via two different pricing strategies, depending on the structure of the death benefits. In the next section we shall derive more explicit solutions in the case of exponential utility. To simplify presentations, we shall assume $d = 1$ throughout the section. However, we should note that all the analysis can be generalized to higher dimensional cases without substantial difficulties. As the benefit is paid at a fixed time, we will drop T in the benefit function g.

To begin with, let us consider an intermediate stochastic control problem with the cost functional being

$$\tilde{J}(t, w, s, z; \pi) \triangleq E_{t,w,s,z}\{u(W_T^\pi - g(S_T, Z_T))\}, \tag{3.1}$$

Namely, we assume that the death has occurred before T, hence $X_T \equiv 1$. The utility maximization problem is then reduced to a standard stochastic control problem. Therefore, denoting the value function to be:

$$\tilde{U}(t, w, s, z) = \sup_{\pi \in A} \tilde{J}(t, w, s, z; \pi), \tag{3.2}$$

it is well known that $\tilde{U}$ is at least the unique viscosity solution to the following HJB equation (cf, Fleming & Rishel, 1975):

$$\begin{cases} 0 = \tilde{U}_t + \max_{\pi}\left\{\frac{1}{2}\sigma^2\pi^2\tilde{U}_{ww} + \pi[\sigma^2 s\tilde{U}_{ws} + \sigma^Z z\sigma\tilde{U}_{wz} + (\mu - r)\tilde{U}_w]\right\} \\ \quad + \frac{1}{2}\sigma^2 s^2\tilde{U}_{ss} + \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{Z^2})z^2\tilde{U}_{zz} + \sigma\sigma^Z sz\tilde{U}_{sz} + rw\tilde{U}_w + \mu s\tilde{U}_s + \mu^Z z\tilde{U}_z; \\ \tilde{U}(w, T, s, z) = u(w - g(s, z)). \end{cases} \tag{3.3}$$

To derive the HJB equation for the original optimization problem, we first argue heuristically. By virtue of the *Bellman Principle* (of dynamic programming, cf. Fleming & Rishel, 1975), and using the total probability formula we have, for any $h > 0$ and any admissible portfolio $\pi \in \mathcal{A}$,

$$\begin{aligned} U(w, t, s, z) &\geq E_{t,x,s,z}\{U(t+h, W^\pi_{t+h}, S_{t+h}, Z_{t+h})\} \\ &= E_{t,w,s,z}\{U(W^\pi_{t+h}, t+h, S_{t+h}, Z_{t+h})|T(x) > t+h\}P\{T(x) > t+h\} \\ &\quad + E_{t,w,s,z}\{U(W^\pi_{t+h}, t+h, S_{t+h}, Z_{t+h})|T(x) \leq t+h\}P\{T(x) \leq t+h\} \\ &= E_{t,w,s,z}\{U(W^\pi_{t+h}, t+h, S_{t+h}, Z_{t+h})\}_h p_{x+t} \\ &\quad + E_{t,w,s,z}\{\tilde{U}(W^\pi_{t+h}, t+h, S_{t+h}, Z_{t+h})\}_h q_{x+t}. \end{aligned} \tag{3.4}$$

Recall that $T(x)$ is the future-life-time random variable given current age x, which is independent of the processes (W^π, S, Z), and that, given $T(x) \leq t+h$, the optimization problem (2.17) on $[t+h, T]$ is the same as (3.2).

Suppose now that both value functions U and $\tilde{U}$ are smooth. Applying Itô's formula to $U(t, W_t, S_t, Z_t)$ and $\tilde{U}(t, W_t, S_t, Z_t)$, respectively, and noting that ${}_hp_{x+t} + {}_h q_{x+t} = 1$, one shows that (suppressing variables)

$$\begin{aligned} &[U(t, w, s, z) - \tilde{U}(t, w, s, z)]_h q_{x+t} \\ &\geq E_{t,w,s,z}\left\{\int_t^{t+h} (U_t + (rW + \pi(\mu - r)U_w) + sU_s\mu + zU_z\mu^Z + \frac{1}{2}\sigma^2(U_{ww}\pi^2 + s^2U_{ss}) \right. \\ &\left. + \frac{1}{2}z^2U_{zz}(\tilde{\sigma}^2 + \sigma^{Z^2}) + sU_{ws}\pi\sigma^2 + zU_{wz}\pi\sigma\sigma^Z + zsU_{sz}\sigma\sigma^Z\Big) du\right\}_h p_{x+t} \\ &+ E_{t,w,s,z}\left\{\int_t^{t+h} \ldots du\right\}_h q_{x+t}. \end{aligned} \tag{3.5}$$

In the above the integrand of the last integral is similar to that of the first one with U being replaced by $\tilde{U}$. Now we divide both sides of (3.5) by h and let $h \to 0$. Noting that ${}_0p_{x+t} = 1$ and ${}_hq_{x+t}/h \to \lambda_x(t)$, we obtain the following HJB equation for U:

$$\begin{cases} 0 = U_t + \max_{\pi}[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} + (U_{ws}S\sigma^2 + U_{wz}Z\sigma^Z\sigma)\pi] \\ \quad + rwU_w + sU_s\mu + zU_z\mu^Z + \frac{1}{2}\sigma^2 s^2 U_{ss} + \frac{1}{2}z^2U_{zz}(\tilde{\sigma}^2 + \sigma^{Z2}) \\ \quad + szU_{sz}\sigma\sigma^Z + \lambda_x(t)(\tilde{U} - U) \\ U(T, w, s, z) = u(w), \end{cases} \tag{3.6}$$

where $\tilde{U}$ satisfies (3.3).

3.1. The special case: g(s, z) ≡ g(s)

First note that if $g = g(S_T)$, then the non-tradable asset Z does not appear in the previous argument, and (3.6) is reduced to

$$\begin{cases} 0 = U_t + \max_\pi \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} + (\sigma^2 s U_{ws})\pi \right] + \frac{1}{2}\sigma^2 s^2 U_{ss} \\ \qquad + rwU_w + s\mu U_s + \lambda_x(t)(\tilde{U} - U) = 0, \\ U(T, w, s) = u(w), \end{cases} \tag{3.7}$$

and (3.3) is reduced to:

$$\begin{cases} 0 = \tilde{U}_t + \max_\pi \left[(\mu - r)\pi \tilde{U}_w + \frac{1}{2}\sigma^2\pi^2 \tilde{U}_{ww} + (\sigma^2 s \tilde{U}_{ws})\pi \right] \\ \qquad + \frac{1}{2}s^2\sigma^2 \tilde{U}_{ss} + rw\tilde{U}_w + \mu s \tilde{U}_s \\ \tilde{U}(T, w, s) = u(w - g(s)) \end{cases} \tag{3.8}$$

We should note that the arguments above actually reflect the following strategy: if death occurs before $t+h$ became known, then one would simply carry out the optimization problem knowing that the terminal wealth will be deducted by the amount of $g(S_T, Z_T)$ at time T. Such a strategy is simple, and works well in the case when the non-tradable assets are involved, as the risk $g(S_T, Z_T)$ is not "hedgable" in general. In the special case when the death benefit takes the form $g(S_t)$, however, the situation is slightly different. In fact, if the market is complete, then considering the benefit payment as a contingent claim, one can actually find out its current market price with which the payment amount $g(S_T)$ can be replicated. One can then simply set aside the current price of the benefit payment, and proceed the optimization problem as if there is no insurance risk involved at all. Such a strategy was actually used in, for example, Young (2003) and Young and Zariphopoulou (2002) in a similar situation (when no non-tradable assets were involved). In what follows we give a brief sketch of the argument, and show that the two strategies will actually produce the same result.

First recall value function V^0 defined by (2.15). Assuming that all the market parameters r, μ and σ are deterministic, continuous functions, then it is well known that the value function V^0 is $C^{1,2}([0, T]) \times \mathbb{R}^d$, and it satisfies the following HJB equation (cf. e.g., Young & Zariphopoulou, 2002):

$$\begin{cases} 0 = V_t^0 + \max_\pi \left\{ \frac{1}{2}|\sigma^T \pi|^2 V_{ww}^0 + \pi(\mu - r)V_w^0 \right\} + rwV_w^0, \\ V^0(T, w) = u(w). \end{cases} \tag{3.9}$$

Next, we consider the value function V, defined by (2.16). Applying the Bellman Principle and the total probability formula again, we can show that a counterpart of (3.4) holds:

$$\begin{aligned} V(t, w, s) \geq\ & E_{t,w,s}\{V(t+h, W_{t+h}, S_{t+h})\}_h p_{x+t} \\ & + E_{t,w,s}\{V^0(t+h, W_{t+h} - c(t+h, S_{t+h}))\}_h q_{x+t}, \end{aligned} \tag{3.10}$$

where $c(t, s)$ is the market price of the contingent claim $g(S_T)$. We note that in the above we used the fact that, by deducting $c(t+h,\ S_{t+h})$ from the wealth at $t+h$ and carrying out the future optimization problem without the insurance risk, the value function becomes $V^0(t+h,\ W_{t+h}-c(t+h,\ S_{t+h}))$ (compared to $\tilde{U}$ in (3.4)!).

Now repeating the same argument as before, assuming that V and V^0 and c are all smooth, and using the fact that $c(\cdot,\cdot)$ satisfies the Black-Scholes PDE (see, e.g., Young & Zariphopoulou, 2002), it is fairly easily checked that V satisfies the following HJB equation (suppressing all variables for V):

$$\begin{cases} 0 = V_t + \max_{\pi}\left\{\frac{1}{2}\sigma^2\pi^2 V_{ww} + (\mu - r)\pi V_w + \sigma^2 \pi s V_{ws}\right\} + rwV_w + \mu s V_s \\ \quad + \frac{1}{2}\sigma^2 s^2 V_{ss} + \lambda_x(t)(V^0(t, w - c(t,s)) - V) \\ V(T, w, s) = u(w) \end{cases} \tag{3.11}$$

At this point it should be clear that strategy will not work in general as the arbitrage price $c(t, s)$ is not uniquely determined if the payoff contains the non-tradable asset Z_T! One should also note that the two different strategies yield almost the same HJB equations (3.7) and (3.11), the only difference is that the last term $\tilde{U}(t, w, s)$ in (3.7) is replaced by $V^0(t, w - c(t, s))$ in (3.11). The following result nevertheless shows that the two strategies are actually the same, that is, $U(t, w, s) = V(t, w, s)$, for all $(t,\ w,\ s)$.

THEOREM 3.1. *Assume (H1), and assume that the benefit function $g \equiv g(s)$. Then, it holds that $V(t,\ w,\ s) \equiv U(t,\ w,\ s)$, for all $(t, w, s) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. Comparing (3.11) and (3.6), it is clear that we need only verify that $\tilde{U}(t, w, s) = V^0(t, w - c(t, s)), \forall (t, w, s)$.

To this end, let us first recall from (2.10) that for the given initial state w and the portfolio π, the wealth process satisfies the SDE

$$W_t = w + \int_0^t (r_u W_u + \langle \pi_u, \mu_u - r_u \mathbf{1}\rangle)du + \int_0^t \langle \pi_u, \sigma_u dB_u \rangle, \tag{3.12}$$

and we denote the solution by $W^{w,\pi}$ to specify the dependence on w and π.

It is by now well known (see, e.g., Ma & Yong, 1999) that under (H1) the Black-Schole price $c(\cdot,\cdot)$ for the contingent claim $g(S_T)$ satisfies the Black-Scholes PDE:

$$\begin{cases} \frac{\partial c}{\partial t} + r_t s \frac{\partial c}{\partial s} + \frac{1}{2}\sigma_t^2 s^2 \frac{\partial^2 c}{\partial s^2} - r_t c(t, s) = 0, \\ c(T, s) = g(s), \end{cases} \tag{3.13}$$

and the process $Y_t \triangleq c(t, S_t),\ t \geq 0$, can be expressed as the unique solution to the following *Backward Stochastic Differential Equation* (BSDE):

$$\begin{cases} dY_t = r_t Y_t dt + \langle \pi_t^0, \mu_t - r_t \mathbf{1}\rangle dt + \langle \pi_t^0, \sigma_t dB_t \rangle, \\ Y_T = g(S_T), \end{cases} \tag{3.14}$$

where $\pi^0 \in \mathcal{A}$ is the *hedging strategy* of the claim $g(S_T)$.

Let us now define a mapping $\mathcal{T}: \mathcal{A} \mapsto \mathcal{A}$ by $\mathcal{T}(\pi) = \pi' \triangleq \pi + \pi^0$, $\pi \in \mathcal{A}$. Then clearly $\mathcal{T}$ is one-to-one mapping, so that $\mathcal{T}(\mathcal{A}) = \mathcal{A}$. Furthermore, the linearity of equation (3.12) and the uniqueness of the solution to SDE implies that $W_t^{w,\pi'} = Y_t + W_t^{w-c(0,s),\pi}$, for all $t \in [0, T]$. In particular, at terminal time T, this becomes

$$W_T^{w-c(0,s),\pi} = W_T^{w,\pi'} - g(S_T), \qquad \forall \pi \in \mathcal{A}.$$

Finally, by the definition of the value functions V^0 and $\tilde{U}$ we have

$$\begin{aligned} V^0(0, w - c(0,s)) &= \sup_{\pi \in \mathcal{A}} E_{0,w}\{u(W_T^{w-c(0,s),\pi})\} \\ &= \sup_{\pi \in \mathcal{A}} E_{0,w}\{u(W_T^{w,\mathcal{T}(\pi)} - g(S_T))\} \\ &= \sup_{\pi' \in \mathcal{A}} E_{0,w}\{u(W_T^{w,\pi'} - g(S_T))\} = \tilde{U}(0, w, s), \end{aligned}$$

proving the theorem. □

4. The case of exponential utility

In this section we consider a special type of utility function – the *exponential utility*. Such a utility function has been widely used in practice, especially in actuarial mathematics. In fact, the premium principle given by (2.20) has been known to have certain very desirable properties if and only if the utility function is exponential (cf. e.g., Gerber, 1979). We should note that the discussion in this section could be considered as a generalized version of the examples in Young (2003) and Young and Zariphopoulou (2002), but as we shall see, the presence of the non-tradable assets does make the problem a little more involved.

Let us be more specific. In what follows we shall assume that the utility function takes the form

$$u(w) \triangleq -\frac{1}{\alpha} e^{-\alpha w}, \qquad w \in \mathbb{R}. \tag{4.1}$$

Then, recall from Young and Zariphopoulou (2002) that in this particular case the HJB equation (3.9) has the following explicit solution.

$$V^0(t, \ w) = -\frac{1}{\alpha} \ \exp\{-\alpha w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\} \tag{4.2}$$

Our discussion below will depend heavily on some classical results in non-linear PDEs, which can be found in, e.g., Ladyzenskaja (1968). We state a lemma, which is essentially Theorem VI-8.1 of Ladyzenskaja (1968), modified to a simpler form but sufficient for our purpose. Consider the following second-order quasilinear PDE with divergence form:

$$\begin{cases} \mathcal{L}u \equiv u_t - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial y_i} a_i(t, y, u, D_y u) + a(t, y, u, D_y u) = 0, & (t, y) \in (0, T] \times \mathbb{R}^n, \\ u(0, y) = \varphi(y), \qquad y \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

LEMMA 4.1. *Suppose that the following conditions hold for the coefficients a,* $(a_1,\dots,a_n)$ *and* φ:

i. *The function* $\varphi(y)$ *is smooth in* $\mathbb{R}^n$, *and such that* $0 \le \varphi(y) \le M$, *for all* $y \in \mathbb{R}^n$;
ii. *There exist constants* b_1, $b_2 \ge 0$, *such that for any* $t \in (0, T]$ *and y, u and* $p \in \mathbb{R}^n$, *the following inequality holds:*

$$u\left\{a(y,t,u,p) - \sum_i \left(\frac{\partial a_i}{\partial u} p_i + \frac{\partial a_i}{\partial y_i}\right)\right\}\Bigg|_{p=0} \ge -b_1 u^2 - b_2^2. \tag{4.4}$$

iii. *The function* $a(t, y, u, p)$ *is smooth, and there exist constants* $0 < \nu < \mu$ *such that the following inequalities hold, for any* $|y| \le N$, $|u| \le M_N$ *and* $i, j = 1,\dots n$:

$$\nu|\xi|^2 \le \sum_{i,j} \frac{\partial a_i}{\partial p_j}\, \xi_i \xi_j \le \mu|\xi|^2, \qquad \forall \xi \in \mathbb{R}^n, \tag{4.5}$$

and

$$\sum_i \left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|\right)(1+|p|) + \sum_{i,j} \left|\frac{\partial a_i}{\partial y_j}\right| + |a| \le \mu(1+|p|)^2. \tag{4.6}$$

iv. *for any* $|y| \le N$, $|u| \le M_N$ *and* $|p| \le M'_N$, *the functions* a_i, *a, and their partial derivatives* $\frac{\partial a_i}{\partial p_j}$, $\frac{\partial a_i}{\partial u}$ *and* $\frac{\partial a_i}{\partial y_j}$ *are all Lipschitz continuous with respect to* (t, y, u, p).

Then the Cauchy problem (4.3) has at least one classical solution. Furthermore, the solution is bounded. □

We shall discuss two cases of benefit functions separately.

4.1. The case $g = g(s)$

This case was studied also in Young (2003). As we mentioned before, in this case the contingent claim $g(S_T)$ is hedgable, and its price $c(\cdot,\cdot)$ satisfies the Black-Scholes PDE (3.13); the optimal price of the UVL is relatively easier to obtain. We have the following theorem.

THEOREM 4.2. *Assume (H1), and assume that utility function u takes the form (4.1). Suppose also that the benefit function* $g \equiv g(s)$, $s \ge 0$, *and that the force of mortality* $\lambda_x(t)$, $t \ge 0$, *are both bounded and deterministic. Then, the solution to (3.11) can be written as*

$$V(t,w,s) = V^0(t,w)\exp\{\alpha c(t,s)e^{r(T-t)} - h(t,s)\}, \tag{4.7}$$

where $c(\cdot,\cdot)$ *is the classical solution to the Black-Scholes equation (3.13), and* $h(\cdot,\cdot)$ *is the classical solution to the following reaction-diffusion system:*

$$\begin{cases} 0 = h_t + srh_s + \frac{1}{2}\sigma^2 s^2 h_{ss} - \lambda_x(t)(e^h - 1) \\ h(T, s) = \alpha g(s) \end{cases} \tag{4.8}$$

Furthermore, the optimal premium of the UVL insurance is given by

$$p = c(0, s) - \frac{h(0, s)}{\alpha} e^{-rT}. \tag{4.9}$$

Proof. First note that the solution to the HJB equation (3.11) is unique (at least in the viscosity sense); we need only show that a classical solution to (3.11) exists.

To this end, recall that the value function $V^0(t, w)$, given explicitly by (4.2), is a smooth function, concave in w, and it satisfies the PDE:

$$\begin{cases} 0 = V_t^0 - \frac{(\mu - r)^2 V_w^{02}}{2\sigma^2 V_{ww}^0} + rwV_w^0 \\ V^0(T, w, s) = u(w). \end{cases} \tag{4.10}$$

We shall look for a classical solution of (3.11) with the special form: $V(t, w, s) = V^0(t, w)\Phi(t, s)$. Note that any solution of such a form will be concave (and C^2) in w, thus we can solve the maximum in (3.11) by choosing

$$\pi^* = -\frac{(\mu - r)V_w + \sigma^2 sV_{ws}}{\sigma^2 V_{ww}},$$

and the equation (3.11) becomes

$$\begin{cases} 0 = V_t - \frac{((\mu - r)V_w + \sigma^2 sV_{ws})^2}{2\sigma^2 V_{ww}} + rwV_w + \mu sV_s + \frac{1}{2}\sigma^2 s^2 V_{ss} + \lambda_x(V^0(t, w - c(t, s)) - V) \\ V(T, w, s) = u(w). \end{cases} \tag{4.11}$$

Plugging in $V = V^0\Phi$ we obtained from (4.11) that

$$\begin{aligned} 0 = \Phi\left\{ V_t^0 - \frac{(\mu - r)^2 V_w^{0^2}}{2\sigma^2 V_{ww}^0} + rwV_w^0 \right\} + V^0\left(\Phi_t + s\mu\Phi_s + \frac{1}{2}\sigma^2 s^2 \Phi_{ss}\right) \\ +\lambda(t)\left(V^0(t, w - c(t, s)) - V^0\Phi) - \frac{V_w^{0^2}}{V_{ww}^0}(\mu - r)s\Phi_s - \frac{s^2\sigma^2\Phi_s^2}{2V_{ww}^0\Phi}\right) \end{aligned}$$

In the above, the first $\{\cdots\}$ vanishes because of (4.10). Also, using the explicit form (4.2) of V^0, and with some straightforward computation we deduce that Φ must satisfy the following PDE:

$$\begin{cases} \Phi_t + rs\Phi_s + \frac{1}{2}\sigma^2 s^2\left(\Phi_{ss} - \frac{\Phi_s^2}{\Phi}\right) + \lambda_x(t)(\exp\{c(t, s)\alpha e^{r(T-t)}\} - \Phi) = 0, \\ \Phi(T, s) = 1. \end{cases} \tag{4.12}$$

We now show that Equation (4.12) has a classical solution. To see this, we first consider the transformation: $h(t, s) = c(t, s)\alpha e^{r(T-t)} - \ln\Phi(t, s), (t, s) \in [0, T] \times \mathbb{R}$. Then it is readily seen that (suppressing variables):

$$h_t = c_t\alpha e^{r(T-t)} - c\alpha r e^{r(T-r)} - \frac{\Phi_t}{\Phi}; \quad h_s = c_s\alpha e^{r(T-t)} - \frac{\Phi_s}{\Phi};$$
$$h_{ss} = c_{ss}\alpha e^{r(T-t)} - \frac{\Phi_{ss}}{\Phi} + \frac{(\Phi_s)^2}{\Phi^2}.$$

Using these relations one easily verifies that h satisfies the PDE:

$$\begin{cases} 0 = \alpha e^{r(T-t)}\left(c_t + rsc_s + \frac{1}{2}\sigma^2 s^2 c_{ss} - rc\right) - h_t - srh_s - \frac{1}{2}\sigma^2 s^2 h_{ss} + \lambda_x(t)(e^h - 1) \\ h(T,s) = \alpha c(T,s) \end{cases} \tag{4.13}$$

Since $c(\cdot,\cdot)$ solves the Black-Scholes PDE (3.13), (4.13) becomes (4.8).

Furthermore, if we make the change of variables: $v = \log s$ and $\tau = T - t$ in (4.8), and denote $\hat{h}(\tau, v) = h(T-\tau, e^v)$, then $\hat{h}$ satisfies the following "reaction-diffusion" equation:

$$\begin{cases} h_\tau - \left(r - \frac{1}{2}\sigma^2\right)h_v - \frac{1}{2}\sigma^2 h_{vv} + \lambda_x(T-\tau)(e^h - 1) = 0 \\ h(0,v) = \alpha g(e^v) \end{cases} \tag{4.14}$$

Thus it suffices to prove that Equation (4.14) has a classical solution. To see this we shall apply Lemma 1. Indeed, note that equation (4.14) is a special case of (4.3) with

$$a_1(t,y,u,p) \triangleq \frac{1}{2}\sigma^2 p, \qquad a(y,t,u,p) \triangleq \left(\frac{1}{2}\sigma^2 - r\right)p + \lambda_x(T-t)(e^u - 1).$$

It can be easily verified that all assumptions of Lemma 4.1, except for (4.4), are trivially satisfied. To verify (4.4) we note that with the coefficients a_1 and a defined above, the left-hand side of (4.4) is reduced to $\lambda_x(T-t)u(e^u - 1)$, which is always non-negative as $\lambda_x \geq 0$ and the function $u(e^u - 1) \geq 0$ for all $u \in \mathbb{R}$. Thus (4.4) holds with $b_1 = b_2 = 0$. Consequently, applying Lemma 4.1 we see that there exists a bounded classical solution $\hat{h}$ to (4.14), hence $h(t,s) \triangleq \hat{h}(T-t, \log s)$ is a solution to (4.8), and

$$V(t,w,s) = V^0(t,w)\Phi(t,s) = V^0(t,w)\exp\{\alpha c(t,s)e^{r(T-t)} - h(t,s)\}$$

is a classical solution to (3.11), proving the first part of the theorem.

To conclude the proof, we recall that by the principle of equivalent utility, the optimal premium is defined by: $V^0(0, w) = V(0, w+p, s)$. By virtue of Theorem 4.2, this relation becomes:

$$V^0(0,w) = V^0(0,w+p)\exp\{\alpha c(0,s)e^{rT} - h(0,s)\}. \tag{4.15}$$

Now the conclusion follows from the explicit form (4.2) of $V^0(0, w)$, and some fairly simple calculations. We leave the details to the interested reader, and the proof is now complete. □

REMARK 4.3. (i) We note that if the force of mortality $\lambda_x(t) \equiv 0$, that is, the death never occurs, then it is easily checked that $h(t, s) = \alpha c(t, s)e^{r(T-t)}$ satisfies (4.8), thus $p \equiv 0$ by (4.9), as it should be.

(ii) The existence of the bounded solution to (4.8) might be a little surprising, as it is a semilinear parabolic PDE with exponential growth, which in general may have a finite time blow-up. The particular form of the non-linear term plays an important role. For a more convincing example, assuming $r = \sigma = 0$, and $g \equiv K < \infty$, then it can be checked that $h(t) = -\log\{1 - (1 - e^{-\alpha K})e^{-\int_0^t \lambda_x(T-u)du}\}$ is a solution to (4.8), which is obviously bounded.

4.2. The case g = g(s, z)

In this case the explicit formula for the price of UVL is a little more complicated, due to the presence of the non-tradable asset Z in the death benefit (hence the contingent claim $g(S_T, Z_T)$ is no longer hedgable in general). But we can still proceed along the same line as before, and possibly with slightly different notations, we have the following theorem.

THEOREM 4.4. *Assume that the utility function is of the form $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$. Assume also that the benefit function $g(\cdot,\cdot)$ and the force of mortality $\lambda_x(\cdot)$ are both smooth and bounded. Then the optimal premium can be written as*

$$p(t, s, z) = \frac{1}{\alpha}e^{-r(T-t)}h(T-t, \log s, \log z), \tag{4.16}$$

where h is a bounded, classical solution to the PDE

$$\begin{cases} h_\tau - \frac{1}{2}\tilde{\sigma}^2 h_{y_2}^2 - \frac{1}{2}\sigma^2 h_{y_1y_1} - \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{z2})h_{y_2y_2} - \sigma\sigma^z h_{y_1y_2} - \left(r - \frac{1}{2}\sigma^2\right)h_{y_1} \\ \quad -\left(\mu^z - \frac{\mu - r}{\sigma}\sigma^z - \frac{\tilde{\sigma}^2 + \sigma^{z2}}{2}\right)h_{y_2} - \lambda_x(T-\tau)(e^{\tilde{h}-h} - 1) = 0; \\ h(0, y_1, y_2) = 0, \end{cases} \tag{4.17}$$

and $\tilde{h}$ is a bounded, classical solution to the PDE:

$$\begin{cases} \tilde{h}_\tau - \frac{1}{2}\tilde{\sigma}^2 \tilde{h}_{y_2}^2 - \frac{1}{2}\sigma^2 \tilde{h}_{y_1y_1} - \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{z2})\tilde{h}_{y_2y_2} - \sigma\sigma^z \tilde{h}_{y_1y_2} \\ \quad -\left(r - \frac{1}{2}\sigma^2\right)\tilde{h}_{y_1} - \left(\mu^z - \frac{\mu - r}{\sigma}\sigma^z - \frac{\tilde{\sigma}^2 + \sigma^{z2}}{2}\right)\tilde{h}_{y_2} = 0; \\ \tilde{h}(0, y_1, y_2) = \alpha g(e^{y_1}, e^{y_2}) \end{cases} \tag{4.18}$$

Proof. The idea of the proof is similar to that of Theorem 4.2. But this time we seek solutions of (3.3) and (3.6) with $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$ such that they are of the special forms

$$\tilde{U}(t, w, s, z) = V^0(t, w)e^{\tilde{h}(T-t, \ln s, \ln z)}, \quad \text{and} \quad U(t, w, s, z) = V^0(t, w)e^{h(T-t, \ln s, \ln z)},$$

respectively. Following the same arguments as in Theorem 4.2, one shows that $\tilde{h}(\tau, s, z)$ and $h(\tau, s, z)$ will have to solve the reaction-diffusion equations (4.18) and (4.17), respectively, after a change of variables $\tau = T - t$, $y_1 = \ln s$ and $y_2 = \ln z$. Thus it remains to be verified that (4.17) and (4.18) both have (bounded) classical solutions, for which we shall make use of Lemma 1 again.

First let us consider (4.18). Define

$$\begin{cases} \tilde{a}_1(t,y,h,p) = \frac{1}{2}\sigma^2 p_1 + \frac{1}{2}\sigma\sigma^z p_2 \\ \tilde{a}_2(t,y,h,p) = \frac{1}{2}\sigma\sigma^z p_1 + \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{z^2})p_2 \\ \tilde{a}(t,y,h,p) = \left(\frac{1}{2}\sigma^2 - r\right)p_1 + \left(\frac{\tilde{\sigma}^2 + \sigma^{z2}}{2} + \frac{\mu - r}{\sigma}\sigma^z - \mu^z\right)p_2 - \frac{1}{2}\tilde{\sigma}^2 p_2^2. \end{cases} \tag{4.19}$$

Conditions (i) and (iv) of Lemma 1 are obviously satisfied. For condition (ii) we note that $\frac{\partial \tilde{a}_i}{\partial h} = 0$, and $\frac{\partial \tilde{a}_i}{\partial y_j} = 0$. Thus

$$\tilde{h}\cdot\left\{\tilde{a}(y,t,\tilde{h},p) - \sum_i \left(\frac{\partial \tilde{a}_i}{\partial h}p_i + \frac{\partial \tilde{a}_i}{\partial y_i}\right)\right\}\Bigg|_{p=0} = 0 \geq -b_1\tilde{h}^2 - b_2 \tag{4.20}$$

holds for any constants b_1, $b_2 \geq 0$, whence (4.4). Moreover, if we define $\tilde{a}_{ij} \triangleq \frac{\partial \tilde{a}_i}{\partial p_j}$, then

$$(\tilde{a}_{ij}) = \begin{pmatrix} \frac{1}{2}\sigma^2 & \frac{1}{2}\sigma\sigma^z \\ \frac{1}{2}\sigma\sigma^z & \frac{1}{2}(\tilde{\sigma}^2 + \sigma^{z^2}), \end{pmatrix} \tag{4.21}$$

then clearly $(\tilde{a}_{ij})$ is positive definite, and $\forall N > 0$ and $|y_i| \leq N$ and $|h| \leq M_N$, one has

$$\begin{aligned} &\sum_i \left(|a_i| + \left|\frac{\partial a_i}{\partial u}\right|\right)(1+|p|) + \sum_{i,j}\left|\frac{\partial a_i}{\partial y_j}\right| + |a| \\ &= ((a_{11}+a_{21})|p_1| + (a_{12}+a_{22})|p_2|)(1+|p|) + \Bigg|\left(\frac{1}{2}\ \sigma^2 - r\right)p_1 \\ &\quad + \left(\frac{\tilde{\sigma}^2 + \sigma^{z2}}{2} + \frac{\mu - r}{\sigma}\ \sigma^z - \mu^z\right)p_2 - \frac{1}{2}\ \tilde{\sigma}^2 p_2^2\Bigg| \leq \mu(1+|p|)^2. \end{aligned} \tag{4.22}$$

Thus (iv) of Lemma 4.1 holds as well, and consequently (4.18) has a bounded classical solution $\tilde{h}$.

We now consider (4.17). In this case we define $a_i = \tilde{a}_i$ and

$$a(t,y,h,p) = \left(\frac{1}{2}\sigma^2 - r\right)p_1 + \left(\frac{\tilde{\sigma}^2 + \sigma^{z^2}}{2} + \frac{\mu - r}{\sigma}\sigma^z - \mu^z\right)p_2 - \frac{1}{2}\tilde{\sigma}^2 p_2^2 - \lambda_x(T-t)(e^{\tilde{h}-h} - 1).$$

Again, all conditions of Lemma 1 can be easily verified, except for (4.4). But we note that in this case the left-hand side of (4.4) is reduced to $-\lambda_x(t)(e^{\tilde{h}(t,s,z)-h} - 1)h$, which is clearly positive when $h \to \pm\infty$, as $\lambda_x(\cdot) \geq 0$. The boundedness of $\tilde{h}$ then implies that

$$-\lambda_x(t)(e^{\tilde{h}(t,s,z)-h} - 1)h \geq -b_2, \quad \forall (t,s,z), \quad \forall h, \tag{4.23}$$

for some constant $b_2 > 0$. To wit, (4.4) holds again with $b_1 = 0$. We can then conclude again that (4.17) admits a bounded classical solution.

Finally, the explicit solution for the price p (4.16) follows from the principle of equivalent utility and solving the equation $V^0(t,w) = U(t,w+p,s,z)$. The proof is now complete. □

5. General life insurance models

In this section, we generalize the insurance model to include the contracts with multiple decrements and random payment times, and derive the HJB equation in this general setting. To be more precise, we shall assume that the payments depend on the different status of the insured and the transition between the status, and also allow the benefit to be paid at a random time (such as the *moment of death*), instead of paying at a pre-determined time T. We note that the term "general life insurance" has been used in most life or pension treaties (see, e.g., Norberg, 1992).

5.1. Properties of payment process and wealth process

Let us begin by recalling some notations from Section 2. Assume that there are m possible status, characterized by the status process $X=\{X_t\}_{t\geq 0}$, which is assumed to be càdlàg and taking values in $\{0, 1,\ldots,m\}$. We denote $I_t^i = 1_{\{X_t=i\}}$, $t \geq 0$, $i=0,\ldots,m$;

$$N_t^{ij} = \#\{\text{transitions from } i \text{ to } j \text{ during time interval } [0,t]\}.$$

Recall from Lemma 2.1 that the intensity λ^{ij} of X is the same as the "force of decrement", and a combination of (2.5) and Lemma 2.1-(i) shows that for $h>0$, $i,j=0,\ldots,m$,

$$_{t+h}\bar{p}_t^i = \exp\left\{-\int_t^{t+h}\sum_{j\neq i}\lambda_s^{ij}ds\right\}; \qquad _{t+h}p_t^{ij} = \int_t^{t+h} {}_\tau\bar{p}_t^i\lambda_\tau^{ij}d\tau. \tag{5.1}$$

Further, recall that the filtration $\mathbf{F}=\{\mathcal{F}_t\}_{t\geq 0} \triangleq \{\mathcal{F}_t^X \vee \mathcal{F}_t^B \vee \mathcal{F}_t^{\tilde{B}}\}_{t\geq 0}$.

In our general life insurance problem we consider two types of payments: one is in the form of "life-annuity" and the other in the form of "life-insurance". More precisely, we shall consider a (cumulative) payment process of the following form:

$$A_t = \sum_i \int_0^t I_u^i b^i(u,S_u,Z_u)du + \sum_{i\neq j}\int_0^t b^{ij}(u,S_u,Z_u)dN_u^{ij}, \qquad t\geq 0, \tag{5.2}$$

where $b^i(t, s, z)$ is the rate of net payment of life annuity at state i, given $S_t=s$, $Z_t=z$ and $b^{ij}(t, s, z)$ is the rate of net payment of life insurance upon transition from state i to state j, given $S_t=s$ and $Z_t=z$. Clearly, the process A_t is $\mathbf{F}$-adapted and càdlàg. In fact, as an accumulated payment up to time t, it is non-decreasing.

Throughout this section we shall make use of the following assumption on the payment rate functions:

(**H3**) The contractual payment rate functions b^i, b^{ij} are continuous, and for some constant $C>0$, such that for all $(t,s)\in[0,T]\times\mathbb{R}^{d+1}$ and $i,j=1,\ldots,k$, it holds that

$$0 \le b^i(t,s) \le C(1+|s|); \quad 0 \le b^{ij}(t,s) \le C(1+|s|). \tag{5.3}$$

Given a payment process A, we see that the wealth process (2.10) should be modified to the following:

$$dW_t^\pi = rW_t^\pi dt + \pi_t^T(\mu_t - r_t 1)dt + \pi_t^T \sigma_t dB_t - dA_t, \qquad t \ge 0. \tag{5.4}$$

We should point out that the non-tradability of the asset Z will not play any essential role in our future discussions, so for notational simplicity from now on we shall consider Z as the $(d+1)$-th component of the vector $S=(S^1,\dots,S^d,S^{d+1})$. If necessary, we shall specify the non-tradability of Z by adding constraints on the portfolio π (such as requiring that the $(d+1)$-th component of $\pi_t \in \mathbb{R}^{d+1}$ vanish). Consequently, from now on we will write the SDE for the risky assets as

$$dS_t = D[S_t]\{\mu_t dt + \sigma_t dB_t\}, \qquad t \ge 0. \tag{5.5}$$

where $D[s]$ denotes the diagonal matrix diag $[s^1,\dots,s^{d+1}]$, and B is a $d+1$-dimensional Brownian motion. In what follows we shall denote, for fixed (t, w, s) and given portfolio π, we denote $W^{t,w,\pi}$ (resp. $S^{t,s}$) to be the solution to (3.12) (resp. (5.5)), such that $W_t = w$ (resp. $S_t = s$). We often simply denote $(W^\pi, S) = (W^{t,w,\pi}, S^{t,s})$ if there is no danger of confusion.

As the utility function under consideration is assumed to be a fairly general one without any growth conditions, we shall make the following extra assumption on the set of admissible strategies $\mathcal{A}$ so that the problem is technically tractable.

(H4) The set of functions $\{E\{|\pi_\cdot|^4\};\pi\in\mathcal{A}\}$ are *uniformly integrable* (in $t\in[0, T]$).

REMARK 5.1. It is immediately seen that (H4) is satisfied if we assume that all the strategies are bounded (the case of compact control space). However, we should note that this assumption is by no means necessary for our original optimization problem.

Let us now recall from Section 2 that the first transition time $\tau_t = \inf\{s \ge t: X_s \ne X_t\}$ and τ_t^i defined by (2.3). Clearly, $\tau_t = \tau_t^1 \wedge \cdots \wedge \tau_t^k$. Furthermore, for any $\pi\in\mathcal{A}$ and $h>0$, $M>0$, we define the stopping times: $\tilde{\tau}_M^\pi \triangleq \inf\{r \ge t: |W_r^\pi - W_t^\pi| + |S_r - S_t| \ge M\}$, and

$$\tau_{h,M}^\pi \triangleq \tau_t \wedge \tilde{\tau}_M^\pi \wedge (t+h). \tag{5.6}$$

Next, let us denote the following subsets of Ω: for $k=0,\dots,m$ and $\pi\in\mathcal{A}$,

$$\begin{aligned} \Lambda_{h,M}^{k,\pi} &\triangleq \{\omega: \tau_{h,M}^\pi(\omega) = t+h, X_t = k\}, \\ \Lambda_{h,M}^{k,i,\pi} &\triangleq \{\omega: \tau_{h,M}^\pi(\omega) = \tau_t^i(\omega), X_t = k\}, \\ \tilde{\Lambda}_{h,M}^{k,\pi} &\triangleq \{\omega: \tau_{h,M}^\pi(\omega) = \tilde{\tau}_M^\pi(\omega), X_t = k\}. \end{aligned} \tag{5.7}$$

The following lemma is useful for our discussion. It is also interesting in its own right.

LEMMA 5.2. *Assume (H1)–(H4), the following convergence holds uniformly for* $\pi \in \mathscr{A}_u$:

i. $\lim_{h\to 0}\frac{1}{h}P\{\tilde{\Lambda}^{k,\pi}_{h,M}|X_t = k, W_t = w, S_t = s\} = 0$;

ii. $\lim_{h\to 0}\frac{1}{h}P\{\Lambda^{k,i,\pi}_{h,M}|X_t = k, W_t = w, S_t = s\} = \lambda^{ki}_t$;

iii. $\lim_{h\to 0}P\{\Lambda^{k,\pi}_{h,M}|X_t = k, W_t = w, S_t = s\} = 1$.

Proof. Let $\pi \in \mathcal{A}$ and assume that $\|\pi\|_\infty < \infty$. To simplify notations we shall drop the superscript "π" from all the notations in (5.6) and (5.7). Moreover, from now on we denote all constants depending only on the coefficients μ, σ, b^i and b^{ij}, etc. by a generic one, $C>0$, which is allowed to vary from line to line. Furthermore, in this proof we shall always assume that $X_t = k$, $W_t = w$ and $S_t = s$, and simply denote the conditional probabilities (resp. expectations) by P (resp. E) without further specification.

(i) Note that on the set $\tilde{\Lambda}^k_{h,M}$ one has

$$\tau_{h,M} = \tilde{\tau}_M \le \tau_t \wedge (t+h) = \tau^1_t \wedge \cdots \wedge \tau^m_t \wedge (t+h).$$

Thus for $v \in [t, \tau_{h,M})$, X_v remains in status k and W is continuous. In fact, on the set $\tilde{\Lambda}^k_{h,M}$ it holds that $W_v = \tilde{W}^{(k)}_v$, where $\tilde{W}^{(k)}$ solves the SDE:

$$\tilde{W}^{(k)}_v = w + \int_t^v \{r_u \tilde{W}^{(k)}_u + \langle \pi_u, \mu_u - r_u 1\rangle - b^k(u, S_u)\}du + \int_t^v \langle \pi_u, \sigma_u dB_u\rangle, \quad v \ge t, \tag{5.8}$$

and that $\tilde{\Lambda}^k_{h,M} \subseteq \{\sup_{t\le u\le t+h}(|\tilde{W}^{(k)}_u - w| + |S_u - s|) \ge M\}$. Therefore, we have

$$\begin{aligned} P\{\tilde{\Lambda}^k_{h,M}\} &\le P\left\{\sup_{t\le u\le t+h}(|\tilde{W}^{(k)}_u - w| + |S_u - s|) \ge M\right\} \\ &\le P\left\{\sup_{t\le u\le t+h}|\tilde{W}^{(k)}_u - w| \ge \frac{M}{2}\right\} + P\left\{\sup_{t\le u\le t+h}|S_u - s| \ge \frac{M}{2}\right\} = I_1 + I_2, \end{aligned} \tag{5.9}$$

where I_1 and I_2 are defined in an obvious way. We shall estimate both terms on the right-hand side above.

First, note that S satisfies the SDE (5.5). By a classic argument using Gronwall inequality, one has $\sup_{0\le u\le T}E|S_u|^{2\alpha} \le C$, $\alpha > 1$, for some constant $C>0$. A simple application of Chebyshev and Burkholder-Davis-Gundy inequality then yields that

$$\begin{aligned} I_2 &= P\left\{\sup_{t\le u\le t+h}|S_u - s| \ge \frac{M}{2}\right\} \\ &\le P_t\left\{\int_t^{t+h}|D[S_r]\mu_r|dr \ge \frac{M}{4}\right\} + P_t\left\{\sup_{t\le u\le t+h}\left|\int_t^u \langle \sigma_r S_r, dB_r\rangle\right| \ge \frac{M}{4}\right\} \\ &\le \frac{C}{M^{2\alpha}}\, E_t\left|\int_t^{t+h}(|\mu_r|^2|S_r|^2 + |\sigma_r S_r|^2)dr\right|^\alpha \le \frac{Ch^\alpha}{M^{2\alpha}}\sup_{0\le u\le T}E|S_u|^{2\alpha} \le \frac{Ch^\alpha}{M^{2\alpha}}. \end{aligned} \tag{5.10}$$

To estimate I_1, we denote $\beta_s \triangleq \exp\{-\int_0^s r_u du\}$ and let $\hat{W}^{(k)}_s = \beta_s \tilde{W}^{(k)}_s$, then from (5.8) we see that $\hat{W}^{(k)}$ satisfies the SDE:

$$\hat{W}_s^{(k)} = \hat{w} + \int_t^s \beta_u \{\langle \pi_u, \mu_u - r_u 1\rangle - b^k(u, S_u)\}du + \int_t^s \beta_u \langle \pi_u, \sigma_u dB_u\rangle. \tag{5.11}$$

Note that $0 < \beta_T \leq \beta_u \leq 1$, and denote $\hat{w} = \beta_t w$, we have

$$\begin{aligned} I_1 &= P\left\{\sup_{t\leq u\leq t+h} |\tilde{W}_u^{(k)} - w| \geq \frac{M}{2}\right\} = P\left\{\sup_{t\leq u\leq t+h} |\beta_u^{-1}\hat{W}_u^{(k)} - w| \geq \frac{M}{2}\right\} \\ &\leq P\left\{\sup_{t\leq u\leq t+h} |\beta_u^{-1}\hat{W}_u^{(k)} - \beta_u^{-1}\hat{w}| + \sup_{t\leq u\leq t+h} |\beta_u^{-1}\hat{w} - w| \geq \frac{M}{2}\right. \\ &\leq P\left\{\sup\nolimits_{t\leq u\leq t+h} |\hat{W}_u^{(k)} - \hat{w}| \geq \frac{M}{2\beta_T^{-1}} - 2w\right\}. \end{aligned}$$

Now following a similar argument as before we obtain that

$$\begin{aligned} I_1 &\leq P\left\{\int_t^{t+h} \beta_u |\langle \pi_u, \mu_u - r_u 1\rangle - b^k(u, S_u)| du \geq \frac{1}{2}\left(\frac{M}{2\beta_T^{-1}} - 2w\right)\right\} \\ &\quad + P\left\{\sup_{t\leq s\leq t+h} \left|\int_t^s \beta_u \langle \pi_u, \sigma_u dB_u\rangle\right| \geq \frac{1}{2}\left(\frac{M}{2\beta_T^{-1}} - 2w\right)\right\} \\ &\leq \frac{C}{(M - 4w\beta_T^{-1})^2}\left(E\left|\int_t^{t+h} |\langle \pi_u, \mu_u - r_u 1\rangle - b_u^k| du\right|^2 + E\left|\int_t^{t+h} \beta_u \langle \pi_u, \sigma_u dB_u\rangle\right|^4\right) \\ &\leq \frac{C}{(M - 4w\beta_T^{-1})^2}\left(h\int_t^{t+h} E|\pi_u|^2 du + h^2(1 + \sup_{t\leq u\leq T} E|S_u|^2) + E\left\{\int_t^{t+h} |\pi_u|^2 du\right\}^2\right) \\ &\leq \frac{Ch}{(M - 4w\beta_T^{-1})^2}\left(\int_t^{t+h} E|\pi_u|^2 du + Ch + \int_t^{t+h} E|\pi_u|^4 du\right). \end{aligned} \tag{5.12}$$

Now for $M \geq 8w\beta_T^{-1}$, combining (5.10) and (5.12), as well as assumption (H4), we proved (i).

(ii) First observe that $P\{\tau^i < t+h\} - P\{\tilde{\Lambda}_{h,M}^k\} \leq P\{\Lambda_{h,M}^{ki}\} \leq P\{\tau^i < t+h\}$. By (5.1) we have $\frac{1}{h} P\{\tau^i < t+h\} = \frac{1}{h}{}_{t+h}p_t^{ki} \to \lambda_t^{ki}$. Moreover, $\frac{1}{h}P\{\tilde{\Lambda}_{h,M}^k\} \to 0$, thanks to (i), thus (ii) follows immediately.

(iii) As $P\{\Lambda_{h,M}^{k,\pi}\} + P\{\tilde{\Lambda}_{h,M}^k\} + \Sigma_{i\neq k} P\{\Lambda_{h,M}^{ki}\} = 1$, (iii) is a direct consequence of (i) and (ii). □

5.2. Dynamic programming and HJB equation

We now turn our attention to the main result of the section, that is, to derive the HJB equation for our general insurance model. We note that in this paper we only give the sufficient conditions under which the HJB equation could be rigorously verified. A more detailed study of the value function, the well-posedness of the HJB equation, and the fact that the value function is, for instance, the viscosity solution to such a HJB equation will be studied in a forthcoming paper (Ma & Yu, 2006).

To accommodate the various status we need to introduce some auxiliary value functions. We shall define, for all $(t, w, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+^{d+1}$, and $k = 0, \ldots, m$,

$$U^k(t,w,s) \triangleq \sup_{\pi\in\mathcal{A}^u} E\{u(W_T^\pi)|W_t = w, S_t = s, X_t = k\}. \tag{5.13}$$

Also, let us denote, for $\varphi \in C^{1,2}([0,T]\times \mathbb{R}^{d+2})$,

$$\begin{aligned}\mathcal{L}[\varphi] = \mathcal{L}_{t,w,s}[\varphi] &= \sum_i \mu_t^i s^i \varphi_{s^i}(t,w,s) + \sum_{i,j,k} \sigma_t^{ik}\sigma_t^{jk} s^i s^j \varphi_{s^i s^j}(t,w,s)\\ &= \langle \varphi_s(t,w,s), D[s]\mu_t\rangle + tr\{D[s](\sigma_t\sigma_t^T)D[s](D_{ss}^2\varphi(t,w,s))\}\end{aligned} \tag{5.14}$$

Further, for $(t,w,s)\in[0,T]\times\mathbb{R}\times\mathbb{R}^{d+1}$, $(\varphi,\psi,p)\in\mathbb{R}\times(-\infty,0)\times\mathbb{R}^{d+1}$, and $k=0,1,\ldots,m$, we define

$$\begin{cases}\mathcal{H}^k(t,w,s,\varphi,\psi,p;\pi) \overset{\Delta}{=} \frac{1}{2}|\sigma_t\pi|^2\psi + [\langle\pi,\mu_t - r_t 1\rangle + r_t w - b^k(t,s)]\varphi\\ \qquad\qquad + \langle\pi, \sigma_t\sigma_t^T D[s]p\rangle\\ H^k(t,w,s,\varphi,\psi,p) \overset{\Delta}{=} \sup_\pi \mathcal{H}^k(t,w,s,\varphi,\psi,p;\pi).\end{cases} \tag{5.15}$$

We should point out that the quadratic nature (in π) of the Hamiltonian in (5.15) and the unrestricted choice of π implies that $H^k < \infty$ if and only if $\psi \in (-\infty, 0)$. We thus have the following theorem.

THEOREM 5.3. *Assume (H1)–(H4). Assume also that for all $k = 0, 1, \ldots m$ the value functions $U^k \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}^n)$. Then, for each k, U^k is strictly concave in w, and $U = (U^0, U^1, \ldots, U^m)$ satisfies the following system of HJB equations:*

$$\begin{cases}0 = U_t^k + \mathcal{L}[U^k] + H^k(t,w,s,U_w^k,U_{ww}^k,U_{ws}^k) + \sum_{j\neq k}\lambda_t^{kj}(U^j(t,w-b^{kj},s) - U^k),\\ U^k(w,T,s) = u(w), \qquad\qquad k = 0,1,\ldots,m.\end{cases} \tag{5.16}$$

Proof. We shall first show that (5.16) holds as an inequality and that all U^k's are strictly concave. To this end, we fix k and let $\pi_t \equiv \pi \in \mathbb{R}^{d+1}$, $t \geq 0$. To simplify notations, in what follows we again drop all the superscript "π" from the notations, and denote $E_t\{\cdot\} = E_{t,w,s,k}\{\cdot\}$. Applying the Bellman Principle to $\tau(h, M)$, we have

$$\begin{aligned}U^k(t,w,s) &\geq E_t\{U^{X_{\tau(h,M)}}(\tau(h,M), W_{\tau(h,M)}, S_{\tau(h,M)})\}\\ &= E_t\{U^k(\tau(h,M), W_{\tau(h,M)-}, S_{\tau(h,M)})\}\\ &\quad + E_t\{U^{X_{\tau(h,M)}}(\tau(h,M), W_{\tau(h,M)}, S_{\tau(h,M)}) - U^k(\tau(h,M), W_{\tau(h,M)-}, S_{\tau(h,M)})\}\end{aligned} \tag{5.17}$$

First applying Itô's formula and then taking expectations, we see that the first term on the right-hand side of (5.17) is (suppressing variables in U's)

$$\begin{aligned}&E_t\{U^k(\tau(h,M), W_{\tau(h,M)-}, S_{\tau(h,M)})\}\\ &= U^k(t,w,s) + E_t\left\{\int_t^{\tau(h,M)}\{U_t^k + \mathcal{L}[U^k] + \frac{1}{2}\pi_u^T\sigma_u|^2 U_{ww}^k + \langle\pi_u, \sigma_u\sigma_u^T(\mathrm{D}[S_u]U_{ws})\rangle\right.\\ &\quad \left. + (r_u W_u - b^k(u,S_u) + \langle\pi_u, \mu_u - r_u 1\rangle)U_w^k\}du + \int_t^{\tau(h,M)}\langle U_w^k\pi_u + \mathrm{D}[S_u]U_s^k, \sigma_u dB_u\rangle\right\}\\ &= U^k(t,w,s) + E_t\left\{\int_t^{\tau(h,M)}\{U_t^k + \mathcal{L}[U^k] + \mathcal{H}^k(u,S_u,W_u,U_w^k,U_{ww}^k,U_{ws}^k;\pi)du\right\}.\end{aligned} \tag{5.18}$$

On the other hand, observe that $X_{\tau(h,M)} = i$ if and only if $\omega \in \Lambda^{k,i}_{h,M}$, we see that

$$\begin{aligned} &E_t\{U^{X_{\tau(h,M)}}(\tau(h,M), W_{\tau(h,M)}, S_{\tau(h,M)}) - U^k(\tau(h,M), W_{\tau(h,M)-}, S_{\tau(h,M)})\} \\ &\quad = \sum_{i\neq k} E_t\{U^i(\tau^i, W_{\tau^i-} - b^{ki}(\tau^i, S_{\tau^i-}), S_{\tau^i-}) - U^k(\tau^i, W_{\tau^i-}, S_{\tau^i}):\Lambda^{k,i}_{h,M}\}. \end{aligned} \tag{5.19}$$

Plugging (5.18) and (5.19) into (5.17) we obtain that

$$\begin{aligned} 0 &\geq E_t\left\{\int_t^{\tau(h,M)} \{U^k_t + \mathcal{L}[U^k](u, W_u, S_u) + \mathcal{H}^k(u, S_u, W_u, U^k_w, U^k_{ww}, U^k_{ws}:\pi)\}du\right\} \\ &\quad + \sum_{i\neq k} E_t\{U^i(\tau^i, W_{\tau^i-} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^k(\tau^i, W_{\tau^i-}, S_{\tau^i}):\Lambda^{k,i}_{h,M}\} \\ &= I_1 + I_2, \end{aligned} \tag{5.20}$$

where I_1 and I_2 are defined in an obvious way. We claim that

$$\lim_{h\to 0} \frac{I_1}{h} = U^k_t(t,w,s) + \mathcal{L}[U^k](t,w,s) + \mathcal{H}^k(t,s,w,U^k_w, U^k_{ww}, U^k_{ws};\pi). \tag{5.21}$$

Indeed, if we denote $F[U^k](r) = U^k_t(r, W_r, S_r) + \mathcal{L}[U^k](r, W_r, S_r) + \mathcal{H}^k(r, W_r, S_r, U^k_w, U^k_{ww}, U^k_{ws}:\pi)$ we have

$$\begin{aligned} &\left|\frac{1}{h}I_1 - (U^k_t(t,w,s) + \mathcal{L}[U^k](t,w,s) + \mathcal{H}^k(t,s,w,U^k_w, U^k_{ww}, U^k_{ws};\pi))\right| \\ &\quad = E_t\left\{\frac{1}{h}\int_t^{t+h} |1_{\{r\leq\tau(h,M)\}}F[U^k](r) - F[U^k(t)|dr\right\} \\ &\quad \leq E_t\left\{\sup_{t\leq r\leq t+h} |1_{\{r\leq\tau(h,M)\}}F[U^k](r) - F[U^k](t)|\right\} \\ &\quad \leq E_t\left\{\sup_{t\leq r\leq \tau(h,M)} |F[U^k](r) - F[U^k](t)|\right\} + E_t\{|F[U^k](t)|:\{\Lambda^k_{h,M}\}^c\}. \end{aligned}$$

As $F[U^k](r)$ is bounded and converges to $F[U^k](t)$, a.s., and $\lim_{h\to 0} P\{\Lambda^k_{h,M}\} = 1$, we see that the right-hand side above vanishes as $h \to 0$, proving (5.21).

To estimate I_2, we note that

$$\begin{aligned} &E_t\{U^i(\tau^i, W_{\tau^{i-}} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^k(\tau^i, W_{\tau^{i-}}, S_{\tau^i}) : \Lambda^{k,i}_{h,M}\} \\ &\quad = \frac{1}{h}E_t\{U^i(\tau^i, W_{\tau^{i-}} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^i(t, w - b^{ki}(t,s), s) : \Lambda^{k,i}_{h,M}\} \\ &\qquad + \frac{1}{h}E_t\{U^k(t,w,s) - U^k(\tau^i, W_{\tau^{i-}}, S_{\tau^i}):\Lambda^{k,i}_{h,M}\} + \frac{P\{\Lambda^{k,i}_{h,M}\}}{h} \\ &\qquad \times (U^i(t, w - b^{ki}(t,s) - s) - U^k(t,w,s)) \end{aligned} \tag{5.22}$$

By virtue of Lemma 5.2-(ii), the last term tends to $\lambda^{ki}(U^i(t,\ w - b^{ki}(t,\ s),\ s) - U^k(t,\ w,\ s))$ as $h \to 0$. We claim that the first two expectations tend to 0 as $h \to 0$. Indeed, as $W_{\tau-} = \tilde{W}_\tau$, we see that the first term can be estimated by

$$\lim_{h\to 0}\frac{1}{h}|E_t\{U^i(\tau^i, W_{\tau^i-} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^i(t, w - b^{ki}(t,s), s):\Lambda^{k,i}_{h,M}\}|$$

$$\le \lim_{h\to 0}\frac{1}{h}E_t\{|U^i(\tau^i, \tilde{W}_{\tau^i} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^i(t, w - b^{ki}(t,s), s)|1_{\{\Lambda^{k,i}_{h,M}\}}\}$$

$$\le \lim_{h\to 0}E_t\left\{\frac{1_{\{\tau^i\le t+h\}}}{h}\sup_{u\in[t,t+h)}|U^i(u, \tilde{W}_u - b^{ki}(u,S_u), S_u) - U^i(t, w - b^{ki}(t,s), s)|\right\}$$

$$\le \lim_{h\to 0}\frac{P\{\tau^i \le t+h\}}{h}E_t\left\{\sup_{u\in[t,t+h)}|U^i(u, \tilde{W}_u - b^{ki}(u,S_u), S_u) - U^i(t, w - b^{ki}(t,s), s)|\right\}$$

$$= \lambda^{ki}\lim_{h\to 0}E_t\left\{\sup_{u\in[t,t+h)}|U^i(u, \tilde{W}_u - b^{ki}(u,S_u), S_u) - U^i(t, w - b^{ki}(t,s), s)|\right\}.$$

In the above, the last inequality is due to the independence of τ^i and $(\tilde{W}, S)$. Clearly, the right-hand side has convergence to zero as $h\to 0$, thanks to the continuity of $\tilde{W}$ and S, as well as the Monotone Convergence Theorem. Similarly, one can argue that the second term on the right-hand side of (5.22) vanishes as $h\to 0$ as well. Combining, we obtain that

$$\lim_{h\to 0}\frac{I_2}{h} = \sum_{i\neq k}\lambda^{ki}(U^i(t, w - b^{ki}(t,s), s) - U^k(t,w,s)) \tag{5.23}$$

Now putting (5.21) and (5.23) together, we obtain from (5.20) that

$$0 \ge \frac{1}{h}(I_1 + I_2) \to U^k_t(t,w,s) + \mathcal{L}[U^k](t,w,s) + \mathcal{H}^k(t,s,w,U^k_w,U^k_{ww},U_{ws};\pi)$$
$$+\sum_{i\neq k}\lambda^{ki}(U^i(t, w - b^{ki}(t,s), s) - U^k(t,w,s)), \quad \text{as } h\to 0.$$

Taking supremum over π we get

$$0 \ge U^k_t + \mathcal{L}[U^k] + \mathcal{H}^k(t,s,w,U^k_w,U^k_{ww},U^k_{ws}) + \sum_{j\neq k}\lambda^{kj}_t(U^j(t, w - b^{kj}(t,s), s) - U^k). \tag{5.24}$$

In particular, (5.24) shows that $H^k(t,s,w,U^k_w,U^k_{ww},U^k_{ws}) < \infty$. Thus we must have $U^k_{ww} < 0$ (see the remark following (5.15)). In other words, U^k is strictly concave for all k, as (t, w, s) are arbitrarily chosen.

It remains to be proven that the opposite direction of the inequality of (5.24) also holds. To see this, note that for any $\varepsilon > 0$, $h > 0$, there exists $\pi^0 \in \mathcal{A}$ (depending on ε, h!), such that

$$U^k(t,w,s) - \varepsilon h < E_t\{u(W^{\pi^0}_T)\}. \tag{5.25}$$

By the uniqueness of the solution to the SDE (5.4), we have

$$E_{\tau, W^{\pi^0}_\tau, S_\tau, X_\tau}(u(W^{\pi^0}_T)) = E_{\tau, W^{\pi^0}_\tau, S_\tau, X_\tau}(u(W^{\tau, W^{\pi^0}_\tau}_T)) \le U^{X_\tau}(\tau, W^{\pi^0}_\tau, S_\tau), \quad P\text{-a.s.}$$

for any stopping time τ. Taking expectation $E_t\{\cdot\}$ on both sides above, one obtains that

$$E_t\{u(W^{\pi^0}_T)\} \le E_t\{U^{X_\tau}(\tau, W^{\pi^0}_\tau, S_\tau)\}, \qquad \forall\, t\le\tau<T. \tag{5.26}$$

Further, for such a fixed π^0, let $\tau = \tau(h, M)$ as before. Then, following the same calculation as in the previous step, and using the definition of H^k, we have

$$
\begin{aligned}
-\varepsilon \leq & \frac{1}{h} E_t \left\{ \int_t^{\tau(h,M)} \{ U_t^k + \mathcal{L}[U^k](u, W_u^{\pi^0}, S_u) + \mathcal{H}^k(u, S_u, W_u^{\pi^0}, U_w^k, U_{ww}^k, U_{ws}^k \, ; \, \pi^0) \} du \right\} \\
& + \frac{1}{h} \sum_{i \neq k} E_t \{ U^i(\tau^i, W_{\tau^i -}^{\pi^0} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^k(\tau^i, W_{\tau^i -}^{\pi^0}, S_{\tau^i}) : \Lambda_{h,M}^{k,i} \} \\
\leq & \frac{1}{h} E_t \left\{ \int_t^{\tau(h,M)} \{ U_t^k + \mathcal{L}[U^k](u, W_u^{\pi^0}, S_u) + H^k(u, S_u, W_u^{\pi^0}, U_w^k, U_{ww}^k, U_{ws}^k) \} du \right\} \\
& + \frac{1}{h} \sum_{i \neq k} E_t \{ U^i(\tau^i, W_{\tau^i -}^{\pi^0} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^k(\tau^i, W_{\tau^i -}^{\pi^0}, S_{\tau^i}) : \Lambda_{h,M}^{k,i} \} \qquad (5.27)
\end{aligned}
$$

Similar to the proof of (5.21), we define a continuous function

$$
\bar{F}[U^k](r) = U_t^k(r, W_r^{\pi^0}, S_r) + L[U^k](r, W_r^{\pi^0}, S_r) + H^k(r, W_r^{\pi^0}, S_r, U_w^k, U_{ww}^k, U_{ws}^k).
$$

We note that this function actually depends on ε and h, as π^0 does. However, using the regularity of all the functions involved, as well as assumption (H4), it is not too hard to show that the following limit still holds: for fixed $\varepsilon > 0$,

$$
\lim_{h \to 0} \frac{1}{h} E_t \left\{ \int_t^{\tau(h,M)} \bar{F}[U^k](r) dr \right\} = \bar{F}[U^k](t). \qquad (5.28)
$$

With the same reason, one shows that the result of (5.23) is also valid for each $\varepsilon > 0$, again thanks to (H4). This, together with (5.28), leads to that, for fixed $\varepsilon > 0$,

$$
\begin{aligned}
& \frac{1}{h} E_t \left\{ \int_t^{\tau(h,M)} \{ U_t^k + \mathcal{L}[U^k](u, W_u^{\pi}, S_u) + H^k(u, S_u, W_u^{\pi}, U_w^k, U_{ww}^k, U_{ws}^k) \} du \right\} \\
& + \frac{1}{h} \sum_{i \neq k} E_t \{ U^i(\tau^i, W_{\tau^i -}^{\pi} - b^{ki}(\tau^i, S_{\tau^i}), S_{\tau^i}) - U^k(\tau^i, W_{\tau^i -}^{\pi}, S_{\tau^i}) : \Lambda_{h,M}^{k,i} \} \\
& \to U_t^k + \mathcal{L}[U^k] + H^k(t, s, w, U_w^k, U_{ww}^k, U_{ws}^k) + \sum_{j \neq k} \lambda_t^{kj} (U^j(t, w - b^{kj}(t,s), s) - U^k).
\end{aligned}
$$

Therefore, by letting $h \to 0$, (5.27) becomes:

$$
-\varepsilon \leq U_t^k + \mathcal{L}[U^k] + H^k(t, s, w, U_w^k, U_{ww}^k, U_{ws}^k) + \sum_{j \neq k} \lambda_t^{kj} (U^j(t, w - b^{kj}(t,s), s) - U^k). \qquad (5.29)
$$

Letting ε go to zero, we obtain the inequality as desired. The proof is now complete. □

Acknowledgements

We would like to thank the anonymous referee for his/her helpful suggestions, as well as providing several interesting references. J.M. is supported in part by NSF grant #0204332.

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