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On Quadratic *g*-Evaluations/Expectations and Related Analysis

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In this article we extend the notion of g-evaluation, in particular g-expectation, of Peng [8, 9] to the case where the generator g is allowed to have a quadratic growth (in the variable "z"). We show that some important properties of the g-expectations, including a representation theorem between the generator and the corresponding g-expectation—and consequently the reverse comparison theorem of quadratic BSDEs as well as the Jensen inequality—remain true in the quadratic case. Our main results also include a Doob–Meyer type decomposition, the optional sampling theorem, and the upcrossing inequality. The results of this article are important in the further development of the general quadratic nonlinear expectations (cf. [5]).

Keywords BMO; Doob–Meyer decomposition; Jensen's inequality; Optional sampling; Quadratic *g*-evaluations; Quadratic *g*-expectations; Reverse comparison theorem; Upcrossing inequality.

Mathematics Subject Classification Primary 60G48; Secondary 60H10, 91B30.

1. Introduction

In this article we extend the notion of *g-evaluations*, introduced by Peng [9], to the case when the generator g is allowed to have quadratic growth in the variable z. This will include the so-called quadratic g-expectation as a special case, as was in the linear growth case initiated in [8]. The notion of g-expectation, as a nonlinear

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extension of the well-known Girsanov transformations and originally motivated by theory of expected utility, has been found to have direct relations with a fairly large class of risk measures in finance. When the nonlinear expectation is allowed to have possible quadratic growth, it is expected that it will lead to the representation theorem that characterizes the general convex, but not necessarily "coherent" risk measures in terms of a class of quadratic BSDEs. The most notable example of such risk measure is the entropic risk measure (see, e.g., [1]), which is known to have a representation as the solution to a quadratic BSDE but falls outside the existing theory of the "filtration-consistent nonlinear expectations" [3], which requires that the generator be only of linear growth. We refer the readers to [2, 3, 8], and the expository article [9] for more detailed account for basic properties of g-evaluations and g-expectations, as well as the relationship between the risk measures and g-expectations. A brief review of the basic properties of g-evaluations and g-expectations will be given in Section 2 for ready references.

The main purpose of this article is to introduce the notion of quadratic g-evaluation and g-expectation, and prove some of the important properties that are deemed as essential. In an accompanying article [5], we shall further extend the notion of filtration consistent nonlinear expectation to the quadratic case, and establish the ultimate relations between a convex risk measure and a BSDE. The main results in this article include the Doob–Meyer decomposition theorem, optional sampling theorem, upcrossing inequality, and Jensen's inequality. We also prove that the quadratic generator can be represented as the limit of the difference quotients of the corresponding g-evaluation, extending the result in linear growth case [2]. With the help of this result, we can then prove the so-called *reversed comparison theorem*, as in the linear case.

Although most of the results presented in this article look similar to those in the linear case, the techniques involved in the proofs are quite different. We combine the techniques used in the study for quadratic BSDEs, initiated by Kobylanski [7] and the by now well-known properties of the BMO martingales. Since many of these results are interesting in their own right, we often present full details of proofs for future references.

This article is organized as follows. In Section 2 we give the preliminaries, and review the existing theory of g-evaluation/expectations and BMO martingales. In Section 3 we define the quadratic g-evaluation and discuss its basic properties. Some fine properties of g-evaluations/expectations are presented in Section 4. These include a representation of quadratic generator via quadratic g-evaluations, a reverse comparison theorem of quadratic BSDE, and the Jensen's inequality. In Section 5 we prove the main results of this paper regarding the quadratic g-martingales: a Doob–Meyer type decomposition, the optional sampling theorem, and the upcrossing inequality.

2. Preliminaries

Throughout this article we consider a filtered, complete probability space $(\Omega, \mathscr{F}, P, \mathbf{F})$ on which is defined a d-dimensional Brownian motion B. We assume that the filtration $\mathbf{F} \stackrel{\triangle}{=} \{\mathscr{F}_t\}_{t\geq 0}$ is generated by the Brownian motion B, augmented by all P-null sets in $\mathscr{F}$, so that it satisfies the *usual hypotheses* (cf. [10]). We denote $\mathscr{P}$ to be the progressively measurable σ-field on $\Omega \times [0, T]$; and $\mathscr{M}_{0,T}$ to be the set of all **F**-stopping times τ such that $0 \le \tau \le T$, P-a.s., where $T > 0$ is some fixed time horizon.

In what follows, we fix a finite time horizon $T > 0$, and denote $\mathbb{E}$ to be a generic Euclidean space, whose inner product and norm will be denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively; and denote $\mathbb{B}$ to be a generic Banach space with norm $\|\cdot\|$. Moreover, the following spaces of functions will be frequently used in the sequel. Let $\mathcal{G}$ be a generic sub-σ-field of $\mathcal{F}$, we denote

- for $0 \le p \le \infty$, $L^p(\mathcal{G}; \mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ, with $E(|\xi|^p) < \infty$. In particular, if $p = 0$, then $L^0(\mathcal{G}; \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables; and if $p = \infty$, then $L^\infty(\mathcal{G}; \mathbb{E})$ denotes the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ such that $\|\xi\|_\infty \stackrel{\triangle}{=} \mathrm{esssup}_{\omega \in \Omega} |\xi(\omega)| < \infty$;
- $0 \le p \le \infty$, $L^p_{\mathbf{F}}([0, T]; \mathbb{B})$ to be all $\mathbb{B}$-valued, **F**-adapted processes ψ, such that $E\int_0^T \|\psi_t\|^p dt < \infty$. In particular, $p = 0$ stands for all $\mathbb{B}$-valued, **F**-adapted processes; and $p = \infty$ denotes all processes $X \in L^0_{\mathbf{F}}([0, T]; \mathbb{B})$ such that $\|X\|_\infty \stackrel{\triangle}{=} \mathrm{esssup}_{t,\omega} |X(t, \omega)| < \infty$;
- $\mathbb{D}^\infty_{\mathbf{F}}([0, T]; \mathbb{B}) = \{X \in L^\infty_{\mathbf{F}}([0, T]; \mathbb{B}) : X \text{ has càdlàg paths}\}$;
- $\mathbb{C}^\infty_{\mathbf{F}}([0, T]; \mathbb{B}) = \{X \in \mathbb{D}^\infty_{\mathbf{F}}([0, T]; \mathbb{B}) : X \text{ has continuous paths}\}$;
- $\mathcal{H}^2_{\mathbf{F}}([0, T]; \mathbb{B}) = \{X \in L^2_{\mathbf{F}}([0, T]; \mathbb{B}) : X \text{ is predictably measurable}\}$.

Finally, if $d = 1$, we shall drop $\mathbb{E} = \mathbb{R}$ from the notation (e.g., $L^p_{\mathbf{F}}([0, T]) = L^p_{\mathbf{F}}([0, T]; \mathbb{R})$, $L^\infty(\mathcal{F}_T) = L^\infty(\mathcal{F}_T; \mathbb{R})$, and so on).

2.1. *g-Evaluations and g-Expectations*

We first recall the notion of g-evaluation introduced in Peng [9]. Given a time duration $[0, T]$, and a "generator" $g = g(t, \omega, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfying the standard conditions (e.g., it is Lipschitz in all spatial variables, and is of linear growth, etc.), consider the following BSDE on $[0, t]$, $t \in [0, T]$:

$$Y_s = \xi + \int_s^t g(r, Y_r, Z_r)dr - \int_s^t Z_r dB_r, \quad s \in [0, t], \tag{2.1}$$

where $\xi \in L^2(\mathcal{F}_t)$. Denote the unique solution by $(Y^{t,\xi}, Z^{t,\xi})$. The *g-evaluation* is defined as the family of operators $\{\mathcal{E}^g_{s,t} : L^2(\mathcal{F}_t) \mapsto L^2(\mathcal{F}_s)\}_{0 \le s \le t \le T}$ such that for any $t \in [0, T]$, $\mathcal{E}^g_{s,t}[\xi] \stackrel{\triangle}{=} Y^{t,\xi}_s$, $s \in [0, t]$.

In particular, for any $\xi \in L^2(\mathcal{F}_T)$, its *g-expectation* is defined by $\mathcal{E}^g(\xi) \stackrel{\triangle}{=} Y^{T,\xi}_0$, and its *conditional g-expectation* is defined by $\mathcal{E}^g[\xi|\mathcal{F}_t] \stackrel{\triangle}{=} \mathcal{E}^g_{t,T}[\xi]$, for any $t \in [0, T]$. We shall denote (2.1) by BSDE(t, ξ, g) in the sequel for notational convenience.

Remark 2.1. An important ingredient in the definition of g-evaluation is its "domain," namely the subset in $L^0(\mathcal{F}_T)$ on which the operator is defined (in the current case being naturally taken as $L^2(\mathcal{F}_T)$). The domain of a g-evaluation/expectation may vary as the conditions on the coefficients change, due to the restrictions on the well-posedness of the BSDE (2.1). For example, owing to the nature of quadratic BSDEs, in the rest of this article we shall choose $L^\infty(\mathcal{F}_T)$ as the domain for quadratic g-evaluations. We refer to our accompanying article [5] for a more detailed discussion on the issue of domains for general nonlinear expectations.

By virtue of the uniqueness of the solution $(Y^{t,\xi}, Z^{t,\xi})$, one can show that the g-evaluation $\mathscr{E}^g_{s,t}$ has the following properties:

(1) (*Monotonicity*) For any $\xi, \eta \in L^2(\mathscr{F}_t)$ with $\xi \geq \eta$, P-a.s., $\mathscr{E}^g_{r,t}[\xi] \geq \mathscr{E}^g_{r,t}[\eta]$, P-a.s.;
(2) (*Time-Consistency*) $\mathscr{E}^g_{r,s}\big[\mathscr{E}^g_{s,t}[\xi]\big] = \mathscr{E}^g_{r,t}[\xi]$, P-a.s., $\xi \in L^2(\mathscr{F}_t)$, $0 \leq r \leq s \leq t \leq T$;
(3) (*Constant-Preserving*) $\mathscr{E}^g_{s,t}[\xi] = \xi$, P-a.s., $\xi \in L^(\mathscr{F}_s)$, if it holds $dt \times dP$-a.s. that

$$g(t, y, 0) = 0, \quad y \in \mathbb{R}; \tag{2.2}$$

(4) ("*Zero-One Law*") For any $\xi \in L^2(\mathscr{F}_t)$ and any $A \in \mathscr{F}_s$, $s \in [0, t]$, it holds that

$$\mathbf{1}_A \mathscr{E}^g_{s,t}[\xi] = \mathbf{1}_A \mathscr{E}^g_{s,t}[\mathbf{1}_A \xi], \quad P\text{-a.s.}$$

Moreover, if $g(t, 0, 0) = 0$, $dt \times dP$-a.s., then $\mathbf{1}_A \mathscr{E}^g_{s,t}[\xi] = \mathscr{E}^g_{s,t}[\mathbf{1}_A \xi]$, P-a.s.;
(5) (*Translation Invariance*) Assume that g is independent of y, then for any $\xi \in L^2(\mathscr{F}_t)$ and $\eta \in L^2(\mathscr{F}_s)$, it holds that $\mathscr{E}^g_{s,t}[\xi + \eta] = \mathscr{E}^g_{s,t}[\xi] + \eta$, P-a.s.

Clearly, if g satisfies (2.2), then one can deduce from (2) and (3) above that

$$\mathscr{E}^g[\xi|\mathscr{F}_s] = \mathscr{E}^g_{s,T}[\xi] = \mathscr{E}^g_{s,t}\big[\mathscr{E}^g_{t,T}[\xi]\big] = \mathscr{E}^g_{s,t}[\xi], \quad P\text{-a.s.}, \;\; \xi \in L^2(\mathscr{F}_t), \;\; 0 \leq s \leq t \leq T; \tag{2.3}$$

and the conditional g-expectation $\mathscr{E}^g\{\cdot \mid \mathscr{F}_t\}$ possesses the following properties that more or less justify its name (assuming (2.2) for (2a) and (3a) below):

(1a) (Monotonicity) For any $\xi, \eta \in L^2(\mathscr{F}_T)$ with $\xi \geq \eta$, P-a.s., $\mathscr{E}^g[\xi|\mathscr{F}_t] \geq \mathscr{E}^g[\eta|\mathscr{F}_t]$, P-a.s.;
(2a) (Time-Consistency) $\mathscr{E}^g\big[\mathscr{E}^g[\xi|\mathscr{F}_t]\big|\mathscr{F}_s\big] = \mathscr{E}^g[\xi|\mathscr{F}_s]$, P-a.s., $\xi \in L^2(\mathscr{F}_T)$, $s \in [0, t]$;
(3a) (Constant-Preserving) $\mathscr{E}^g[\xi|\mathscr{F}_t] = \xi$, P-a.s., $\xi \in L^2(\mathscr{F}_t)$;
(4a) (Zero-One Law) For any $\xi \in L^2(\mathscr{F}_T)$ and $A \in \mathscr{F}_t$, it holds that $\mathbf{1}_A \mathscr{E}^g[\mathbf{1}_A \xi|\mathscr{F}_t] = \mathbf{1}_A \mathscr{E}^g[\xi|\mathscr{F}_t]$, P-a.s.; Moreover, if $g(t, 0, 0) = 0$, $dt \times dP$-a.s., then $\mathbf{1}_A \mathscr{E}^g[\xi|\mathscr{F}_t] = \mathscr{E}^g[\mathbf{1}_A \xi|\mathscr{F}_t]$, P-a.s.;
(5a) (Translation Invariance) Assume that g is independent of y, then for any $\xi \in L^2(\mathscr{F}_T)$ and $\eta \in L^2(\mathscr{F}_t)$ it holds that $\mathscr{E}^g[\xi + \eta|\mathscr{F}_t] = \mathscr{E}^g[\xi|\mathscr{F}_t] + \eta$, P-a.s.

2.2. *BMO Martingales and BMO Processes*

An important tool for studying the quadratic BSDEs, whence the quadratic g-expectations, is the so-called "BMO martingales" and the related stochastic exponentials (see, e.g., [4]). We refer to the monograph of Kazamaki [6] for a complete exposition of the theory of continuous BMO and exponential martingales. In what follows, we list some of the important facts that are useful in our future discussions for ready references.

To begin with, we recall that a uniformly integrable martingale M null at zero is called a "BMO martingale" on $[0, T]$ if for some $1 \leq p < \infty$, it holds that

$$\|M\|_{\mathrm{BMO}_p} \stackrel{\triangle}{=} \sup_{\tau \in \mathscr{M}_{0,T}} \big\| E\{|M_T - M_{\tau-}|^p \big| \mathscr{F}_\tau\}^{1/p} \big\|_\infty < \infty. \tag{2.4}$$

In such a case we denote $M \in \mathrm{BMO}(p)$. It is important to note that $M \in \mathrm{BMO}(p)$ if and only if $M \in \mathrm{BMO}(1)$, and all the $\mathrm{BMO}(p)$ norms are equivalent (cf. [6]).

Therefore, in what follows we say that a martingale M is BMO without specifying the index p; and we shall use only the BMO(2) norm and denote it simply by $\|\cdot\|_{\mathrm{BMO}}$. Note also that for a *continuous* martingale M one has

$$\|M\|_{\mathrm{BMO}} = \|M\|_{\mathrm{BMO}_2} = \sup_{\tau\in\mathscr{M}_{0,T}} \left\| E\{\langle M\rangle_T - \langle M\rangle_\tau | \mathscr{F}_\tau\}^{1/2} \right\|_\infty.$$

For a given Brownian motion B, we say that a process $Z \in L^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ is a BMO process, denoted by $Z \in \mathrm{BMO}$ by a slight abuse of notations, if the stochastic integral $M \stackrel{\triangle}{=} Z\cdot B = \int Z_t dB_t$ is a BMO martingale.

Next, for a continuous martingale M, the Doléans–Dade stochastic exponential of M, denoted customarily by $\mathscr{E}(M)$, is defined as $\mathscr{E}(M)_t \stackrel{\triangle}{=} \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$, $t\ge 0$. If M is further a BMO martingale, then the stochastic exponential $\mathscr{E}(M)$ is itself a uniformly integrable martingale (see [6, Theorem 2.3]).

The theory of BMO was brought into the study of quadratic BSDEs for the following reason. Consider, for example, the BSDE(T,ξ,g) (see (2.1)) where the generator g has a quadratic growth. Assume that there is some $k>0$ (we may assume without loss of generality that $k\ge \frac{1}{2}$) such that for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$,

$$|g(t,\omega,y,z)| \le k(1+|z|^2), \quad (y,z)\in\mathbb{R}\times\mathbb{R}^d, \tag{2.5}$$

and denote $(Y,Z)\in\mathbb{C}^\infty_{\mathbf{F}}([0,T])\times\mathscr{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ be a solution of the BSDE(T,ξ,g). For any $\tau\in\mathscr{M}_{0,T}$, applying Itô's formula to e^{4kY_t} from τ to T one has

$$\begin{aligned} e^{4kY_\tau} + 8k^2\int_\tau^T e^{4kY_s}|Z_s|^2 ds &= e^{4kY_T} + 4k\int_\tau^T e^{4kY_s} g(s,Y_s,Z_s)ds - 4k\int_\tau^T e^{4kY_s}Z_s dB_s \\ &\le e^{4kY_T} + 4k^2\int_\tau^T e^{4kY_s}\big(1+|Z_s|^2\big)ds - 4k\int_\tau^T e^{4kY_s}Z_s dB_s. \end{aligned}$$

It is then not hard to derive, using some standard arguments, the following estimate:

$$E\left[\int_\tau^T |Z_s|^2 ds \,|\, \mathscr{F}_\tau\right] \le e^{4k\|Y\|_\infty} E\big[e^{4k\xi} - e^{4kY_\tau}\,|\,\mathscr{F}_\tau\big] + e^{8k\|Y\|_\infty}(T-\tau). \tag{2.6}$$

In other words, we conclude that $Z\in\mathrm{BMO}$, and that

$$\|Z\|^2_{\mathrm{BMO}} \le (1+T)e^{8k\|Y\|_\infty}. \tag{2.7}$$

3. Quadratic g-Evaluations on $L^\infty(\mathscr{F}_T)$

Our study of the g-evaluation/expectation benefited greatly from the techniques used to treat the quadratic BSDEs, initiated by Kobylanski [7]. We first list some results regarding the existence, uniqueness, and comparison theorems for the quadratic BSDEs. Throughout the rest of the article we assume that the generator g in BSDE(T,ξ,g) (2.1) takes the form:

$$g(t,\omega,y,z) = g_1(t,\omega,y,z)y + g_2(t,\omega,y,z), \quad \forall (t,\omega,y,z)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d,$$

and satisfies the following *Standing Assumptions*:

(H1) Both g_1 and g_2 are $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable and both $g_1(t, \omega, \cdot, \cdot)$ and $g_2(t, \omega, \cdot, \cdot)$ are continuous for any $(t, \omega) \in [0, T] \times \Omega$;

(H2) There exist a constant $k > 0$ and an increasing function $\ell : \mathbb{R}^+ \mapsto \mathbb{R}^+$, such that for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$,

$$|g_1(t, \omega, y, z)| \le k \quad \text{and} \quad |g_2(t, \omega, y, z)| \le k + \ell(|y|)|z|^2, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d;$$

(H3) With the same increasing function ℓ, for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$,

$$\left| \frac{\partial g}{\partial z}(t, \omega, y, z) \right| \le \ell(|y|)(1 + |z|), \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d;$$

(H4) For any $\varepsilon > 0$, there exists a positive function $h_\varepsilon(t) \in L^1[0, T]$ such that for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$,

$$\frac{\partial g}{\partial y}(t, \omega, y, z) \le h_\varepsilon(t) + \varepsilon |z|^2, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d.$$

Under the assumptions (H1)–(H4), it is known (cf. [7, Theorems 2.3 and 2.6]) that for any $\xi \in L^\infty(\mathscr{F}_T)$, the BSDE (2.1) admits a unique solution $(Y, Z) \in \mathbb{C}^\infty_{\mathbf{F}}([0, T]) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$. In fact, this result can be extended to the following more general form, which will be useful in our future discussion.

Proposition 3.1. *Assume that g satisfies* (H1)–(H4). *For any* $\xi \in L^\infty(\mathscr{F}_T)$ *and any* $V \in \mathbb{D}^\infty_{\mathbf{F}}([0, T])$, *the BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + V_T - V_t - \int_t^T Z_s dB_s, \quad t \in [0, T], \tag{3.1}$$

admits a unique solution $(Y, Z) \in \mathbb{D}^\infty_{\mathbf{F}}([0, T]) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$.

Proof. We define a new generator $\tilde{g}$ by $\tilde{g}(t, \omega, y, z) \stackrel{\triangle}{=} g(t, \omega, y - V_t(\omega), z)$, $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Then it is easy to see that for any $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$

$$\begin{aligned} \tilde{g}_1(t, \omega, y, z) &= g_1(t, \omega, y - V_t(\omega), z), \\ \tilde{g}_2(t, \omega, y, z) &= g_2(t, \omega, y - V_t(\omega), z) - g_1(t, \omega, y - V_t(\omega), z)V_t(\omega). \end{aligned}$$

It can be easily verified that $\tilde{g}$ also satisfies (H1)–(H4). We can then conclude (see, [7]) that the BSDE$(T, \xi + V_T, \tilde{g})$ admits a unique solution $(\widetilde{Y}, Z) \in \mathbb{C}^\infty_{\mathbf{F}}([0, T]) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$. But this amounts to saying that $(\widetilde{Y} - V, Z)$ is the unique solution of (3.1), proving the corollary. □

Proposition 3.1 indicates that if g satisfies (H1)–(H4), then we can again define a g-evaluation $\mathscr{E}^g_{s,t} : L^\infty(\mathscr{F}_t) \mapsto L^\infty(\mathscr{F}_s)$ for $0 \le s \le t \le T$, as in the previous section. We shall name it as the "*quadratic g-evaluation/expectation*" for obvious reasons. More generally, for any $\sigma, \tau \in \mathscr{M}_{0,T}$ such that $\sigma \le \tau$, P-a.s., we can define the quadratic g-evaluation $\mathscr{E}^g_{\sigma,\tau} : L^\infty(\mathscr{F}_\tau) \mapsto L^\infty(\mathscr{F}_\sigma)$ by $\mathscr{E}^g_{\sigma,\tau}[\xi] \overset{\triangle}{=} Y^\xi_\sigma$, where $\xi \in L^\infty(\mathscr{F}_\tau)$, and Y^ξ satisfies the BSDE:

$$Y^\xi_t = \xi + \int_t^T \mathbf{1}_{\{s<\tau\}} g(s, Y^\xi_s, Z^\xi_s)ds - \int_t^T Z^\xi_s dB_s, \quad t \in [0, T]. \tag{3.2}$$

with $Z^\xi \in \mathscr{H}^2_{\mathbf{F}}([0,T]; \mathbb{R}^d)$, and $Y^\xi_t = Y^\xi_{t\wedge\tau}$ and $Z^\xi_t = \mathbf{1}_{\{t<\tau\}} Z^\xi_t$, P-a.s. In particular, if $\tau = T$, we define the quadratic g-expectation of ξ for any $\sigma \in \mathscr{M}_{0,T}$ by $\mathscr{E}^g[\xi|\mathscr{F}_\sigma] \overset{\triangle}{=} \mathscr{E}^g_{\sigma,T}[\xi]$.

We note that, similar to the deterministic-time case, $\mathscr{E}^g_{\sigma,\tau}$ has the following properties:

(1) *Time-Consistency:* For any $\rho, \sigma, \tau \in \mathscr{M}_{0,T}$ with $\rho \le \sigma \le \tau$, P-a.s., we have

$$\mathscr{E}^g_{\rho,\sigma}\big[\mathscr{E}^g_{\sigma,\tau}[\xi]\big] = \mathscr{E}^g_{\rho,\tau}[\xi], \quad P\text{-a.s.} \quad \forall \xi \in L^\infty(\mathscr{F}_\tau);$$

(2) *Constant-Preserving:* Assume (2.2), $\mathscr{E}^g_{\sigma,\tau}[\xi] = \xi$, P-a.s., $\forall \xi \in L^\infty(\mathscr{F}_\sigma)$;

(3) *"Zero-One Law":* For any $\xi \in L^\infty(\mathscr{F}_\tau)$ and $A \in \mathscr{F}_\sigma$, we have $\mathbf{1}_A \mathscr{E}^g_{\sigma,\tau}[\mathbf{1}_A \xi] = \mathbf{1}_A \mathscr{E}^g_{\sigma,\tau}[\xi]$, P-a.s.; Moreover, if $g(t,0,0) = 0$, $dt \times dP$-a.s., then $\mathscr{E}^g_{\sigma,\tau}[\mathbf{1}_A \xi] = \mathbf{1}_A \mathscr{E}^g_{\sigma,\tau}[\xi]$, P-a.s.;

(4) *"Translation Invariant":* If g is independent of y, then

$$\mathscr{E}^g_{\sigma,\tau}[\xi + \eta] = \mathscr{E}^g_{\sigma,\tau}[\xi] + \eta, \quad P\text{-a.s.} \quad \forall \eta \in L^\infty(\mathscr{F}_\sigma), \ \xi \in L^\infty(\mathscr{F}_\tau).$$

(5) *Strict Monotonicity:* For any $\xi, \eta \in L^\infty(\mathscr{F}_\tau)$ with $\xi \ge \eta$, P-a.s., we have $\mathscr{E}^g_{\sigma,\tau}[\xi] \ge \mathscr{E}^g_{\sigma,\tau}[\eta]$, P-a.s.; Moreover, if $\mathscr{E}^g_{\sigma,\tau}[\xi] = \mathscr{E}^g_{\sigma,\tau}[\eta]$, P-a.s., then $\xi = \eta$, P-a.s.

We remark that the last property (5) above is not completely obvious. In fact this will be a consequence of so-called "strict comparison theorem" for quadratic BSDEs, a strengthened version of the usual comparison theorem (see, for example, [7, Theorem 2.6]). For completeness we shall present such a version, under the following conditions that are similar to those in [7], but slightly weaker than (H1)–(H4).

(A1) g is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)$-measurable and $g(t, \omega, \cdot, \cdot)$ is continuous for any $(t, \omega) \in [0,T] \times \Omega$;

(A2) For any $M > 0$, there exist $\ell \in L^1[0,T]$, $k \in L^2[0,T]$ and $C > 0$ such that for $dt \times dP$-a.s. $(t, \omega) \in [0,T] \times \Omega$ and any $(y, z) \in [-M, M] \times \mathbb{R}^d$,

$$\big|g(t,\omega,y,z)\big| \le \ell(t) + C|z|^2 \quad \text{and} \quad \left|\frac{\partial g}{\partial z}(t,\omega,y,z)\right| \le k(t) + C|z|;$$

(A3) For any $\varepsilon > 0$, there exists a positive function $h_\varepsilon \in L^1[0,T]$ such that for $dt \times dP$-a.s. $(t, \omega) \in [0,T] \times \Omega$ and any $(y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\frac{\partial g}{\partial y}(t,\omega,y,z) \le h_\varepsilon(t) + \varepsilon |z|^2.$$

Theorem 3.2. *Assume* (A1)–(A3). *Let* $\xi^1, \xi^2 \in L^\infty(\mathscr{F}_T)$ *and* V^i, $i = 1,2$ *be two adapted, integrable, right-continuous processes null at* 0. *Let* $(Y^i_t, Z^i_t) \in \mathbb{D}^\infty_{\mathbf{F}}([0,T]) \times$

$\mathscr{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$, $i=1,2$ *be solutions to the BSDEs*:

$$Y^i_t = \xi^i + \int_t^T g(s, Y^i_s, Z^i_s)ds + \int_t^T dV^i_s - \int_t^T Z^i_s dB_s, \quad t\in[0,T],\ i=1,2,$$

respectively. If $\xi^1 \ge \xi^2$, *P-a.s. and* $V^1_t - V^2_t$ *is increasing, then it holds P-a.s. that*

$$Y^1_t \ge Y^2_t, \quad t\in[0,T]. \tag{3.3}$$

Moreover, if $Y^1_\tau = Y^2_\tau$ *for some* $\tau\in\mathscr{M}_{0,T}$, *then it holds P-a.s. that*

$$\xi^1 = \xi^2, \quad \textit{and} \quad V^1_T - V^2_T = V^1_\tau - V^2_\tau. \tag{3.4}$$

Proof. It is not hard to see that (3.3) is a mere generalization of [7, Theorem 2.6], thus we only need to prove (3.4). Let $M \stackrel{\triangle}{=} \|Y^1\|_\infty + \|Y^2\|_\infty$, and define $\Delta\eta = \eta^1 - \eta^2$ for $\eta = Y, Z, V$, respectively. Then ΔY satisfies:

$$\begin{aligned} d\Delta Y_t &= -\big(g(t,Y^1_t,Z^1_t) - g(t,Y^2_t,Z^2_t)\big)dt - d\Delta V_t + \Delta Z_t dB_t \\ &= -\int_0^1 \left(\frac{\partial g}{\partial y}(\Xi^\lambda_t)\Delta Y_t + \frac{\partial g}{\partial z}(\Xi^\lambda_t)\Delta Z_t\right)d\lambda dt - d\Delta V_t + \Delta Z_t dB_t \\ &= -a_t\Delta Y_t dt - d\Delta V_t + \Delta Z_t(-b_t dt + dB_t), \end{aligned} \tag{3.5}$$

where $\Xi^\lambda_t \stackrel{\triangle}{=} (t, \lambda\Delta Y_t + Y^2_t, \lambda\Delta Z_t + Z^2_t)$, and

$$a_t \stackrel{\triangle}{=} \int_0^1 \frac{\partial g}{\partial y}(\Xi^\lambda_t)d\lambda \quad \text{and} \quad b_t \stackrel{\triangle}{=} \int_0^1 \frac{\partial g}{\partial z}(\Xi^\lambda_t)d\lambda, \quad t\in[0,T].$$

Note that $|\lambda\Delta Y_t + Y^2_t| \le M$, $\forall t\in[0,T]$, P-a.s., by using some standard arguments with the help of assumptions (A1)–(A3) as well as the Burkholder–Davis–Gundy inequality we deduce from (3.5) that

$$E\left\{\sup_{t\in[0,T]}\int_0^t a_s ds + \sup_{t\in[0,T]}\left|\int_0^t b_s dB_s\right|\right\} < \infty. \tag{3.6}$$

Define $Q_t \stackrel{\triangle}{=} \exp\big\{\int_0^t a_s ds - \frac{1}{2}\int_0^t |b_s|^2 ds + \int_0^t b_s dB_s\big\}$, $t\ge 0$, and

$$\tau_n \stackrel{\triangle}{=} \inf\big\{t\in[\tau,T] : Q_t > n\big\} \wedge T, \quad n\in\mathbb{N},$$

we see that $\tau_n \uparrow T$, P-a.s., and (3.6) indicates that there exists a null set $\mathscr{N}$ such that for each $\omega\in\mathscr{N}^c$, $T = \tau_m(\omega)$ for some $m\in\mathbb{N}$. On the other hand, for any $n\in\mathbb{N}$, integrating by parts on $[\tau,\tau_n]$ yields that

$$\begin{aligned} Q_{\tau_n}\Delta Y_{\tau_n} &= Q_\tau\Delta Y_\tau - \int_\tau^{\tau_n} Q_t\Delta Y_t a_t dt - \int_\tau^{\tau_n} Q_t\Delta Z_t b_t dt - \int_\tau^{\tau_n} Q_t d\Delta V_t \\ &\quad + \int_\tau^{\tau_n} Q_t\Delta Z_t dB_t + \int_\tau^{\tau_n}\Delta Y_t Q_t a_t dt + \int_\tau^{\tau_n}\Delta Y_t Q_t b_t dB_t + \int_\tau^{\tau_n} Q_t\Delta Z_t b_t dt \\ &= -\int_\tau^{\tau_n} Q_t d\Delta V_t + \int_\tau^{\tau_n} Q_t\Delta Z_t dB_t + \int_\tau^{\tau_n}\Delta Y_t Q_t b_t dB_t. \end{aligned}$$

Taking expectation on both sides gives:

$$E\left\{Q_{\tau_n}\Delta Y_{\tau_n}+\int_{\tau}^{\tau_n}Q_t d\Delta V_t\right\}=0,$$

which implies that there exists a null set $\mathcal{N}_n$ such that for any $\omega\in\mathcal{N}_n{}^c$, it holds that $\Delta Y_{\tau_n(\omega)}(\omega)=0$ and $\Delta V_{\tau_n(\omega)}(\omega)=\Delta V_{\tau(\omega)}(\omega)$. Therefore, for any $\omega\in\Big\{\mathcal{N}\cup\big(\bigcup_{n\in\mathbb{N}}\mathcal{N}_n\big)\Big\}^c$, one has

$$\Delta Y_T(\omega)=0\quad\text{and}\quad\Delta V_T(\omega)=\Delta V_{\tau(\omega)}(\omega).$$

This completes the proof. □

In most of the discussion below, we assume the generator g satisfies (H1)–(H4) (hence (A1)–(A3)). We first extend a property of g-expectations [2, Proposition 3.1] to the case of quadratic g-evaluations.

Proposition 3.3. *Assume* (H1)–(H4). *Assume further that the generator g is deterministic. For any $t\in[0,T]$ and $\xi\in L^\infty(\mathcal{F}_t)$, if ξ is independent of $\mathcal{F}_s$ for some $s\in[0,t)$, then the random variable $\mathcal{E}^g_{s,t}[\xi]$ is deterministic.*

Proof. Let $0\le s<t\le T$ be such that $\xi\in L^\infty(\mathcal{F}_t)$ and that it is independent of $\mathcal{F}_s$. It suffices to show that $\mathcal{E}^g_{s,t}[\xi]=c$, P-a.s. for some constant c. To see this, for any $r\in[0,t-s]$, we define $B'_r=B_{s+r}-B_s$, $\mathcal{F}'_r=\sigma\big(B'_u,u\in[0,r]\big)$, and $\mathbf{F}'=\{\mathcal{F}'_r\}_{r\in[0,t-s]}$. Clearly, B' is an $\mathbf{F}'$-Brownian motion on $[0,t-s]$. Since $\xi\in\mathcal{F}_t$ is independent of $\mathcal{F}_s$, one can easily deduce that $\xi\in\mathcal{F}'_{t-s}$. Now we denote by $\{(Y'_r,Z'_r)\}_{r\in[0,t-s]}$ the unique solution to the BSDE:

$$Y'_r=\xi+\int_r^{t-s}g(s+u,Y'_u,Z'_u)du-\int_r^{t-s}Z'_u dB'_u,\quad r\in[0,t-s].$$

The simple change of variables $r=v-s$ and $w=s+u$ yields that

$$\begin{aligned}Y'_{v-s}&=\xi+\int_v^t g(w,Y'_{w-s},Z'_{w-s})dw-\int_v^t Z'_{w-s}dB'_{w-s}\\&=\xi+\int_v^t g(w,Y'_{w-s},Z'_{w-s})dw-\int_v^t Z'_{w-s}dB_w,\quad v\in[s,t].\end{aligned}$$

In other words, $\{(Y'_{v-s},Z'_{v-s})\}_{v\in[s,t]}$ is a solution to BSDE(t,ξ,g) on $[s,t]$. The uniqueness of the solution to BSDE then leads to that $Y'_{v-s}=\mathcal{E}^g_{v,t}[\xi]$, $v\in[s,t]$. In particular, one has $\mathcal{E}^g_{s,t}[\xi]=Y'_0$, P-a.s., which is a constant by the definition of $\mathbf{F}'$ and the Blumenthal 0-1 law, completing the proof. □

As we can see from the discussion so far, so long as the corresponding quadratic BSDE is well-posed, the resulting g-evaluation/expectation should behave very similarly to those with linear growth generators, with almost identical proofs using the properties obtained so far. We therefore conclude this section by listing some further properties of the g-evaluation/expection in one proposition for ready references, and leave the proofs to the interested reader.

Proposition 3.4. *Let g_i, $i = 1, 2$, be two generators both satisfy* (H1)–(H4).

1) *Suppose that $g_i(t, 0, 0) = 0$, $i = 1, 2$, and that*

$$\mathscr{E}^{g_1}_{0,t}[\xi] = \mathscr{E}^{g_2}_{0,t}[\xi], \quad \forall t \in [0, T], \ \forall \xi \in L^\infty(\mathscr{F}_t), \tag{3.7}$$

then for any $\xi \in L^\infty(\mathscr{F}_T)$, it holds P-a.s. that $\mathscr{E}^{g_1}_{t,T}[\xi] = \mathscr{E}^{g_2}_{t,T}[\xi]$, $\forall t \in [0, T]$.

2) *Suppose further that g_i, $i = 1, 2$ are independent of y, For any $t \in [0, T]$, if $\mathscr{E}^{g_1}_{0,t}[\xi] \le \mathscr{E}^{g_2}_{0,t}[\xi]$, $\forall \xi \in L^\infty(\mathscr{F}_t)$, then for any $\xi \in L^\infty(\mathscr{F}_t)$, it holds P-a.s. that $\mathscr{E}^{g_1}_{s,t}[\xi] \le \mathscr{E}^{g_2}_{s,t}[\xi]$, $\forall s \in [0, t]$.*

To end this section, we state a stability result of quadratic BSDEs which is a slight generalization of Theorem 2.8 in [7]. Since there is no substantial difference in the proof, we omit it.

Theorem 3.5. *Let $\{g_n\}$ be a sequence of generators satisfying* (H1) *and* (H2) *with the same constant $k > 0$ and increasing function ℓ. Denote, for each $n \in \mathbb{N}$, $(Y^n, Z^n) \in \mathbb{C}^\infty_{\mathbf{F}}([0, T]) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ to be a solution of BSDE(T, ξ_n, g_n) with $\xi_n \in L^\infty(\mathscr{F}_T)$.*

Suppose that $\{\xi_n\}$ is a bounded sequence in $L^\infty(\mathscr{F}_T)$, and converges P-a.s. to some $\xi \in L^\infty(\mathscr{F}_T)$; and that for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$, $\{g_n(t, \omega, y, z)\}$ converges to $g(t, \omega, y, z)$ locally uniformly in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ with g satisfying (H1)–(H4). *Then BSDE(T, ξ, g) admits a unique solution $(Y, Z) \in \mathbb{C}^\infty_{\mathbf{F}}([0, T]) \times \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that P-a.s. Y^n_t converges to Y_t uniformly in $t \in [0, T]$ and that Z^n converges to Z in $\mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$.*

4. Some Fine Properties of Quadratic *g*-Evaluations

In this section we extend some fine properties of *g*-evaluation to the quadratic case. These properties have been discovered for different reasons in the linear growth cases, and they form an integral part of the theory of nonlinear expectation. In the quadratic case, however, the proofs need to be adjusted, sometimes significantly. We collect some of them here for the distinguished importance.

We begin by a representation theorem for the generators via quadratic *g*-expectation.

Theorem 4.1. *Assume* (H1)–(H4). *Let $(t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d$. If g satisfies*

(g1) $\lim_{(s,y')\to(t^+,y)} g(s, y', z) = g(t, y, z)$, *P-a.s. and*

(g2) *For some $\varepsilon_0 \in (0, T - t]$ and some $\delta > 0$, there exists an integrable process $\{\tilde{h}_s\}_{s\in[t,t+\varepsilon_0]}$ such that for $dt \times dP$-a.s. $(s, \omega) \in [t, t + \varepsilon_0] \times \Omega$,*

$$\frac{\partial g}{\partial y'}(s, y', z) \ge \tilde{h}_s, \quad \forall y' \in \mathbb{R} \text{ with } |y' - y| \le \delta,$$

then it holds P-a.s. that

$$g(t, y, z) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\left(\mathscr{E}^g_{t,(t+\varepsilon)\wedge\tau}[y + z(B_{(t+\varepsilon)\wedge\tau} - B_t)] - y\right),$$

where $\tau \stackrel{\triangle}{=} \inf\left\{s > t : |B_s - B_t| > \frac{\delta}{1+|z|}\right\} \wedge T$.

Proof. We set $M \triangleq 1 + |y| + \frac{\delta|z|}{1+|z|}$, and $\widetilde{M} \triangleq kM + 2\ell(4M)|z|^2$. By reducing ε_0, we may assume that $\widetilde{M}\varepsilon_0 e^{k\varepsilon_0} \le \frac{\delta}{1+|z|} \wedge \frac{1}{4\ell(4M)}$.

Fix $\varepsilon \in (0, \frac{\ln 2}{k} \wedge \varepsilon_0]$. Since $\|z(B_{(t+\varepsilon)\wedge\tau} - B_t)\|_\infty \le \frac{\delta|z|}{1+|z|}$, there exists a unique solution $\big\{(Y^\varepsilon_s, Z^\varepsilon_s)\big\}_{s\in[t,t+\varepsilon]} \in \mathbb{C}^\infty_{\mathbf{F}}([t, t+\varepsilon]) \times \mathscr{H}^2_{\mathbf{F}}([t, t+\varepsilon]; \mathbb{R}^d)$ to the following BSDE:

$$Y^\varepsilon_s = y + z(B_{(t+\varepsilon)\wedge\tau} - B_t) + \int_s^{t+\varepsilon} \mathbf{1}_{\{r<\tau\}} g(r, Y^\varepsilon_r, Z^\varepsilon_r)dr - \int_s^{t+\varepsilon} Z^\varepsilon_r dB_r, \quad s \in [t, t+\varepsilon].$$

We know from Corollary 2.2 of [7] that $\|Y^\varepsilon\|_\infty \le (|y| + \frac{\delta|z|}{1+|z|} + k\varepsilon)e^{k\varepsilon} \le 2M$. Now let

$$\widetilde{Y}^\varepsilon_s \triangleq Y^\varepsilon_s - y - z(B_{s\wedge\tau} - B_t), \quad \widetilde{Z}^\varepsilon_s \triangleq Z^\varepsilon_s - \mathbf{1}_{\{s<\tau\}} z, \ \ \forall s \in [t, t+\varepsilon].$$

It is easy to check that $\{(\widetilde{Y}^\varepsilon_s, \widetilde{Z}^\varepsilon_s)\}_{s\in[t,t+\varepsilon]}$ is a solution of the BSDE:

$$\widetilde{Y}^\varepsilon_s = \int_s^{t+\varepsilon} \tilde{g}(r, \widetilde{Y}^\varepsilon_r, \widetilde{Z}^\varepsilon_r)dr - \int_s^{t+\varepsilon} \widetilde{Z}^\varepsilon_r dB_r, \quad s \in [t, t+\varepsilon] \tag{4.1}$$

with $\tilde{g}(s, \omega, y', z') \triangleq \varphi(y')\mathbf{1}_{\{s<\tau\}} g\big(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z' + z\big)$, $(s, \omega, y', z') \in [t, t+\varepsilon] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ where $\varphi : \mathbb{R} \mapsto [0, 1]$ is an arbitrary $C^1(\mathbb{R})$ function that equals to 1 inside $[-3M, 3M]$, vanishes outside $(-3M-1, 3M+1)$ and satisfies $\sup_{3M<|x|<3M+1} |\varphi'(x)| \le 2$. For any $(s, \omega, y', z') \in [t, t+\varepsilon] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, we see that

$$\tilde{g}(s, \omega, y', z') = \tilde{g}_1(s, \omega, y', z')y' + \tilde{g}_2(s, \omega, y', z')$$

with

$$\begin{aligned}
\tilde{g}_1(s, \omega, y', z') &= \varphi(y')\mathbf{1}_{\{s<\tau\}} g_1\big(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z' + z\big), \\
\tilde{g}_2(s, \omega, y', z') &= \varphi(y')\mathbf{1}_{\{s<\tau\}} g_1\big(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z' + z\big) \\
&\quad \times \big(y + z(B_{s\wedge\tau}(\omega) - B_t(\omega))\big) \\
&\quad + \varphi(y')\mathbf{1}_{\{s<\tau\}} g_2\big(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z' + z\big).
\end{aligned}$$

One can easily deduce from (H2) and (H3) that for $dt \times dP$-a.s. $(s, \omega) \in [t, t+\varepsilon] \times \Omega$, it holds for any $(y', z') \in \mathbb{R} \times \mathbb{R}^d$ that

$$|\tilde{g}_1(s, \omega, y', z')| \le k \tag{4.2}$$

$$|\tilde{g}_2(s, \omega, y', z')| \le kM + 2\ell(4M)\big(|z|^2 + |z'|^2\big) = \widetilde{M} + 2\ell(4M)|z'|^2 \tag{4.3}$$

$$\text{and} \quad \left|\frac{\partial \tilde{g}}{\partial z'}(s, \omega, y', z')\right| \le \ell(4M)\big(1 + |z'| + |z|\big). \tag{4.4}$$

Corollary 2.2 of [7] once again shows that $\|\widetilde{Y}^\varepsilon\|_\infty \le \widetilde{M}\varepsilon e^{k\varepsilon} \le \widetilde{M}\varepsilon_0 e^{k\varepsilon_0} \le \frac{\delta}{1+|z|} \wedge \frac{1}{4\ell(4M)}$. Applying Itô's formula to $|\widetilde{Y}^\varepsilon_s|^2$ we obtain that

$$|\widetilde{Y}^\varepsilon_s|^2 + \int_s^{t+\varepsilon} |\widetilde{Z}^\varepsilon_r|^2 dr = 2\int_s^{t+\varepsilon} \widetilde{Y}^\varepsilon_r\, \tilde{g}(r, \widetilde{Y}^\varepsilon_r, \widetilde{Z}^\varepsilon_r)dr - 2\int_s^{t+\varepsilon} \widetilde{Y}^\varepsilon_r \widetilde{Z}^\varepsilon_r dB_r, \quad s \in [t, t+\varepsilon]. \tag{4.5}$$

Using (4.2)–(4.4) and some standard manipulations one derives easily that

$$\begin{aligned}
&2\int_s^{t+\varepsilon} \widetilde{Y}^\varepsilon_r \tilde{g}(r, \widetilde{Y}^\varepsilon_r, \widetilde{Z}^\varepsilon_r)dr \\
&= 2\int_s^{t+\varepsilon} \widetilde{Y}^\varepsilon_r\, \tilde{g}(r, \widetilde{Y}^\varepsilon_r, 0)dr + 2\int_s^{t+\varepsilon} \widetilde{Y}^\varepsilon_r \widetilde{Z}^\varepsilon_r \left(\int_0^1 \frac{\partial \tilde{g}}{\partial z'}(r, \widetilde{Y}^\varepsilon_r, \lambda \widetilde{Z}^\varepsilon_r)d\lambda\right)dr \\
&\le 2\int_s^{t+\varepsilon} |\widetilde{Y}^\varepsilon_r|\big(k|\widetilde{Y}^\varepsilon_r| + \widetilde{M}\big)dr + 2\ell(4M)\int_s^{t+\varepsilon} |\widetilde{Y}^\varepsilon_r|\,|\widetilde{Z}^\varepsilon_r|\left(1 + |z| + \frac{1}{2}|\widetilde{Z}^\varepsilon_r|\right)dr \\
&\le \int_s^{t+\varepsilon} |\widetilde{Y}^\varepsilon_r|\big(2k|\widetilde{Y}^\varepsilon_r| + 2\widetilde{M} + \ell(4M)(1+|z|)^2\big)dr + 2\ell(4M)\int_s^{t+\varepsilon} |\widetilde{Y}^\varepsilon_r|\,|\widetilde{Z}^\varepsilon_r|^2 dr \\
&\le C\varepsilon^2 + \frac{1}{2}\int_s^{t+\varepsilon} |\widetilde{Z}^\varepsilon_r|^2 dr, \quad s \in [t, t+\varepsilon],
\end{aligned}$$

where C is a generic constant depending on $|y|, |z|, \varepsilon_0, \delta, k$ and $\ell(4M)$, which may vary from line to line. Taking the conditional expectation $E[\ |\mathscr{F}_s]$ on both sides of (4.5) we have

$$E\left\{\int_s^{t+\varepsilon} |\widetilde{Z}^\varepsilon_r|^2 dr \,\Big|\, \mathscr{F}_s\right\} \le C\varepsilon^2, \quad s \in [t, t+\varepsilon]. \tag{4.6}$$

Now, taking the conditional expectation in the BSDE (4.1) we have

$$\begin{aligned}
\frac{1}{\varepsilon}\widetilde{Y}^\varepsilon_t - \tilde{g}(t,0,0) &= \frac{1}{\varepsilon}E\left\{\int_t^{t+\varepsilon} \big(\tilde{g}(r, \widetilde{Y}^\varepsilon_r, \widetilde{Z}^\varepsilon_r) - \tilde{g}(t,0,0)\big)dr \,\Big|\, \mathscr{F}_t\right\} \\
&= \frac{1}{\varepsilon}E\Bigg\{\int_t^{t+\varepsilon}\Bigg[\widetilde{Z}^\varepsilon_r \int_0^1 \frac{\partial \tilde{g}}{\partial z'}(r, \widetilde{Y}^\varepsilon_r, \lambda \widetilde{Z}^\varepsilon_r)d\lambda \\
&\qquad + \widetilde{Y}^\varepsilon_r \int_0^1 \frac{\partial \tilde{g}}{\partial y'}(r, \lambda \widetilde{Y}^\varepsilon_r, 0)d\lambda + \tilde{g}(r,0,0) - \tilde{g}(t,0,0)\Bigg]dr \,\Bigg|\, \mathscr{F}_t\Bigg\}.
\end{aligned}$$

We know from (g2) and (H4) that for $dt \times dP$-a.s. $(s, \omega) \in [t, t+\varepsilon] \times \Omega$,

$$\tilde{h}_s \le \frac{\partial g}{\partial y'}(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z) \le h_1(s) + |z|^2$$

holds for any $y' \in \mathbb{R}$ with $|y'| \le \frac{\delta}{1+|z|}$. It follows that for $dt \times dP$-a.s. $(s, \omega) \in [t, t+\varepsilon] \times \Omega$,

$$\left|\frac{\partial \tilde{g}}{\partial y'}(s, \omega, y', 0)\right| = \left|\varphi'(y')\mathbf{1}_{\{s<\tau\}} g(s, \omega, y' + y + z(B_{s\wedge\tau}(\omega) - B_t(\omega)), z)\right|$$

$$
\begin{aligned}
&+\left|\varphi(y')\mathbf{1}_{\{s<\tau\}}\frac{\partial g}{\partial y'}(s,\omega,y'+y+z(B_{s\wedge\tau}(\omega)-B_t(\omega)),z)\right|\\
&\leq 2k(1+4M)+\big(1+2\ell(4M)\big)|z|^2+|\tilde{h}_s|+h_1(s)\triangleq h_s
\end{aligned}
$$

holds for any $y'\in\mathbb{R}$ with $|y'|\leq\frac{\delta}{1+|z|}$. Clearly, $\{h_s\}_{s\in[t,t+\varepsilon_0]}$ is an integrable process. Then applying (4.4), (4.6) and the Hölder Inequality we have

$$
\begin{aligned}
\left|\frac{1}{\varepsilon}\widetilde{Y}_t^\varepsilon-\tilde{g}(t,0,0)\right| &\leq \frac{1}{\varepsilon}E\left\{\int_t^{t+\varepsilon}\left[\ell(4M)\left((1+|z|)|\widetilde{Z}_r^\varepsilon|+\frac{1}{2}|\widetilde{Z}_r^\varepsilon|^2\right)+|\widetilde{Y}_r^\varepsilon|h_r\right]dr\,\Big|\,\mathcal{F}_t\right\}\\
&\quad+E\left\{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\big|\tilde{g}(r,0,0)-\tilde{g}(t,0,0)\big|dr\,\Big|\,\mathcal{F}_t\right\}\\
&\leq C\big(\varepsilon+\sqrt{\varepsilon}\big)+\widetilde{M}e^{k\varepsilon}E\left[\int_t^{t+\varepsilon}h_r dr\,\Big|\,\mathcal{F}_t\right]\\
&\quad+E\left\{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\big|\tilde{g}(r,0,0)-\tilde{g}(t,0,0)\big|dr\,\Big|\,\mathcal{F}_t\right\}. \qquad (4.7)
\end{aligned}
$$

As $\lim_{s\to t^+}\mathbf{1}_{\{s<\tau\}}=1$ and $\lim_{s\to t^+}(B_{s\wedge\tau}-B_t)=0$, P-a.s., one can deduce from (g1) that

$$
\lim_{s\to t^+}\tilde{g}(s,0,0)=\lim_{s\to t^+}g(s,y+z(B_{s\wedge\tau}-B_t),z)=g(t,y,z)=\tilde{g}(t,0,0),\quad P\text{-a.s.},
$$

which implies that

$$
\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\big|\tilde{g}(r,0,0)-\tilde{g}(t,0,0)\big|dr=0,\quad P\text{-a.s.}
$$

Since $|\tilde{g}(s,\omega,0,0)|\leq\widetilde{M}$ for $dt\times dP$-a.s. $(s,\omega)\in[t,t+\varepsilon]\times\Omega$, Lebesgue Convergence Theorem implies that the right hand side of (4.7) converges P-a.s. to 0 as $\varepsilon\to 0^+$. Therefore,

$$
\begin{aligned}
g(t,y,z)=\tilde{g}(t,0,0)&=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\widetilde{Y}_t^\varepsilon=\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}(Y_t^\varepsilon-y)\\
&=\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,(t+\varepsilon)\wedge\tau}[y+z(B_{(t+\varepsilon)\wedge\tau}-B_t)]-y\big),\quad P\text{-a.s.},
\end{aligned}
$$

where (3.2) was used in the last equality. The proof is now complete. □

A simple application of the theorem above gives rise to a reverse to the Comparison Theorem of quadratic BSDE:

Theorem 4.2. *Assume that g_i, $i=1,2$ satisfy* (H1)–(H4) *and* (2.2). *Let $t\in[0,T)$. If $\mathcal{E}^{g_1}[\xi|\mathcal{F}_t]\leq\mathcal{E}^{g_2}[\xi|\mathcal{F}_t]$, P-a.s. for any $\xi\in L^\infty(\mathcal{F}_T)$, and if both g_i satisfy* (g1) *and* (g2) *for any $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, then it holds P-a.s. that*

$$
g_1(t,y,z)\leq g_2(t,y,z),\quad\forall(y,z)\in\mathbb{R}\times\mathbb{R}^d.
$$

We also have the following corollary of Theorem 4.1.

Proposition 4.3. *Assume that g satisfies* (H1)–(H4) *and* (2.2). *We also assume that P-a.s., $g(\cdot, y, z)$ is continuous for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. If g satisfies* (g1) *and* (g2) *for any $(t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d$, then g is independent of y if and only if*

$$\mathscr{E}^g[\xi + c] = \mathscr{E}^g[\xi] + c, \quad \forall \xi \in L^\infty(\mathscr{F}_T), \ \forall c \in \mathbb{R}.$$

Proof:. "⇒": A simply application of translation invariance of quadratic g-expectations.

"⇐": For any $c \in \mathbb{R}$, we define a new generator $g^c(t, \omega, y, z) \stackrel{\triangle}{=} g(t, \omega, y - c, z)$, $\forall (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. It is easy to check that g^c satisfies (H1)–(H4) as well as the other assumptions on g in this proposition. For any $\xi \in L^\infty(\mathscr{F}_T)$, let (Y, Z) denote the unique solution to BSDE(T, ξ, g). Setting $\widetilde{Y}_t = Y_t + c, \ t \in [0, T]$ one obtains that

$$\widetilde{Y}_t = \xi + c + \int_t^T g^c(s, \widetilde{Y}_s, Z_s) ds - \int_t^T Z_s dB_s, \quad \forall t \in [0, T].$$

Thus, it holds P-a.s. that

$$\mathscr{E}^{g^c}[\xi + c | \mathscr{F}_t] = \widetilde{Y}_t = Y_t + c = \mathscr{E}^g[\xi | \mathscr{F}_t] + c, \quad \forall t \in [0, T].$$

In particular, taking $t = 0$ gives that $\mathscr{E}^{g^c}[\xi] = \mathscr{E}^g[\xi]$ for any $\xi \in L^\infty(\mathscr{F}_T)$. Since g satisfies (2.2), it easy to see that the condition (3.7) is satisfied for $g^1 \stackrel{\triangle}{=} g$ and $g^2 \stackrel{\triangle}{=} g^c$. Hence, Proposition 3.4 implies that for any $\xi \in L^\infty(\mathscr{F}_T)$, it holds P-a.s. that $\mathscr{E}^g[\xi | \mathscr{F}_t] = \mathscr{E}^{g^c}[\xi | \mathscr{F}_t]$, $\forall t \in [0, T]$. Applying Theorem 4.1 we see that for any $(t, z) \in [0, T) \times \mathbb{R}^d$, it holds P-a.s. that $g(t, c, z) = g^c(t, c, z) = g(t, 0, z)$. Then (H1) implies that for any $t \in [0, T)$, it holds P-a.s. that $g(t, y, z) = g(t, 0, z)$, $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$. Eventually, by our assumption, it holds P-a.s. that $g(t, y, z) = g(t, 0, z)$, $\forall (t, y, z) \in [0, T) \times \mathbb{R} \times \mathbb{R}^d$. This proves the proposition. □

To end this section we extend another important feature of the g-expectation to the quadratic case: The Jensen's inequality. We begin by recalling some basic facts for convex functions, and we refer to Rockafellar [11] for all the notions to appear below.

Recall that if $F : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function, then by considering the convex real function $f(\lambda) \stackrel{\triangle}{=} F(\lambda x) - \big(\lambda F(x) + (1 - \lambda) F(0)\big)$, $\lambda \in \mathbb{R}$, with $f(0) = f(1) = 0$, it is easy to check that for any $x \in \mathbb{R}^n$, it holds that

$$\begin{cases} F(\lambda x) \le \lambda F(x) + (1 - \lambda) F(0), & \text{if } \lambda \in [0, 1], \\ F(\lambda x) \ge \lambda F(x) + (1 - \lambda) F(0), & \text{if } \lambda \in (0, 1)^c. \end{cases} \tag{4.8}$$

Next, if $F : \mathbb{R} \mapsto \mathbb{R}$ is a convex (real) function, then we denote by ∂F the *subdifferential* of F (see [11]). In particular, for any $x \in \mathbb{R}$, $\partial F(x)$ is simply an interval $[F'_-(x), F'_+(x)]$, where F'_- and F'_+ are left-, and right-derivatives of F, respectively. The following result is an extension of the linear growth case (cf. [2, Proposition 5.2]).

Theorem 4.4. *Assume that g is independent of y and satisfies* (H1)–(H4) *and* (2.2). *Let* $t \in [0, T)$. *If* $g(s, \omega, z)$ *is convex in z for* $dt \times dP$*-a.s.* $(s, \omega) \in [t, T] \times \Omega$, *then*

$$F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big) \le \mathscr{E}^g[F(\xi)|\mathscr{F}_t], \quad P\text{-}a.s.$$

for any $\xi \in L^\infty(\mathscr{F}_T)$ *with* $\partial F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big) \cap (0,1)^c \neq \emptyset$, *P-a.s.*

Proof. Since both $F'_-(x)$ and $F'_+(x)$ are nondecreasing functions, we can define another non-decreasing function:

$$\beta(x) \triangleq \mathbf{1}_{\{F'_-(x)\le 0\}} F'_-(x) + \mathbf{1}_{\{F'_-(x)>0\}} F'_+(x), \quad x \in \mathbb{R}.$$

Thus, $\beta_t \triangleq \beta\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big)$ is an $\mathscr{F}_t$-measurable random variable. Since $\beta(x) \in (0,1)^c$ for any $x \in \mathbb{R}$ with $\partial F(x) \cap (0,1)^c \neq \emptyset$, it follows that

$$\beta_t \in (0,1)^c, \quad P\text{-a.s.} \tag{4.9}$$

One can deduce from the convexity of F that

$$\beta_t\big(\xi - \mathscr{E}^g[\xi|\mathscr{F}_t]\big) \le F(\xi) - F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big). \tag{4.10}$$

Since $\xi \in L^\infty(\mathscr{F}_T)$, it is clear that $F(\xi)$, $\mathscr{E}^g[\xi|\mathscr{F}_t]$, $F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big)$ as well as $\beta_t\big(\xi - \mathscr{E}^g[\xi|\mathscr{F}_t]\big)$ are all of $L^\infty(\mathscr{F}_T)$. Taking $\mathscr{E}^g[\ |\mathscr{F}_t]$ on both side of (4.10), and using Translation Invariance of quadratic g-expectation we have

$$\begin{aligned}\mathscr{E}^g[\beta_t\xi|\mathscr{F}_t] - \beta_t\mathscr{E}^g[\xi|\mathscr{F}_t] &= \mathscr{E}^g\big[\beta_t\big(\xi - \mathscr{E}^g[\xi|\mathscr{F}_t]\big)\big|\mathscr{F}_t\big] \\ &\le \mathscr{E}^g\big[F(\xi) - F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big)\big|\mathscr{F}_t\big] \\ &= \mathscr{E}^g[F(\xi)|\mathscr{F}_t] - F\big(\mathscr{E}^g[\xi|\mathscr{F}_t]\big), \quad P\text{-a.s.}\end{aligned}$$

Hence, it suffices to show that $\beta_t\mathscr{E}^g[\xi|\mathscr{F}_t] \le \mathscr{E}^g[\beta_t\xi|\mathscr{F}_t]$, P-a.s. To see this, let $Y_t \triangleq \mathscr{E}^g[\xi|\mathscr{F}_t]$, $t \in [0, T]$. As $\beta_t \in \mathscr{F}_t$, one has

$$\beta_t Y_s = \beta_t\xi + \int_s^T \beta_t g(r, Z_r)dr - \int_s^T \beta_t Z_r dB_r, \quad \forall s \in [t, T].$$

Since g is convex and satisfies (2.2), using (4.8) and (4.9) we obtain

$$\beta_t Y_s \le \beta_t\xi + \int_s^T g(r, \beta_t Z_r)dr - \int_s^T \beta_t Z_r dB_r = \mathscr{E}^g[\beta_t\xi|\mathscr{F}_s], \quad \forall s \in [t, T].$$

In particular, we have $\beta_t\mathscr{E}^g[\xi|\mathscr{F}_t] \le \mathscr{E}^g[\beta_t\xi|\mathscr{F}_t]$, P-a.s., proving the theorem. □

5. Main Results

In this section we prove the main results of this paper regarding the *quadratic g-martingales*. To begin with, we give the following definition. Recall that $\mathscr{E}^g_{s,t}[\cdot]$, $0 \le s \le t \le T$ denotes the g-evaluation.

Definition 5.1. An $X \in L^\infty_{\mathbf{F}}([0,T])$ is called a "g-submartingale" (resp. g-supermartingale) if for any $0 \le s \le t \le T$, it holds that

$$\mathscr{E}^g_{s,t}[X_t] \ge \text{(resp. } \le) X_s, \quad P\text{-a.s.}$$

X is called a g-martingale if it is both a g-submartingale and a g-supermartingale.

We should note here that, in the above the martingale is defined in terms of quadratic g-evaluation, instead of quadratic g-expectation as we have usually seen. This slight relaxation is merely for convenience in applications. It is clear, however, that if g satisfies (2.2), then the quadratic g-martingale defined above should be the same as the one defined via quadratic g-expectations, thanks to (2.3).

We shall extend three main results for g-expectation to the quadratic case: the Doob–Meyer decomposition, the optional sampling theorem, and the upcrossing theorem. Although the results look similar to the existing one in the g-expectation literature, the proofs are more involved due to the special nature of the quadratic BSDEs. We present these results separately.

We begin by proving a Doob–Meyer type decomposition theorem for g-martingales.

Theorem 5.2 (Doob–Meyer Decomposition Theorem). *Assume* (H1)–(H4). *Let Y be any g-submartingale* (*resp. g-supermartingale*) *that has right-continuous paths. Then there exist a càdlàg increasing* (*decreasing*) *process A null at* 0 *and a process $Z \in \mathscr{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ such that*

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - A_T + A_t - \int_t^T Z_s dB_s, \quad t \in [0,T].$$

Proof. We first assume that Y is a g-submartingale. Set $M \triangleq (\|Y\|_\infty + kT)e^{kT}$ and $K \triangleq \ell(M+1)$, we let $\phi : \mathbb{R} \mapsto [0,1]$ be any $C^2(\mathbb{R})$ function that equals to 1 inside $\left[e^{-2KM}, e^{2KM}\right]$ and vanishes outside $\left(e^{-2K(M+1)}, e^{2K(M+1)}\right)$. Let us construct a new generator: For any $(t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$,

$$\tilde{g}(t,\omega,y,z) \triangleq \phi(y)\left[2Ky\, g\left(t,\omega,\frac{\ln(y)}{2K},\frac{z}{2Ky}\right) - \frac{|z|^2}{2y}\right].$$

One can deduce from (H2) that for $dt \times dP$-a.s. $(t,\omega) \in [0,T] \times \Omega$,

$$\tilde{g}(t,y,z) \le 2(M+2)kK\phi(y)y, \quad (y,z) \in \mathbb{R} \times \mathbb{R}^d.$$

Since $2(M+2)kK\phi(y)y$ is Lipschitz continuous in y, we can construct (cf. [7]) a decreasing sequence $g_n(t,y,z)$ of generators uniformly Lipsichitz in (y,z) such that P-a.s.

$$g_n(t,y,z) \searrow \tilde{g}(t,y,z), \quad \forall (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d.$$

Now fix $t \in [0,T]$, for any $\xi \in L^\infty(\mathscr{F}_t)$ with $\|\xi\|_\infty \le \|Y\|_\infty$, we define $y_s \triangleq \mathscr{E}^g_{s,t}[\xi]$, $s \in [0,t]$. It follows from [7, Corollary 2.2] that $\|y\|_\infty \le (\|Y\|_\infty + kT)e^{kT} = M$.

Applying Itô's formula we see that $\tilde{y}_s \triangleq e^{2Ky_s}$, $s \in [0, t]$ together with a process $\tilde{z} \in \mathscr{H}^2_{\mathbf{F}}([0, t]; \mathbb{R}^d)$ is a solution of the following BSDE:

$$\tilde{y}_s = e^{2K\xi} + \int_s^t \tilde{g}(r, \tilde{y}_r, \tilde{z}_r)dr - \int_s^t \tilde{z}_r dB_r, \quad \forall s \in [0, t].$$

Since g_n is Lipschitz, a standard comparison theorem implies that

$$e^{2K\mathscr{E}^g_{s,t}[\xi]} = \tilde{y}_s \leq \mathscr{E}^{g_n}_{s,t}[e^{2K\xi}], \quad s \in [0, t], \ P\text{-a.s.}$$

In particular, taking $\xi = Y_t$ shows that

$$e^{2KY_s} \leq e^{2K\mathscr{E}^g_{s,t}[Y_t]} \leq \mathscr{E}^{g_n}_{s,t}[e^{2KY_t}], \quad s \in [0, t], \ P\text{-a.s.}$$

Namely, $\widetilde{Y} = e^{2KY}$ is a right-continuous g_n-submartingale in the sense of g^n-evaluation for any $n \in \mathbb{N}$. Applying the known g-submartingale decomposition theorem for the Lipschitz case (see [9, Theorem 3.9]), we can find a càdlàg increasing process A^n null at 0 and a process $Z^n \in \mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$ such that

$$\widetilde{Y}_t = \widetilde{Y}_T + \int_t^T g_n(s, \widetilde{Y}_s, Z^n_s)ds - A^n_T + A^n_t - \int_t^T Z^n_s dB_s, \quad t \in [0, T], \tag{5.1}$$

from which we see that $\widetilde{Y}$, whence Y is càdlàg. Note that, in the representation (5.1), the martingale parts must coincide for any m and n. In other words, one must have $Z^m = Z^n$ as the elements in $\mathscr{H}^2_{\mathbf{F}}([0, T]; \mathbb{R}^d)$. Thus, for any $n \in \mathbb{N}$, (5.1) can be rewritten as

$$\widetilde{Y}_t = \widetilde{Y}_T + \int_t^T g_n(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - A^n_T + A^n_t - \int_t^T \widetilde{Z}_s dB_s, \quad t \in [0, T].$$

Since $g_n \searrow \tilde{g}$, the Lebesgue Convergence Theorem implies that

$$\int_0^T \big[g_n(s, \widetilde{Y}_s, \widetilde{Z}_s) - \tilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s)\big]ds \to 0, \quad P\text{-a.s.}$$

Consequently, it holds P-a.s. that

$$A^n_t \to \widetilde{A}_t \triangleq \widetilde{Y}_t - \widetilde{Y}_0 + \int_0^t \tilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - \int_0^t \widetilde{Z}_s dB_s, \quad \forall t \in [0, T].$$

It is easy to check that $\widetilde{A}$ is also a càdlàg increasing process null at 0. Now let us define a new $C^2(\mathbb{R})$ function ψ by $\psi(y) \triangleq \frac{\phi(y)\ln(y)}{2K}$, $y \in \mathbb{R}$. Applying Itô's formula to $\psi(\tilde{Y}_t)$ from t to T one has

$$\begin{aligned} Y_t = Y_T &+ \int_{t+}^T \frac{1}{2K\widetilde{Y}_{s-}}\big[\tilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - d\widetilde{A}_s - \widetilde{Z}_s dB_s\big] \\ &+ \frac{1}{2}\int_{t+}^T \frac{|\widetilde{Z}_s|^2}{2K\widetilde{Y}^2_{s-}}ds - \sum_{s\in(t,T]} \left\{\Delta Y_s - \frac{\Delta\widetilde{Y}_s}{2K\widetilde{Y}_{s-}}\right\} \end{aligned}$$

$$= Y_T + \int_t^T \frac{1}{2K\widetilde{Y}_s}\big[\tilde{g}(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - d\widetilde{A}_s^c - \widetilde{Z}_s dB_s\big] + \frac{1}{2}\int_t^T \frac{|\widetilde{Z}_s|^2}{2K\widetilde{Y}_s^2}ds - \sum_{s\in(t,T]} \Delta Y_s$$

$$= Y_T + \int_t^T g\Big(s, Y_s, \frac{\widetilde{Z}_s}{2K\widetilde{Y}_s}\Big)ds - \int_t^T \frac{1}{2K\widetilde{Y}_s} d\widetilde{A}_s^c - \int_t^T \frac{\widetilde{Z}_s}{2K\widetilde{Y}_s}dB_s - \sum_{s\in(t,T]} \Delta Y_s,$$

where the second equality is due to the fact that $\Delta\widetilde{Y}_s = \Delta\widetilde{A}_s > 0$ and $\widetilde{A}^c$ denotes the continuous part of $\widetilde{A}$. Clearly, $A_t \triangleq \int_0^t \frac{1}{2K\widetilde{Y}_s} d\widetilde{A}_s^c + \sum_{s\in(0,t]} \Delta Y_s$ is a càdlàg increasing process null at 0, finally we get

$$Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - A_T + A_t - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

On the other hand, if Y is a g-supermartingale, then one can easily check that $-Y$ is correspondingly a g^--submartingale with

$$g^-(t, \omega, y, z) \triangleq -g(t, \omega, -y, -z), \quad \forall (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d. \quad (5.2)$$

Clearly, g^- also satisfies (H1)–(H4), thus there exist a càdlàg increasing process A null at 0 and a process $Z \in \mathscr{H}_{\mathbf{F}}^2([0, T]; \mathbb{R}^d)$ such that

$$-Y_t = -Y_T + \int_t^T g^-(s, -Y_s, Z_s)ds - A_T + A_t - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

We can rewrite this BSDE as:

$$Y_t = Y_T + \int_t^T g(s, Y_s, -Z_s)ds - (-A_T) + (-A_t) - \int_t^T (-Z_s)dB_s, \quad t \in [0, T].$$

The proof is now complete. □

We now turn our attention to the *optional sampling theorem*. We begin by presenting a lemma that will play an important role in the proof of the optional sampling theorem.

Lemma 5.3. *Let $\tau \in \mathscr{M}_{0,T}$ be finite valued in a set $0 = t_0 < t_1 < \cdots < t_n = T$. If $t_i \le s < t \le t_{i+1}$ for some $i \in \{0, 1, \ldots, n-1\}$, then for any $\xi \in \mathscr{F}_{t\wedge\tau}$*

$$\mathscr{E}^g_{s\wedge\tau, t\wedge\tau}[\xi] = \mathbf{1}_{\{\tau\le t_i\}}\xi + \mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathscr{E}^g_{s,t}[\xi], \quad P\text{-a.s.} \quad (5.3)$$

Proof. For any $\xi \in \mathscr{F}_{t\wedge\tau}$, let (Y, Z) be the unique solution to the BSDE (3.2) with $\tau = t \wedge \tau$. Then we have

$$\mathscr{E}^g_{r\wedge\tau, t\wedge\tau}[\xi] = Y_{r\wedge\tau} = \xi + \int_{r\wedge\tau}^T \mathbf{1}_{\{u<t\wedge\tau\}} g(u, Y_u, Z_u)du - \int_{r\wedge\tau}^T \mathbf{1}_{\{u<t\wedge\tau\}} Z_u dB_u$$

$$= \xi + \int_r^t \mathbf{1}_{\{u<\tau\}} g(u, Y_{u\wedge\tau}, Z_u)du - \int_r^t \mathbf{1}_{\{u<\tau\}} Z_u dB_u, \quad \forall r \in [0, t].$$

For any $r \in [s, t]$, since $\{\tau \le t_i\} = \{\tau \ge t_{i+1}\}^c \in \mathscr{F}_{t_i} \subset \mathscr{F}_r$, one can deduce that

$$\begin{aligned}\mathbf{1}_{\{\tau\le t_i\}} Y_{r\wedge\tau} &= \mathbf{1}_{\{\tau\le t_i\}}\xi + \int_r^t \mathbf{1}_{\{\tau\le t_i\}}\mathbf{1}_{\{u<\tau\}} g(u, Y_{u\wedge\tau}, Z_u)du - \int_r^t \mathbf{1}_{\{\tau\le t_i\}}\mathbf{1}_{\{u<\tau\}} Z_u dB_u \\ &= \mathbf{1}_{\{\tau\le t_i\}}\xi, \end{aligned} \tag{5.4}$$

and that

$$\begin{aligned}\mathbf{1}_{\{\tau\ge t_{i+1}\}} Y_{r\wedge\tau} &= \mathbf{1}_{\{\tau\ge t_{i+1}\}}\xi + \int_r^t \mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathbf{1}_{\{u<\tau\}} g(u, Y_{u\wedge\tau}, Z_u)du - \int_r^t \mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathbf{1}_{\{u<\tau\}} Z_u dB_u \\ &= \mathbf{1}_{\{\tau\ge t_{i+1}\}}\xi + \int_r^t \mathbf{1}_{\{\tau\ge t_{i+1}\}} g(u, Y_{u\wedge\tau}, Z_u)du - \int_r^t \mathbf{1}_{\{\tau\ge t_{i+1}\}} Z_u dB_u. \end{aligned} \tag{5.5}$$

On the other hand, we let $Y'_r = \mathscr{E}^g_{r,t}[\xi]$, $r \in [0, t]$. Then for any $r \in [s, t]$, by the definition of quadratic g-evaluation, one has

$$\mathbf{1}_{\{\tau\le t_i\}} Y'_r = \mathbf{1}_{\{\tau\le t_i\}}\xi + \int_r^t \mathbf{1}_{\{\tau\le t_i\}} g(u, Y'_u, Z'_u)du - \int_r^t \mathbf{1}_{\{\tau\le t_i\}} Z'_u dB_u. \tag{5.6}$$

Adding (5.6) to (5.5) shows that $\widetilde{Y}_r \triangleq \mathbf{1}_{\{\tau\ge t_{i+1}\}} Y_{r\wedge\tau} + \mathbf{1}_{\{\tau\le t_i\}} Y'_r$ and $\widetilde{Z}_r \triangleq \mathbf{1}_{\{\tau\ge t_{i+1}\}} Z_r + \mathbf{1}_{\{\tau\le t_i\}} Z'_r$ solve the following BSDE

$$\widetilde{Y}_r = \xi + \int_r^t g(u, \widetilde{Y}_u, \widetilde{Z}_u)du - \int_r^t \widetilde{Z}_u dB_u, \quad \forall r \in [s, t].$$

Then it is not hard to check that $\widehat{Y}_r = \mathbf{1}_{\{r\ge s\}}\widetilde{Y}_r + \mathbf{1}_{\{r<s\}}\mathscr{E}^g_{r,s}[\widetilde{Y}_s]$, $r \in [0, t]$ is the unique solution of BSDE(t, ξ, g). Hence we can rewrite $\widehat{Y}_r = \mathscr{E}^g_{r,t}[\xi]$, $r \in [0, t]$. In particular, it holds P-a.s. that

$$\mathbf{1}_{\{\tau\ge t_{i+1}\}} Y_{s\wedge\tau} = \mathbf{1}_{\{\tau\ge t_{i+1}\}}\widetilde{Y}_s = \mathbf{1}_{\{\tau\ge t_{i+1}\}}\hat{Y}_s = \mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathscr{E}^g_{s,t}[\xi]. \tag{5.7}$$

Letting $r = s$ in (5.4) and then adding it to (5.7), the lemma follows. □

We are now ready to prove the optional sampling theorem.

Theorem 5.4. *Assume* (H1)–(H4). *For any g-submartingale X (resp., g-supermartingale, g-martingale) such that* $\operatorname{esssup}_{\omega\in\Omega}\sup_{t\in[0,T]}|X(t,\omega)| < \infty$, *and for any* $\sigma, \tau \in \mathscr{M}_{0,T}$ *with* $\sigma \le \tau$, *P-a.s. Assume either that σ and τ are finitely valued or that X is right-continuous, then*

$$\mathscr{E}^g_{\sigma,\tau}[X_\tau] \ge (\textit{resp.}\ \le, =)\, X_\sigma, \quad P\text{-a.s.}$$

Proof. We shall consider only the g-submartingale case, as the other cases can be deduced easily by standard argument. To begin with, we assume that τ takes values in a finite set $0 = t_0 < t_1 < \cdots < t_n = T$. Note that if $t \ge t_n$, then it is clear that $\mathscr{E}^g_{t\wedge\tau,\tau}[X_\tau] = \mathscr{E}^g_{\tau,\tau}[X_\tau] = X_\tau$, P-a.s. We can then argue inductively that for any $t \in [0, T]$,

$$\mathscr{E}^g_{t\wedge\tau,\tau}[X_\tau] \ge X_{t\wedge\tau}, \quad P\text{-a.s.} \tag{5.8}$$

In fact, assume that for some $i \in \{1, \ldots, n\}$, (5.8) holds for any $t \geq t_i$. Then for any $t \in [t_{i-1}, t_i)$, the time-consistence and the monotonicity of quadratic g-evaluations as well as (5.3) imply that

$$\begin{aligned}\mathscr{E}^g_{t\wedge\tau,\tau}[X_\tau] &= \mathscr{E}^g_{t\wedge\tau,t_i\wedge\tau}\big[\mathscr{E}^g_{t_i\wedge\tau,\tau}[X_\tau]\big] \geq \mathscr{E}^g_{t\wedge\tau,t_i\wedge\tau}[X_{t_i\wedge\tau}] \\ &= \mathbf{1}_{\{\tau\leq t_{i-1}\}}X_{t_i\wedge\tau} + \mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[X_{t_i\wedge\tau}] \\ &= \mathbf{1}_{\{\tau\leq t_{i-1}\}}X_{t\wedge\tau} + \mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[X_{t_i\wedge\tau}], \quad P\text{-a.s.}\end{aligned}$$

Since $\{\tau \geq t_i\} = \{\tau \leq t_{i-1}\}^c \in \mathscr{F}_t$, the "zero-one law" of quadratic g-evaluations shows that

$$\begin{aligned}\mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[X_{t_i\wedge\tau}] &= \mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[\mathbf{1}_{\{\tau\geq t_i\}}X_{t_i\wedge\tau}] = \mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[\mathbf{1}_{\{\tau\geq t_i\}}X_{t_i}] \\ &= \mathbf{1}_{\{\tau\geq t_i\}}\mathscr{E}^g_{t,t_i}[X_{t_i}] \geq \mathbf{1}_{\{\tau\geq t_i\}}X_t = \mathbf{1}_{\{\tau\geq t_i\}}X_{t\wedge\tau}, \quad P\text{-a.s.}\end{aligned}$$

Hence, (5.8) holds for any $t \geq t_{i-1}$, this completes the inductive step. If σ is also finitely valued, for example in the set $0 = s_0 < s_1 < \cdots < s_m = T$, then it holds P-a.s.

$$\begin{aligned}\mathscr{E}^g_{\sigma,\tau}[X_\tau] = \mathscr{E}^g_{\sigma\wedge\tau,\tau}[X_\tau] &= \sum_{j=0}^{m}\mathbf{1}_{\{\sigma=s_j\}}\mathscr{E}^g_{s_j\wedge\tau,\tau}[X_\tau] \\ &\geq \sum_{j=0}^{m}\mathbf{1}_{\{\sigma=s_j\}}X_{s_j\wedge\tau} = X_{\sigma\wedge\tau} = X_\sigma.\end{aligned} \tag{5.9}$$

For a general $\tau \in \mathscr{M}_{0,T}$, we define two sequences $\{\sigma_n\}$ and $\{\tau_n\}$ of finite valued stopping times such that P-a.s.

$$\sigma_n \searrow \sigma, \quad \tau_n \searrow \tau, \quad \text{and} \quad \sigma_n \leq \tau_n, \quad \forall n \in \mathbb{N}.$$

Fix $n \in \mathbb{N}$ and let (Y^n, Z^n) be the unique solution to the BSDE (3.2) with $\xi = X_{\tau_n}$ and $\tau = \tau_n$. We know from (5.9) that P-a.s.

$$Y^n_{\sigma_m} = \mathscr{E}^g_{\sigma_m,\tau_n}[X_{\tau_n}] \geq X_{\sigma_m}, \quad \forall m \geq n.$$

In light of the right-continuity of X and Y^n, letting $m \to \infty$ gives that

$$Y^n_\sigma \geq X_\sigma, \quad P\text{-a.s.}$$

Now let (Y, Z) be the unique solution to the BSDE (3.2) with $\xi = X_\tau$. It is easy to see that for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$, $\mathbf{1}_{\{t\leq\tau_n\}}g(t, \omega, y, z)$ converges to $\mathbf{1}_{\{t\leq\tau\}}g(t, \omega, y, z)$ uniformly in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Theorem 3.5 then implies that P-a.s. Y^n_t converges to Y_t uniformly in $t \in [0, T]$. Thus, we have

$$\mathscr{E}^g_{\sigma,\tau}[X_\tau] = Y_\sigma = \lim_{n\to\infty} Y^n_\sigma \geq X_\sigma, \quad P\text{-a.s.},$$

proving the theorem. □

Finally, we study the so-called *upcrossing inequality* for quadratic g-submartingales, which would be essential for the study of path regularity of g-submartingales.

Theorem 5.5. *Given a g-submartingale X, we set* $J \triangleq \big(\|X\|_\infty + kT\big)e^{kT}$ *and denote* $\widetilde{X}_t = X_t + k(J+1)t,\ t \in [0,T]$. *As usual, for any finite set* $\mathscr{D} = \{0 \le t_0 < t_1 < \cdots < t_n \le T\}$, *we let* $U_a^b(\widetilde{X}, \mathscr{D})$ *denote the number of upcrossings of the interval* $[a,b]$ *by* $\widetilde{X}$ *over* $\mathscr{D}$. *Then there is a BMO process* $\big\{\beta_{\mathscr{D}}(t)\big\}_{t\in[0,t_n]}$ *such that*

$$E\bigg[U_a^b(\widetilde{X}, \mathscr{D}) \exp\bigg(\int_0^{t_n} \beta_{\mathscr{D}}(s)dB_s - \frac{1}{2}\int_0^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds\bigg)\bigg] \le \frac{\|X\|_\infty + k(J+1)T + |a|}{b-a},$$

and that $E\int_0^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds \le C$, *a constant independent of the choice of* $\mathscr{D}$.

Proof. For any $j \in \{1, \ldots, n\}$ we consider the following BSDE:

$$Y_t^j = X_{t_j} + \int_t^{t_j} g(s, Y_s^j, Z_s^j)ds - \int_t^{t_j} Z_s^j dB_s, \quad \forall t \in [t_{j-1}, t_j].$$

Applying Corollary 2.2 of [7] one has

$$\|Y^j\|_\infty \le \big(\|X_{t_j}\|_\infty + k(t_j - t_{j-1})\big)e^{k(t_j - t_{j-1})} \le J. \tag{5.10}$$

Now let us define a d-dimensional process $\beta_{\mathscr{D}}(t) = (\beta_t^1, \ldots, \beta_t^d)$, $t \in [0, t_n]$ by

$$\beta_t^l \triangleq \sum_{j=1}^n \mathbf{1}_{t\in(t_{j-1},t_j]} \int_0^1 \frac{\partial g}{\partial z_l}\big(t, Y_t^j, (Z_t^{j,1}, \ldots, \lambda Z_t^{j,l}, 0, \ldots 0)\big)d\lambda, \quad l \in \{1, \ldots, d\}.$$

It is easy to see from Mean Value Theorem that for any $t \in (t_{j-1}, t_j]$,

$$\begin{aligned} &g(t, Y_t^j, Z_t^j) - g(t, Y_t^j, 0) \\ &= \sum_{l=1}^d \Big\{g\big(t, Y_t^j, (Z_t^{j,1}, \ldots, Z_t^{j,l}, 0, \ldots, 0)\big) - g\big(t, Y_t^j, (Z_t^{j,1}, \ldots, Z_t^{j,l-1}, 0, \ldots, 0)\big)\Big\} \\ &= \sum_{l=1}^d Z_t^{j,l}\beta_t^l = \langle Z_t^j, \beta_{\mathscr{D}}(t)\rangle. \end{aligned} \tag{5.11}$$

Moreover, (H3) implies that

$$\big|\beta_t^l\big| \le \ell(J)\sum_{j=1}^n \mathbf{1}_{t\in(t_{j-1},t_j]}(1 + |Z_t^j|), \quad t \in [0, t_n],\ l \in \{1, \cdots, d\}. \tag{5.12}$$

We see from (2.7) that each Z^j is a BMO process, thus so is $\beta_{\mathscr{D}}$. In fact, for any $\tau \in \mathscr{M}_{0,t_n}$, one can deduce from (5.12) that

$$\begin{aligned} E\bigg[\int_\tau^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds \,\Big|\, \mathscr{F}_\tau\bigg] &\le Ct_n + C\sum_{j=1}^n E\bigg[\int_{(\tau\vee t_{j-1})\wedge t_j}^{t_j} |Z_s^j|^2 ds \,\Big|\, \mathscr{F}_\tau\bigg] \\ &\le CT + C\sum_{j=1}^n \bigg\{\mathbf{1}_{\{\tau\le t_{j-1}\}} E\bigg[\int_{t_{j-1}}^{t_j} |Z_s^j|^2 ds \,\Big|\, \mathscr{F}_{\tau\wedge t_{j-1}}\bigg] \\ &\qquad + \mathbf{1}_{\{t_{j-1}<\tau\le t_j\}} E\bigg[\int_{(\tau\vee t_{j-1})\wedge t_j}^{t_j} |Z_s^j|^2 ds \,\Big|\, \mathscr{F}_{(\tau\vee t_{j-1})\wedge t_j}\bigg]\bigg\} \end{aligned}$$

$$\leq CT + C\sum_{j=1}^{n}\bigg\{\mathbf{1}_{\{\tau\leq t_{j-1}\}}E\Big[E[\int_{t_{j-1}}^{t_j}|Z_s^j|^2ds\,|\,\mathscr{F}_{t_{j-1}}]\,\Big|\,\mathscr{F}_{\tau\wedge t_{j-1}}\Big]$$

$$+\mathbf{1}_{\{t_{j-1}<\tau\leq t_j\}}\|Z_s^j\|^2_{BMO_2}\bigg\}$$

$$\leq CT + C\sum_{j=1}^{n}\|Z_s^j\|^2_{BMO_2}, \tag{5.13}$$

where $C \stackrel{\triangle}{=} 2d\ell(J)^2$. Thus, $\{\mathscr{E}(\beta_{\mathscr{D}} \bullet B)_t\}_{t\in[0,t_n]}$ is a uniformly integrable martingale. By Girsanov's theorem we can find an equivalent probability $Q^{\mathscr{D}}$ such that $dQ^{\mathscr{D}}/dP = \mathscr{E}(\beta_{\mathscr{D}} \bullet B)_{t_n}$. Then (5.11) and (H2) show that for any $j \in \{1, \ldots, n\}$ and any $t \in [t_{j-1}, t_j]$,

$$\begin{aligned} Y_t^j &= X_{t_j} + \int_t^{t_j}\big[g(s, Y_s^j, 0) + \langle Z_s^j, \beta_{\mathscr{D}}(s)\rangle\big]ds - \int_t^{t_j} Z_s^j dB_s \\ &= X_{t_j} + \int_t^{t_j} g(s, Y_s^j, 0)ds - \int_t^{t_j} Z_s^j dB_s^{\mathscr{D}} \\ &\leq X_{t_j} + k(J+1)(t_j - t) - \int_t^{t_j} Z_s^j dB_s^{\mathscr{D}}, \end{aligned}$$

where $B^{\mathscr{D}}$ denotes the Brownian Motion under $Q^{\mathscr{D}}$. Taking the conditional expectation $E_{Q^{\mathscr{D}}}[\cdot|\mathscr{F}_t]$ on both sides of the above inequality one can obtain that

$$\mathscr{E}^g_{t,t_j}[X_{t_j}] = Y_t^j \leq E_{Q^{\mathscr{D}}}[X_{t_j}|\mathscr{F}_t] + k(J+1)(t_j - t), \quad P\text{-a.s. } \forall t \in [t_{j-1}, t_j].$$

In particularly, taking $t = t_{j-1}$ we have

$$X_{t_{j-1}} \leq \mathscr{E}^g_{t_{j-1},t_j}[X_{t_j}] \leq E_{Q^{\mathscr{D}}}[X_{t_j}|\mathscr{F}_{t_{j-1}}] + k(J+1)(t_j - t_{j-1}), \quad P\text{-a.s.}$$

Hence, $\{\widetilde{X}_{t_j}\}_{j=0}^n$ is a $Q^{\mathscr{D}}$-submartingale. Applying the classical upcrossing theorem one has

$$E_{Q^{\mathscr{D}}}\big[U_a^b(\widetilde{X}, \mathscr{D})\big] \leq \frac{E_{Q^{\mathscr{D}}}\big[(\widetilde{X}_{t_n} - a)^+\big]}{b-a} \leq \frac{\|X\|_\infty + k(J+1)T + |a|}{b-a}$$

Furthermore, we denote $C > 0$ to be a generic constant depending only on $d, T, J, k, \|X\|_\infty$, and is allowed to vary from line to line. Letting $\tau = 0$ in (5.13) one can deduce that

$$\begin{aligned} E\int_0^{t_n}|\beta_{\mathscr{D}}(s)|^2ds &\leq C + C\sum_{j=1}^{n}E\int_{t_{j-1}}^{t_j}|Z_s^j|^2ds \\ &\leq C + C\sum_{j=1}^{n}\Big\{e^{4\widetilde{K}J}E\big[e^{4\widetilde{K}Y_{t_j}^j} - e^{4\widetilde{K}Y_{t_{j-1}}^j}\big] + e^{8\widetilde{K}J}(t_j - t_{j-1})\Big\} \\ &\leq C + C\sum_{j=1}^{n}E\big[e^{4\widetilde{K}X_{t_j}} - e^{4\widetilde{K}X_{t_{j-1}}}\big] = C + CE\big[e^{4\widetilde{K}X_{t_n}} - e^{4\widetilde{K}X_{t_0}}\big] \leq C, \end{aligned}$$

where we applied (2.6) and (5.10) with $\widetilde{K} \triangleq \frac{1}{2} \vee k(J+1) \vee \ell(J)$ to derive the second inequality and the third inequality is due to the fact that $Y^j_{t_{j-1}} = \mathscr{E}^g_{t_{j-1},t_j}[X_{t_j}] \geq X_{t_{j-1}}$. The proof is now complete. □

With the above upcrossing inequality, we can discuss the continuity of the quadratic g-sub(super)martingales.

Corollary 5.6. *If X is a g-submartingale (resp. g-supermartingale), then for any denumerable dense subset $\mathscr{D}$ of $[0, T]$, it holds P-a.s. that*

$$\lim_{r \nearrow t, r \in \mathscr{D}} X_r \text{ exists for any } t \in (0, T] \text{ and } \lim_{r \searrow t, r \in \mathscr{D}} X_r \text{ exists for any } t \in [0, T).$$

Proof. If X is a g-supermartingale, then $-X$ is correspondingly a g^--submartingale with g^- defined in (5.2). Hence, it suffices to assume that X is a g-submartingale. Let $\{\mathscr{D}_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of $\mathscr{D}$ such that $\bigcup_n \mathscr{D}_n = \mathscr{D}$. For any two real numbers $a < b$, Theorem 5.5 and Jensen's Inequality imply that:

$$\begin{aligned}
\widetilde{C} &\triangleq 1 + \frac{\|X\|_\infty + k(J+1)T + |a|}{b-a} \\
&\geq 1 + E\bigg[U^b_a(\widetilde{X}, \mathscr{D}_n) \exp\bigg\{\int_0^{t_n} \beta_{\mathscr{D}}(s)dB_s - \frac{1}{2}\int_0^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds\bigg\}\bigg] \\
&= E\bigg[\big(1 + U^b_a(\widetilde{X}, \mathscr{D}_n)\big) \exp\bigg\{\int_0^{t_n} \beta_{\mathscr{D}}(s)dB_s - \frac{1}{2}\int_0^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds\bigg\}\bigg] \\
&\geq \exp\bigg\{E\bigg[\ln\big(1 + U^b_a(\widetilde{X}, \mathscr{D}_n)\big) + \int_0^{t_n} \beta_{\mathscr{D}}(s)dB_s - \frac{1}{2}\int_0^{t_n} |\beta_{\mathscr{D}}(s)|^2 ds\bigg]\bigg\},
\end{aligned}$$

from which one can deduce that

$$E\big[\ln\big(1 + U^b_a(\widetilde{X}, \mathscr{D}_n)\big)\big] \leq \ln \widetilde{C} + \frac{1}{2} + \|\beta_{\mathscr{D}}\|^2_{L^2_{\mathbf{F}}([0,t_n];\mathbb{R}^d)} \leq C',$$

where C' is a constant independent of the choice of $\mathscr{D}$. Since $U^b_a(\widetilde{X}, \mathscr{D}_n) \nearrow U^b_a(\widetilde{X}, \mathscr{D})$ as $\mathscr{D}_n \nearrow \mathscr{D}$, Monotone Convergence Theorem implies that $\ln\big(1 + U^b_a(\widetilde{X}, \mathscr{D})\big)$ is integrable, thus $U^b_a(\widetilde{X}, \mathscr{D}) < \infty$, P-a.s. Then a classical argument yields the conclusion for $\widetilde{X}$, thus for X. The proof is now complete. □

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