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In this paper we study the ruin problem for insurance models that involve investments. Our risk reserve process is an extension of the classical Cramér–Lundberg model, which will contain stochastic interest rates, reserve-dependent expense loading, diffusion perturbed models, and many others as special cases. By introducing a new type of exponential martingale parametrized by a general rate function, we put various Cramér–Lundberg type estimations into a unified framework. We show by examples that many existing Lundberg-type bounds for ruin probabilities can be recovered by appropriately choosing the rate functions. *Key words: risk reserve*, *ruin probability*, *exponential martingales*, *Lundberg bound*.

1. INTRODUCTION

In this paper we study the classical ruin problem for a mixed insurance-finance model in which the risk reserve is connected to a financial market. More precisely, we will extend the classical Cramér–Lundberg model to the following (see §2 for detailed derivation):

$$X_t = x + \int_0^t [r_s X_s + c(1+\rho(s, X_s)) + \langle \pi_s, \mu_s - r_s \mathbf{1}\rangle]\, ds + \int_0^t \langle \pi_s, \sigma_s\, dW_s\rangle - \int_0^{+} \int_{\mathbb{R}_+} f(s, x, \cdot) N_p(ds\, dx), \tag{1.1}$$

where ρ is the *expense loading* function, f is the *claim rate*, p is a stationary Poisson point process, W is a standard Brownian motion, r is the *interest rate* of a riskless bank account, $\mu = (\mu_1, \ldots, \mu_n)$ is the *mean return* of the risky securities, σ is the *volatility matrix* of the financial market, $\pi = (\pi_1, \ldots, \pi_n)$ is the *portfolio* of the insurer, and finally, $\mathbf{1} \triangleq (1, 1, \ldots, 1)$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$.

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The ruin problems in stochastic environments have been studied by many authors, we refer the readers to, e.g., Asmussen and Nielsen [1], Djehiche [4], Kalashnikov & Norberg [7], Nyrhinen [10], Paulsen [11–13], Petersen [15], etc., for results in various situations, and to the recent survey of Paulsen [14] for more general accounts in this regard. In this paper we propose a more unified framework for ruin probability estimation, via the method of *exponential martingales*. To be more precise, we consider a class of exponential martingales parametrized by a "rate function", and we show that various types of existing Lundberg bounds of ruin probabilities, even those that do not have "exponential tails", can all be obtained from this exponential martingale by choosing appropriate rate functions.

To better illustrate these points, we shall study several examples in details. These examples contain different types of Cramér–Lundberg bounds that have been studied separately before with different methods. Our purpose here is to show that all these bounds can be derived from our exponential martingales by carefully choosing the rate functions. For instance, by taking the rate function to be of the form $I_\delta(t, x) = \delta x \exp\{-\int_0^t r_s\, ds\}$, parametrized by the constant δ, we can easily derive the Lundberg bounds for classical models, discounted reserve models, and perturbed reserve models. In particular, our method can be used to determine the *adjustment coefficient* even in the case of random coefficients, which does not seem to be amenable by the existing methods. By setting the rate function to be of the form $I(x) = \int_0^x \gamma(z)\, dz$, we recover the Lundberg bound of Asmussen & Nielsen [1] and determine the "local adjustment coefficients". In one example we show that, if instead of solving the Lundberg equation for γ, we solve a general first order integro-differential equation for I, then we can even derive the "exact" ruin probability, whence the "sharp" Lundberg bound, in a situation studied in [1], which seems to be beyond the scope of the local adjustment coefficient method. In another example we show how to derive the "power tailed" ruin probability in the presence of "proportional investments", as was seen in the recent works of Nyrhinen [10] and Kalashnikov–Norberg [7], by constructing a C^2 solution of a second order integro-differential inequality.

We remark that in this paper we mainly concentrate on the upper bounds. Some of the bounds presented in the examples are by no means sharp, and due to the generality of our risk model (1.1), obtaining the "right" rate function in some cases may seem to be more technical than the existing methods. But we nevertheless believe that by finding the "right" rate function, especially via solving the integro-differential inequalities up to the second order, our method has the potential to produce better Lundberg-type bounds for a fairly large class of reserve models, at least within the realm of exponential martingales. In these cases the technicality could be the worthy "trade off" for the versatility of the method.

The rest of the paper is organized as follows. In Section 2 we formulate the problem, and give basic definitions. In Section 3 we study the exponential martingale; and in Section 4 we use the exponential martingales to derive two main theorems for the Lundberg-type bounds. Finally in Section 5 we study several examples in details.

2. PROBLEM FORMULATION

We assume that all the uncertainties or randomness under consideration are from a common, complete probability space $(\Omega, \mathscr{F}, P)$. Consider the following classical Cramér–Lundberg model: an insurance company has a (sole) income from premiums, which is collected "continuously" at an instantaneous rate $c_t > 0$ at each time $t \geqslant 0$; the debt the company is obliged to, at each time t, is described by a claim process S_t, which is often simply assumed to be a compound Poisson process. The *risk reserve* of the company is then defined by

$$X_t = x + \int_0^t c_s \, ds - S_t, \quad t \geqslant 0. \tag{2.1}$$

We generalize the model in the following way. First, we assume that the premium rate takes the form $c_t = c(1 + \rho(t, X_t))$, $t \geqslant 0$, where c is some constant and $\rho(t, X_t)$ is the so-called *expense/safty loading* at time t. Second, we assume that the claim process S_t is a general càdlàg pure jump process that does not have any downward jumps. To be more precise, we assume that S_t takes the form of the following stochastic integral:

$$S_t = \int_0^{t+} \int_{\mathbb{R}_+} f(s, x, \omega) N_p(ds\, dx), \tag{2.2}$$

where p is a stationary Poisson point process of class (QL) and N_p is its corresponding counting measure on $(0, \infty) \times \mathbb{R}_+$ (see, e.g., Ikeda-Watanabe [6] for the definitions of all these terms).

Now let us assume that the insurance company will invest all its reserve in a financial market. To specify the financial market structure, we assume that there are n risky assets (stocks) and 1 riskless asset (bond/money market account) in the market, and we denote the price per share of the bond at time t to be P_t^0, while that of the i-th stock at time t is P_t^i, $i = 1, \ldots, n$. We assume that these prices are described by the (stochastic) differential equations:

$$\begin{cases} dP_t^0 = r_t P_t^0 \, dt; \\ dP_t^i = P_t^i [\mu_t^i \, dt + \sum_{j=1}^n \sigma_t^{ij} \, dW_t^j], \quad i = 1, \ldots, n, \end{cases} \quad t \in [0, T], \tag{2.3}$$

where $\{r_t\}$ is the *interest rate*; $\mu = (\mu^1, \ldots, \mu^n)$ is the *appreciation rate*; $\sigma = (\sigma^{ij})_{i,j=1}^n$ is the *volatility matrix*, and $W = \{W_t : t \geqslant 0\}$ is a standard Brownian motion, which is the source of uncertainty in the stock market. We assume that the Brownian motion W and the claim process S are independent. For mathematical clarity we define the following filtrations

$$\mathbf{F}^p \triangleq \{\mathscr{F}_t^p : t \geqslant 0\}, \quad \mathbf{F}^W \triangleq \{\mathscr{F}_t^W : t \geqslant 0\}, \quad \mathbf{F} \triangleq \{\mathscr{F}_t^p \vee \mathscr{F}_t^W : t \geqslant 0\},$$

where $\mathbf{F}^p$ and $\mathbf{F}^W$ are the natural filtrations generated by p and W, respectively, with standard augmentations, and they are refined so that $\mathbf{F}^p$, $\mathbf{F}^W$, and $\mathbf{F}$ all satisfy the *usual hypotheses* (cf. Protter [16]).

In order to carry out our analysis, let us give a more detailed characterization of the claim process S. Let us recall from [6] the following notions. Note that p is a stationary Poisson point process, we know that the *compensator* of N_p, denoted by $\hat{N}_p(dt\,dx)$, is given by $\hat{N}_p(dt\,dx) = E(N_p(dt\,dx)) = \nu(dx)\,dt$, where $\nu(dx)$ is called the *characteristic measure* of p, defined on $(\mathbb{R}_+, \mathscr{B}_{\mathbb{R}_+})$. Furthermore, we define the *compensated* random measure $\tilde{N}_p \triangleq N_p - \hat{N}_p$. Then, for any $U \in \mathscr{B}(\mathbb{R}_+)$, $(t, \omega) \mapsto \tilde{N}_p(t, U)$ is an $\mathbf{F}^p$-martingale. Let us now consider the following spaces:

- F_p is the space of all random fields $f(t, x, \omega)$; $[0, T] \times \mathbb{R} \times \Omega \mapsto \mathbb{R}_+$ such that for fixed x, $f(\cdot, x, \cdot)$ is $\mathbf{F}^p$-predictable, and that $\int_0^t \int_{\mathbb{R}_+} |f(s, x, \cdot)| \nu(dx)\,ds < \infty$ a.s.
- F_p^1 is the subset of F_p such that

$$E\left\{\int_0^t \int_{\mathbb{R}_+} |f(s, x, \cdot)| \nu(dx)\,ds\right\} < \infty, \quad \forall t > 0.$$

- F_p^2 is the subset of F_p such that

$$E\left\{\int_0^t \int_{\mathbb{R}_+} |f(s, x, \cdot)|^2 \nu(dx)\,ds\right\} < \infty, \quad \forall t > 0.$$

- $F_p^{1,loc}$ is the subset of F_p such that there exists a sequence of $\mathbf{F}^p$-stopping times τ_n such that $\tau_n \uparrow \infty$, P-a.s., and that $f(\cdot \wedge \tau_n, \cdot, \cdot) \in F_p^1$, $n = 1, 2, \ldots$.

Recall from [6] that the stochastic integral (2.2) is well-defined for all $f \in F_p$, and furthermore, the stochastic integral

$$\int_0^t \int_{\mathbb{R}_+} f(s, x, \cdot) \tilde{N}_p(ds\,dx) \triangleq \int_0^t \int_{\mathbb{R}_+} f(s, x, \cdot) N_p(ds\,dx) - \int_0^t \int_{\mathbb{R}_+} f(s, x, \cdot) \nu(dx)\,ds$$

is an $\mathbf{F}$-local martingale if $f \in {F_p}^{1,loc}$, and a true martingale if $f \in F_p^1$.

Throughout this paper we shall make use of the following *standing assumptions*.

(**A1**) All the market parameters τ, μ, σ are $\mathbf{F}^W$-adapted stochastic processes.

(**A2**) The expense loading function ρ: $[0, T] \times \mathbb{R}$ is continuous, such that there exists a constant $C > 0$ such that

$$\begin{cases} 0 < \rho(t, 0) \leqslant C; \\ |\rho(t, x) - \rho(t, y)| \leqslant C|x - y|, \end{cases} \quad \text{for all } t \geqslant 0. \tag{2.4}$$

(**A3**) The random field $f(\cdot, \cdot, \cdot) \in F_p$, such that $f(t, x, \cdot) \geqslant 0$, $\forall (t, x)$, a.s.; and that there exists a $\delta_0 > 0$,

$$\int_0^t \int_{\mathbb{R}_+} \exp\{\delta_0 f(s, x, \cdot)\} \nu(dx)\,ds < \infty, \quad \forall t \geq 0, \ P\text{-a.s.}$$

Now let us suppose that the insurance company can trade continuously, and at each time t it invests π_t^i-dollars into the i-th stock, $i = 1, \ldots, n$, and puts the rest of its reserve (i.e., $X_t - \Sigma_{i=1}^n \pi_t^i$) into the money market. Then it is easy to show (cf. e.g, [8]), using (2.3), that in this case the risk reserve will follow the stochastic differential equation:

$$X_t = x + \int_0^t [r_s X_s + c(1 + \rho(s, X_s)) + \langle \pi_s, \mu_s - r_s \mathbf{1} \rangle] \, ds + \int_0^t \langle \pi_s, \sigma_s \, dW_s \rangle - S_t. \tag{2.5}$$

where $\mathbf{1} \triangleq (1, 1, \ldots, 1)$, and $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$. Our future discussion will all be based on this generalized risk model. We remark here that under our setting, the market should always be *incomplete* in general, due to the presence of the claim process, which is not "investable".

The classical ruin problem concerns the following two probabilities

$$\psi(x, T) = P\{X_t < 0 \colon \exists t \in (0, T]\}; \quad \psi(x) = P\{X_t < 0 \colon \exists t > 0\}, \tag{2.6}$$

where X is the risk process in (2.5). The probability $\psi(x, T)$ is called the *finite horizon ruin probability*; while $\psi(x)$ is called the *infinite horizon ruin probability*, for obvious reasons. Since finding the closed form expression of ruin probabilities is rather remote in general, our main purpose is to find the Lundberg-type bounds, as usual.

We should note that the immediate difficulty for extending the results from the classical model (2.1), or any *piecewise deterministic Markov processes* (PDMP) model, to the general model (2.5) is the presence of the Brownian motion W and the nonlinear function ρ. For example, a simple calculation shows that the usual methods of deriving the so-called *adjustment coefficient* and exponential martingales do not seem to work. Therefore, in the following sections we will introduce a new construction of the exponential martingale, which will lead to the desired Lundberg bounds.

3. EXPONENTIAL (LOCAL) MARTINGALES

Recall that our risk process is of the following form:

$$\begin{aligned} X_t = x &+ \int_0^t \{(r_s X_s + c(1 + \rho(s, X_s)) + \langle \pi_s, \mu_s - \tau_s \mathbf{1} \rangle)\} \, ds \\ &+ \int_0^t \langle \pi_s, \sigma_s \, dW_s \rangle - \int_0^{t+} \int_{\mathbb{R}_+} f(s, x, \cdot) N_p(ds \, dx). \end{aligned} \tag{3.1}$$

In what follows we denote $C^{1,2}(\mathbb{R}_+, \mathbb{R})$ to be the space of all continuous functions $\varphi(t, x)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, such that $\varphi(\cdot, \cdot)$ is continuously differentiable in t and twice continuously differentiable in x. Furthermore, we shall require that the *portfolio process* π satisfy the following "admissibility condition":

$$\int_0^t |\sigma_s^T \pi_s|^2\, ds < \infty, \quad \forall t \in [0, T], \text{ a.s.} \tag{3.2}$$

To construct the exponential martingale we proceed as follows. First, for any function $I \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ we define

$$Z_t^I \triangleq \int_0^t \int_{\mathbb{R}+} [\exp\{I(s, X_s) - I(s, X_s - f(s, x, \cdot))\} - 1]\nu(dx)\, ds \tag{3.3}$$

$$V_t^I \triangleq \int_0^t \{\partial_x I(s, X_s)[r_s X_s + c(1 + \rho(s, X_s)) + \langle \pi_s, \mu_s - r_s \mathbf{1} \rangle] + \partial_t I(s, X_s)\}\, ds \tag{3.4}$$

$$Y_t^I \triangleq \int_0^t \{(\partial_x I(s, X_s))^2 - \partial_{xx}^2 I(s, X_s)\} |\sigma_s^T \pi_s|^2\, ds, \tag{3.5}$$

and

$$K_t^I \triangleq -V_t^I + \frac{1}{2} Y_t^I + Z_t^I, \tag{3.6}$$

$$L_t^I \triangleq \exp\{-I(t, X_t) - K_t^I\}, \quad t \geqslant 0. \tag{3.7}$$

Since the process X has RCLL (right-continuous and left-limit) paths, it is not hard to check that $V_t^I < \infty$ and $Y_t^I < \infty$, $\forall t$, P-a.s.

For processes Z^I, K^I and L^I, however, we shall make use of the following technical assumption:

HYPOTHESIS A. *A function $I \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ is said to satisfy Hypothesis A if the process Z^I defined by* (3.3) *satisfies $Z_t^I < \infty$, $\forall t \geqslant 0$, P-almost surely.*

Clearly, if I satisfies Hypothesis A, then processes K^I and L^I will be almost surely finite for all t as well. We shall call such function I the "*rate function*" in the sequel, for the obvious reason that we will see in a moment. The main result of this section is the following.

THEOREM 3.1 *Suppose that the rate function I satisfies Hypothesis A. Then the process $\{L_t^I: t \geqslant 0\}$ is an* **F**-*local martingale.*

Proof. For each I satisfying Hypothesis A, we define a function

$$F^I(t, v, x, y, z) = \exp\left(v - I(t, x) - \frac{1}{2} y - z\right).$$

Then, by definitions (3.3)–(3.7) we see that $L_t^I = F^I(t, V_t^I, X_t, Y_t^I, Z_t^I)$, $\forall t$. Further, it is easily checked that (suppressing all variables)

$$\partial_t F^I = -(\partial_t I) F^I, \quad \partial_v F^I = F^I, \quad \partial_x F^I = -(\partial_x I) F^I, \quad \partial_y F^I = -\frac{1}{2} F^I,$$

$$\partial_z F^I = -F^I, \quad \partial^2_{xx} F^I = \{(\partial_x I)^2 - \partial^2_{xx} I\} F^I.$$

Applying Itô's formula and noting that V^I, Y^I, and Z^I all have bounded variation paths, we have

$$\begin{aligned}
&F^I(t, V^I_t, X_t, Y^I_t, Z^I_t) - F^I(V^I_0, X_0, Y^I_0, Z^I_0)\\
&= \int_0^t \partial_t F^I\, ds + \int_0^t \partial_v F^I\, dV^I_s + \int_0^t \partial_x F^I (r_s X_s + c(1+\rho(s, X_s)) + \langle \pi_s, \mu_s - r_s \mathbf{1}\rangle)\, ds\\
&\quad + \int_0^t \partial_y F^I\, dY^I_s + \int_0^t \partial_z F^I\, dZ^I_s + \int_0^t \partial_x F^I \langle \pi_s, \sigma_s\, d\omega_s\rangle + \frac{1}{2}\int_0^t \partial^2_{xx} F^I |\sigma^T_s \pi_s|^2\, ds\\
&\quad + \int_0^{t+} \int_{\mathbb{R}_+} \{F^I(s, V^I_s, X_{s-} - f(s, x, \cdot), Y^I_s, Z^I_s) F^I(s, V^I_s, X_{s-}, Y^I_s, Z^I_s)\} N_p(dx\, ds)\\
&= -\int_0^t F^I\, dZ^I_s - \int_0^t \partial_x F^I \langle \pi_s, \sigma_s\, dW_s\rangle\\
&\quad + \int_0^{t+} \int_{\mathbb{R}+} \{F^I(s, V^I_s, X_{s-} - f(s, x, \cdot), Y^I_s, Z^I_s) - F^I(s, V^I_s, X_{s-}, Y^I_s, Z^I_s)\} N_p(dx\, ds).
\end{aligned} \tag{3.8}$$

Since for each (t, x), it holds P-almost surely that

$$\begin{aligned}
&F^I(t, V^I_t, X_{t-} - f(t, x, \cdot), Y^I_t, Z^I_t) - F^I(t, V^I_t, X_{t-}, Y^I_t, Z^I_t)\\
&\quad = \{\exp(I(t, X_{t-}) - I(t, X_{t-} - f(t, x, \cdot))) - 1\} F^I(t, V^I_t, X_{t-}, Y^I_t, Z^I_t);
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t+} F^I(s, V^I_s, X_s, Y^I_s, Z^I_s)\, dZ^I_s\\
&\quad = \int_0^t \int_{\mathbb{R}_+} \{\exp(I(s, X_s) - I(s, X_s - f(s, x, \cdot))) - 1\} F^I(s, V^I_s, X_s, Y^I_s, Z^I_s) \nu(dx)\, ds,
\end{aligned}$$

we deduce from (3.8) that

$$\begin{aligned}
F^I(t, V^I_t, X_t, Y^I_t, Z^I_t) = F^I(0, X_0, Y^I_0, Z^I_0) &- \int_0^t \partial_x F^I \langle \pi_s, \sigma_s\, dW_s\rangle\\
&+ \int_0^{t+} \int_{\mathbb{R}_+} \{\exp(I(s, X_{s-}) - I(s, X_{s-} - f(s, x, \cdot))) - 1\}\\
&\times F^I(s, V^I_s, X_{s-}, Y^I_s, Z^I_s) \tilde{N}_p(dx\, ds).
\end{aligned} \tag{3.9}$$

Since the last two terms are local martingales, the theorem follows. □

An important example of the rate function I satisfying Hypothesis A is

$$I(t,x)=I_\delta(t,x)=\delta x\, e^{-\int_0^t r_s\,ds}, \tag{3.10}$$

for some constant $\delta\in\mathbb{R}$. In fact, as we will see below, many "classical" Lundberg bounds can be derived by taking I_δ in this form with appropriately chosen δ. We now study this case in detail.

We first make the following observations: let $\beta_t=-\int_0^t r_s\,ds$ be the discount factor, and let $\tilde{X}_t=e^{\beta_t}X_t$, $t\geq 0$, be the discounted risk reserve. Then we see that for fixed δ, $I_\delta(t,X_t)=\delta\tilde{X}_t$, and an easy application of Itô's formula shows that $\tilde{X}$ satisfies the SDE:

$$\begin{aligned}\tilde{X}_t=x&+\int_0^t e^{\beta_s}(\tilde{\rho}(s,\beta_s,\tilde{X}_s)+\langle\pi_s,\mu_s-r_s\mathbf{1}\rangle)\,ds\\&+\int_0^t e^{\beta_s}\langle\pi_s,\sigma_s\,dW_s\rangle-\int_0^{t+}\int_{\mathbb{R}_+}e^{\beta_s}f(s,x,\cdot)N_p(dt\,dx),\end{aligned} \tag{3.11}$$

where $\tilde{\rho}(t,\beta_t,\tilde{X}_t)=c(1+\rho(t,e^{-\beta_t}\tilde{X}_t))=c(1+\rho(t,X_t))$.

Now assume that f is a random field satisfying (A3). For any $\delta>0$, $\gamma\geq 0$, let us define two processes:

$$m_t^f(\gamma)\triangleq\int_{\mathbb{R}_+}[\exp\{\gamma f(t,x,\omega)\}-1]\nu(dx),$$

$$Z_t^\delta\triangleq\int_0^t m_s^f(\delta\,e^{\beta_s})\,ds,\ \ Z_t^{\delta,0}\triangleq\int_0^t m_s^f(\delta)\,ds,\quad t\geq 0. \tag{3.12}$$

Then clearly both $m^f(\gamma)$ and Z^δ are $\mathbf{F}$-predictable processes, and $m^f(\gamma)$ is increasing in γ and integrable for all $\gamma\leq\delta_0$, thanks to assumption (A3). Furthermore, it is easily seen that the process Z^δ above is exactly Z^I defined by (3.3), with I being replaced by (3.10).

Next, let us define the following two subsets of $\mathbb{R}_+$:

$$\begin{cases}\mathscr{D}=\{\delta\geq 0:\ Z_t^\delta<\infty,\ P\text{-a.s.},\ \forall t\geq 0\};\\ \mathscr{D}_0=\{\delta\geq 0:\ Z_t^{\delta,0}<\infty\ \ P\text{-a.s},\ \forall t\geq 0\}.\end{cases} \tag{3.13}$$

Since $\gamma\geq 0$ and $\beta_s\leq 0$, the monotonicity of $m^f(\gamma)$ (in γ) shows that $\mathscr{D}_0\subseteq\mathscr{D}$. Now for each $\delta\in\mathscr{D}$, and I_δ as in (3.10), we can rewrite V^I, Y^I, and K^I of (3.4)–(3.6) as

$$V_t^\delta=\delta\int_0^t e^{\beta_s}[\tilde{\rho}(s,\beta_s,\tilde{X}_s)+\langle\pi_s,\mu_s-r_s\mathbf{1}\rangle]\,ds; \tag{3.14}$$

$$Y_t^\delta=\delta^2\int_0^t e^{2\beta_s}|\sigma_s^T\pi_s|^2\,ds; \tag{3.15}$$

$$K_t^\delta=-V_t^\delta+\frac{1}{2}Y_t^\delta+Z_t^\delta. \tag{3.16}$$

Thus L^I of (3.7) can be expressed as

$$L_t^\delta \triangleq \exp\{-\delta \tilde{X}_t - K_t^\delta\}, \quad t \geqslant 0. \tag{3.17}$$

Our second main result of the section is then the following

THEOREM 3.2 *Suppose that the assumptions* (A1)–(A3) *hold. Then the process* $\{L_t^\delta : t \geqslant 0\}$ *defined by* (3.17) *enjoys the following properties:*

(i) *For every* $\delta \in \mathscr{D}$, $\{L_t^\delta : t \geqslant 0\}$ *is an* **F**-*local martingale.*
(ii) *If the processes* π, σ, μ, *and* r *are all bounded, and the random field* f *is deterministic, then for every* $\delta \in \mathscr{D}_0$, $\{L_t^\delta : t \geqslant 0\}$ *is an* **F**-*martingale.*
(iii) *If in addition to* (*ii*), r *is also deterministic, then* (*ii*) *holds for all* $\delta \in \mathscr{D}$.

Proof. (i) By definition of the set $\mathscr{D}$ we see that for each $\delta \in \mathscr{D}$, the function I of (3.10) satisfies Hypothesis A. Therefore (i) follows from Theorem 3.1.

(ii) We now assume that the processes π, μ, σ, and r are all bounded; and we denote the bounds by a common constant $K > 0$. (In the sequel we shall denote all generic constant by K, which is allowed to vary from line to line.) Further, we assume that f is deterministic and $\delta \in \mathscr{D}_0$. Note that with the choice (3.10), equation (3.9) becomes

$$\begin{aligned} F^\delta(t, V_t^\delta, X_t, Y_t^\delta, Z_t^\delta) = F^\delta(0, X_0, Y_0^\delta, Z_0^\delta) - \delta \int_0^t F^\delta\, e^{\beta_s} \langle \pi_s, \sigma_s\, dW_s \rangle \\ + \int_0^{t+} \int_{\mathbb{R}_+} \{\exp(\delta\, e^{\beta_s} f(s, x, \cdot\,) - 1)\} \\ \times F^\delta(s, V_s^\delta, X_{s-}, Y_s^\delta, Z_s^\delta) \tilde{N}_p(dx\, ds). \end{aligned} \tag{3.18}$$

Denoting the two stochastic integrals on the right side of (3.18) to be I^1 and I^2, respectively, we shall prove that both of them are in fact true martingales.

To see this, first note that

$$E\langle I^1, I^1 \rangle_t^{1/2} = \delta E\left\{\int_0^t \exp(-2\delta \tilde{X}_s + 2V_s^\delta - Y_s^\delta - 2Z_s^\delta) e^{2\beta_s} |\sigma_s^T \pi_s|^2\, ds\right\}^{1/2}. \tag{3.19}$$

We shall prove that the right side of (3.19) is finite for each t. Indeed, recall that $\beta_t \leqslant 0$, $Y_t^\delta \geqslant 0$, $\forall t \geqslant 0$, a.s.; also, since $f \geqslant 0$, one has $m_t^f(\delta\, e^{\beta t}) \geqslant 0$, $\forall t \geqslant 0$, a.s. Thus, for all $\delta \geqslant 0$, $Z_t^\delta \geqslant 0$ as well. Consequently, if we denote

$$M_t = \exp\{-2\delta \int_0^t e^{\beta_s} \langle \pi_s, \sigma_s\, dW_s \rangle\}; \quad \tilde{S}_t = \int_0^{t+} \int_{\mathbb{R}_+} e^{\beta_s} f(s, x) N_p(ds\, dx),$$

then $\exp\{-2\delta \tilde{X}_s + 2V_s^\delta\} = e^{-2\delta x} M_s\, e^{2\delta \tilde{S}_s}$, and (3.19) becomes

$$E\langle I^1, I^1\rangle_t^{1/2} \leqslant \delta K E\left\{\int_0^t \exp(-2\delta \tilde{X}_s + 2V_s^\delta)\, ds\right\}^{1/2}$$
$$\leqslant \delta\, e^{-\delta x} K E\left\{\int_0^t (M_s\, e^{2\delta \tilde{S}_s})\, ds\right\}^{1/2}$$
$$\leqslant \delta\, e^{-\delta x} K E\left\{\left(\int_0^t M_s\, ds\right)^{1/2} e^{\delta \tilde{S}_t}\right\}. \quad (3.20)$$

Here we have used the fact that $\tilde{S}$ is non-decreasing.

Further, note that $0 \leqslant e^{\beta_t} f(t, x) \leqslant f(t, x)$, we have $e^{\delta \tilde{S}_t} \leqslant e^{\delta S_t}$. Since r, π, σ are all $\mathbf{F}^W$-adapted and the σ-fields $\mathbf{F}^p$ and $\mathbf{F}^W$ are independent, M_t and S_t are independent processes. Therefore we deduce from (3.20) that

$$E\langle I^1, I^1\rangle_t^{1/2} \leqslant \delta\, e^{-\delta x} K E\left\{\left(\int_0^t M_s\, ds\right)^{1/2} e^{\delta S_t}\right\} \leqslant \delta\, e^{-\delta x} K \left\{E\int_0^t M_s\, ds\right\}^{1/2} E\, e^{\delta S_t}. \quad (3.21)$$

It now suffices to prove that both expectations in the right hand side above are finite. To see this, note that M satisfies the following SDE:

$$dM_t = 4\delta M_t\, e^{2\beta_t} |\sigma_t^T \pi_t|^2\, dt - 2\delta M_t\, e^{\beta_t} \langle \pi_t, \sigma_t\, dW_t\rangle. \quad (3.22)$$

It is then standard to show, using the boundedness of the processes π, σ, and β, that $E \sup_{0 \leqslant s \leqslant t} M_s \leqslant K\, e^{Kt}$, for some generic constant $K > 0$. Consequently one has

$$E\int_0^t M_s\, ds < \infty. \quad (3.23)$$

Finally, let us introduce a sequence of $\mathbf{F}$-stopping times:

$$\tau_n = \inf\{t \geqslant 0;\ e^{\delta S_t} \geqslant n\}.$$

Applying Itô's formula one shows that

$$e^{\delta S_{t \wedge \tau_n}} = 1 + \int_0^{(t \wedge \tau_n)+} \int_{\mathbb{R}_+} e^{\delta S_{s-}} \{\exp(\delta f(s, x)) - 1\} N_p(ds\, dx). \quad (3.24)$$

Taking expectation on both sides of (3.24) we have

$$Ee^{\delta S_{t \wedge \tau_n}} = 1 + E\int_0^{t \wedge \tau_n} \int_{\mathbb{R}_+} e^{\delta S_s} \{\exp(\delta f(s, x)) - 1\} \nu(dx)\, ds$$
$$= 1 + E\int_0^t 1_{\{s \leqslant \tau_n\}}\, e^{\delta S_s} m_s^f(\delta)\, ds \leqslant 1 + \int_0^t E\{e^{\delta S_{s \wedge \tau_n}}\} m_s^f(\delta)\, ds. \quad (3.25)$$

The Gronwall inequality then leads to that

$$E\, e^{\delta S_{t \wedge \tau_n}} \leqslant \exp\left\{\int_0^t m_s^f(\delta)\, ds\right\} = e^{Z_t^{\delta,0}} < \infty. \quad (3.26)$$

Since S_t is nondecreasing, letting $n \to \infty$ in (3.26) we get $E\, e^{\delta S_t} < e^{Z_t^{\delta,0}} < \infty$, thanks to the Monotone Convergence Theorem. This, together with (3.23) and (3.21), shows that $E\langle I^1, I^1\rangle_t^{1/2} < \infty$. Now we can apply the Burkholder–Davis–Gundy inequality to conclude that I^1 is a true martingale (see, e.g., Protter [16]).

To show that I^2 is also a true martingale, we use the same argument as before. Note again that M and S are independent, we have

$$\begin{aligned} E &\int_0^t \int_{\mathbb{R}_+} \left|\{\exp(\delta\, e^{\beta_s} f(s,x)) - 1\} F^\delta(V_s, \tilde{X}_s, Y_s, Z_s^\delta)\right| \nu(dx)\, ds \\ &\leqslant K \int_0^t m_s^f(\delta) E\{e^{\delta S_s} M_s^{1/2}\}\, ds \\ &\leqslant K \int_0^t [m_s^f(\delta)]\{E\, e^{\delta S_t}\}\{E M_s\}^{1/2}\, ds \\ &\leqslant K^{3/2} \exp\left\{Z_t^{\delta,0} + \frac{1}{2} K t\right\} \int_0^t m_s^f(\delta)\, ds < \infty. \end{aligned} \tag{3.27}$$

The last inequality is due to the fact that $E\, e^{\delta S_t} \leqslant e^{Z_t^{\delta,0}}$ and $EM_s < K\, e^{Kt}$, for $0 \leqslant s \leqslant t$. This means that the integrand of I^2 belongs to F_p^1. Thus, I^2 (whence L^δ itself) is a true martingale, proving (ii).

(iii) Let us now assume that the interest rate r is also a deterministic function. In this case we note that the processes M and $\tilde{S}$ become independent. Therefore the second last inequality in (3.20) can be improved to

$$\delta\, e^{-\delta x} K E\left\{\left(\int_0^t M_s\, ds\right)^{1/2} e^{\delta \tilde{S}_t}\right\} = \delta\, e^{-\delta x} K E\left\{\int_0^t M_s\, ds\right\}^{1/2} E\, e^{\delta \tilde{S}_t}. \tag{3.28}$$

Using the same argument as before, we now obtain that:

$$E\, e^{\delta \tilde{S}_t} \leqslant \exp\left\{\int_0^T m^f(\delta\, e^{\beta_s})\, ds\right\} \leqslant e^{Z_t^\delta} < \infty,$$

for all t, which leads to that I^1 is a true martingale for all $\delta \in \mathscr{D}$. The argument for I^2 is similar, we leave it to the interested readers. □

4. LUNDBERG-TYPE BOUNDS

In this section we use the exponential martingales L^I and L^δ to derive various Lundberg-type bounds. Recall that

$$\psi(x, T) = P\{X_t < 0,\ \exists t \in (0, T]\}; \quad \text{and} \quad \psi(x) = P\{X_t < 0,\ \exists t > 0\}.$$

We first give a result using general rate function I.

THEOREM 4.1 (Lundberg Bounds). *Assume that the rate function I satisfies Hypothesis A, such that $I(t, x) \leqslant 0$, for all t and $x \leqslant 0$. Then, it holds that*

$$\psi(x, T) \leqslant e^{-I(0,x)} E \sup_{0 \leqslant t \leqslant T} \exp(K_t^I), \tag{4.1}$$

$$\psi(x) \leqslant e^{-I(0,x)} E \sup_{t \geqslant 0} \exp(K_t^I), \tag{4.2}$$

where K^I is defined by (3.6).

Proof. The proof is more or less standard. First, we define a stopping time $\tau \triangleq \inf\{t, X_t < 0\}$. Since L_t^I is a local martingale by Theorem 3.1, and it is obviously nonnegative, it is a supermartingale. Applying the Optional Sampling Theorem to L_t^I, we see that

$$
\begin{aligned}
e^{-I(0,x)} &\geqslant E\{\exp(-I(\tau \wedge T, X_{\tau \wedge T}) + V^I_{\tau \wedge T} - \frac{1}{2} Y^I_{\tau \wedge T} - Z^I_{\tau \wedge T})\} \\
&\geqslant E\{\exp(-I(\tau, X_\tau) - K^I_\tau | \tau < T\} P\{\tau < T\}. \qquad (4.3)
\end{aligned}
$$

Since X has RCLL paths, one has $X_r \leqslant 0$. Thus, using the assumption on the rate function I we have $I(\tau, X_\tau) \leqslant 0$, and the last inequality can be reduced to

$$
e^{-I(0,x)} \geqslant E\{\exp(-K^I_\tau) | \tau < T\} P\{\tau < T\} \geqslant E\left\{\inf_{0 \leqslant t \leqslant T} \exp(-K^I_t)\right\} \psi(x, T). \qquad (4.4)
$$

Applying Jensen's inequality, we derive from (4.4) that

$$
\begin{aligned}
\psi(x, T) &\leqslant e^{-I(0,x)} / E\left\{\inf_{0 \leqslant t \leqslant T} \exp(-K^I_t)\right\} \leqslant e^{-I(0,x)} E\left\{\frac{1}{\inf_{0 \leqslant t \leqslant T} \exp(-K^I_t)}\right\} \\
&= e^{-I(0,x)} E\left\{\sup_{0 \leqslant t \leqslant T} \exp(K^I_t)\right\},
\end{aligned}
$$

proving (4.1). The estimate (4.2) follows from letting $T \to \infty$ in (4.1), thus the proof is complete. □

It is sometimes more convenient to use the following slightly modified form of Theorem 4.1.

COROLLARY 4.2. *Assume all assumptions of Theorem* 4.1 *are in force. Then the following Lundberg bounds hold:*

$$
\psi(x, T) \leqslant e^{-I(0,x)} E \sup_{0 \leqslant t \leqslant T} \exp(K^I_t(X^+)), \qquad (4.5)
$$

$$
\psi(x) \leqslant e^{-I(0,x)} E \sup_{t \geqslant 0} \exp(K^I_t(X^+)), \qquad (4.6)
$$

where $K^I(X^+)$ *is the same as* K^I *except that all* X_s*'s in* Z^I, V^I, *and* Y^I (3.3)–(3.5) *are replaced by* $X_s^+ \triangleq X_s \vee 0,\ s \geqslant 0$.

Proof. Recall that $\tau = \inf\{t > 0\colon X_t < 0\}$. It is easy to see that X^+ and X are the same for $t < \tau$. Since the Lebesgue measure does not charge single points, from (3.3)–(3.5), and (3.6) we see that $K^I_\tau = K^I_\tau(X^+)$, a.s. Thus, we can rewrite (4.3) as

$$
e^{-I(0,x)} \geqslant E\{\exp(-K^I_\tau(X^+)) | \tau < T\} P\{\tau < T\}. \qquad (4.7)
$$

The inequalities (4.5) and (4.6) then follow from the same arguments at those in Theorem 4.1. □

Next, we consider the special case when the rate function $I(t, x) = I_\delta(t, x)$ is defined by (3.10). Recall from the last section that in this case $\tilde{X}_t \triangleq \beta_t X_t = e^{-\int_0^t r_s ds} X_t$ is the discounted risk reserve process, and for each $\delta \in \mathscr{D}$, the processes V^I, Y^I, Z^I, K^I, and L^I are replaced by V^δ, Y^δ, Z^δ, K^δ, and L^δ defined by (3.12)–(3.17). We have the following theorem.

THEOREM 4.3. *Assume* (A1)–(A3). *Then, for every* $\delta \in \mathscr{D}$, *the ruin probabilities* $\psi(x, T)$ *and* $\psi(x)$ *have the following upper bounds:*

$$\psi(x, T) \leqslant e^{-\delta x} E \sup_{0 \leqslant t \leqslant T} \exp(K_t^\delta), \tag{4.8}$$

$$\psi(x) \leqslant e^{-\delta x} E \sup_{t \geqslant 0} \exp(K_t^\delta). \tag{4.9}$$

Furthermore, if the loading function $\tilde{\rho}$ *satisfies that* $\tilde{\rho}(t, x) \geqslant c_\rho$, *for some constant* $c_\rho > 0$, *then in*, (4.8) and (4.9) *the process* K^δ *can be replaced by*

$$\tilde{K}_t^\delta \triangleq -\tilde{V}_t^\delta + \frac{1}{2} Y_t^\delta + Z_t^\delta, \tag{4.10}$$

where

$$\tilde{V}_t^\delta \triangleq \delta \int_0^t e^{\beta_s}(c_\rho + \langle \pi_s, \mu_s - r_s \mathbf{1} \rangle)\, ds.$$

Finally, define $\tilde{\delta} = \sup\{\delta \in \mathscr{D}: E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta)\} < \infty\}$. *Then for all* $\varepsilon > 0$ *it holds that*

$$\lim_{x \to \infty} \psi(x)\, e^{(\tilde{\delta} - \varepsilon)x} = 0. \tag{4.11}$$

Proof: The inequalities (4.8) and (4.9) are the consequences of Theorem 4.1, we need only prove the second part of the theorem. To this end let us assume that $\tilde{\rho}$ has a lower bound $c_\rho > 0$. Then, since $K_t^\delta \leqslant \tilde{K}_t^\delta$, the similar argument also leads to (4.8) and (4.9) with K^δ being replaced by $\tilde{K}^\delta$.

To prove (4.11), we first prove that the mapping $\delta \mapsto E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta)\}$ is convex, and that $\mathscr{D}' \triangleq \{\delta \in \mathscr{D}: E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta)\} < \infty\}$ is a convex set. Indeed, for fixed f, t, and ω it is easily checked that the mapping $\gamma \mapsto m_t^f(\gamma)$ is convex, thus so is the mapping $\delta \mapsto Z_t^\delta$. This implies that $\mathscr{D}$ is a convex set. Moreover, from (4.10) we see that the mapping $\delta \mapsto \tilde{K}_t^\delta$ is convex as well, thus so is the mapping $\delta \to \exp(\tilde{K}_t^\delta)$, and consequently, since the convexity is preserved under supreme, we see that $\delta \mapsto E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta)\}$ is a convex mapping. This, together with the convexity of $\mathscr{D}$, shows that $\mathscr{D}'$ is convex. Furthermore, by definition of $\tilde{\delta}$, it is readily seen that $[0, \tilde{\delta}) \subseteq \mathscr{D}'$, since clearly $0 \in \mathscr{D}'$ and $\mathscr{D}'$ is convex.

We can now prove (4.11). Note that if $\tilde{\delta} = 0$, then there is nothing to prove. So let us assume that $\tilde{\delta} > \varepsilon > 0$. Since $[0, \tilde{\delta}) \subseteq \mathscr{D}'$ letting $\delta = \tilde{\delta} - (\varepsilon/2)$ we have

$$c \triangleq E\left\{\sup_{t \geqslant 0} \exp(K_t^\delta)\right\} < \infty, \quad \text{and} \quad \psi(x) \leqslant c\, e^{-\delta x}.$$

Consequently, we have

$$\lim_{x \to \infty} \psi(x)\, e^{(\tilde{\delta} - \varepsilon)x} \leqslant \lim_{x \to \infty} c\, e^{-\varepsilon x/2} = 0.$$

This proves the theorem. □

5. EXAMPLES

In this section we try to apply Theorems 4.1 and 4.3 to derive several existing Lundberg-type bounds. As we pointed out before, the purpose here is to show the versatility of our method.

EXAMPLE 5.1 (*Classical model*). Assume that $\pi_t \equiv 0$, $r_t \equiv 0$, $\rho(t, x) \equiv 0$, $\mu_t \equiv 0$, $\sigma_t \equiv 0$, and S_t is a compound Poisson process with Poisson intensity λ and jump size distribution $F(\cdot)$. In other words, we have $f(t, x, \cdot) = x$, and $\nu(dx) = \lambda F(dx)$. Then our model (2.5) is reduced to the classical Cramér–Lundberg model.

Applying Theorem 4.3 we see that

$$\psi(x) \leqslant e^{-\delta x} \sup_{0 \leqslant t} \exp(\tilde{K}_t^\delta),$$

where $\tilde{K}_t^\delta = t(\int_0^\infty (e^{\delta x} - 1)\lambda F(dx) - c\delta)$, and thus

$$\tilde{\delta} = \sup\{\delta \in \mathscr{D}\colon\ E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta)\} < \infty\} = \sup\left\{\delta\colon\ \int_0^\infty (e^{\delta x} - 1)\lambda F(dx) - c\delta \leqslant 0\right\}. \tag{5.1}$$

Therefore we must have $E\{\sup_{t \geqslant 0} \exp(\tilde{K}_t^{\tilde{\delta}})\} \leqslant 1$, and our Lundberg bound becomes

$$\psi(x) \leqslant e^{-\tilde{\delta} x}, \quad \forall x. \tag{5.2}$$

It is well-known (see, for example, [5, pp. 11]) that (5.2) is exactly the classical Cramér–Lundberg inequality, and $\tilde{\delta}$ defined by (5.1) is exactly the so-called *Lundberg exponent*. □

EXAMPLE 5.2 (*Discounted risk reserve*). Assume that $\pi_t \equiv 0$, $\rho(t, x) \equiv 0$, $\mu_t \equiv 0$, $\sigma_t \equiv 0$, and r_t is a deterministic function. Assume also that S_t is a compound Poisson process with $f(t, x, \cdot) \equiv x$ and $\nu(dx) = \lambda F(dx)$. Then we have a risk model in an economic environment with deterministic interest rates (also called "discounted risk reserve" model, see, for example, [17, pp. 474]).

Applying Theorem 4.3 we have

$$\psi(x) \leqslant e^{-\tilde{\delta} x} \sup_{t \geqslant 0} \exp(\tilde{K}_t^\delta),$$

where $\tilde{K}^\delta$ is now of the form

$$\tilde{K}_t^\delta = \int_0^t \left\{\int_0^\infty [\exp(\delta\, e^{\beta_s} x) - 1]\lambda F(dx) - c\, e^{\beta_s}\right\} ds,$$

and $\tilde{\delta} = \sup\{\delta \geqslant 0: \sup_{t \geqslant 0} \tilde{K}_t^{\delta} < \infty\}$. It is clear that in this case our Lundberg bound and the Lundberg coefficient are again the same as the standard ones, see, e.g., Theorem 11.4.1, Rokski et al. [17]. □

EXAMPLE 5.3 (*Perturbed risk reserve*). We now assume that $\pi_t \equiv 1$, $\rho(t, x) \equiv 0$, $r_t \equiv 0$, $\mu_t \equiv 0$, $\sigma_t \equiv \varepsilon$, and S_t is a compound Poisson process with $f(t, x, \cdot) \equiv x$ and $\nu(dx) = \lambda F(dx)$. Such a model is known as "Perturbed risk reserve" (cf. e.g., [17, pp. 570]), since the reserve process X_t is perturbed by a Brownian motion εW_t.

By Theorem 4.3 we see that the Lundberg bound in this case is

$$\psi(x) \leqslant e^{-\delta x} \sup_{0 \leqslant t} \exp(\tilde{K}_t^{\delta}), \quad \forall \delta \in \mathscr{D},$$

where

$$\tilde{K}_t^{\delta} = t\left(-c\delta + \frac{1}{2}\delta^2\varepsilon^2 + \int_0^{\infty} (e^{\delta x} - 1)\lambda F(dx)\right).$$

To figure out the Lundberg exponent, we consider the function

$$k(\delta) \triangleq -c\delta + \frac{1}{2}\delta^2\varepsilon^2 + \int_0^{\infty} (e^{\delta x} - 1)\lambda F(dx). \tag{5.3}$$

Since $k(\delta) \to \infty$ as $\delta \to \infty$, it is clear that $\tilde{\delta} \triangleq \sup\{\delta > 0: k(\delta) = 0\} < \infty$, provided the set $\{\delta > 0: k(\delta) = 0\}$ is not empty. Therefore, assuming that $0 < \tilde{\delta} < \infty$ we then have $\psi(x) \leqslant e^{-\tilde{\delta} x}$, which is again the same as the standard Lundberg bound and Lundberg exponent (cf. e.g., [17, pp. 570]). □

We remark that a sufficient condition for the existence of "adjustment coefficient", i.e., the function $k(\cdot)$ defined by (5.3) to have positive root, is the following

$$c > \lambda E[U_1], \tag{5.4}$$

where U_1 is the jump size random variable of the compound Poisson process S. This condition is known as the "*net profit condition*" (see, e.g., Asmussen–Nielsen [1]). Indeed, it is easy to check that $k(0) = 0$, $k(\delta) \to +\infty$ as $\delta \to +\infty$, and

$$k'(0) = -c + \lambda E[U_1] < 0,$$

by the net profit condition (5.4). We conclude that $k(\cdot)$ must have a positive root.

The net profit condition (5.4) is also useful in the following example.

EXAMPLE 5.4 (*Asmussen–Nielsen bound*). We still assume that $\pi \equiv 0$, $\mu \equiv 0$, $\sigma \equiv 0$, but $r_t = r$ is a constant, and $\rho(t, x) \equiv \rho(x)$ is an increasing function in x. We also assume that $f(t, x, \cdot) = x$, and $\nu(dx) = \lambda F(dx)$ where F is the claim size distribution.

Then we will have the same situation discussed in Asmussen–Nielsen [1], that is, risk reserve process is now given by

$$X_t = x + \int_0^t p(X_s)\,ds + \int_0^t \int_{\mathbb{R}_+} x N_p(dx\,ds), \tag{5.5}$$

where $p(x) \triangleq rx + c(1+\rho(x))$, $x \in \mathbb{R}$. The net profit condition in this case is

$$\inf_{x \geqslant 0} p(x) > \lambda E[U_1]. \tag{5.6}$$

This time we shall apply Corollary 4.2. Namely, we try to find a function I satisfying Hypothesis A, and $I(t, x) \leqslant 0$, $x \leqslant 0$. Then the Lundberg inequalities (4.5) and (4.6) hold, with

$$K_t^I(X^+) = \int_0^t \Big\{ -\{\partial_x I(s, X_s^+) p(X_s^+) + \partial_t I(s, X_s^+)\} + \int_{\mathbb{R}_+} [\exp\{I(s, X_s^+) - I(s, X_s^+ - x)\} - 1]\lambda F(dx) \Big\}\,ds, \quad t \geqslant 0. \tag{5.7}$$

In light of Asmussen–Nielsen [1], we consider the rate function I of the following form

$$I(x) = \int_0^x \gamma(y)\,dy \quad x \geqslant 0. \tag{5.8}$$

If we assume further that $\gamma(\cdot)$ is non-decreasing, then we derive from (5.7) that

$$K_t^I(X^+) = \int_0^t \Big\{ -\gamma(X_s^+) p(X_s^+) + \int_{\mathbb{R}_+} \Big[\exp\Big\{\int_{X_s^+ - x}^{X_s^+} \gamma(y)\,dy\Big\} - 1\Big]\lambda F(dx)\Big\}\,ds$$
$$\leqslant \int_0^t \Big\{ -\gamma(X_s^+) p(X_s^+) + \int_{\mathbb{R}_+} [\exp\{\gamma(X_s^+)x\} - 1]\lambda F(dx)\Big\}\,ds, \tag{5.9}$$

for all $t \geqslant 0$. Thus a natural choice of $\gamma(\cdot)$ is a positive, non-decreasing solution of the following Lundberg equation:

$$-\gamma p(y) + \int_{\mathbb{R}_+} [e^{\gamma x} - 1]\lambda F(dx) = 0, \quad y \geqslant 0. \tag{5.10}$$

Using elementary analysis for implicit functions one can show that such a solution, called the "*local adjustment coefficient*" by Asmussen and Nielsen [1], does exist under the net profit condition (5.6) and the assumption that $\rho(\cdot)$ (or $p(\cdot)$) is non-decreasing; and it satisfies that $\inf_{y \geqslant 0} \gamma(y) > 0$ (see also [1]). Furthermore, it can be checked that if $p(\cdot)$ is continuously differentiable, then so is $\gamma(\cdot)$, and hence $I(\cdot) \in C^2(\mathbb{R}_+)$. Since $I(0) = 0$, and $I'(0) = \gamma(0) > 0$, we see that $I(x) < 0$ when $x < 0$ with $|x|$ small. Therefore, we can choose $\delta > 0$, and extend $I(\cdot)$ to whole $\mathbb{R}$ in such way that I is C^2, $I(x) < 0$ for $x \in (-\delta, 0)$, and $I(x) = 0$, $\forall x \leqslant -\delta$.

Note that by definition of γ (5.9) now leads to

$$K_t^I(X^+) \leqslant \int_0^t \Big\{ -\gamma(X_s^+) p(X_s^+) + \int_{\mathbb{R}_+} [\exp\{\gamma(X_s^t)x\} - 1]\lambda F(dx)\Big\}\,ds = 0, \quad t \geqslant 0. \tag{5.11}$$

Thus I satisfies Hypothesis A, and $I(x) \leqslant 0$ for all $x \leqslant 0$. Applying Corollary 4.2 and noting that $\exp\{K_t^I(X^+)\} \leqslant 1$, $\forall t \geqslant 0$ by (5.11), we obtain the Lundberg inequalities

$$\psi(x, T) \leqslant e^{-I(x)} \quad \text{and} \quad \psi(x) \leqslant e^{-I(x)},$$

which are the same as the results of Asmusssen and Nielsen [1]. □

It is worth noting that the inequality in (5.11) indicates a possibility of improving the Asmussen–Nielsen bound using our method, because there is still room to manipulate the rate function without changing the non-positivity of $K^I(X^+)$. In the following example we shall consider a special case of Example 5.4, and derive the sharpest Lundberg bound from Theorem 4.1 (or Corollary 4.2), which does not seem to be amendable by the local adjustment coefficient method of [1].

EXAMPLE 5.5. Let us assume that in Example 5.4 the loading function $\rho(x) \equiv 0$, and the claim size has exponential distribution with rate θ. That is, $F(x) = 1 - e^{-\theta x}$, $x \geqslant 0$. We note that this case was studied by Asmussen–Nielsen [1, Example 2] using the local adjustment coefficient method, and a Lundberg bound was proved to be

$$\psi(x) \leqslant e^{-\theta x}\left(1 + \frac{r}{c}x\right)^{\lambda/r}, \quad x \geqslant 0.$$

Let us now try to use Theorem 4.1 (or Corollary 4.2) to derive a sharper bound.
First note that in this case, the process $K^I(X^+)$ takes the form

$$K_t^I(X^+) = \int_0^t \left\{ -I'(X_s^+)(rX_s^+ + c) + \int_0^\infty [\exp\{I(X_s^+) - I(X_s^+ - x)\} - 1]\lambda\theta\, e^{-\theta x}\, dx \right\} ds, \tag{5.12}$$

for $t \geqslant 0$. Now instead of considering I of the form (5.8) let us look directly at the integro-differential equation

$$-I'(y)[ry + c] + \int_0^\infty [\exp\{I(y) - I(y - x)\} - 1]\lambda\theta\, e^{-\theta x}\, dx = 0. \tag{5.13}$$

A direct computation shows that the following function

$$I(y) = -\log\left[\frac{\displaystyle\int_y^\infty e^{-\theta z}\left(1 + \frac{rz}{c}\right)^{(\lambda/r) - 1} dz}{\displaystyle\frac{c}{\lambda} + \int_0^\infty e^{-\theta z}\left(1 + \frac{rz}{c}\right)^{(\lambda/r) - 1} dz}\right]$$

is a solution to (5.13) for $y \geqslant 0$. Clearly, $I(y) > 0$ for $y > 0$. Thus, for each n let us molify I to a nonnegative, C^2-function $I^{(n)}$ such that

$$I^{(n)}(y) = \begin{cases} I(y), & y \geqslant 0; \\ 0, & y \leqslant -1/n, \end{cases}$$

and we require that $\sup_n \sup_{y \in [-1/n, 0]} I^{(n)}(y) \leqslant C$, for some constant $C > 0$.

We note that the function $I^{(n)}$ does not satisfy the nonpositivity condition on $(-\infty, 0]$, so we cannot apply Theorem 4.1 (or Corollary 4.2) directly. But this technical difficulty can be fixed by modifying the proof there as follows. Since $I^{(n)}$ is C^2, we still have (4.3):

$$e^{-I^{(n)}(x)} \geqslant E\{\exp\{-I^{(n)}(X_\tau) - K_\tau^{I^{(n)}}\} \mid \tau < T\} P\{\tau < T\},$$

where $\tau = \inf\{t > 0: X_\tau < 0\}$. Since for $t \leqslant \tau$, $K_t^{I^{(n)}} = K_t^{I^{(n)}}(X^+) = K_t^I(X^+) = 0$ by our choice of I and $I^{(n)}$, we have $\exp\{-K_\tau^{I^{(n)}}\} = 1$. Hence for $x \geqslant 0$,

$$\begin{aligned} e^{-I(x)} = e^{-I^{(n)}(x)} &\geqslant E\{\exp\{-I^n(X_\tau)\} \mid \tau < T\}\{\tau < T\} \\ &= E\{\exp\{-I^{(n)}(X_\tau)\} \mid \tau < T, X_\tau \leqslant -1/n\} P\{X_\tau \leqslant -1/n\} P\{\tau < T\} \\ &\quad + E\{\exp\{-I^{(n)}(X_\tau)\} \mid \tau < T, X_\tau > -1/n\} P\{X_\tau > -1/n\} P\{\tau < T\} \\ &\geqslant P\{X_\tau \leqslant -1/n\} P\{\tau < T\} + e^{-C} P\{X_\tau > -1/n\} P\{\tau < T\}. \end{aligned} \tag{5.14}$$

Now from (5.5) it is not hard to see that $X_t < 0$ only when $\Delta X_t = X_{t-} - X_t < 0$, we must have $X_\tau < 0$, a.s. Thus letting $n \to \infty$ in the above we obtain that

$$e^{-I(x)} \geqslant P\{\tau < T\} = \psi(x, T), \quad x \geqslant 0.$$

Letting $T \to \infty$, we obtain that $\psi(x) \leqslant e^{-I(x)}$, $x \geqslant 0$. We should point out that this bound is the sharpest. In fact, in this case it is well-known (see, for example, Segerdahl [18]) that $\psi(x) = e^{-I(x)}$, $x > 0$. □

EXAMPLE 5.6. (*Proportional investments*). We now change Example 5.5 slightly by allowing the so-called "Proportional investments", that is, we assume that the portfolio $\pi_t = \alpha X_t$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a constant vector with $\alpha_j > 0$, $\forall i$, such that $|\alpha| \triangleq \Sigma_i \alpha_i > 0$. (Note: if $|\alpha| = 0$, then we return to Example 5.5.) We also assume that $\tau_t = \mu_t = r$, and of $\sigma_t = \sigma$, thus the reserve process becomes

$$X_t = x + \int_0^t p(X_s)\, ds + \int_0^t \langle \alpha X_s, \sigma\, dW_s \rangle - \int_0^{t+} \int_{\mathbb{R}_+} x N_p(dx\, ds), \tag{5.15}$$

where $p(x) = rx + c$. For the sake of argument in what follows we make use of the following extra *compatibility condition*:

$$r > \frac{1}{2} |\sigma^T \alpha|^2 > 0. \tag{5.16}$$

In light of Examples 5.4 and 5.5, this time we shall construct a rate function $I \in C^2(\mathbb{R})$ satisfying

$$G^I(y) \triangleq -I'(y)\{ry + C\} + \frac{1}{2}(I'(y)^2 - I''(y))y^2|\sigma^T\alpha|^2$$
$$+ \int_{\mathbb{R}_+} [\exp\{I(y) - I(y-x)\} - 1]\lambda\theta\, e^{-\theta x}\, dx \leqslant 0, \quad y \geqslant 0, \tag{5.17}$$

such that $I(y) \sim k \ln y + C$ for some constant k, C as $y \to \infty$, so that $K_t^I(X^+) = \int_0^t G^I(X_s^+)\, ds \leqslant 0$, for $t \geqslant 0$, a.s. It is clear that (5.17) is a second order integrodifferential inequality. We follow the idea of the so-called "*principle of smooth-fit*" to find a C^2-solution to this inequality which will serve as our rate function. The derivation of such C^2 solution is a little lengthy, we give only a sketch here, and refer to Sun [19] for complete details.

We begin with the two-parameter family: $I_{\beta,k}(y) = k(\ln(y+\beta) - \ln 2\beta)\mathbf{1}_{[\beta,\infty)}(y)$, where k and β are to be determined.

Step 1. Choose $k = (2r/|\sigma^T\alpha|) - 1$, and denote $B = \beta r + c$, and $C = c\beta$. Then $k > 0$, thanks to (5.16); and it can be checked that (denoting $I = I_{\beta,k}$)

$$G^I(y) = \frac{-k\{By + C\}}{(y+\beta)^2} + \lambda\theta \int_{y-\beta}^{\infty} \exp\{I(y)\}\, e^{-\theta x}\, dx$$
$$+ \lambda\theta \int_0^{y-\beta} \exp\{I(y) - I(y-x)\}\, e^{-\theta x}\, dx - \lambda, \quad y \geqslant \beta, \tag{5.18}$$

Step 2. We prove that (5.17) holds for all $y \geqslant \beta$ when β is sufficiently large. To see this we first let $\delta \in (0, (\theta kr/(kr + 8\lambda)))$, and let $\beta = \beta(\delta) = k/\delta$. Clearly for any $y \geqslant \beta$ we have $I'(y) = k/(y+\beta) \leqslant k/(2\beta) < \delta$. Thus, using the fact that $I(b) - I(a) = \int_a^b I'(y)\, dy$ and $I(\beta) = 0$ one shows that

$$\lambda\theta \int_{y-\beta}^{\infty} \exp\{I(y)\}\, e^{-\theta x}\, dx + \lambda\theta \int_0^{y-\beta} \exp\{I(y) - I(y-x)\}\, e^{-\theta x}\, dx - \lambda$$
$$\leqslant \lambda\theta \int_{\mathbb{R}_+} e^{\delta(x \wedge (y-\beta))}\, e^{-\theta x}\, dx - \lambda \leqslant \lambda\theta \int_{\mathbb{R}_+} e^{-(\theta-\delta)x}\, dx - \lambda = \frac{\delta\lambda}{\theta - \delta}, \quad y \geqslant \beta.$$

Furthermore, note that

$$-\frac{k(By + C)}{(y+\beta)^2} \leqslant -\frac{k\beta r y}{(y+\beta)^2} \leqslant -\frac{kr\beta}{4y}, \quad y \geqslant \beta. \tag{5.19}$$

Thus if $\beta \leqslant y < 2\beta$, then we deduce from (5.18) and the definition of δ that

$$G^I(y) \leqslant \frac{-k\{(\beta r + c)y + c\beta\}}{(y+\beta)^2} + \frac{\delta\lambda}{\theta-\delta} \leqslant \frac{-kr\beta}{4y} + \frac{\delta\lambda}{\theta-\delta} < 0, \tag{5.20}$$

that is, (5.17) holds for $y \in [\beta, 2\beta)$. To see the case when $y \geqslant 2\beta$, we observe that $y - \beta \geqslant y/2$, $y + \beta \leqslant \frac{3}{2}y$, and

$$\lambda\theta\int_{y-\beta}^{\infty}\exp\{I(y)\}\,e^{-\theta x}\,dx=\lambda\theta\int_{y-\beta}^{\infty}\left(\frac{y+\beta}{2\beta}\right)^{k}e^{-\theta x}\,dx\leqslant\lambda\left(\frac{3y}{4}\right)^{k}e^{-\theta y/2}\triangleq h_1(y). \tag{5.21}$$

Therefore, using the fact that the mapping $z\mapsto z/(z-x)$ is decreasing, and integrating by parts twice one has

$$\begin{aligned}\lambda\theta\int_{0}^{y-\beta}\exp\{I(y)-I(y-x)\}\,e^{-\theta x}\,dx-\lambda&=\lambda\theta\int_{0}^{y-\beta}\left(\frac{y+\beta}{y+\beta-x}\right)^{k}e^{-\theta x}\,dx-\lambda\\ &\leqslant\frac{\lambda k}{\theta}\frac{1}{y+1}-\left(\lambda\,e^{-\theta y}+\frac{\lambda k}{\theta}\right)(y+1)^{k}\,e^{-\theta y}\\ &\quad+\frac{k(k+1)\lambda}{\theta}\int_{0}^{y}\frac{(y+1)^{k}}{(y+1-x)^{k+2}}\,e^{-\theta x}\,dx\\ &\triangleq\frac{\lambda k}{\theta}\frac{1}{y+1}+h_2(y).\end{aligned} \tag{5.22}$$

Thus, combining (5.19), (5.21), and (5.22), we derive from (5.18) that, for $y\geqslant 2\beta$,

$$G^{I}(y)\leqslant\frac{-kr\beta}{4y}+\frac{\lambda k}{\theta}\frac{1}{y+1}+h_1(y)+h_2(y)\leqslant\frac{1}{y}\left\{-\frac{kr\beta}{4}+\frac{\lambda k}{\theta}+y[h_1(y)+h_2(y)]\right\}. \tag{5.23}$$

Now, some careful estimation shows that $y[h_1(y)+h_2(y)]\to 0$ as $y\to\infty$. Thus we can first choose $M>0$ so that $0<y[h_1(y)+h_2(y)]\leqslant 1$, whenever $y>M$, and then choose $\delta\in(0,\ (\theta kr/kr+8\lambda))$, such that

$$\beta=\beta(\delta)=\frac{k}{\delta}>\max\left\{M,\frac{4}{kr}\left(\frac{\lambda k}{\theta}+1\right)\right\}.$$

Combining (5.18)–(5.23) we conclude that $G^{I}(y)\leqslant 0$, for all $y\geqslant\beta$.

Step 3. In order to apply Corollary 4.2, we modify $I_{\beta,k}$ slightly so that it is also C^2 at $y=\beta$. This can be done by choosing I to be of the form

$$I(y)=\begin{cases}I_{\beta,k}(y)+a_3 & y\geqslant\beta+1,\\ -\dfrac{a_1}{12}(y-\beta)^4+\dfrac{a_2}{6}(y-\beta)^3 & \beta\leqslant y\leqslant\beta+1,\\ 0 & y\leqslant\beta,\end{cases}$$

where a_1, a_2 and a_3 are determined by solving a linear system algebraic equations. We leave it to the interested readers.

Now applying Corollary 4.2 we get

$$\psi(x)\leqslant e^{-I(x)}=K(x+\beta)^{-k},\quad \text{for } x \text{ large},$$

where K is some constant. In other words, this is a case of so-called "power ruin probability" (see, e.g., Nyrhinen [10] and Kalashnikov & Norberg [7]). We remark that this is still only a upper bound, which may not be sharp. □

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