

ON THE PRINCIPLE OF SMOOTH FIT FOR A CLASS OF SINGULAR STOCHASTIC CONTROL PROBLEMS FOR DIFFUSIONS*

JIN MA†

Abstract. This paper considers the principle of smooth fit for a class of one-dimensional singular stochastic control problems allowing the system to be of nonlinear diffusion type. The existence and the uniqueness of a convex C^2 -solution to the corresponding variational inequality are obtained. It is proved that this solution gives the value function of the control problem, and the optimal control process is constructed. As an example of the degenerate case, it is proved that the conclusion is also true for linear systems, and the explicit formula for the smooth fit points is derived.

Key words. singular stochastic control, principle of smooth fit, variational inequality, free boundary problem, diffusion with reflections

AMS(MOS) subject classifications. 93E20, 49A10, 49A60

1. Introduction. Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a complete probability space with filtration $\{\mathcal{F}_t\}$, which is assumed to be right-continuous, and \mathcal{F}_0 contains all the P -null sets in \mathcal{F} . We assume that a one-dimensional standard Brownian motion $W = \{W(t) : t \geq 0\}$ with respect to $\{\mathcal{F}_t\}$ is given on this probability space.

Consider the system described by the stochastic differential equation

$$(1.1) \quad dX(t) = a(X(t))dt + \sigma(X(t))dW(t) + d\xi(t), \quad X(0) = x,$$

or, equivalently, by the stochastic integral equation

$$(1.2) \quad X(t) = x + \int_0^t a(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \xi(t),$$

where $\xi = \{\xi(t) : t \geq 0\}$ is a *left-continuous, $\{\mathcal{F}_t\}$ -adapted process with locally bounded variation paths*. The process ξ is to be chosen by the decision maker as the *control process*, and the objective is to minimize the following cost function:

$$(1.3) \quad V_\xi(x) = E \int_{[0, \infty)} e^{-\alpha t} [cd\check{\xi}(t) + h(X(t))dt],$$

where $\check{\xi} = \{\check{\xi}(t) : t \geq 0\}$ is the *total variation process* of ξ ; the constant $\alpha > 0$ is called the *discount factor*; h is a nonnegative, strictly convex, C^2 -function; and $c > 0$.

Problems of similar type have been studied by many authors (cf. [1], [2], [6], [8]–[10], [12]–[14]). In the case when $c = 0$, $h(x) = x^2$, $a(x) \equiv 0$, $\sigma(x) \equiv 1$, the problem was solved explicitly by Beněš, Shepp, and Witsenhausen [1] under the constraints that either ξ has bounded derivatives (bounded velocity follower problem) or it has bounded total variation (finite-fuel follower problem). Under the same setting but without the extra restriction on ξ , and allowing h to be a general strictly convex function and $c = 1$, the result was generalized by Karatzas [8]. Almost simultaneously, Harrison and Taksar [6] treated the case with a more general cost function but restricted (compact) state space and, also, they assumed the drift and the diffusion coefficients to be constants.

* Received by the editors December 19, 1990; accepted for publication (in revised form) May 24, 1991. This work is a part of the author's Ph.D. dissertation at the University of Minnesota.

† Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

For the case when a, σ are nonconstant, the problem was developed by Menaldi and Robin [9], Chow, Menaldi, and Robin [2], and Shreve, Lehoczky, and Gaver [13], among others. In [9], however, no control entered the cost function explicitly (i.e., $c = 0$), and, for the convex case (i.e., h is convex), the result there was only valid when a and σ are constants. In [2] the horizon was assumed to be finite, and only the monotone follower problem (i.e., ξ is monotone) was considered. We note that the bounded variation control problem (optimal correction problem) was considered there only when some special symmetric conditions were satisfied by h , so that the problem could be reduced to the monotone follower problem. In general, however, these conditions are not satisfied in our setting. Finally, in [13], it was essentially the homogeneous problem (i.e., $h(\cdot) \equiv 0$), so the problem is quite different from ours. We note that, for the convexity of the value function, all the above work required the coefficients of the system to be constant or linear (in spatial variables), so that the convexity of the function h would imply the convexity of value function immediately. However, this requirement is not satisfied, in general, in our setting.

The problem is also studied for a higher-dimension case by Soner and Shreve [14] and Menaldi and Taksar [10]; some regularity results for the free boundary, as well as the convexity of the value function, were obtained. However, the difficulties that arise in higher dimensions seem to restrict the problem only to the case when a, σ are constants.

In this paper, we are interested in the system when $a(x) = ax + b$ and when σ is any nonvanishing, Lipschitz continuous, C^2 -function of linear growth. Under some conditions on the discount factor α and the function σ , we prove that the *principle of smooth fit* always holds in this case. Namely, we prove that there exists a unique *convex C^2 -solution* to the variational inequality that is *linear* outside a certain finite interval (even though the data of the system, e.g., σ , could be nonlinear), which, as was pointed out by Shreve [12], gives the value function and leads to the existence of the optimal policy for such problems. Consequently, the optimal policy can then be chosen to be the proper local times to make the dynamics to be the reflected diffusion on a certain region. Compared to the usual way of treating variational inequalities, our approach is direct and elementary but strongly restricted to the one-dimensional case.

An interesting question then is how this setting includes the linear case, namely, when $\sigma(\cdot)$ is also linear. An immediate problem is that the related ordinary differential equation (ODE) becomes singular at some point (the zero of σ). In §5 we treat this case specifically to get an explicit solution.

The paper is organized as follows. In §2 we give the formulation of the problem and the verification theorems. In §3 we study the ODE related to the H-J-B equation and give some basic results as lemmas for the main theorems. Section 4 is devoted to the main results, and, finally, in §5, we study the linear case, which can also be treated as an example for our setting.

2. Formulation of the problem and the verification theorems. We will henceforth consider the system

$$(2.1) \quad X(t) = x + \int_0^t (aX(s) + b)ds + \int_0^t \sigma(X(s))dW(s) + \xi(t),$$

where a, b are constants, $W(\cdot)$ is a one-dimensional Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$.

As in §1, for a given ξ , the cost function is defined by

$$(2.2) \quad V_\xi(x) = E \int_{[0, \infty)} e^{-\alpha t} [cd\check{\xi}(t) + h(X(t))] dt.$$

We will assume that $c = 1$ for simplicity.

The value function is defined by

$$(2.3) \quad V^*(x) = \inf_{\xi \in \mathcal{B}} V_\xi(x), \quad x \in \mathbf{R},$$

where \mathcal{B} is a class of processes called *admissible controls*, which will be described later.

We make the following basic assumptions:

(A1) The function $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is of class C^2 such that, for some $K > 0$,

$$(2.4) \quad |\sigma'(x)| + |(\sigma^2(x))''| \leq K, \quad x \in \mathbf{R};$$

$$(2.5) \quad \sigma(x) \neq 0, \quad x \in \mathbf{R}.$$

Clearly, (2.4) implies that $\sigma(\cdot)$ is globally Lipschitz and of linear growth; i.e., for some $K_1 > 0, K_2 > 0$,

$$(2.6) \quad |\sigma(x) - \sigma(y)| \leq K_1|x - y|, \quad x, y \in \mathbf{R},$$

and

$$(2.7) \quad |\sigma(x)| \leq K_2(1 + |x|), \quad x \in \mathbf{R},$$

where the constants K_1, K_2 depend only on K and $\sigma(0)$.

(A2) The function $h : \mathbf{R} \rightarrow [0, \infty)$ is of class C^2 such that, for some k, K_3 with $0 < k < K_3$,

$$(2.8) \quad 0 < k \leq h''(x) \leq K_3, \quad x \in \mathbf{R},$$

and there exists $\bar{x} \in \mathbf{R}$ such that

$$(2.9) \quad (x - \bar{x})h'(x) \geq 0, \quad x \in \mathbf{R}; \quad h'(\bar{x}) = 0.$$

Also, for simplicity, we assume that $\bar{x} = 0$.

(A3) The discount factor $\alpha > 0$ satisfies

$$(2.10) \quad \alpha > \frac{1}{2} \sup_{x \in \mathbf{R}} |(\sigma^2(x))''| + 2|a|.$$

Remark 2.1. (i) Condition (A3) seems to be a little strong, since it actually requires that the discount factor be sufficiently large. This is to compensate for the fact that the coefficients are not constant. In fact, without this assumption, the convexity of the value function, which is essential in the smooth fit technique, may be false. The similar condition has also been used in [2], [9], [13], and others.

(ii) By the definition of the function h , the cost function satisfies $V_\xi(x) \geq 0$ for all $x \in \mathbf{R}, \xi \in \mathcal{B}$. Also, (2.8) and (2.9) imply that h is strictly convex and (recall that $\bar{x} = 0$), for any $\delta > 0$, there exist $-\infty < r_1 < 0 < r_2 < \infty$ such that $|h'(x)| < \delta, x \in (r_1, r_2)$, and $|h'(r_1)| = |h'(r_2)| = \delta$.

(iii) Throughout the paper, instead of using constants K_1, K_2, K_3, \dots , we use a generic constant $K > 0$, which may vary line by line if no confusion occurs.

As we mentioned before, $\xi = \{\xi(t) : t \geq 0\}$ is an $\{\mathcal{F}_t\}$ -adapted, left-continuous process such that, for each $\omega \in \Omega$, the path $\xi(\cdot, \omega)$ is of locally bounded variation on $[0, +\infty)$ and $\xi(0) = 0$. We may write ξ in its canonical form $\xi = \xi^+ - \xi^-$ as the difference of two nondecreasing processes ξ^+ and ξ^- with $\xi^\pm(0) = 0$. If we assume that the decomposition is minimal, then the total variation process $\check{\xi}$ can be written as $\check{\xi}(t) = \xi^+(t) + \xi^-(t), t \geq 0$. We denote the totality of such ξ 's by \mathcal{B} (*admissible controls*), and denote, for each $[A, B] \subseteq \mathbf{R}$, $\mathcal{B}_{[A, B]} = \{\xi \in \mathcal{B} : X_\xi(t) \in [A, B] \text{ for } t > 0, \text{ almost surely}\}$. Observe that, for $\xi \in \mathcal{B}_{[A, B]}$, $X_\xi(0) = x$ could be outside the interval $[A, B]$, but, after an initial jump, the trajectories of $X_\xi(\cdot)$ will remain in $[A, B]$, (P)-almost surely. It is known that, under our basic assumptions, (2.1) has a (pathwise) unique solution for $t \geq 0$ and every $\xi \in \mathcal{B}(\mathcal{B}_{[A, B]})$ (cf. [5]).

Due to the results for the Brownian motion case (cf. [1], [8]), to get a nontrivial lower bound for the cost functions and the sufficient conditions for a cost function to be optimal, we should seek a convex solution of the following *variational inequality*:

$$(2.11) \quad \alpha V(x) - \frac{1}{2}\sigma^2(x)V''(x) - (ax + b)V'(x) - h(x) \vee [|V'(x)| - 1] = 0, \quad x \in \mathbf{R}.$$

The following theorem verifies this fact.

THEOREM 2.1. *Suppose that $V : \mathbf{R} \rightarrow \mathbf{R}$ is a C^2 -function satisfying*

$$(2.12) \quad V''(x) \geq 0, \quad x \in \mathbf{R};$$

$$(2.13) \quad |V'(x)| \leq 1, \quad x \in \mathbf{R};$$

$$(2.14) \quad \alpha V(x) \leq \frac{1}{2}\sigma^2(x)V''(x) + (ax + b)V'(x) + h(x), \quad x \in \mathbf{R};$$

then, under assumptions (A1)–(A3), for all $x \in \mathbf{R}$ and all $\xi \in \mathcal{B}$, we have that $V(x) \leq V_\xi(x)$. Consequently, if there exists a $\xi^* \in \mathcal{B}$ such that $V(x) = V_{\xi^*}(x)$, for all $x \in \mathbf{R}$, then

$$V(x) = V_{\xi^*}(x) = V^*(x), \quad x \in \mathbf{R}.$$

Before proving the theorem, we first give a lemma that may be of independent interest. Soner and Shreve [14, Thm. 3.1] used an easier version to prove their result. We note that their version would suffice for the proof of our theorem as well.

LEMMA 2.2. *Let $\xi \in \mathcal{B}$ and $X(\cdot) = X_\xi(\cdot)$ be the corresponding solution of (2.1). If $E\check{\xi}(t+) = o(e^{\alpha t})$ and $E \int_0^\infty e^{-\alpha t} |X(t)|^2 dt < \infty$, then*

$$E|X(t+)| = o(e^{\alpha t}), \quad \text{as } t \rightarrow \infty,$$

where here (and in the following) $o(\rho)$ means $\lim_\rho o(\rho)/\rho = 0$.

Proof. Since

$$|X(t+)| \leq |X(t)| + |X(t+) - X(t)| = |X(t)| + |\xi(t+) - \xi(t)| \leq |X(t)| + |\check{\xi}(t+)|,$$

it suffices to prove that $E|X(t)| = o(e^{\alpha t})$, as $t \rightarrow \infty$. By (2.1), we have that

$$E|X(t)| \leq |x| + \int_0^t [|a|E|X(s)| + |b|] ds + E \left| \int_0^t \sigma(X(s)) dW(s) \right| + E\check{\xi}(t)$$

$$\begin{aligned}
 (2.15) \quad &\leq |x| + |b|t + E\check{\xi}(t+) + \left[E \int_0^t \sigma^2(X(s)) ds \right]^{\frac{1}{2}} + \int_0^t |a|E|X(s)| ds \\
 &\leq |x| + |b|t + E\check{\xi}(t+) + \left[1 + E \int_0^t \sigma^2(X(s)) ds \right] + \int_0^t |a|E|X(s)| ds \\
 &\leq [|x| + 1 + (|b| + 2K^2)t] + E\check{\xi}(t+) + 2K^2 \int_0^t E|X(s)|^2 ds \\
 &\quad + |a| \int_0^t E|X(s)| ds.
 \end{aligned}$$

Let $p(t) = [1 + |x| + (|b| + 2K^2)t] + E\check{\xi}(t+) + 2K^2 \int_0^t E|X(s)|^2 ds$. We claim that $p(t) = o(e^{\alpha t})$, as $t \rightarrow \infty$. Indeed, by assumption, $E\check{\xi}(t+) = o(e^{\alpha t})$, so we must only show that

$$E \int_0^t |X(s)|^2 ds = o(e^{\alpha t}), \quad \text{as } t \rightarrow \infty.$$

Define $\phi(t) = e^{-\alpha t} E|X(t)|^2$; then the assumption implies that $\int_0^\infty \phi(t) dt < \infty$. Therefore, a simple application of dominated convergence theorem leads to

$$e^{-\alpha t} E \int_0^t |X(s)|^2 ds = \int_0^\infty 1_{[0,t]}(s) e^{-\alpha(t-s)} \phi(s) ds \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

This proves the claim.

Now applying Gronwall’s inequality (e.g., cf. [7, eq. (2.7.1)]) to (2.15), we obtain that

$$(2.16) \quad E|X(t)| \leq p(t) + \int_0^t e^{|a|(t-s)} p(s) ds.$$

Note that $p(t) = o(e^{\alpha t})$ and $\alpha > |a|$, so, given $\epsilon > 0$, we can choose $T > 0$ so that $e^{-\alpha s} p(s) < \epsilon$ for $s \geq T$; hence, for some $k_1 > 0$,

$$\begin{aligned}
 \int_0^t e^{|a|(t-s)} p(s) ds &\leq \int_0^T e^{|a|(t-s)} p(s) ds + \epsilon \int_T^t e^{|a|(t-s)} \cdot e^{\alpha s} ds \\
 &\leq e^{|a|t} k_1 T + e^{|a|t} \cdot \epsilon \cdot \int_T^t e^{(\alpha - |a|)s} ds \\
 &\leq e^{|a|t} k_1 T + e^{|a|t} \cdot \epsilon \cdot \frac{e^{(\alpha - |a|)t}}{\alpha - |a|}.
 \end{aligned}$$

Since ϵ is arbitrary, we obtain that $\int_0^t e^{|a|(t-s)} p(s) ds = o(e^{\alpha t})$; the consequence then follows from (2.16). \square

Proof of the theorem. Our approach is typical. Let $\xi \in \mathcal{B}$ and write $\xi = \xi^+ - \xi^-$. Let $X(\cdot) = X_\xi^x(\cdot)$ be the corresponding solution of (2.1). Denote the right-continuous version of ξ by $\{\check{\xi}(t+), t \geq 0\}$. (The right-continuous version of an adapted left-continuous process $\eta(\cdot)$ is a process $\zeta(\cdot)$ such that, for each $\omega \in \Omega$, $\zeta(t, \omega) = \eta(t+, \omega)$, for all $t \geq 0$ and $\zeta(0-, \omega) = \eta(0, \omega)$. The right-continuity of the filtration $\{\mathcal{F}_t\}$ guarantees that ζ is also adapted.) Define $F(t, x) = e^{-\alpha t} V(x)$, for $(t, x) \in [0, \infty) \times \mathbf{R}$. By the generalized Itô formula (Meyer [11]), we have that

$$e^{-\alpha t} V(X(t+)) = V(X(0+))$$

$$\begin{aligned}
 & + \int_0^t e^{-\alpha s} [-\alpha V(X(s)) + (aX(s) + b)V'(X(s)) + \frac{1}{2}\sigma^2(X(s))V''(X(s))] ds \\
 (2.17) \quad & + \int_{(0,t]} e^{-\alpha s} V'(X(s)) d\xi(s+) + \int_0^t e^{-\alpha s} V'(X(s)) \sigma(X(s)) dW(s) \\
 & + \sum_{0 < s \leq t} [V(X(s+)) - V(X(s)) - V'(X(s))(X(s+) - X(s))].
 \end{aligned}$$

By (2.13) we see that the second term on the above right-hand side is no less than $-\int_0^t e^{-\alpha s} h(X(s)) ds$. The convexity of V implies that

$$\sum_{0 < s \leq t} [V(X(s+)) - V(X(s)) - V'(X(s))(X(s+) - X(s))] \geq 0, \quad \text{a.s.}$$

So (2.17) becomes

$$\begin{aligned}
 (2.18) \quad e^{-\alpha t} V(X(t+)) & \geq V(X(0+)) - \int_0^t e^{-\alpha s} h(X(s)) ds \\
 & + \int_{(0,t]} e^{-\alpha s} V'(X(s)) d\xi(s+) \\
 & + \int_0^t e^{-\alpha s} V'(X(s)) \sigma(X(s)) dW(s).
 \end{aligned}$$

Note that the convexity of V also implies that

$$\begin{aligned}
 0 & \leq V(X(0+)) - V(X(0)) - V'(X(0))(X(0+) - X(0)) \\
 & = V(X(0+)) - V(X(0)) - V'(X(0))(\xi(0+) - \xi(0)) \\
 & = V(X(0+)) - V(X(0)) - \int_{\{0\}} V'(X(s)) d\xi(s+).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 (2.19) \quad e^{-\alpha t} V(X(t+)) & \geq V(X(0)) - \int_0^t e^{-\alpha s} h(X(s)) ds \\
 & + \int_{[0,t]} e^{-\alpha s} V'(X(s)) d\xi(s+) \\
 & + \int_0^t e^{-\alpha s} V'(X(s)) \sigma(X(s)) dW(s).
 \end{aligned}$$

Define

$$(2.20) \quad M(t) = e^{-\alpha t} V(X(t+)) + \int_{[0,t]} e^{-\alpha s} [d\check{\xi}(s+) + h(X(s)) ds],$$

$$(2.21) \quad m(t) = \int_0^t e^{-\alpha s} V'(X(s)) \sigma(X(s)) dW(s).$$

Some computation from (2.19) yields that

$$\begin{aligned}
 (2.22) \quad EM(t) & \geq V(x) + E \int_{[0,t]} e^{-\alpha s} [1 + V'(X(s))] d\xi^+(s+) \\
 & + E \int_{[0,t]} e^{-\alpha s} [1 - V'(X(s))] d\xi^-(s+) + Em(t).
 \end{aligned}$$

Since $|V'(x)| \leq 1$, (2.22) gives $EM(t) \geq V(x) + Em(t)$.

Observe that, by the definition of $\check{\xi}(t+)$, $\int_{[0,t]} e^{-\alpha s} d\check{\xi}(s+) = \int_{[0,t]} e^{-\alpha s} d\check{\xi}(s)$ for all $t \geq 0$, since the integrand $e^{-\alpha t}$ is continuous. So the expectation of the second term on the right-hand side of (2.20) converges to $V_\xi(x)$ as $t \rightarrow \infty$. Therefore, to finish the proof, we must only show that $\lim_{t \rightarrow \infty} EM(t) = \lim_{t \rightarrow \infty} e^{-\alpha t} EV(X(t+)) + V_\xi(x) = V_\xi(x)$ and $Em(t) = 0$, whenever $V_\xi(x) < \infty$. (If $V_\xi(x) = \infty$, there is nothing to prove.) It is readily seen, however, that $V_\xi(x) < \infty$ implies that $E\check{\xi}(t+) = o(e^{\alpha t})$ and $E \int_0^\infty e^{-\alpha t} |X(t)|^2 dt < \infty$; i.e., the assumptions of Lemma 2.2 are satisfied. The latter, together with (2.13) and (2.7), implies that $m(t)$ is a L^2 -martingale, so $Em(t) = 0$ for each $t \geq 0$. On the other hand, by Lemma 2.2, we have that $E|X(t+)| = o(e^{\alpha t})$. It follows immediately that $EV(X(t+)) = o(e^{\alpha t})$, since $V(\cdot)$ is at most of linear growth by (2.13). This leads to the conclusion that $\lim_{t \rightarrow \infty} e^{-\alpha t} EV(X(t+)) = 0$. Therefore $V_\xi(x) \geq V(x)$, $x \in \mathbf{R}$. The remainder of the theorem is obvious, so we are done. \square

Finally, we give a local version of Theorem 2.1, which will be very useful in this paper. Since the proof is virtually identical to that of Theorem 2.1, we omit it.

THEOREM 2.3. *Let V be a C^2 -function defined on \mathbf{R} satisfying (2.14). Let $-\infty < L < B < \infty$ and suppose that V satisfies (2.12), (2.13) on $[L, B]$. Then under assumptions (A1)–(A3), we have that*

$$V(x) \leq \inf_{\xi \in \mathcal{B}_{[L,B]}} V_\xi(x), \quad x \in [L, B].$$

Furthermore, if there exists a $\xi^* \in \mathcal{B}_{[L,B]}$ such that $V(x) = V_{\xi^*}(x)$ for all $x \in [L, B]$, then

$$V(x) = V_{\xi^*}(x) = \inf_{\xi \in \mathcal{B}_{[L,B]}} V_\xi(x), \quad x \in [L, B].$$

3. Some basic results for the ODE related to H-J-B equation. In this section, we study the following ODE related to the H-J-B equation (2.11) under assumptions (A1)–(A3):

$$(3.1) \quad \alpha V(x) = (ax + b)V'(x) + \frac{1}{2}\sigma^2(x)V''(x) + h(x), \quad x \in \mathbf{R}$$

and give some results that serve as lemmas for the main theorem.

We consider the following *free boundary problem*. Find a pair of real numbers $-\infty < L < B < \infty$ and a solution V of (3.1) that is *convex* on $[L, B]$, satisfying the boundary conditions

$$(3.2) \quad V'(L) = -1, \quad V'(B) = 1;$$

$$(3.3) \quad V''(L) = V''(B) = 0.$$

Remark 3.1. For the boundary conditions (3.2) and (3.3), it should be understood first that all the derivatives there are one-sided in the appropriate direction. Then observe that once a solution exists on $[L, B]$, it can actually be extended to be defined on the whole real line by our assumptions on the data. Hence, in the following, the derivatives at the boundary will be the usual two-sided derivatives.

The other observation is that, since σ, h are of class C^2 and since σ is nonvanishing, we can easily check by directly differentiating (3.1) that any solution of (3.1) will be of class C^4 (on the whole real line).

We claim that, under our basic assumptions, the solution to (3.1), (3.2) exists and is unique for any given $L < B$. Indeed, let f, g be two independent solutions to the homogeneous equation

$$(3.4) \quad \frac{1}{2}\sigma^2(x)V''(x) + (ax + b)V'(x) - \alpha V(x) = 0$$

with the boundary conditions

$$(3.5) \quad \begin{aligned} f(0) &= 1; & g(0) &= 0; \\ f'(0) &= 0; & g'(0) &= 1; \end{aligned}$$

then a general solution of (3.1) can be written as

$$(3.6) \quad V(x) = C_1f(x) + C_2g(x) - 2 \int_0^x \varphi(x, s) \frac{h(s)}{\sigma^2(s)} ds,$$

where $\varphi(\cdot, s)$ is the solution of (3.4) for $x \geq s$, satisfying

$$(3.7) \quad \varphi(s, s) = 0; \quad \varphi_x(s, s) = 1$$

(cf. [3]). Clearly, the existence and uniqueness of the solution to the boundary problem (3.1), (3.2) for given $L < B$ is equivalent to the fact that

$$(3.8) \quad \begin{vmatrix} f'(L) & g'(L) \\ f'(B) & g'(B) \end{vmatrix} = f'(L)g'(B) - g'(L)f'(B) \neq 0.$$

Let $\Psi(x) = f'(L)g(x) - g'(L)f(x)$; then Ψ is a solution to (3.4) with $\Psi'(L) = 0$. So it follows from the following lemma quoted from Shreve [12] that $\Psi'(B) \neq 0$, i.e., (3.8) holds. (We outline the proof of this lemma in the Appendix for the benefit of the reader.)

LEMMA 3.1. *Suppose that $\alpha > |a|$ and let V be a nonconstant solution to (3.4) defined on some interval $[L, B]$; then*

- (a) *If V has a zero in $[L, B]$, then V' has no zero in $[L, B]$;*
- (b) *If $V'(\bar{x}) = 0$ for some $\bar{x} \in [L, B]$, then $(x - \bar{x})V(x)V'(x) > 0$, for all $x \in [L, B]$ such that $x \neq \bar{x}$.*

We can now write the explicit fomula for C_1, C_2 to solve the boundary problem (3.1), (3.2) for given $L < B$, as follows:

$$(3.9) \quad C_1 = \frac{1}{\Delta} \det \begin{bmatrix} 2I_1(L) - 1 & g'(L) \\ 2I_1(B) + 1 & g'(B) \end{bmatrix}, \quad C_2 = \frac{1}{\Delta} \det \begin{bmatrix} f'(L) & 2I_1(L) - 1 \\ f'(B) & 2I_1(B) + 1 \end{bmatrix},$$

where $I_1(x) = \int_0^x \varphi_x(x, s)(h(s)/\sigma^2(s))ds$, and

$$\Delta = \det \begin{bmatrix} f'(L) & g'(L) \\ f'(B) & g'(B) \end{bmatrix}.$$

We will henceforth denote, for given $L < B$, the solution to (3.1), (3.2) by $V_{L,B}$. (Recall from Remark 3.1 that it is actually defined on \mathbf{R} and is of class C^4 .) The following lemmas give the crucial properties of such solutions.

LEMMA 3.2. *Let $V_{L,B}$ be the solution to (3.1), (3.2) on some interval $[L, B] \subset \mathbf{R}$; then the following statements are equivalent:*

- (1) $V_{L,B}$ is convex on $[L, B]$;
- (2) $|V'_{L,B}(x)| \leq 1$ for all $x \in [L, B]$;
- (3) $V''_{L,B}(L) \geq 0, V''_{L,B}(B) \geq 0$.

Proof. We denote $V = V_{L,B}$.

(1) \implies (2). If V is convex, then V' is increasing, so the boundary condition (3.2) gives $|V'(x)| \leq 1$ for all $x \in [L, B]$.

(2) \implies (3). This is obvious by (3.2).

(3) \implies (1). We prove that $V''(x) \geq 0$ for all $x \in [L, B]$. Differentiating both sides of (3.1) twice and letting $W = V''$, we have that

$$0 = \frac{1}{2}\sigma^2(x)W''(x) + [(ax + b) + (\sigma^2(x))']W'(x) + (2a + \frac{1}{2}(\sigma^2(x))'' - \alpha)W(x) + h''(x).$$

Let $c(x) = 2a + \frac{1}{2}(\sigma^2(x))'' - \alpha$ and

$$(LW)(x) = \frac{1}{2}\sigma^2(x)W''(x) + [(ax + b) + (\sigma^2(x))']W'(x) + c(x)W(x);$$

then we have that $LW = -h'' < 0$ by (A2) and $c < 0$ by (A3). Therefore, by the *maximum principle* (cf. [4]), $W = V''$ has no negative minimum on $[L, B]$. Thus $V''(x) \geq 0$ on $[L, B]$, since otherwise V'' must have a negative minimum by the assumption. The proof is now completed. \square

Note that assumption (A2) implies that there exists a unique pair of real numbers $-\infty < r_1 < 0 < r_2 < \infty$ with $|h'(r_1)| = |h'(r_2)| = \alpha - a$ such that $|h'(x)| < \alpha - a$ for all $x \in (r_1, r_2)$ (see also Remark 2.1). We have the following lemma.

LEMMA 3.3. *Suppose that $[L, B] \subseteq \mathbf{R}$ and $V_{L,B}$ are the same as those in Lemma 3.2 and suppose that $V_{L,B}$ is convex on $[L, B]$; then*

- (i) $V''_{L,B}(B) = 0 \implies h'(B) \geq \alpha - a > 0$ and $B \geq r_2 > 0$;
- (ii) $V''_{L,B}(L) = 0 \implies h'(L) \leq -(\alpha - a) < 0$ and $L \leq r_1 < 0$.

Proof. We only prove (i) (the proof of (ii) is similar). Again, denote $V = V_{L,B}$; as already observed, $V'''(x)$ exists for all x and satisfies

$$\frac{1}{2}\sigma^2(x)V'''(x) + [(ax + b) + \sigma(x)\sigma'(x)]V''(x) + (a - \alpha)V'(x) + h'(x) = 0.$$

Now, letting $x \nearrow B$, we get that

$$(3.10) \quad \frac{1}{2}\sigma^2(B)V'''(B) + (a - \alpha) + h'(B) = 0.$$

Since $V''(x) \geq 0$ for all $x \in [L, B]$ and $V''(B) = 0$, we get that $V'''(B) \leq 0$. Hence the result follows from (3.10), condition (2.9), and the definition of r_2 . \square

We now give a lemma concerning the continuous dependence of the solution on the boundary data. It is easily seen that $C_1 = C_1(L, B), C_2 = C_2(L, B)$ given by (3.9) are continuous functions of L and B for $L < B$, since $\Delta \neq 0$ for all $L < B$. However, it is not clear that, if $V_{L,B}$ is convex on some $[L, B]$, then $V_{L',B'}$ should also be convex on $[L', B']$ for those L' close to L and B' close to B . We have the following lemma.

LEMMA 3.4. *Suppose that, for some $-\infty < L < B < \infty$, the function $V_{L,B}$ is convex on $[L, B]$. If $V_{L,B}''(B) > 0$ (respectively, $V_{L,B}''(L) > 0$), then, for any $\epsilon > 0$, there exist $L', B' \in \mathbf{R}$ with $B < B'; L \leq L' < L + \epsilon < B$ (respectively, $L' < L; L < B - \epsilon < B' \leq B$), such that $V_{L',B'}$ is convex on $[L', B']$.*

Proof. Since $V_{L,B}''(B) > 0$, by the continuity of C_1, C_2 in L, B , we can find that $\delta_0 > 0$ such that the solution $V_{L,B+\delta}$ satisfies $V''_{L,B+\delta}(B + \delta) > 0$ for $0 < \delta < \delta_0$. We may assume that $\delta_0 < \epsilon < 1$.

Now pick $x_0 \in (L, L + \epsilon)$ such that $V'_{L,B}(x_0) > -1$. Such an x_0 always exists; otherwise, by part (2) of Lemma 3.2, $V'_{L,B}(x) \equiv -1$ on $[L, L + \epsilon)$, which implies that $V_{L,B}$ is linear on $[L, L + \epsilon)$. Then, however, $h(x) = \alpha V_{L,B}(x) + (ax + b)$, $x \in [L, L + \epsilon)$ is also linear. This contradicts (2.8).

Let $\epsilon_1 = V'_{L,B}(x_0) + 1 > 0$. Note that the solution family $\{V_{L,B+\delta}, 0 \leq \delta < \delta_0\}$ are all defined on \mathbf{R} , so the explicit form of the solution (3.6), (3.9) and the continuity of $V''_{L,B}$ shows that we can choose $0 < \delta_1 < \delta_0$ such that

$$|V'_{L,B+\delta_1}(x) - V'_{L,B}(x)| < \epsilon_1, \quad x \in [L, B + 1]$$

and

$$V''_{L,B}(x) > 0, \quad x \in [B, B + \delta_1],$$

since $V''_{L,B}(B) > 0$. It follows that $V'_{L,B}(x) \geq 1$ for all $x \in [B, B + \delta_1]$ and therefore $V'_{L,B+\delta_1}(x) > -1$ for all $x \in [x_0, B + \delta_1]$. Let

$$L' = \inf\{u \geq L : V'_{L,B+\delta_1}(x) > -1, u \leq x \leq B + \delta_1\};$$

then $L \leq L' < x_0 < L + \epsilon$.

It is easily seen from the definition of L' that we must have that $V''_{L',B+\delta_1}(L') \geq 0$ and $V'_{L',B+\delta_1}(L') = -1$. By Lemma 3.2, $V_{L',B+\delta_1}$ is convex on $[L', B + \delta_1]$. Therefore, with $B' = B + \delta_1$, the solution $V_{L',B'}$ is just what we want. The case when $V''_{L,B}(L) > 0$ is similar, so we are done. \square

The next question is: When does a convex solution $V_{L,B}$ satisfying (3.1), (3.2) exist? We can prove the following lemma.

LEMMA 3.5. *For any $[L, B] \subset \mathbf{R}$, there exist $L \leq L' < B' \leq B$ such that $V_{L',B'}$ is convex on $[L', B']$.*

Proof. Let $V_{L,B}$ be the solution to (3.1), (3.2) on $[L, B]$. Define

$$B' = \sup\{u : V'_{L,B}(x) \leq 1, L \leq x \leq u\} \wedge B;$$

$$L' = \inf\{u : V'_{L,B}(x) \geq -1, u \leq x \leq B'\} \vee L.$$

By (3.2) and the continuity of $V'_{L,B}$, it is easily seen that $L \leq L' < B' \leq B$; $V'_{L,B}(L') = -1$, $V'_{L,B}(B') = 1$ and $|V'_{L,B}(x)| \leq 1$ for $L' \leq x \leq B'$.

Replacing L by L' and B by B' , we obtain a solution $V_{L',B'}$, which is convex on $[L', B']$ by Lemma 3.2. \square

By Lemma 3.5, we see that, for any $[L, B] \in \mathbf{R}$, the solution $V_{L,B}$ to (3.1), (3.2) has a *convex portion*, which also satisfies (3.2). We concentrate on the totality of such convex portions. Define, for each $x_0 \in \mathbf{R}$,

$$(3.11) \quad \mathcal{A}_{x_0} = \{[L, B] : x_0 \in (L, B) \text{ and there exists a } V_{L,B} \text{ convex on } [L, B]\}.$$

Apparently, for r_1, r_2 defined as those in Lemma 3.3 (and the argument preceding it), there exists $r_1 < x_0 < r_2$ such that $\mathcal{A}_{x_0} \neq \emptyset$. Denote $\mathcal{A} = \mathcal{A}_{x_0}$. (As we see in §4, we may actually take $x_0 = 0$.) We find a unique $[L^*, B^*] \in \mathcal{A}$ such that the corresponding V_{L^*,B^*} satisfies (3.3).

The following lemma is a basic property of \mathcal{A} . We endow a partial order “ \prec ” on \mathcal{A} by usual inclusion; i.e.,

$$[L, B] \prec [L', B'] \iff [L, B] \subseteq [L', B'].$$

LEMMA 3.6. (a) \mathcal{A} is “closed” in the following sense: if $\{[L_n, B_n]\} \subseteq \mathcal{A}$ such that $L_n \rightarrow L; B_n \rightarrow B$, and $-\infty < L < x_0 < B < \infty$, then $[L, B] \in \mathcal{A}$.

(b) Every totally ordered subset of \mathcal{A} has an upper bound.

Proof. (a) Let $\{[L_n, B_n]\} \subseteq \mathcal{A}$ such that $L_n \rightarrow L; B_n \rightarrow B$ for some $-\infty < L < x_0 < B < \infty$. Let C_1^n, C_2^n be the constants in (3.6) with respect to V_{L_n, B_n} determined by (3.11) (with corresponding Δ^n); then it is easily seen that there exist some C_1, C_2 , and Δ such that $C_i^n \rightarrow C_i, i = 1, 2$, and, $\Delta^n \rightarrow \Delta$, since $L < B$. It can then be checked that C_1, C_2 determine a solution $V_{L, B}$ to (3.1), (3.2) on $[L, B]$ via (3.6) such that $V''_{L, B}(L) \geq 0$ and $V''_{L, B}(B) \geq 0$, since $V''_{L_n, B_n}(L_n) \geq 0$ and $V''_{L_n, B_n}(B_n) \geq 0$ for every n . Therefore $V''_{L, B}$ is convex on $[L, B]$ by Lemma 3.2; i.e. $[L, B] \in \mathcal{A}$.

(b) Let $\{[L_\lambda, B_\lambda] : \lambda \in \Lambda\}$ be a totally ordered subset of \mathcal{A} ; then there exist $L < B$ such that $(L, B) = \cup_\lambda (L_\lambda, B_\lambda)$.

It can be proved that $-\infty < L < B < \infty$ (we defer the proof to next section, Lemma 4.2). Moreover, since, for each $\lambda, L_\lambda < x_0 < B_\lambda$, then $L < x_0 < B$.

If Λ is a finite set or $\{[L_\lambda, B_\lambda]\}$ has a maximum element, then there is nothing to prove. So assume that Λ is infinite and that there is no maximum element in the family. Then we can find a sequence

$$[L_1, B_1] \subseteq [L_2, B_2] \subseteq \dots$$

such that $L_n \searrow L, B_n \nearrow B$. By part (a), $[L, B] \in \mathcal{A}$. It is clear that $[L, B]$ is the upper bound of the family $\{[L_\lambda, B_\lambda]\}$. \square

Now, by Lemma 3.6 and Zorn’s lemma, we see that \mathcal{A} has a maximal element. We should note that the maximal element is not unique, since \mathcal{A} is only a partially ordered set. However, we may now define a subset of \mathcal{A} as follows:

$$(3.12) \quad \mathcal{A}_{\max} = \{ \text{all maximal elements in } \mathcal{A} \}.$$

The previous argument shows that $\mathcal{A}_{\max} \neq \emptyset$. We are mostly interested in this set later.

To end this section, we present a simple but important property of \mathcal{A}_{\max} .

LEMMA 3.7. For any $[L, B] \in \mathcal{A}_{\max}$, we have that

$$(3.13) \quad V''_{L, B}(L) \cdot V''_{L, B}(B) = 0.$$

Proof. First, note that, for any $[L, B] \in \mathcal{A}$, we have that $V''_{L, B}(L) \geq 0; V''_{L, B}(B) \geq 0$. So, if the conclusion is not true, then we can find an $[L, B] \in \mathcal{A}_{\max} \subseteq \mathcal{A}$ such that

$$V''_{L, B}(L) > 0; \quad V''_{L, B}(B) > 0.$$

Then, by the continuous dependence of the solution on L, B , we can find an $\epsilon > 0$, so that $V_{L-\epsilon, B+\epsilon}$ exists on $[L - \epsilon, B + \epsilon]$ and satisfies

$$V''_{L-\epsilon, B+\epsilon}(L - \epsilon) \geq 0; \quad V''_{L-\epsilon, B+\epsilon}(B + \epsilon) \geq 0.$$

So Lemma 3.2 implies that $V_{L-\epsilon, B+\epsilon}$ is convex, and then $[L - \epsilon, B + \epsilon] \in \mathcal{A}$, since $x_0 \in (L, B) \subset (L - \epsilon, B + \epsilon)$, but this contradicts the maximality of $[L, B]$. \square

4. Main theorems. In this section, we give our main results. The first theorem is relatively simple, but we still prove it for completeness. The remainder of the section is devoted to the second theorem, which is more involved. We prove that the

principle of smooth fit always holds under our setting, and then the first theorem leads to the existence of the optimal control. Finally, we give a brief description of the optimal reflecting barriers.

First, let $[L, B] \subset \mathbf{R}$ and let $X^x(\cdot)$ denote the diffusion process starting at $x \in [L, B]$, satisfying

$$dX^x(t) = (aX^x(s) + b)ds + \sigma(X^x(s))dW(s),$$

with reflection at L and B . Then, following §23 in [5], we have two adapted, continuous, nondecreasing processes $\xi_L(\cdot)$ and $\xi_B(\cdot)$, which are zero at $t = 0$, such that, for all $t \geq 0$,

$$(4.1) \quad X^x(t) = x + \int_0^t (aX^x(s) + b)ds + \int_0^t \sigma(X^x(s))dW(s) + \xi_L(t) - \xi_B(t)$$

and

$$\xi_L(t) = \int_0^t 1_{\{X(s)=L\}}d\xi_L(s), \quad \xi_B(t) = \int_0^t 1_{\{X(s)=B\}}d\xi_B(s).$$

Denote such solution by $X_{\xi}^x(\cdot)$. Let f be a solution to (3.1) on $[L, B]$ and let $F(t, x) = e^{-\alpha t}f(x)$. Applying Itô's formula to the function F , we obtain that

$$(4.2) \quad \begin{aligned} f(x) = & f'(B)E \int_0^\infty e^{-\alpha t}d\xi_B(t) - f'(L)E \int_0^\infty e^{-\alpha t}d\xi_L(t) \\ & + E \int_0^\infty e^{-\alpha t}h(X_{\xi_{L,B}}^x(t))dt, \end{aligned}$$

where $\xi_{L,B} = \xi_L - \xi_B$. (See also [13, Lemma 2.1]. Note that it was also proved there that both $E \int_0^\infty e^{-\alpha t}d\xi_B(t)$ and $E \int_0^\infty e^{-\alpha t}d\xi_L(t)$ are finite.) If $V_{L,B}$ is a solution to (3.1), (3.2), then, with $f = V_{L,B}$, (4.2) becomes

$$(4.3) \quad V_{L,B}(x) = E \int_0^\infty e^{-\alpha t}[d\check{\xi}_{L,B}(t) + h(X_{\xi_{L,B}}^x(t))dt].$$

Namely, $\xi_{L,B}$ yields the cost function $V_{L,B}$. We now state our main theorems.

THEOREM 4.1. *Suppose that there exists $[L^*, B^*] \subset \mathbf{R}$ and a solution V_{L^*,B^*} to (3.1)–(3.3) on $[L^*, B^*]$; then*

$$(4.4) \quad V^*(x) = \begin{cases} (L^* - x) + V_{L^*,B^*}(L^*), & x < L^*; \\ V_{L^*,B^*}(x), & L^* \leq x \leq B^*; \\ (x - B^*) + V_{L^*,B^*}(B^*), & x > B^* \end{cases}$$

is the value function, and the optimal control is given by $\xi_x^* = \{\xi_x^*(t) : t \geq 0\}$ satisfying $\xi_x^*(0) = 0$, and, for $t > 0$,

$$(4.5) \quad \xi_x^*(t) = \begin{cases} (L^* - x) + \xi_{L^*}(t) - \xi_{B^*}(t), & x < L^*; \\ \xi_{L^*}(t) - \xi_{B^*}(t), & L^* \leq x \leq B^*; \\ (B^* - x) + \xi_{L^*}(t) - \xi_{B^*}(t), & x > B^*. \end{cases}$$

It is obvious that ξ_x^* is left-continuous, and, for $x < L^*$ ($x > B^*$), ξ_x^* has an initial jump, which makes the process $X^* = X_{\xi_x^*}^x(\cdot)$ jump to L^* (B^*) and then proceeds as a reflected diffusion on $[L^*, B^*]$. This is just the usual idea used by many authors (cf. [1], [6], [8], [13]). Observe also that Theorem 4.1 depends heavily on the existence of the interval $[L^*, B^*]$ and the corresponding convex solution V_{L^*, B^*} . In some cases, the nonexistence of such an interval leads to the nonexistence of the optimal policy (cf. Shreve, Lehoczky, and Gaver [13]). However, the next theorem gives an affirmative answer to the question of the existence of such interval in our setting as well as the existence of the convex C^2 -solution to the variational inequality (2.11).

THEOREM 4.2. *Let assumptions (A1)–(A3) hold. Then there exists a unique interval $[L^*, B^*] \subset \mathbf{R}$ on which there exists a unique, convex solution of (3.1)–(3.3). Furthermore, the variational inequality (2.11) admits a unique convex C^2 -solution, which gives the value function of the control problem (2.1)–(2.3).*

Remark. By setting $a = 0$, $\sigma(\cdot) \equiv \sigma$ (constant), we see that our result contains the corresponding one in [8] as a special case.

Proof of Theorem 4.1. It is readily seen that the control ξ_x^* yields the cost function V^* defined by (4.4), so we need only show that V^* is the optimal cost.

Since $\xi^* \in \mathcal{B}$, we have that

$$(4.6) \quad V^*(x) \geq \inf_{\xi \in \mathcal{B}} V_\xi(x), \quad x \in \mathbf{R}.$$

On the other hand, by the assumption of the theorem, we see that $V^* \in C^2(\mathbf{R})$ and

$$(4.7) \quad V^{*'}(x) = \begin{cases} -1, & x < L^*; \\ V'_{L^*, B^*}(x), & L^* \leq x \leq B^*; \\ 1, & x > B^*; \end{cases}$$

$$(4.8) \quad V^{*''}(x) = \begin{cases} 0, & x < L^* \text{ or } x > B^*; \\ V''_{L^*, B^*}(x), & L^* \leq x \leq B^*. \end{cases}$$

By Lemma 3.2, (2.12) and (2.13) are satisfied. We now verify (2.14). If $x \in [L^*, B^*]$, there is nothing to prove. Let $x > B^*$; then, by the definition of V^* , we have that

$$(4.9) \quad \alpha V^*(x) = \alpha(x - B^*) + \alpha V_{L^*, B^*}(B^*).$$

Since V_{L^*, B^*} is a solution of (3.1)–(3.3), we have, at $x = B^*$, that

$$(4.10) \quad \alpha V_{L^*, B^*}(B^*) = \frac{1}{2}\sigma^2(B^*)V^{*''}(B^*) + (aB^* + b)V^{*'}(B^*) + h(B^*) = (aB^* + b) + h(B^*).$$

Thus (4.9) becomes $\alpha V^*(x) = \alpha(x - B^*) + (aB^* + b) + h(B^*)$. Therefore a simple computation shows that

$$(4.11) \quad \alpha V^*(x) \leq (ax + b)V^{*'}(x) + \frac{1}{2}\sigma^2(x)V^{*''}(x) + h(x), \quad x > B^*$$

is equivalent to

$$(4.12) \quad (\alpha - a) \leq \frac{h(x) - h(B^*)}{x - B^*}, \quad x > B^*.$$

Since h is strictly convex, h' is increasing. Therefore, for any $x > B^*$,

$$\frac{h(x) - h(B^*)}{x - B^*} = h'(\theta) > h'(B^*) \geq \alpha - a,$$

where $\theta \in (B^*, x)$ and the last inequality is due to Lemma 3.3 (i). Thus we have proved (2.14) for $x > B^*$. The case when $x < L^*$ is similar, so (2.14) is verified. By Theorem 2.1, we have that

$$V^*(x) \leq \inf_{\xi \in \mathcal{B}} V_\xi(x), \quad x \in \mathbf{R}.$$

The proof is now complete. \square

For the proof of Theorem 4.2, we first prove some lemmas. Our purpose here is to find an interval $[L^*, B^*] \subseteq \mathbf{R}$ satisfying the conditions of Theorem 4.1. The candidate is chosen from the set \mathcal{A}_{\max} defined by (3.12). We now take a closer look at the sets $\mathcal{A}, \mathcal{A}_{\max}$ defined by (3.11), (3.12).

Define

$$\begin{aligned} \mu &= \sup\{B : \exists [L, B] \in \mathcal{A}\}; \\ \nu &= \inf\{L : \exists [L, B] \in \mathcal{A}\}. \end{aligned}$$

Then we have the following lemma.

LEMMA 4.3. *It holds that*

$$(4.13) \quad -\infty < \nu < x_0 < \mu < \infty,$$

where x_0 is such that $x_0 \in (r_1, r_2)$ and $\mathcal{A} = \mathcal{A}_{x_0} \neq \emptyset$.

Proof. $\nu < x_0 < \mu$ is obvious by the definition of \mathcal{A} ; the proof of the first inequality is the same as that of the last one, so we only prove $\mu < \infty$.

Suppose not; then there exists a sequence $\{[L_n, B_n]\}_{n=0}^\infty \subseteq \mathcal{A}$ such that $B_n \nearrow \infty$. Therefore we can choose a $\delta > 0$ such that $[x_0, x_0 + \delta] \subseteq [L_n, B_n]$ for all $n > 0$. Since, for each n , V_{L_n, B_n} is the convex solution to (3.1), (3.2) on $[L_n, B_n]$, by Theorem 2.3, with the argument in the beginning of this section and part (1) of Remark 2.1, we have that

$$0 \leq V_{L_n, B_n}(x) = \inf_{\xi \in \mathcal{B}_{[L_n, B_n]}} V_\xi(x), \quad n \geq 0, \quad x \in [L_n, B_n].$$

Since $\mathcal{B}_{[x_0, x_0 + \delta]} \subset \mathcal{B}_{[L_n, B_n]}$, for all n , we also have that

$$\inf_{\xi \in \mathcal{B}_{[L_n, B_n]}} V_\xi(x) \leq \inf_{\xi \in \mathcal{B}_{[x_0, x_0 + \delta]}} V_\xi(x) \quad n \geq 0; \quad x \in [x_0, x_0 + \delta].$$

Therefore

$$0 \leq V_{L_n, B_n}(x) \leq \inf_{\xi \in \mathcal{B}_{[x_0, x_0 + \delta]}} V_\xi(x), \quad x \in [x_0, x_0 + \delta].$$

In particular, we have that $0 \leq V_{L_n, B_n}(x_0) \leq v = \inf_{\xi \in \mathcal{B}_{[x_0, x_0 + \delta]}} V_\xi(x_0)$. Since $|V'_{L_n, B_n}(x)| \leq 1$ for $x \in [L_n, B_n]$, we get that

$$(4.14) \quad |V_{L_n, B_n}(x)| \leq v + |x - x_0|, \quad x \in [L_n, B_n].$$

However, the convexity of V_{L_n, B_n} (i.e., $V''_{L_n, B_n} \geq 0$) on $[L_n, B_n]$ gives

$$\begin{aligned} \alpha V_{L_n, B_n}(x) &= (ax + b)V'_{L_n, B_n}(x) + \frac{1}{2}\sigma^2 V''_{L_n, B_n}(x) + h(x) \\ &\geq (ax + b)V'_{L_n, B_n}(x) + h(x), \quad x \in [L_n, B_n]. \end{aligned}$$

Hence, by (4.13) and the fact that $|V'_{L_n, B_n}(x)| \leq 1$ for $x \in [L_n, B_n]$, for some $K > 0$,

$$0 \leq h(x) \leq K(1 + |x|), \quad x \in [L_n, B_n].$$

This is impossible, since $B_n \nearrow \infty$ and h is of at least quadratic growth by (2.8). The contradiction shows that $\mu < \infty$. Similarly, we have that $\nu > -\infty$. This completes the proof. \square

Now define

$$(4.15) \quad \mathcal{A}_1 = \{[L, B] \in \mathcal{A}_{\max} : V''_{L, B}(B) = 0\};$$

$$(4.16) \quad \mathcal{A}_2 = \{[L, B] \in \mathcal{A}_{\max} : V''_{L, B}(L) = 0\}.$$

By Lemma 3.7, $\mathcal{A}_{\max} = \mathcal{A}_1 \cup \mathcal{A}_2$.

LEMMA 4.4. *It holds that $\mathcal{A}_1 \neq \emptyset$, $\mathcal{A}_2 \neq \emptyset$.*

Proof. Since $\mathcal{A}_{\max} \neq \emptyset$, one of \mathcal{A}_1 or \mathcal{A}_2 must be nonempty. Suppose that $\mathcal{A}_2 \neq \emptyset$, but $\mathcal{A}_1 = \emptyset$; then $\mathcal{A}_{\max} = \mathcal{A}_2$, and, for any $[L, B] \in \mathcal{A}_{\max}$, we must have that $V''_{L, B}(L) = 0; V''_{L, B}(B) > 0$. Let

$$b = \sup\{B : \exists [L, B] \in \mathcal{A}_{\max}\};$$

then $b \leq \mu < \infty$, and there exists a sequence $\{[L_n, B_n]\} \subseteq \mathcal{A}_{\max}$ such that $B_n \nearrow b$. Since $\nu \leq L_n < x_0$ for any n , along a subsequence (may assume itself), we have that $L_n \rightarrow l$ for some $l \leq x_0$. Observe that if $V''_{L_n, B_n}(L_n) = 0$ for all n , we must have that $V''_{l, b}(l) = 0$. By part (ii) of Lemma 3.3 and part (a) Lemma 3.6, $V_{l, b} \in \mathcal{A}$.

Now let $[L^*, B^*]$ be a maximum element containing $[l, b]$; then we must have that $B^* = b$ and $V''_{L^*, B^*}(B^*) = 0$ by the definition of b and Lemma 3.4. However, this contradicts $\mathcal{A}_1 = \emptyset$. The similar argument shows that $\mathcal{A}_1 \neq \emptyset$, but $\mathcal{A}_2 = \emptyset$ is also impossible; so the lemma is proved. \square

It is now clear that we may succeed in proving Theorem 4.2 if we can find an element in $\mathcal{A}_1 \cap \mathcal{A}_2$. To this end, we need the following lemma.

LEMMA 4.5. *Suppose that $[L, B], [L', B'] \in \mathcal{A}_{\max}$ such that $L \leq L' < B \leq B'$ and $V''_{L, B}(B) = V''_{L', B'}(L') = 0$; then $L = L'; B = B'$. Consequently, $[L, B] = [L', B'] \in \mathcal{A}_1 \cap \mathcal{A}_2$.*

Proof. First, note that either $L < L', B < B'$ or $L = L', B = B'$ must hold, since both intervals are maximal elements. So we only must prove that the first case is impossible.

Suppose that $L < L', B < B'$. We show that this leads to a contradiction. Let

$$V(x) = \begin{cases} V_{L, B}(x), & x \leq B; \\ (x - B) + V_{L, B}(B), & B \leq x. \end{cases}$$

Then we have that $V \in C^2$, since $V''_{L, B}(B) = 0$. By Lemma 3.3 (ii), we have that $h'(B) \geq \alpha - a$, which leads to

$$\alpha V(x) \leq (ax + b)V'(x) + \frac{1}{2}\sigma^2(x)V''(x) + h(x), \quad x \in \mathbf{R}$$

following the argument in the proof of Theorem 4.1. Clearly, $V(x)$ satisfies (2.11), (2.12) on $[L, B']$; therefore, by Theorem 2.3, we obtain that

$$V(x) \leq V_\xi(x); \quad \xi \in \mathcal{B}_{[L, B']}, \quad x \in [L, B'].$$

Now, for $x \in [L', B'] \subset [L, B']$, let $\xi = \xi_{L'} - \xi_{B'} \in \mathcal{B}_{[L', B']} \subseteq \mathcal{B}_{[L, B']}$, where $\xi_{L'}, \xi_{B'}$ are defined in the beginning of this section. Then we see that $V_\xi(x) = V_{L', B'}(x) = E \int_0^\infty e^{-\alpha t} [d\check{\xi}(t) + h(X_\xi^x(t))dt]$. So we get that $V(x) \leq V_{L', B'}(x)$, for all $x \in [L', B']$. In particular, we have that

$$V_{L, B}(x) \leq V_{L', B'}(x), \quad x \in [L', B'] \cap [L, B].$$

Similarly, replacing V by the function

$$V(x) = \begin{cases} (L' - x) + V_{L', B'}(L'), & x \leq L'; \\ V_{L', B'}(x), & L' \leq x, \end{cases}$$

we can also show that

$$V_{L', B'}(x) \leq V_{L, B}(x), \quad x \in [L', B'] \cap [L, B],$$

which gives $V_{L, B} \equiv V_{L', B'}$ on $[L', B'] \cap [L, B] = [L', B] \supseteq [r_1, r_2]$ by Lemma 3.3. Hence the uniqueness of the solution to the Cauchy problem of the ODE implies that $V_{L, B} \equiv V_{L', B'}$, but this implies that $[L, B'] \in \mathcal{A}$, contradicting the maximality of $[L', B']$ and $[L, B]$. \square

Define

$$(4.17) \quad \mathbf{B}_1 = \inf\{B : \exists [L, B] \in \mathcal{A}_1\};$$

$$(4.18) \quad \mathbf{L}_2 = \sup\{L : \exists [L, B] \in \mathcal{A}_2\}.$$

Then there exists a sequence $\{[L_n, B_n]\} \subseteq \mathcal{A}_{\max}$ such that $B_n \searrow \mathbf{B}_1$. Since every $[L_n, B_n]$ is a maximal element, $\{L_n\}$ must also be decreasing, and is bounded below by μ . Therefore $L_n \searrow L_1$ for some $L_1 < x_0$. By Lemma 3.6 (a), we have that $[L_1, \mathbf{B}_1] \in \mathcal{A}$. Let $[L_1, \mathbf{B}_1]$ be the maximal element in \mathcal{A} containing $[L_1, \mathbf{B}_1]$; we claim that $\mathbf{B}_1 = \mathbf{B}_1$. Indeed, suppose that $\mathbf{B}_1 > \mathbf{B}_1$; then, for n large enough, we should have that $L_n > L_1 \geq L_1; \mathbf{B}_1 < B_n < \mathbf{B}_1$, namely, $[L_n, B_n]$ is properly contained in $[L_1, \mathbf{B}_1]$. This contradicts their maximalities. Therefore we may now write $[L_1, \mathbf{B}_1]$ as the maximal element containing $[L_1, \mathbf{B}_1]$.

Similarly, we can find a $\mathbf{B}_2 > \mathbf{L}_2$ such that $[L_2, \mathbf{B}_2]$ is a maximal element. The following lemma is final.

LEMMA 4.6. *It holds that $[\mathbf{L}_1, \mathbf{B}_1] = [\mathbf{L}_2, \mathbf{B}_2]$.*

Proof. Since $[\mathbf{L}_1, \mathbf{B}_1]$ and $[\mathbf{L}_2, \mathbf{B}_2]$ are both maximal, we either have Case 1: $\mathbf{L}_1 < \mathbf{L}_2, \mathbf{B}_1 < \mathbf{B}_2$ or Case 2: $\mathbf{L}_1 > \mathbf{L}_2, \mathbf{B}_1 > \mathbf{B}_2$, if they are not identical.

Suppose that $\mathbf{L}_1 < \mathbf{L}_2$ and $\mathbf{B}_1 < \mathbf{B}_2$; then, by the definition of \mathbf{B}_1 , we can find that $[L, B] \in \mathcal{A}_1 \subseteq \mathcal{A}_{\max}$ such that $\mathbf{B}_1 \leq B < \mathbf{B}_2$. By the maximality of $[\mathbf{L}_1, \mathbf{B}_1]$ and $[\mathbf{L}_2, \mathbf{B}_2]$, we must have that $\mathbf{L}_1 \leq L < \mathbf{L}_2$. Similarly, by the definition of \mathbf{L}_2 , we can now find that $[L', B'] \in \mathcal{A}_2 \subseteq \mathcal{A}_{\max}$ such that $L < L' \leq \mathbf{L}_2$, and then $B < B' \leq \mathbf{B}_2$ because $[L, B]$ is also maximal. Now, by the definition of \mathcal{A}_1 and \mathcal{A}_2 , we have that $V''_{L', B'}(L') = 0, V''_{L, B}(B) = 0$, which contradicts Lemma 4.5. So Case 1 is impossible.

Suppose that $\mathbf{L}_1 > \mathbf{L}_2$ and $\mathbf{B}_1 > \mathbf{B}_2$. Let $V_1 = V_{\mathbf{L}_1, \mathbf{B}_1}; V_2 = V_{\mathbf{L}_2, \mathbf{B}_2}$. By the definitions of \mathbf{B}_1 and \mathbf{L}_2 , we must have that $V_1''(\mathbf{L}_1) > 0; V_2''(\mathbf{B}_2) > 0$. Then,

by Lemma 3.4, however, there exists a $[L, B] \in \mathcal{A}$ such that $\mathbf{L}_2 < L < \mathbf{L}_1; \mathbf{B}_2 < B < \mathbf{B}_1$. Let $[\hat{L}, \hat{B}]$ be the maximal element containing $[L, B]$; then it is easily seen that $\mathbf{L}_1 > \hat{L} > \mathbf{L}_2$ and $\mathbf{B}_1 > \hat{B} > \mathbf{B}_2$ still hold, since $[\mathbf{L}_i, \mathbf{B}_i], i = 1, 2$ and $[\hat{L}, \hat{B}]$ are all maximal elements. Now, however, by the definition of \mathbf{B}_1 and \mathbf{L}_1 , we must have that $V''_{\hat{L}, \hat{B}}(\hat{B}) > 0$, since $\hat{B} < \mathbf{B}_1$, and $V''_{\hat{L}, \hat{B}}(\hat{L}) > 0$ since $\hat{L} > \mathbf{L}_2$. Therefore $V''_{\hat{L}, \hat{B}}(\hat{L}) \cdot V''_{\hat{L}, \hat{B}}(\hat{B}) > 0$, which contradicts Lemma 3.7. So Case 2 is also impossible. Namely, $[\mathbf{L}_1, \mathbf{B}_1]$ and $[\mathbf{L}_2, \mathbf{B}_2]$ must be identical. \square

Proof of Theorem 4.2. Let $L^* = \mathbf{L}_1 = \mathbf{L}_2; B^* = \mathbf{B}_1 = \mathbf{B}_2$ and let $V^*(x) = V_{L^*, B^*}(x)$, $x \in [L^*, B^*]$. We first prove that V^* is the convex solution of (3.1)–(3.3) on $[L^*, B^*]$.

That V^* is the convex solution of (3.1), (3.2) is clear by the definition of $[L^*, B^*]$. So we must only verify (3.3). By Lemma 3.8, we have that $V^{**}(L^*) \cdot V^{**}(B^*) = 0$. We assume that $V^{**}(L^*) = 0$. The convexity of V^* implies that $V^{**}(B^*) \geq 0$. Suppose that $V^{**}(B^*) > 0$. Recall that $B^* = \mathbf{B}_1$; hence there exists a sequence $\{[L_n, B_n]\} \subseteq \mathcal{A}_1$ such that $B_n \searrow B^*$, and there exists an $\bar{L} \in (-\infty, x_0]$ such that $L_n \searrow \bar{L}$. If $\bar{L} > L^*$, then Lemma 3.4 and the maximality of $[L^*, B^*]$ allow the existence of an element $[L, B] \in \mathcal{A}_{\max}$ such that $L^* < L < \bar{L}$ and $B^* < B$, which is impossible because then we can find that $[L_n, B_n] \subset [L, B]$ for n large enough, which contradicts the maximality of $[L_n, B_n]$. Therefore $\bar{L} = L^*$, and so $V^{**}(B^*) = \lim_{n \rightarrow \infty} V''_{L_n, B_n}(B_n) = 0$, a contradiction. Thus the existence of the interval $[L^*, B^*]$ is proved.

To see the uniqueness, let $[L^{**}, B^{**}]$ be another such interval. By Lemma 3.3, we have that $x_0 \in (L^{**}, B^{**})$; so $[L^{**}, B^{**}] \in \mathcal{A}$. The same proof as that of Lemma 4.6 shows that neither $L^* < L^{**}; B^* < B^{**}$ nor $L^* > L^{**}; B^* > B^{**}$ is possible. The maximality of $[L^*, B^*]$ shows that it cannot be contained in $[L^{**}, B^{**}]$; so the only possible case is $[L^{**}, B^{**}] \subseteq [L^*, B^*]$. Since $\mathcal{B}_{[L^{**}, B^{**}]} \subseteq \mathcal{B}_{[L^*, B^]}$, we have that

$$V_{L^*, B^*}(x) \leq V_{L^{**}, B^{**}}(x), \quad x \in [L^{**}, B^{**}].$$

On the other hand, let

$$V^{**}(x) = \begin{cases} (L^{**} - x) + V_{L^{**}, B^{**}}(L^{**}), & L^* \leq x \leq L^{**}; \\ V_{L^{**}, B^{**}}(x), & L^{**} \leq x \leq B^{**}; \\ (x - B^{**}) + V_{L^{**}, B^{**}}(B^{**}), & B^{**} \leq x \leq L^*. \end{cases}$$

Then Theorem 2.3 shows that

$$V_{L^{**}, B^{**}}(x) \leq V_{L^*, B^*}(x), \quad x \in [L^{**}, B^{**}].$$

It follows immediately from the uniqueness of the ODE that $V^* \equiv V_{L^{**}, B^{**}}$, and so $L^* = L^{**}; B^* = B^{**}$. Thus the first part of the theorem is proved.

To prove the second part, let $[L^*, B^*], V_{L^*, B^*}$ be those in the first part; then (4.4) in Theorem 4.1 presents a solution to the variational inequality (2.11). So we must only prove the uniqueness.

Let $V(\cdot)$ be any convex C^2 -solution to (2.11); then, for any $x \in \mathbf{R}$, either $|V'(x)| = 1$ or $V(x)$ satisfy (3.1). Let $\mathcal{C} = \{x : |V'(x)| < 1\}$; then V must satisfy (3.1) on \mathcal{C} . So the growth condition of h implies that \mathcal{C} is bounded (see also Lemma 4.3). Moreover, the monotonicity of $V'(\cdot)$ shows that $\mathcal{C} = (L, B)$, where

$$L = \inf\{x : x \in \mathcal{C}\}, \quad B = \sup\{x : x \in \mathcal{C}\};$$

so that $V'(L) = -1$, $V'(B) = 1$. Then, however, we must have that $V'(x) \equiv -1$ for $x \leq L$ and $V'(x) \equiv 1$ for $x \geq B$, which leads to $V''(L) = 0$ and $V''(B) = 0$ because V is C^2 . Therefore, by the first part, $[L, B]$ is unique ($= [L^*, B^*]$), and V must be linear outside $[L, B]$; thus V must be of the form (4.4). This proves the uniqueness, and then the theorem. \square

4.1. A discussion of determining the optimal reflecting barriers. Having worked diligently to get the existence and uniqueness of the optimal reflecting barriers L^* and B^* , we now present a somewhat "explicit" way of determining these two points via a system of (maybe transcendental) equations. The scheme that we use is similar to that in [6].

For any given $-\infty < L < B < \infty$, let ϕ_1, ϕ_2 be two independent solutions to (3.4) satisfying the boundary conditions

$$\begin{aligned} \phi_1'(L) &= 1; & \phi_2'(L) &= 0; \\ \phi_1'(B) &= 0; & \phi_2'(B) &= 1, \end{aligned}$$

and let G be a special solution of (3.1) satisfying the boundary condition $G(L) = G(B) = 0$. Such solutions exist by the argument given in the beginning of §3. Set

$$(4.19) \quad \Psi(x) = G(x) - [1 + G'(L)]\phi_1(x) + [1 - G'(B)]\phi_2(x).$$

We can easily check that Ψ is the solution to (3.1) and (3.2). Differentiating (4.19) twice, using the facts that G satisfies (3.1) and ϕ_1, ϕ_2 satisfy (3.4), and setting $\Psi''(L) = \Psi''(B) = 0$, $a(x) = ax + b$, we get that

$$\begin{aligned} 0 = \Psi''(L) &= \frac{2}{\sigma^2(L)} \{-h(L) + a(L) + \alpha[(1 - G'(B))\phi_2(L) - (1 + G'(L))\phi_1(L)]\}, \\ 0 = \Psi''(B) &= \frac{2}{\sigma^2(B)} \{-h(B) - a(B) + \alpha[(1 - G'(B))\phi_2(B) - (1 + G'(L))\phi_1(B)]\}, \end{aligned}$$

or, equivalently,

$$(4.20) \quad h(L) = a(L) + \alpha[(1 - G'(B))\phi_2(L) - (1 + G'(L))\phi_1(L)],$$

$$(4.21) \quad h(B) = -a(B) + \alpha[(1 - G'(B))\phi_2(B) - (1 + G'(L))\phi_1(B)].$$

Theorem 4.2 shows that (4.20), (4.21) admit a unique solution (L^* and B^*), which gives the optimal reflecting barriers. In the case when $a(x) \equiv a$; $\sigma(x) \equiv \sigma$ are both constants, we can write

$$(4.22) \quad \phi_1(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}; \quad \phi_2(x) = \hat{C}_1 e^{\lambda_1 x} + \hat{C}_2 e^{\lambda_2 x}$$

with

$$\begin{aligned} \lambda_1 &= \frac{-a + \sqrt{a^2 + 2\alpha\sigma^2}}{\sigma^2}; & \lambda_2 &= \frac{-a - \sqrt{a^2 + 2\alpha\sigma^2}}{\sigma^2}, \\ C_1 &= \frac{\exp(\lambda_2 B)}{\lambda_1 \Delta}; & C_2 &= -\frac{\exp(\lambda_1 B)}{\lambda_2 \Delta}; \\ \hat{C}_1 &= -\frac{\exp(\lambda_2 L)}{\lambda_1 \Delta}; & \hat{C}_2 &= \frac{\exp(\lambda_1 L)}{\lambda_2 \Delta}, \end{aligned}$$

where $\Delta = \exp(\lambda_2 B + \lambda_1 L) - \exp(\lambda_1 B + \lambda_2 L)$, and, finally,

$$G(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} - \frac{2}{\sqrt{a^2 + 2\alpha\sigma^2}} \int_0^x e^{-(a/\sigma^2)(x-s)} \sinh\left(\frac{\sqrt{a^2 + 2\alpha\sigma^2}}{\sigma^2}(x-s)\right) h(s) ds,$$

where c_1, c_2 are chosen so that $G(L) = G(B) = 0$. Of course, we can pose more conditions on a, σ , and h (e.g., $a = 0, \sigma = 1$, or h is symmetric and so on) to make (4.20), (4.21) more explicit. For example, if $a = 0, \sigma = 1$, (4.22) becomes

$$\phi_1(x) = -\frac{1}{\sqrt{2\alpha}} \frac{\cosh \sqrt{2\alpha}(B-x)}{\sinh \sqrt{2\alpha}(B-L)}; \quad \phi_2(x) = \frac{1}{\sqrt{2\alpha}} \frac{\cosh \sqrt{2\alpha}(x-L)}{\sinh \sqrt{2\alpha}(B-L)},$$

and so on. For the simpler case—when h is an even function—Karatzas [8] had a transcendental equation to determine B^* and L^* ($= -B^*$) by a slightly different method. However, by the uniqueness of such solution, (4.20) and (4.21) would also give the same answer.

5. The linear case. In this section, we consider the case when σ is also linear. More precisely, we assume that $a(x) = ax + b; \sigma(x) = \theta(ax + b)$, where a, b, θ are constants and $a \neq 0, \theta \neq 0$.

Clearly, the basic assumption (A1) is partially violated, since now σ possesses a zero at $x = -b/a$. The major disadvantage of this violation is that the ODE related to the H-J-B equation now has a singularity at the zero of σ . We then wonder whether the value function is still C^2 . However, we prove directly that, under the extra condition on the position of the “vertex” of function h (condition (5.4)), such a singularity is removable. Namely, there still exists a convex C^2 -solution to the variational inequality (2.11), which is now of the form

$$(5.1) \quad [\alpha V(x) - \frac{1}{2}\theta^2(ax+b)^2 V''(x) - (ax+b)V'(x) - h(x)] \vee [|V'(x)| - 1] = 0, \quad x \in \mathbf{R}.$$

We will also derive the explicit formula for determining the smooth fitting points (it might be a transcendental equation). Consequently, we still conclude that the value function is C^2 , convex, and that the optimal policy exists in the manner that was discussed in the previous sections.

Note that condition (2.10) now becomes

$$(5.2) \quad \alpha > 2|a| + \theta^2 a^2.$$

We modify condition (2.9) (of assumption (A2)) by

$$(5.3) \quad \left(x + \frac{b}{a}\right) h'(x) \geq 0, \quad x \in \mathbf{R}; \quad h'\left(-\frac{b}{a}\right) = 0;$$

i.e., we restrict the vertex of h to the point $x = -b/a$ so as to “kill” the singularity caused by σ .

Observe that, if we set $Y(t) = Y^{x+b/a}(t) = X^x(t) + b/a$, where $X^x(\cdot)$ is the solution of the Stochastic differential equation (S.D.E.) (2.1) with $\sigma = \theta(ax + b)$, then $Y(\cdot)$ will satisfy

$$(5.4) \quad Y(t) = \tilde{x} + a \int_0^t Y(s) ds + \theta a \int_0^t Y(s) dW(s) + \xi(t),$$

Downloaded 12/01/16 to 128.125.208.57. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

where $\tilde{x} = (x + b/a)$. Therefore, the cost function (2.2) becomes

$$(5.5) \quad V_\xi(x) = E \int_{[0,\infty)} e^{\alpha t} [d\check{\xi}(t) + h(Y^{\tilde{x}}(t) - \frac{b}{a})dt] \stackrel{\text{def}}{=} \tilde{V}_\xi(\tilde{x}).$$

Define $\tilde{h}(\cdot) = h(\cdot - b/a)$; then, by (5.3), \tilde{h} satisfies (2.8) and

$$(5.6) \quad x\tilde{h}'(x) \geq 0, \quad x \in \mathbf{R}; \quad \tilde{h}'(0) = 0.$$

So, without loss of generality, we may just consider system (5.4) with the cost function (5.5). Namely, we will henceforth assume that $b = 0$ and h satisfies (5.6).

The ODE (3.1) now becomes

$$(5.7) \quad \alpha V(x) = axV'(x) + \frac{1}{2}\theta^2 a^2 x^2 V''(x) + h(x).$$

We see that it is now symmetric with respect to the origin and is in the form of the *Euler equation*. So we may solve it explicitly to get the smooth fitting points.

To begin, we first solve the equation for $x > 0$. Letting $U = V'$ and differentiating (5.7), we get that

$$(5.8) \quad \frac{1}{2}\theta^2 a^2 x^2 U''(x) + (a + \theta^2 a^2)xU'(x) + (a - \alpha)U(x) + h'(x) = 0.$$

Set $x = e^t, t \in \mathbf{R}$, let $W(t) = U(e^t) = U(x)$, and denote $\dot{W} = dW/dt, \ddot{W} = d^2W/dt^2$; (5.8) becomes

$$(5.9) \quad \frac{1}{2}\theta^2 a^2 \ddot{W}(t) + (a + \frac{1}{2}\theta^2 a^2)\dot{W}(t) + (a - \alpha)W(t) + h'(e^t) = 0.$$

We may easily write the solution of (5.9) as

$$(5.10) \quad W(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} - \frac{2}{\theta^2 a^2} \int_{\ln B}^t \varphi(t - \tau)h'(e^\tau)d\tau,$$

where $B > 0$ is arbitrarily chosen and φ is the solution of the homogeneous equation

$$(5.11) \quad \frac{1}{2}\theta^2 a^2 \ddot{W}(t) + (a + \frac{1}{2}\theta^2 a^2)\dot{W}(t) + (a - \alpha)W(t) = 0$$

with the initial condition

$$(5.12) \quad \varphi(0) = 0; \quad \dot{\varphi}(0) = 1,$$

and λ_1, λ_2 are the solutions of the characteristic equation

$$(5.13) \quad \frac{1}{2}\theta^2 a^2 \lambda^2 + (a + \frac{1}{2}\theta^2 a^2)\lambda + (a - \alpha) = 0.$$

Namely,

$$(5.14) \quad \lambda_1 = \frac{-(a + \frac{1}{2}\theta^2 a^2) + \sqrt{(a - \frac{1}{2}\theta^2 a^2)^2 + 2\theta^2 a^2 \alpha}}{\theta^2 a^2};$$

$$(5.15) \quad \lambda_2 = \frac{-(a + \frac{1}{2}\theta^2 a^2) - \sqrt{(a - \frac{1}{2}\theta^2 a^2)^2 + 2\theta^2 a^2 \alpha}}{\theta^2 a^2}.$$

Clearly, if $\alpha > |a|$, then $\lambda_1 > 0 > \lambda_2$, and, if we rewrite (5.13) as

$$(5.16) \quad 0 = [\frac{1}{2}\theta^2 a^2 \lambda^2 + (a + \theta^2 a^2)](\lambda - 1) + (2a + \theta^2 a^2 - \alpha),$$

then it follows from (5.2) and $\lambda_1 > 0$ that $\lambda_1 > 1$. Furthermore, if we let $\varphi(t) = \hat{C}_1 e^{\lambda_1 t} + \hat{C}_2 e^{\lambda_2 t}$, then (5.12) gives

$$(5.17) \quad \hat{C}_1 = -\frac{1}{\lambda_2 - \lambda_1}; \quad \hat{C}_2 = \frac{1}{\lambda_2 - \lambda_1}.$$

Therefore (5.10) becomes

$$(5.18) \quad W(t) = \sum_{i=1}^2 [C_i e^{\lambda_i t} - \frac{2\hat{C}_i}{\theta^2 a^2} \int_{\ln B}^t e^{\lambda_i(t-\tau)} h'(e^\tau) d\tau].$$

In terms of the original variable, i.e., $t = \ln x; x > 0$, we get that, for each $B > 0$,

$$(5.19) \quad U_B(x) = \sum_{i=1}^2 x^{\lambda_i} [C_i - \frac{2\hat{C}_i}{\theta^2 a^2} \int_B^x \frac{h'(u)}{u^{\lambda_i+1}} du].$$

We now choose C_1, C_2 so that $U_B(0+) = 0; U_B(B) = 1$. To do this, we first give the following lemma.

LEMMA 5.1. For any C^2 -function h satisfying (2.8), (5.6) and λ_1, λ_2 given by (5.14), (5.15), we have that

- (a) $\lim_{x \rightarrow 0+} x^{\lambda_1} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du = 0;$
- (b) $\lim_{x \rightarrow 0+} x^{\lambda_2} \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du = 0;$
- (c) $\lim_{x \rightarrow 0+} x^{\lambda_1-1} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du = \frac{h''(0)}{1-\lambda_1};$
- (d) $\lim_{x \rightarrow 0+} x^{\lambda_2-1} \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du = \frac{h''(0)}{1-\lambda_2}.$

Proof. (a) and (c). Since $\lambda_1 > 1$, $\lim_{x \rightarrow 0+} x^{-\lambda_1} = \lim_{x \rightarrow 0+} x^{-\lambda_1+1} = +\infty$. Thus, by L'Hospital's rule, we have that

$$\begin{aligned} \lim_{x \rightarrow 0+} x^{\lambda_1} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du &= \lim_{x \rightarrow 0+} \frac{h'(x)/x^{\lambda_1+1}}{(-\lambda_1)x^{-\lambda_1-1}} = \frac{h'(0)}{(-\lambda_1)} = 0; \\ \lim_{x \rightarrow 0+} x^{\lambda_1-1} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du &= \lim_{x \rightarrow 0+} \frac{h'(x)/x^{\lambda_1+1}}{(1-\lambda_1)x^{-\lambda_1}} = \frac{1}{1-\lambda_1} \lim_{x \rightarrow 0+} \frac{h'(x)}{x}, \\ &= \frac{h''(0)}{1-\lambda_1}, \end{aligned}$$

here we use the fact that $h'(0) = 0$.

(b) and (d). The proof is similar to the previous one, except that now we have that $\lim_{x \rightarrow 0+} x^{-\lambda_2} = \lim_{x \rightarrow 0+} x^{-\lambda_2+1} = 0$, since $\lambda_2 < 0$. So we can apply the previous argument to get the result, provided that we can show that $\lim_{x \rightarrow 0+} \int_0^x (h'(u)/u^{\lambda_2+1}) du = 0$.

Observe that, by integration by parts,

$$(5.20) \quad \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du = \frac{1}{(-\lambda_2)} \left[\frac{h'(u)}{u^{\lambda_2}} \Big|_0^x - \int_0^x \frac{h''(u)}{u^{\lambda_2}} du \right].$$

So the result follows from $\lambda_2 < 0$. \square

By (5.20) we see that $0 \leq \int_0^B (h'(u)/u^{\lambda_2+1}) du < \infty$. Let $C_2 = -(2\hat{C}_2/\theta^2 a^2) \times \int_0^B (h'(u)/u^{\lambda_2+1}) du$, then (5.19) becomes

$$(5.21) \quad U_B(x) = x^{\lambda_1} \left[C_1 - \frac{2\hat{C}_1}{\theta^2 a^2} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du \right] - x^{\lambda_2} \frac{2\hat{C}_2}{\theta^2 a^2} \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du.$$

Hence Lemma 5.1 (a), (b) imply that $U_B(0+) = 0$. Moreover, by (5.21),

$$U_B(B) = B^{\lambda_1} C_1 - B^{\lambda_2} \frac{2\hat{C}_2}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du.$$

So $U_B(B) = 1$ if and only if

$$(5.22) \quad C_1 = B^{-\lambda_1} \left[1 + \frac{2B^{\lambda_2} \hat{C}_2}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du \right].$$

Furthermore, (5.21) also gives

$$(5.23) \quad \begin{aligned} U'_B(x) &= -\lambda_1 x^{\lambda_1-1} \frac{2\hat{C}_1}{\theta^2 a^2} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du + C_1 \lambda_1 x^{\lambda_1-1} \\ &\quad - \lambda_2 x^{\lambda_2-1} \frac{2\hat{C}_2}{\theta^2 a^2} \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du - \frac{2h'(x)}{\theta^2 a^2 x} [\hat{C}_1 + \hat{C}_2] \\ &= -\lambda_1 x^{\lambda_1-1} \frac{2\hat{C}_1}{\theta^2 a^2} \int_B^x \frac{h'(u)}{u^{\lambda_1+1}} du + C_1 \lambda_1 x^{\lambda_1-1} \\ &\quad - \lambda_2 x^{\lambda_2-1} \frac{2\hat{C}_2}{\theta^2 a^2} \int_0^x \frac{h'(u)}{u^{\lambda_2+1}} du, \end{aligned}$$

since $\hat{C}_1 + \hat{C}_2 = 0$.

Using (5.17), (5.23), Lemma 5.1 (c) and (d), and the fact that $\lambda_1 > 1$, we get that

$$(5.24) \quad \begin{aligned} 0 < U'_B(0+) &= -\lambda_1 \hat{C}_1 \frac{2h''(0)}{\theta^2 a^2 (1 - \lambda_1)} - \lambda_2 \hat{C}_2 \frac{2h''(0)}{\theta^2 a^2 (1 - \lambda_2)} \\ &= \frac{2h''(0)[(\lambda_1 \hat{C}_1 + \lambda_2 \hat{C}_2) - \lambda_1 \lambda_2 (\hat{C}_1 + \hat{C}_2)]}{\theta^2 a^2 (\lambda_1 - 1)(1 - \lambda_2)} \quad (5.24) \\ &= \frac{2h''(0)}{\theta^2 a^2 (\lambda_1 - 1)(1 - \lambda_2)} < \infty. \end{aligned}$$

Finally, setting $x = B$ and substituting (5.22) into (5.23), we get that

$$(5.25) \quad \begin{aligned} U'_B(B) &= \lambda_1 B^{\lambda_1-1} \left\{ B^{-\lambda_1} \left[1 + \frac{2B^{\lambda_2} \hat{C}_2}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du \right] \right\} \\ &= \frac{1}{B} \left\{ \lambda_1 - (\lambda_2 - \lambda_1) B^{\lambda_2} \frac{2\hat{C}_2}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du \right\} \\ &= \frac{1}{B} \left\{ \lambda_1 - \frac{2B^{\lambda_2}}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du \right\} \end{aligned}$$

by (5.17). Therefore $U'_B(B) = 0$ if and only if

$$(5.26) \quad \frac{2B^{\lambda_2}}{\theta^2 a^2} \int_0^B \frac{h'(u)}{u^{\lambda_2+1}} du - \lambda_1 = 0.$$

Let $F(B) = (2B^{\lambda_2}/\theta^2 a^2) \int_0^B (h'(u)/u^{\lambda_2+1}) du$. Lemma 5.1(b) gives that $\lim_{B \rightarrow 0+} F(B) = 0$, and the same argument will show that $\lim_{B \rightarrow +\infty} F(B) = +\infty$, since $\lim_{B \rightarrow +\infty} h'(B) = +\infty$ by (2.8). Hence there must be a $B^* > 0$ such that $F(B^*) = \lambda_1$, i.e., $U'_{B^*}(B^*) = 0$ by (5.25).

Since, on $(0, B^*]$, the differential equation (5.7) has no singularity, (5.24) and Lemma 3.2 give that $U'_{B^*}(x) \geq 0$ for all $x \in (0, B^*]$. We now consider that $V_{B^*}(x) = C + \int_0^x U_{B^*}(t) dt$, $0 < x \leq B^*$, where C is some constant. Then V_{B^*} is a solution of (5.7) if and only if $h(0) = \alpha C$. Therefore

$$(5.27) \quad V_{B^*}(x) = \frac{h(0)}{\alpha} + \int_0^x U_{B^*}(t) dt$$

is the solution to (5.7) with the properties

$$(5.28) \quad \begin{aligned} V_{B^*}(0+) &= \frac{h(0)}{\alpha}; \\ V'_{B^*}(0+) &= U_{B^*}(0+) = 0; \quad V'_{B^*}(B^*) = U_{B^*}(B^*) = 1; \\ V''_{B^*}(0+) &= \frac{2h''(0)}{\theta^2 a^2 (\lambda_1 - 1)(1 - \lambda_2)}; \quad V''_{B^*}(B^*) = U'_{B^*}(B^*) = 0; \\ V''_{B^*}(x) &\geq 0, \quad x \in (0, B^*]. \end{aligned}$$

This solves the smooth fitting problem on \mathbf{R}^+ .

To solve (5.7) for $x < 0$, we first consider the following equation:

$$(5.29) \quad \alpha V^1(x) = axV^{1'}(x) + \frac{1}{2}\theta^2 a^2 x^2 V^{1''}(x) + h(-x), \quad x > 0.$$

Conditions (2.8), (5.6) allow us to repeat the previous argument to find a real number $B_1 > 0$ determined by

$$(5.30) \quad \frac{2B_1^{\lambda_2}}{\theta^2 a^2} \int_0^{B_1} \frac{h'(-u)}{u^{\lambda_2+1}} du - \lambda_1 = 0,$$

and a solution $V^1_{B_1}$ to (5.29) for $x > 0$ such that

$$(5.31) \quad \begin{aligned} V^1_{B_1}(0+) &= \frac{h(0)}{\alpha}; \\ V^1_{B_1}'(0+) &= 0; \quad V^1_{B_1}'(B_1) = 1; \\ V^1_{B_1}''(0+) &= \frac{2h''(0)}{\theta^2 a^2 (\lambda_1 - 1)(1 - \lambda_2)}; \quad V^1_{B_1}''(B_1) = 0; \\ V^1_{B_1}''(x) &\geq 0, \quad x \in (0, B_1]. \end{aligned}$$

We can now define

$$(5.32) \quad V_{L^*, B^*}(x) = \begin{cases} V^1(-x), & x \in [L^*, 0); \\ \frac{h(0)}{\alpha}, & x = 0; \\ V(x), & x \in (0, B^*], \end{cases}$$

where $L^* = -B_1$. It is easily checked, by using (5.27), (5.28), and (5.31), that V_{L^*, B^*} is a C^2 -solution to (5.7) and is convex on $[L^*, B^*]$ satisfying (3.2), (3.3). Therefore Theorem 4.1 applies. We have actually proved the following theorem.

THEOREM 5.2. *For the linear system*

$$X(t) = x + \int_0^t (aX(s) + b)ds + \int_0^t \theta(aX(s) + b)dW(s) + \xi(t),$$

where a, b, θ are constants, $a \neq 0, \theta \neq 0$, there always exist $-\infty < L^* < B^* < \infty$ and an optimal control given by (4.5), provided that (5.2), (5.3) hold. The value function V^* is convex, and C^2 and is given by (4.4) with V_{L^*, B^*} given by (5.32). Moreover, the “smooth fit points” $L^*(= -B_1), B^*$ are determined by (5.30) and (5.26), along with (5.14), (5.15).

6. Appendix. We now outline the proof of Lemma 3.1. We can always refer to [12, Lemmas 4.1, 4.2] for complete details.

Proof of Lemma 3.1. By (2.5), σ is nonvanishing; so we can rewrite (3.4) as

$$(6.1) \quad V''(x) = \gamma(x)V(x) + \delta(x)V'(x),$$

where

$$\gamma(x) = \frac{2\alpha}{\sigma^2(x)}, \quad \delta(x) = -\frac{ax + b}{\sigma^2(x)}.$$

Introduce a change of variable $\phi(x) = \int_0^x [\exp \int_0^u \delta(v)dv]du$ and define $U(y) = V \circ \phi^{-1}(y)$; then U satisfies

$$(6.2) \quad U''(y) = [\phi'(\phi^{-1}(y))]^{-2}\gamma(\phi^{-1}(y))U(y) = \tilde{\gamma}(y)U(y), \quad y \in [\tilde{L}, \tilde{B}],$$

where $\tilde{\gamma}(y) = [\phi'(\phi^{-1}(y))]^{-2}\gamma(\phi^{-1}(y)) > 0; \tilde{L} = \phi(L), \tilde{B} = \phi(B)$. It is readily seen that U (respectively, U') and V (respectively, V') have the same sign; hence V inherits the desired properties from U . Namely, without loss of generality, we may assume that $a = b = 0$, and then (6.1) becomes

$$(6.3) \quad V''(x) = \gamma(x)V(x), \quad x \in [L, B],$$

where $\gamma > 0$. Observe now that V is strictly convex (strictly concave) on any interval where it is positive (negative).

To prove (i), suppose that $V(\bar{x}) = 0$, for some $\bar{x} \in [L, B]$. Then we must have that $V'(\bar{x}) \neq 0$; otherwise, $V \equiv 0$ by the uniqueness of the solution to (6.3). Suppose that $V'(\bar{x}) > 0$. Define

$$\bar{w} = \sup \{w \in [\bar{x}, B] : V'(x) > 0, \text{ for } \bar{x} \leq x < w\}.$$

By a simple analysis on the signs of V', V , and V'' on the interval (\bar{x}, \bar{w}) , we show that V is convex on (\bar{x}, \bar{w}) , which implies that $V'(\bar{w}) \geq V'(\bar{x}) > 0$. Then the definition of \bar{w} and the continuity of V' lead to $\bar{w} = B$. Therefore V' has no zero on $[\bar{x}, B]$. Similarly, we can show that V' has no zero on $[L, \bar{x}]$. The case where $V'(\bar{x}) < 0$ is treated similarly. This proves (i).

To prove (ii), assume that $V(\bar{x}) = 0$ for some $\bar{x} \in [L, B]$. Again, we must have that $V(\bar{x}) \neq 0$. Without loss of generality, assume that $V(\bar{x}) > 0$. By (i), V would

Downloaded 12/01/16 to 128.125.208.57. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

have no zero on $[L, B]$ as V' has a zero at $\bar{x} \in [L, B]$. So $V(x) > 0$ for all $x \in [L, B]$; i.e., V is strictly convex on $[L, B]$, which implies that

$$(x - \bar{x})V'(x) > 0, \quad \text{for all } x \in [L, B], \quad x \neq \bar{x},$$

since $V'(\bar{x}) = 0$. Then

$$(x - \bar{x})V'(x)V(x) > 0, \quad x \in [L, B]. \quad \square$$

Acknowledgments. I would like to express my deep gratitude to my advisor Professor Naresh Jain for many helpful discussions. I also thank Professors S. E. Shreve and I. Karatzas for providing information on recent research in this area, along with their papers. Finally, I thank the referees for their useful comments.

REFERENCES

- [1] V. E. BENÈS, L. A. SHEPP, AND H. S. WITSENHAUSEN, *Some solvable stochastic control problems*, *Stochastics*, 4 (1980), pp. 39–83.
- [2] P. L. CHOW, J. L. MENALDI, AND M. ROBIN, *Additive control of stochastic linear systems with finite horizon*, *SIAM. J. Control Optim.*, 23 (1985), pp. 858–899.
- [3] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw–Hill, New York, Toronto, London, 1955.
- [4] A. FRIEDMANN, *Partial Differential Equations of Parabolic Type*, Prentice–Hall, Englewood Cliffs, NJ, 1965.
- [5] I. I. GIKHMAN AND A. V. SKOROHOD, *Stochastic Differential Equations*, Springer, New York, 1972.
- [6] J. M. HARRISON AND M. I. TAKSAR, *Instantaneous control of Brownian motion*, *Math. Oper. Res.*, 8 (1983), pp. 439–453.
- [7] U. G. HAUSSMANN, *A Stochastic Maximum Principle for Optimal control of Diffusions*, Pitman Research Notes in Math. Series, Longman Scientific & Technical, London, 1986.
- [8] I. KARATZAS, *A class of singular stochastic control problems*, *Adv. Appl. Probab.*, 15 (1983), pp. 225–254.
- [9] J. L. MENALDI AND M. ROBIN, *On some cheap control problems for diffusion processes*, *Trans. Amer. Math. Soc.*, 278 (1983), pp. 771–802.
- [10] J. L. MENALDI AND M. I. TAKSAR, *Optimal correction problem of a multidimensional stochastic system*, *Automatica*, 25 (1989), pp. 223–232.
- [11] P. A. MEYER, *Un cours sur les intégral stochastiques*, *Séminaire de Probabilité, X.*, Lecture Notes in Math., 511, Springer, New York, 1976.
- [12] S. E. SHREVE, *An Introduction To Singular Stochastic Control*, in *Stochastic Differential Systems, Stochastic Control Theory and Applications*, IMA Volumes in Math and Its Applications, 10, Springer, New York, 1986.
- [13] S. E. SHREVE, J. P. LEHOCZKY, AND D. P. GAVER, *Optimal consumption for general diffusions with absorbing and reflecting barriers*, *SIAM. J. Control Optim.*, 22 (1984), pp. 55–75.
- [14] H. M. SONER AND S. E. SHREVE, *Regularity of the value function for a two-dimensional singular stochastic control problem*, *SIAM. J. Contr. Optim.*, 27 (1989), pp. 876–907.