Equilibrium Model of Limit Order Books – A Mean-field Game View

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Abstract In this paper, we propose a continuous time equilibrium model of the (sell-side) limit order book (LOB) in which the liquidity dynamics follows a non-local, reflected mean-field stochastic differential equation (SDE) with state-dependent intensity. To motivate the model we first study an N-seller static mean-field type Bertrand game among the liquidity providers. We shall then formulate the continuous time model as the limiting mean-field dynamics of the representative seller, and argue that the frontier of the LOB (e.g., the best ask price) is the value function of a mean-field stochastic control problem by the representative seller. Using a dynamic programming approach, we show that the value function is a viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation, which can be used to determine the equilibrium density function of the LOB, in the spirit of [32].

1 Introduction

With the rapid growth of electronic trading, the study of order-driven markets has become an increasingly prominent focus in quantitative finance. Indeed, in the current financial world more than half of the markets use a limit order book (LOB) mechanism to facilitate trade. There has been a large amount of literature studying LOB from various angles, combined with some associated optimization problems such as placement, liquidation, executions, etc. (see, e.g. [1, 3, 4, 5, 7, 14, 18, 20, 30, 35, 36, 39, 40] to mention a few). Among many important structural issues of LOB, one of the focuses has been the dynamic move-

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ment of the LOB, both its frontier and its “density" (or “shape"). The latter was shown to be a determining factor of the “liquidity cost" (cf. [32]), an important aspect that impacts the pricing of the asset. We refer to, e.g., [2, 19, 26, 32] for the study of LOB particularly concerning its shape formation.

In this paper, we assume that all sellers are *patient* and all buyers are *impatient*. We extend dynamic model of LOB proposed in [32] in two major aspects. The guiding idea is to specify the *expected equilibrium utility function*, which plays an essential role in the modeling of the shape of the LOB in that it endogenously determines both the dynamic density of the LOB and its frontier. More precisely, instead of assuming, more or less in an ad hoc manner, that the equilibrium price behaves like an “utility function", we shall consider it as the consequence of a Bertrand-type game among a large number of liquidity providers (sellers who set limit orders). Following the argument of [13], we first study an N-seller static Bertrand game, where a profit function of each seller involves not only the limit order price less the waiting cost, but also the average of the other sellers' limit order prices observed. We show that the Nash equilibrium exists in such a game. With an easy randomization argument, we can then show that, as $N \to \infty$, the Nash equilibrium converges to an optimal strategy of a single player's optimization problem with a mean-field nature, as expected.

We note that the Bertrand game in finance can be traced back to as early as 1800s, when Cournot [15] and Bertrand [8] first studied oligopoly models of markets with a small number of competitive players. We refer to [17] and [41] for background and references. Since Cournot's model uses quantity as a strategic variable to determine the price, while Bertrand model does the opposition, we choose to use the Bertrand game as it fits our problem better. We shall assume that the sellers use the same marginal profit function, but with different choices of the price-waiting cost preference to achieve the optimal outcome (see Sect. 3 for more detailed formulation).

We would like to point out that our study of Bertrand game is in a sense “motivational" for the second main feature of this paper, that is, the continuous time, mean-field type dynamic liquidity model. More precisely, we assume that the liquidity dynamics is a pure-jump Markov process, with a mean-field type state dependent jump intensity. Such a dynamic game is rather complicated, and is expected to involve systems of nonlinear, mean-field type partial differential equations (see, e.g., [23, 27]). We therefore consider the limiting case as the number of sellers tends to infinity, and argue that the dynamics of the total liquidity follows a pure jump SDE with reflecting boundary conditions and mean-field-type state-dependent jump intensity. We note that such SDE is itself new and therefore interesting in its own right.

We should point out that the special features of our underlying liquidity dynamics (mean-field type; state-dependent intensity; and reflecting boundary conditions) require the combined technical tools in mean-field games, McKean-Vlasov SDEs with state-dependent jump intensity, and SDEs with discontinuous paths and reflecting boundary conditions. In particular, we refer to the works [10, 11, 12, 21, 22, 24, 25, 29, 31, 33, 37] (and the references cited therein) for the technical foundation of this paper. Furthermore, apart from justifying the underlying liquidity dynamics, another main task of this paper is to substantiate the corresponding stochastic control problem, including validating the dynamic programming principle (DPP) and showing that the value function is a viscosity solution to the corresponding Hamilton-Jacobi-Bellman (HJB) equation.

This paper is organized as follows. In Sect. 2, we introduce necessary notations and preliminary concepts, and study the well-posedness of a reflected mean-field SDEs with jumps that will be essential in our study. We shall also provide an Itô's formula involving reflected mean-field SDEs with jumps for ready reference. In Sect. 3 we investigate a static Bertrand game with N sellers, and its limiting behavior as N tends to infinity. Based on the results, we then propose in Sect. 4 a continuous time mean-field type dynamics of the representative seller, as well as a mean-field stochastic control problem as the limiting version of dynamic Bertrand game when the number of sellers becomes sufficiently large. In Sect. 5 and Sect. 6 we validate the *Dynamic Programming Principle* (DPP), derive the HJB equation, and show that the value function is a viscosity solution to the corresponding HJB equation.

2 Preliminaries

Throughout this paper we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined two standard Brownian motions $W = \{W_t : t \geq 0\}$ and $B = \{B_t : t \geq 0\}$. Let $(\mathcal{A}, \mathscr{B}_{\mathcal{A}})$ and $(\mathcal{B}, \mathscr{B}_{\mathcal{B}})$ be two measurable spaces. We assume that there are two Poisson random measures $\mathcal{N}^s$ and $\mathcal{N}^b$, defined on $\mathbb{R}_+ \times \mathcal{A} \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathcal{B}$, and with Lévy measures $\nu^s(\cdot)$ and $\nu^b(\cdot)$, respectively. In other words, we assume that the Poisson measures $\mathcal{N}^s$ and $\mathcal{N}^b$ have mean measures $\widehat{\mathcal{N}}^s(\cdot) := (m \times \nu^s \times m)(\cdot)$ and $\widehat{\mathcal{N}}^b(\cdot) := (m \times \nu^b)(\cdot)$, respectively, where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}_+$, and we denote the *compensated* random measures $\widetilde{\mathcal{N}}^s(A) := (\mathcal{N}^s - \widehat{\mathcal{N}}^s)(A) = \mathcal{N}^s(A) - (m \times \nu^s \times m)(A)$ and $\widetilde{\mathcal{N}}^b(B) := (\mathcal{N}^b - \widehat{\mathcal{N}}^b)(B) = \mathcal{N}^b(B) - (m \times \nu^b)(B)$, for any $A \in \mathscr{B}(\mathbb{R}_+ \times \mathcal{A} \times \mathbb{R}_+)$ and $B \in \mathscr{B}(\mathbb{R}_+ \times \mathcal{B})$. For simplicity, throughout this paper we assume that both ν^s and ν^b are finite, that is, $\nu^s(\mathcal{A}), \nu^b(\mathcal{B}) < \infty$, and we assume the Brownian motions and Poisson random measures are mutually independent. We note that for any $A \in \mathscr{B}(\mathcal{A} \times \mathbb{R}_+)$ and $B \in \mathscr{B}(\mathcal{B})$, the processes $(t, \omega) \mapsto \widetilde{\mathcal{N}}^s([0,t] \times A, \omega)$, $\widetilde{\mathcal{N}}^b([0,t] \times B, \omega)$ are both $\mathbb{F}^{\mathcal{N}^s, \mathcal{N}^b}$-martingales. Here $\mathbb{F}^{\mathcal{N}^s, \mathcal{N}^b}$ denotes the filtration generated by $\mathcal{N}^s$ and $\mathcal{N}^b$.

For a generic Euclidean space E and for $T > 0$, we denote $C([0,T]; E)$ and $\mathbb{D}([0,T]; E)$ to be the spaces of continuous and càdlàg functions, respectively. We endow both spaces with "sup-norms", so that both of them are complete metric spaces. Next, for $p \geq 1$ we denote $L^p(\mathcal{F}; E)$ to be the space of all E-valued $\mathcal{F}$-measurable random variable ξ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[|\xi|^p] < \infty$. Also, for $T \geq 0$, we denote $L^p_{\mathbb{F}}([t,T]; E)$ to be all E-valued $\mathbb{F}$-adapted process η on $[t,T]$, such that $\|\eta\|_{p,T} := \mathbb{E}[\int_t^T |\eta_s|^p ds]^{1/p} < \infty$. We often use the notations $L^p(\mathbb{F}; C([0,T]; E))$ and $L^p(\mathbb{F}; \mathbb{D}([0,T]; E))$ when we need to specify the path properties for elements in $L^p_{\mathbb{F}}([0,T]; E)$.

For $p \geq 1$ we denote by $\mathscr{P}_p(E)$ the space of probability measures μ on $(E, \mathscr{B}(E))$ with finite p-th moment, i.e. $\|\mu\|_p^p := \int_E |x|^p \mu(dx) < \infty$. Clearly, for $\xi \in L^p(\mathcal{F}; E)$, its law $\mathcal{L}(\xi) = \mathbb{P}_\xi := \mathbb{P} \circ \xi^{-1} \in \mathscr{P}_p(E)$. We endow $\mathscr{P}_p(E)$ with the following p-Wasserstein metric:

$$\begin{aligned} W_p(\mu, \nu) &:= \inf\Big\{\Big(\int_{E\times E} |x-y|^p \pi(dx, dy)\Big)^{\frac{1}{p}} : \pi \in \mathscr{P}_p(E\times E) \\ &\qquad\qquad \text{with marginals } \mu \text{ and } \nu\Big\} \qquad (1)\\ &= \inf\Big\{\|\xi - \xi'\|_{L^p(\Omega)} : \xi, \xi' \in L^p(\mathcal{F}; E) \text{ with } \mathbb{P}_\xi = \mu,\ \mathbb{P}_{\xi'} = \nu\Big\}. \end{aligned}$$

Furthermore, we suppose that there is a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ such that (i) the Brownian motion W and Poisson random measures $\mathcal{N}^s, \mathcal{N}^b$ are independent of $\mathcal{G}$; and (ii) $\mathcal{G}$ is "rich enough" in the sense that for every $\mu \in \mathscr{P}_2(\mathbb{R})$, there is a random variable $\xi \in L^2(\mathcal{G}; E)$ such that $\mu = \mathbb{P}_\xi$. Let $\mathbb{F} = \mathbb{F}^{W,B,\mathcal{N}^s,\mathcal{N}^b \vee \mathcal{G}} = \{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \mathcal{F}_t^{\mathcal{N}^s} \vee \mathcal{F}_t^{\mathcal{N}^b} \vee \mathcal{G}$, $t \geq 0$, be the filtration generated by W, B, $\mathcal{N}^s$, $\mathcal{N}^b$, and $\mathcal{G}$, augmented by all the $\mathbb{P}$-null sets so that it satisfies the *usual hypotheses* (cf. [38]).

Let us introduce two spaces that are useful for our analysis later. We write $C_b^{1,1}(\mathscr{P}_2(\mathbb{R}))$ to denote the space of all differentiable functions $f : \mathscr{P}_2(\mathbb{R}) \to \mathbb{R}$ such that $\partial_\mu f$ exists, and is bounded and Lipschitz continuous. That is, for some constant $C > 0$, it holds

(i) $|\partial_\mu f(\mu, x)| \leq C, \quad \mu \in \mathscr{P}_2(\mathbb{R}),\ x \in \mathbb{R}$;

(ii) $|\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \leq C\{|x - x'| + W_2(\mu, \mu')\}, \quad \mu, \mu' \in \mathscr{P}_2(\mathbb{R}),\ x, x' \in \mathbb{R}$.

We shall denote $C_b^{2,1}(\mathscr{P}_2(\mathbb{R}))$ to be the space of all functions $f \in C_b^{1,1}(\mathscr{P}_2(\mathbb{R}))$ such that

(i) $\partial_\mu f(\cdot, x) \in C_b^{1,1}(\mathscr{P}_2(\mathbb{R}))$ for all $x \in \mathbb{R}$;

(ii) $\partial_\mu^2 f : \mathscr{P}_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ is bounded and Lipschitz continuous;

(iii) $\partial_\mu f(\mu, \cdot) : \mathbb{R} \to \mathbb{R}$ is differentiable for every $\mu \in \mathscr{P}_2(\mathbb{R})$, and its derivative $\partial_y \partial_\mu f : \mathscr{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ is bounded and Lipschitz continuous.

2.1 Mean-field SDEs with reflecting boundary conditions

In this subsection we consider the following (discontinuous) SDE with reflection, which will be a key element of our discussion: for $t \in [0, T]$,

$$\begin{aligned} X_s = x &+ \int_t^s \int_{A\times\mathbb{R}_+} \theta(r, X_{r-}, \mathbb{P}_{X_r}, z) \mathbf{1}_{[0, \lambda(r, X_{r-}, \mathbb{P}_{X_r})]}(y) \widetilde{\mathcal{N}}^s(dr dz dy) \qquad (2)\\ &+ \int_t^s b(r, X_r, \mathbb{P}_{X_r}) dr + \int_t^s \sigma(r, X_r, \mathbb{P}_{X_r}) dB_r + \beta_s + K_s, \qquad s \in [t, T], \end{aligned}$$

where θ, λ, b, σ are measurable functions defined on appropriate subspaces of $[0,T]\times\Omega\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})\times\mathbb{R}$, β is an $\mathbb{F}$-adapted process with càdlàg paths, and K is a "reflecting process". That is, it is an $\mathbb{F}$-adapted, non-decreasing, and càdlàg process so that

(i) $X_\cdot \geq 0$, $\mathbb{P}$-a.s.;

(ii) $\int_0^T \mathbf{1}_{\{X_r>0\}} dK_r^c = 0$, $\mathbb{P}$-a.s. (K^c denotes the continuous part of K); and

(iii) $\Delta K_t = (X_{t-} + \Delta Y_t)^-$ for all $t \in [0,T]$, where $Y = X - K$.

We call SDE (2) a *mean-field SDE with discontinuous paths and reflections* (MFSDEDR), and we denote the solution by $(X^{t,x}, K^{t,x})$, although the superscript is often omitted when context is clear. If $b, \sigma = 0$ and β is pure jump, then the solution (X,K) becomes pure jump as well (i.e., $dK^c \equiv 0$). We note that the main feature of this SDE is that the jump intensity $\lambda(\cdots)$ of the solution X is "state-dependent" with mean-field nature. Its well-posedness thus requires some attention since, to the best of our knowledge, it has not been studied in the literature.

We shall make use of the following *Standing Assumptions*.

Assumption 2.1 *The mappings $\lambda : [0,T]\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R}) \mapsto \mathbb{R}_+$, $b : [0,T]\times\Omega\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R}) \mapsto \mathbb{R}$, $\sigma : [0,T]\times\Omega\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R}) \mapsto \mathbb{R}$, and $\theta : [0,T]\times\Omega\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})\times\mathbb{R} \mapsto \mathbb{R}$ are all uniformly bounded and continuous in (t,x), and satisfy the following conditions, respectively:*

(i) For fixed $\mu \in \mathscr{P}_2(\mathbb{R})$ and $x, z \in \mathbb{R}$, the mappings $(t,\omega) \mapsto \theta(t,\omega,x,\mu,z)$, $(b,\sigma)(t,\omega,x,\mu)$ are $\mathbb{F}$-predictable;

(ii) For fixed $\mu \in \mathscr{P}_2(\mathbb{R})$, $(t,z) \in [0,T]\times\mathbb{R}$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, the functions $\lambda(t,\cdot,\mu)$, $b(t,\omega,\cdot,\mu)$, $\sigma(t,\omega,\cdot,\mu)$, $\theta(t,\omega,\cdot,\mu,z) \in C_b^1(\mathbb{R})$;

(iii) For fixed $(t,x,z) \in [0,T]\times\mathbb{R}^2$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, the functions $\lambda(t,x,\cdot)$, $b(t,\omega,x,\cdot)$, $\sigma(t,\omega,x,\cdot)$, $\theta(t,\omega,x,\cdot,z) \in C_b^{1,1}(\mathscr{P}_2(\mathbb{R}))$;

(iv) There exists $L > 0$, such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, it holds that

$$
\begin{aligned}
&|\lambda(t,x,\mu) - \lambda(t,x',\mu')| + |b(t,\omega,x,\mu) - b(t,\omega,x',\mu')| \\
&+|\sigma(t,\omega,x,\mu) - \sigma(t,\omega,x',\mu')| + |\theta(t,\omega,x,\mu,z) - \theta(t,\omega,x',\mu',z)| \\
&\qquad \leq L\left(|x-x'| + W_1(\mu,\mu')\right), \qquad t \in [0,T],\ x,x',z \in \mathbb{R},\ \mu,\mu' \in \mathscr{P}_2(\mathbb{R}).
\end{aligned}
$$

Remark 2.2 (i) The requirements on the coefficients in Assumption 2.1 (such as boundedness) are stronger than necessary, only to simplify the arguments. More general (but standard) assumptions are easily extendable without substantial difficulties.

(ii) Throughout this paper, unless specified, we shall denote $C > 0$ to be a generic constant depending only on T and the bounds in Assumption 2.1. Furthermore, we shall allow it to vary from line to line.

It is well-known that (see, e.g., [9]), as a mean-field SDE, the solution to (2) may not satisfy the so-called "flow property", in the sense that $X_r^{t,x} \neq X_r^{s,X_s^{t,x}}$, $0 \leq t \leq s \leq r \leq T$. It is also noted in [9] that if we consider the following accompanying SDE of (2): for $s \in [t,T]$,

$$
\begin{aligned}
X_s^{t,\xi} &= \xi + \int_t^s b(r, X_r^{t,\xi}, \mathbb{P}_{X_r^{t,\xi}})dr + \int_t^s \sigma(r, X_r^{t,\xi}, \mathbb{P}_{X_r^{t,\xi}})dB_r + \beta_s + K_s^{t,\xi} \\
&\quad + \int_t^s \int_{A\times\mathbb{R}_+} \theta(r, X_{r-}^{t,\xi}, \mathbb{P}_{X_r^{t,\xi}}, z)\mathbf{1}_{[0,\lambda(r,X_{r-}^{t,\xi},\mathbb{P}_{X_r^{t,\xi}})]}(y)\widetilde{\mathcal{N}}^s(drdzdy) \quad (3)
\end{aligned}
$$

and then using the law $\mathbb{P}_{X^{t,\xi}}$ to consider a slight variation of (3):

$$
\begin{aligned}
X_s^{t,x,\xi} &= x + \int_t^s b(r, X_r^{t,x,\xi}, \mathbb{P}_{X_r^{t,\xi}})dr + \int_t^s \sigma(r, X_r^{t,x,\xi}, \mathbb{P}_{X_r^{t,\xi}})dB_r + \beta_s + K_s^{t,x,\xi} \\
&\quad + \int_t^s \int_{A\times\mathbb{R}_+} \theta(r, X_{r-}^{t,x,\xi}, \mathbb{P}_{X_r^{t,\xi}}, z)\mathbf{1}_{[0,\lambda(r,X_{r-}^{t,x,\xi},\mathbb{P}_{X_r^{t,\xi}})]}(y)\widetilde{\mathcal{N}}^s(drdzdy), (4)
\end{aligned}
$$

where $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$, then we shall argue below that the following flow property holds:

$$
\left(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}\right) = (X_r^{t,x,\xi}, X_r^{t,\xi}), \qquad 0 \le t \le s \le r \le T, \qquad (5)
$$

for all $(x,\xi) \in \mathbb{R} \times L^2(\mathcal{F}_t;\mathbb{R})$. We should note that although both SDEs (3) and (4) resemble the original equation (2), the process $X^{t,x,\xi}$ has the full information of the solution given the initial data (x,ξ), where ξ provides the initial distribution $\mathbb{P}_\xi$, and x is the actual initial state.

To prove the well-posedness of SDEs (3) and (4), we first recall the so-called “Discontinuous Skorohod Problem" (DSP) (see, e.g., [16, 31]). Let $Y \in \mathbb{D}([0,T])$, $Y_0 \ge 0$. We say that a pair $(X,K) \in \mathbb{D}([0,T])^2$ is a solution to the DSP(Y) if

(i) $X = Y + K$;

(ii) $X_t \ge 0$, $t \ge 0$; and

(iii) K is nondecreasing, $K_0 = 0$, and $K_t = \int_0^t \mathbf{1}_{\{X_{s-}=0\}} dK_s$, $t \ge 0$.

It is well-known that the solution to DSP exists and is unique, and it can be shown (see [31]) that the condition (iii) amounts to saying that $\int_0^t \mathbf{1}_{\{X_{s-}>0\}} dK_s^c = 0$, where K^c denotes the continuous part of K, and $\Delta K_t = (X_{t-} + \Delta Y_t)^-$. Furthermore, it is shown in [16] that solution mapping of the DSP, $\Gamma : \mathbb{D}([0,T]) \mapsto \mathbb{D}([0,T])$, defined by $\Gamma(Y) = X$, is Lipschitz continuous under uniform topology. That is, there exists a constant $L > 0$ such that

$$
\sup_{t\in[0,T]} |\Gamma(Y^1)_t - \Gamma(Y^2)_t| \le L \sup_{t\in[0,T]} |Y_t^1 - Y_t^2|, \qquad Y^1, Y^2 \in \mathbb{D}([0,T]). \qquad (6)
$$

Before we proceed to prove the well-posedness of (3) and (4), we note that the two SDEs can be argued separately. Moreover, while (3) is a mean-field (or McKean-Vlasov)-type of SDE, (4) is actually a standard SDE (although with state-dependent intensity) with discontinuous paths and reflection, given the law of the solution to (3), $\mathbb{P}_{X^{t,\xi}}$, and it can be argued similarly but much simpler. Therefore, in what follows we shall focus only on the well-posedness of SDE (3). Furthermore, for simplicity we shall assume $b \equiv 0$, as the general case can be argued similarly without substantial difficulty.

The scheme of solving the SDE (3) is more or less standard (see, e.g., [31]). We shall first consider an SDE without reflection: for $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$ and $s \in [t,T]$,

$$\begin{aligned} Y^{t,\xi}_s &= \xi + \int_t^s \int_{A\times\mathbb{R}_+} \theta(r,\Gamma(Y^{t,\xi})_{r-},\mathbb{P}_{\Gamma(Y^{t,\xi})_r},z)\mathbf{1}_{[0,\lambda^{\Gamma(t,\xi)}_{r-}]}(y)\widetilde{\mathcal{N}}^s(drdzdy) \\ &\quad + \int_t^s \sigma(r,\Gamma(Y^{t,\xi})_r,\mathbb{P}_{\Gamma(Y^{t,\xi})_r})dB_r + \beta_s, \end{aligned} \tag{7}$$

where $\lambda^{\Gamma(t,\xi)}_{r-} := \lambda(r,\Gamma(Y^{t,\xi})_{r-},\mathbb{P}_{\Gamma(Y^{t,\xi})_r})$. Clearly, if we can show that (7) is well-posed, then by simply setting $X^{t,\xi}_s = \Gamma(Y^{t,\xi})_s$ and $K^{t,\xi}_s = X^{t,\xi}_s - Y^{t,\xi}_s$, $s \in [t,T]$, we see that $(X^{t,\xi}, K^{t,\xi})$ would solve SDE (3)(!). We should note that a technical difficulty caused by the presence of the state-dependent intensity is that the usual L^2-norm does not work as naturally as expected, as we shall see below. We nevertheless have the following result.

Theorem 2.3 *Assume that Assumptions 2.1 is in force. Then, there exists a solution $Y^{t,\xi} \in L^2_{\mathbb{F}}(\mathbb{D}([t,T]))$ to SDE (7). Furthermore, such solution is pathwisely unique.*

Proof Assume $t = 0$. For a given $T_0 > 0$, and $y \in L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$, consider a mapping $\mathscr{T}$:

$$\begin{aligned} \mathscr{T}(y)_s &:= \xi + \int_0^s \int_{A\times\mathbb{R}_+} \theta(r,\Gamma(y)_{r-},\mathbb{P}_{\Gamma(y)_r},z)\mathbf{1}_{[0,\lambda(r,\Gamma(y)_{r-},\mathbb{P}_{\Gamma(y)_r})]}(u)\widetilde{\mathcal{N}}^s(drdzdu) \\ &\quad + \int_0^s \sigma(r,\Gamma(y)_r,\mathbb{P}_{\Gamma(y)_r})dB_r + \beta_s, \qquad s \ge 0. \end{aligned} \tag{8}$$

We shall argue that $\mathscr{T}$ is a contraction mapping on $L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$ for $T_0 > 0$ small enough.

To see this, denote, for $\eta \in \mathbb{D}([0,T_0])$, $|\eta|^*_s := \sup_{0\le r\le s}|\eta_r|$, and define $\theta_s(z) := \theta(s,\Gamma(y)_s,\mathbb{P}_{\Gamma(y)_s},z)$, $\lambda_s := \lambda(s,\Gamma(y)_s,\mathbb{P}_{\Gamma(y)_s})$, $\sigma_s := \sigma(s,\Gamma(y)_s,\mathbb{P}_{\Gamma(y)_s})$, $s \in [0,T_0]$. Then, we have

$$\begin{aligned} \mathbb{E}[|\mathscr{T}(y)|^*_{T_0}] &\le C\Big\{\mathbb{E}|\xi| + \mathbb{E}\Big[\int_0^{T_0}\int_{A\times\mathbb{R}_+}\big|\theta_r(z)\mathbf{1}_{[0,\lambda_r]}(u)\big|\nu^s(dz)dudr\Big] \\ &\qquad + \mathbb{E}\Big[\Big(\int_0^{T_0}|\sigma_r|^2dr\Big)^{1/2}\Big]\Big\} \\ &\le C\mathbb{E}|\xi| + C\mathbb{E}\Big[\int_0^{T_0}\int_A|\theta_r(z)\lambda_r|\nu^s(dz)dr\Big] \\ &\qquad + C\mathbb{E}\Big[\Big(\int_0^{T_0}|\sigma_r|^2dr\Big)^{1/2}\Big] < \infty, \end{aligned}$$

thanks to Assumption 2.1. Hence, $\mathscr{T}(y) \in L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$.

We now show that $\mathscr{T}$ is a contraction mapping on $L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$. For $y_1, y_2 \in L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$, we denote θ^i, λ^i, and σ^i, respectively, as before, and denote $\Delta\varphi :=$

$\varphi^1 - \varphi^2$, for $\varphi = \theta, \lambda, \sigma$, and $\Delta\mathscr{T}(s) := \mathscr{T}(y_1)_s - \mathscr{T}(y_2)_s$, $s \geq 0$. Then, we have, for $s \in [0, T_0]$,

$$\Delta\mathscr{T}(s) = \int_0^s \Delta\sigma_r dB_r + \int_0^s \int_{A\times\mathbb{R}_+} \Big[\Delta\theta_r(z)\mathbf{1}_{[0,\lambda_r^1]}(y) + \theta_r^2(z)\big(\mathbf{1}_{[0,\lambda_r^1]}(y) - \mathbf{1}_{[0,\lambda_r^2]}(y)\big)\Big]\widetilde{\mathcal{N}}^s(drdzdy).$$

Clearly, $\Delta\mathscr{T} = \mathscr{T}(y_1) - \mathscr{T}(y_2)$ is a martingale on $[0, T_0]$. Since $\widetilde{\mathcal{N}} = \mathcal{N} - \widehat{\mathcal{N}}$ and $|\mathbf{1}_{[0,a]}(\cdot) - \mathbf{1}_{[0,b]}(\cdot)| \leq \mathbf{1}_{[a\wedge b, a\vee b]}(\cdot)$ for any $a, b \in \mathbb{R}$, we have, for $0 \leq s \leq T_0$,

$$\mathbb{E}|\Delta\mathscr{T}|_s^* \leq \mathbb{E}\Big[\Big(\int_0^s |\Delta\sigma_r|^2 dr\Big)^{\frac{1}{2}}\Big] + 2\mathbb{E}\Big[\int_0^s \int_{A\times\mathbb{R}_+} \big|\theta_r^2(z)\big(\mathbf{1}_{[0,\lambda_r^1]}(y) - \mathbf{1}_{[0,\lambda_r^2]}(y)\big) + \Delta\theta_r(z)\mathbf{1}_{[0,\lambda_r^1]}(y)\big|\nu^s(dz)dydr\Big] := I_1 + I_2. \tag{9}$$

Recalling from Remark 2.2-(ii) for the generic constant $C > 0$, and by Assumption 2.1-(iv), (6), and the definition of $W_1(\cdot,\cdot)$ (see (1)), we have

$$\begin{aligned}
I_1 &\leq C\mathbb{E}\Big[\Big(\int_0^s \{|y_1 - y_2|_r^{*,2} + W_1(\mathbb{P}_{\Gamma(y_1)_r}, \mathbb{P}_{\Gamma(y_2)_r})^2\}dr\Big)^{1/2}\Big] \\
&\leq C\mathbb{E}\big[\sqrt{s}\big(|y_1 - y_2|_s^* + \mathbb{E}|y_1 - y_2|_s^*\big)\big] \leq C\sqrt{T_0}\|y_1 - y_2\|_{L^1(\mathbb{D}([0,T_0]))} \\
I_2 &\leq C\Big(\mathbb{E}\Big[\int_0^s \int_A |\Delta\theta_r(z)|\nu^s(dz)dr\Big] + \mathbb{E}\Big[\int_0^s |\Delta\lambda_r|dr\Big]\Big) \\
&\leq C\mathbb{E}\Big[\int_0^s \Big(|\Gamma(y_1)_r - \Gamma(y_2)_r| + W_1(\mathbb{P}_{\Gamma(y_1)_r}, \mathbb{P}_{\Gamma(y_2)_r})\Big)dr\Big] \\
&\leq C\mathbb{E}\Big[\int_0^s |\Gamma(y_1) - \Gamma(y_2)|_r^* dr\Big] \leq CT_0\|y_1 - y_2\|_{L^1(\mathbb{D}([0,T_0]))}.
\end{aligned} \tag{10}$$

Combining (9) and (10), we deduce that

$$\|\Delta\mathscr{T}\|_{L^1(\mathbb{D}([0,T_0]))} \leq C(T_0 + \sqrt{T_0})\|y_1 - y_2\|_{L^1(\mathbb{D}([0,T_0]))}, \qquad s \in [0, T_0]. \tag{11}$$

Therefore, by choosing T_0 such that $C(T_0 + \sqrt{T_0}) < 1$, we see that the mapping $\mathscr{T}$ is a contraction on $L^1(\mathbb{D}([0,T_0]))$, which implies that (7) has a unique solution in $L^1_{\mathbb{F}}(\mathbb{D}([0,T_0]))$. Moreover, we note that T_0 depends only on the universal constant in Assumption 2.1. We can repeat the argument for the time interval $[T_0, 2T_0], [2T_0, 3T_0], \cdots$, and conclude that (7) has a unique solution in $L^1_{\mathbb{F}}(\mathbb{D}([0,T]))$ for any given $T > 0$.

Finally, we claim that the solution $Y \in L^2_{\mathbb{F}}(\mathbb{D}([0,T]))$. Indeed, by Burkholder-Davis-Gundy's inequality and Assumption 2.1, we have

$$
\begin{aligned}
\mathbb{E}[|Y|_s^{*,2}] &\le C\Big\{\mathbb{E}|\xi|^2+\mathbb{E}\Big[\int_0^s\int_{A\times\mathbb{R}_+}\big|\theta_r(z)\mathbf{1}_{[0,\lambda_r]}(y))\big|^2\nu^s(dz)dydr\Big]\\
&\quad+\mathbb{E}[\int_0^s|\sigma_r|^2dr]+\mathbb{E}|\beta|_T^{*,2}\Big\} \qquad (12)\\
&\le C\Big\{\mathbb{E}|\xi|^2+\mathbb{E}\Big[\int_0^s\big[1+|Y_r|^2+W_1(0,\Gamma(Y)_r)\big]^2dr\Big]+\mathbb{E}|\beta|_T^{*,2}\Big\}\\
&\le C\Big\{\mathbb{E}|\xi|^2+\int_0^s(1+\mathbb{E}[|Y|_r^{*,2}])dr+\mathbb{E}|\beta|_T^{*,2}\Big\}, \qquad s\in[0,T].
\end{aligned}
$$

Here, in the last inequality above we used the fact that

$$
W_1(0,\Gamma(Y)_r)^2\le(\|\Gamma(Y)_r\|_{L^1(\Omega)})^2\le(\mathbb{E}|\Gamma(Y)|_r^*)^2\le C\mathbb{E}[|Y|_r^{*,2}], \qquad r\in[0,s].
$$

Applying the Gronwall inequality, we obtain that $\mathbb{E}[|Y|_T^{*,2}]<\infty$. The proof is now complete. □

Remark 2.4 (i) It is worth noting that once we solved $X^{t,\xi}$, then we know $\mathbb{P}_{X^{t,\xi}}$, and (4) can be viewed as a standard SDEDR with coefficient $\tilde{\lambda}(s,x):=\lambda(s,x,\mathbb{P}_{X_s^{t,\xi}})$, which is Lipschitz in x. This guarantees the existence and uniqueness of the solution $(X^{t,x,\xi},K^{t,x,\xi})$ to (4).

(ii) The uniqueness of the solutions to (3) and (4) implies that $X_s^{t,x,\xi}\ |_{x=\xi}= X_s^{t,\xi},\ s\in[t,T]$. That is, $X_s^{t,x,\xi}\ |_{x=\xi}$ solves the same SDE as $X_s^{t,\xi},\ s\in[t,T]$. (See more detail in [34].)

(iii) Given $(t,x)\in[0,T]\times\mathbb{R}$, if $\mathbb{P}_{\xi_1}=\mathbb{P}_{\xi_2}$ for $\xi_1,\xi_2\in L^2(\mathcal{F}_t;\mathbb{R})$, then X^{t,x,ξ_1} and X^{t,x,ξ_2} are indistinguishable. So, $X^{t,x,\mathbb{P}_\xi}:=X^{t,x,\xi}$, i.e. $X^{t,x,\xi}$ depends on ξ only through its law.

2.2 An Itô's formula

We shall now present an Itô's formula that will be frequently used in our future discussion. We note that a similar formula for mean-field SDE can be found in [9], and the one involving jumps was given in the recent work [22]. The one presented below is a slight modification of that of [22], taking the particular state-dependent intensity feature of the dynamics into account. Since the proof is more or less standard but quite lengthy, we refer to [34] for the details.

In what follows we let $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ be a copy of the probability space $(\Omega,\mathcal{F},\mathbb{P})$, and denote $\tilde{\mathbb{E}}[\cdot]$ to be the expectation under $\tilde{\mathbb{P}}$. For any random variable ϑ defined on $(\Omega,\mathcal{F},\mathbb{P})$, we denote, when there is no danger of confusion, $\tilde{\vartheta}\in(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ to be a copy of ϑ such that $\tilde{\mathbb{P}}_{\tilde{\vartheta}}=\mathbb{P}_\vartheta$. We note that that $\tilde{\mathbb{E}}[\cdot]$ acts only on the variables of the form $\tilde{\vartheta}$.

We first define the following classes of functions.

Definition 2.5 *We say that* $F\in C_b^{1,2,(2,1)}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R}))$, *if* *(i)* $F(t,v,\cdot,\cdot) \in C_b^{2,1}(\mathbb{R}\times\mathscr{P}_2(\mathbb{R}))$, *for all* $t\in[0,T]$ *and* $v\in\mathbb{R}$*;*

(ii) $F(\cdot,v,x,\mu)\in C_b^1([0,T])$, *for all* $(v,x,\mu)\in\mathbb{R}\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})$*;*

(iii) $F(t,\cdot,x,\mu)\in C_b^2(\mathbb{R})$, *for all* $(t,x,\mu)\in[0,T]\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})$*;*

(iv) All derivatives involved in the definitions above are uniformly bounded over $[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})$ *and Lipschitz continuous in* (x,μ)*, uniformly with respect to* t*.*

We are now ready to state the Itô's formula. Let $V^{t,v}$ be an Itô process given by

$$V_s^{t,v} = v + \int_t^s b^V(r, V_r^{t,v})dr + \int_t^s \sigma^V(r, V_r^{t,v})dB_r^V, \tag{13}$$

where $v\in\mathbb{R}$ and $(B_t^V)_{t\in[0,T]}$ is a standard Brownian motion independent of $(B_t)_{t\in[0,T]}$. For notational simplicity, in what follows for the coefficients $\varphi = b,\sigma,\beta,\lambda$, we denote $\varphi_s^{t,x,\xi} := \varphi(s, X_s^{t,x,\xi}, \mathbb{P}_{X_s^{t,\xi}})$, $\theta_s^{t,x,\xi}(z) := \theta(s, X_s^{t,x,\xi}, \mathbb{P}_{X_s^{t,\xi}}, z)$, $\tilde{\varphi}_s^{t,\xi} := \varphi(s, \tilde{X}_s^{t,\tilde{\xi}}, \mathbb{P}_{X_s^{t,\xi}})$, and $\tilde{\theta}_s^{t,\xi}(z) := \theta(s, \tilde{X}_s^{t,\tilde{\xi}}, \mathbb{P}_{X_s^{t,\xi}}, z)$. Similarly, denote $b_s^{t,v} := b^V(s, V_s^{t,v})$ and $\sigma_s^{t,v} := \sigma^V(s, V_s^{t,v})$. Also, let us write $\Theta_s^t := (s, V_s^{t,v}, X_s^{t,x,\xi}, \mathbb{P}_{X_s^{t,\xi}})$, from which we have $\Theta_t^t = (t,v,x,\mathbb{P}_\xi)$.

Proposition 2.6 (Itô's Formula) *Let* $\Phi\in C_b^{1,2,(2,1)}([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R}))$*, and* $(X^{t,\xi}, X^{t,x,\xi}, V^{t,v})$ *be the solutions to (3), (4) and (13), respectively, on* $[t,T]$*. Then, for* $0\le t\le s\le T$*, it holds*

$$\begin{aligned}
&\Phi(\Theta_s^t) - \Phi(\Theta_t^t)\\
=&\int_t^s \Big(\partial_t\Phi(\Theta_r^t) + \partial_x\Phi(\Theta_r^t)b_r^{t,x,\xi} + \frac{1}{2}\partial_{xx}^2\Phi(\Theta_r^t)(\sigma_r^{t,x,\xi})^2 + \partial_v\Phi(\Theta_r^t)b_r^{t,v}\\
&\quad + \frac{1}{2}\partial_{vv}^2\Phi(\Theta_r^t)(\sigma_r^{t,v})^2\Big)dr\\
&+ \int_t^s \partial_x\Phi(\Theta_r^t)\sigma_r^{t,x,\xi}dB_r + \int_t^s \partial_v\Phi(\Theta_r^t)\sigma_r^{t,v}dB_r^V + \int_t^s \partial_x\Phi(\Theta_{r-}^t)\mathbf{1}_{\{X_{r-}=0\}}dK_r\\
&+ \int_t^s\int_A \Big(\Phi(r, V_r^{t,v}, X_{r-}^{t,x,\xi} + \theta_r^{t,x,\xi}(z), \mathbb{P}_{X_r^{t,\xi}}) - \Phi(\Theta_{r-}^t)\\
&\qquad -\partial_x\Phi(\Theta_{r-}^t)\theta_{r-}^{t,x,\xi}(z)\Big)\lambda_r^{t,x,\xi}\nu^s(dz)dr \qquad\qquad (14)\\
&+\int_t^s\int_{A\times\mathbb{R}_+}\Big(\Phi(r, V_r^{t,v}, X_{r-}^{t,x,\xi} + \theta_r^{t,x,\xi}(z), \mathbb{P}_{X_r^{t,\xi}}) - \Phi(\Theta_{r-}^t)\Big)\mathbf{1}_{[0,\lambda_r^{t,x,\xi}]}(y)\widetilde{\mathcal{N}}^s(drdzdy)\\
&+ \int_t^s \widetilde{\mathbb{E}}\Big[\partial_\mu\Phi(\Theta_r^t, \tilde{X}_r^{t,\tilde{\xi}})\tilde{b}_r^{t,\xi} + \frac{1}{2}\partial_y(\partial_\mu\Phi)(\Theta_r^t, \tilde{X}_r^{t,\tilde{\xi}})(\tilde{\sigma}_r^{t,\xi})^2\\
&\quad +\int_0^1\int_A\Big(\partial_\mu\Phi(\Theta_r^t, \tilde{X}_r^{t,\tilde{\xi}} + \rho\tilde{\theta}_r^{t,x,\xi}(z)) - \partial_\mu\Phi(\Theta_r^t, \tilde{X}_s^{t,\tilde{\xi}})\Big)\tilde{\theta}_r^{t,\xi}(z)\tilde{\lambda}_r^{t,\xi}\nu^s(dz)d\rho\Big]dr.
\end{aligned}$$

3 A Bertrand game among the sellers (static case)

In this section we analyze a price setting mechanism among liquidity providers (investors placing limit orders), and use it as the basis for our continuous time model in the rest of the paper. To begin with, recall that in this paper we assume all sellers are patient and all buyers are impatient. We therefore consider only the sell-side LOB. Following the ideas of [13, 27, 28], we shall consider the process of the (static) price setting as a Bertrand-type of game among the sellers, each placing a certain number of sell limit orders at a specific price, and trying to maximize her expected utility. To be more precise, we assume that sellers use the price at which they place limit orders as their *strategic variable*, and the number of shares submitted would be determined accordingly. Furthermore, we assume that there is a *waiting cost*, also as a function of the price. Intuitively, a higher price will lead to a longer execution time, hence a higher waiting cost. Thus, there is a competitive game among the sellers for better total reward. Finally, we assume that the sellers are homogeneous in the sense that they have the same subjective probability measure, so that they share the same degree of risk aversion.

We now give a brief description of the problem. We assume that there are N sellers, and the jth seller places limit orders at price $p_j = X + l_j$, $j = 1, 2, \cdots, N$, where X is the fundamental price. Without loss of generality, we may assume $X = 0$. As a main element in an *oligopolistic competitions* (cf. e.g., [28]), we assume that each seller i is equipped with a *demand function*, denoted by $h_i^N(p_1, p_2, \cdots, p_N)$, for a given price vector $p = (p_1, p_2, \cdots, p_N)$, reflecting the seller's perceived demand from the buyers. The seller i will determine the number of shares of limit orders to be placed in the LOB based on the values of his/her demand function, given the price vector. Hence this is a *Bertrand-type game*[1]. More specifically, we assume that the demand functions h_i^N, $i = 1, 2, \cdots, N$, are smooth and satisfy the following properties:

$$\frac{\partial h_i^N}{\partial p_i} < 0, \quad \text{and} \quad \frac{\partial h_i^N}{\partial p_j} > 0, \quad \text{for } j \neq i. \tag{15}$$

We note that (15) simply amounts to saying that each seller expects less demand (for her orders) when her own price increases, and more demand when other seller(s) increase their prices. Furthermore, we shall assume that the demand functions are invariant under permutations of the other sellers' prices, in the sense that, for fixed $p_1, \cdots, p_N$ and all $i, j \in \{1, \cdots, N\}$,

$$h_i^N(p_1, \cdots, p_i, \cdots, p_j, \cdots, p_N) = h_j^N(p_1, \cdots, p_j, \cdots, p_i, \cdots, p_N). \tag{16}$$

It is worth noting that the combination of (15) and (16) is the following fact: if a price vector p is ordered by $p_1 \leq p_2 \leq \cdots \leq p_N$, then for any $i < j$, it holds that

[1] A *Cournot game* is one such that the price p_i is the function of the numbers of shares $q = (q_1, \cdots, q_N)$ through a demand function. The two games are often exchangeable if the demand functions are invertible (see, e.g., [28]).

$$h_j^N(p) = h_j^N(p_1, \cdot\cdot, p_j, \cdot\cdot, p_i, \cdot\cdot, p_N) \le h_i^N(p_1, \cdot\cdot, p_i, \cdot\cdot, p_i, \cdot\cdot, p_N) \le h_i^N(p) \tag{17}$$

That is, the demand functions are ordered in a reversed way, which in a sense indicates that the shape of LOB should be a non-increasing function of prices in the LOB. Finally, for each i, we denote the price vector for "other" prices for seller i by p_{-i}. For seller i, the "least favorable" price for given p_{-i} is one that would generate zero demand, which is often referred to as the *choke price*. We shall assume such a price exists, and denote it by $\hat{p}_i(p_{-i}) < \infty$, namely,

$$h_i^N(p_1, \cdots, p_{i-1}, \hat{p}_i, p_{i+1}, \cdots, p_N) = 0. \tag{18}$$

We note that the existence of the choke price, together with the monotonicity property (15), indicates the possibility that $h_j^N(p) < 0$, for some j and some price vector p. But since the size of order placement cannot be negative, such scenario becomes unpractical. To amend this, we introduce the notion of *actual demand*, denoted by $\{\widehat{h}_i(p)\}$, which we now describe.

Consider an ordered price vector $p = (p_1, \cdots, p_N)$, with $p_i \le p_j$, $i \le j$, and we look at $h_N^N(p)$. If $h_N^N(p) \ge 0$, then by (17) we have $h_i^N(p) \ge 0$ for all $i = 1, \cdots, N$. In this case, we denote $\widehat{h}_i(p) = h_i^N(p)$ for all $i = 1, \cdots, N$. If $h_N^N(p) < 0$, then we set $\widehat{h}_N(p) = 0$. That is, the N-th seller does not act at all. We assume that the remaining $N-1$ sellers will observe this fact and modify their strategy as if there are only $N-1$ sellers. More precisely, we first choose a choke price $\widehat{p}_N$ so that $h_N^N(p_1, \cdots, p_{N-1}, \widehat{p}_N) = 0$, and define

$$h_i^{N-1}(p_1, p_2, \cdots, p_{N-1}) := h_i^N(p_1, p_2, \cdots, p_{N-1}, \widehat{p}_N), \qquad i = 1, \cdots, N-1,$$

and continue the game among the $N-1$ sellers.

In general, for $1 \le n \le N-1$, assume the $(n+1)$-th demand functions $\{h_i^{n+1}\}_{i=1}^{n+1}$ are defined. If $h_{n+1}^{n+1}(p_1, \cdots, p_{n+1}) < 0$, then other n sellers will assume $(n+1)$-th seller sets a price at $\hat{p}_{n+1}$ with zero demand (i.e., $h_{n+1}^{n+1}(p_1, p_2, \cdots, p_n, \hat{p}_{n+1}) = 0$), and modify their demand functions to

$$h_i^n(p_1, p_2, \cdots, p_n) := h_i^{n+1}(p_1, p_2, \cdots, p_n, \hat{p}_{n+1}), \qquad i = 1, \cdots, n. \tag{19}$$

We can now define the actual demand function $\{\widehat{h}_i\}_{i=1}^N$.

Definition 3.1 (*Actual demand function*) *Assume that $\{h_i^N\}_{i=1}^N$ is a family of demand functions. The family of "actual demand functions", denoted by $\{\widehat{h}_i\}_{i=1}^N$, are defined in the following steps: for a given ordered price vector p,*

(i) if $h_N^N(p) \ge 0$, then we set $\widehat{h}_i(p) = h_i^N(p)$ for all $i = 1, \cdots, N$;

(ii) if $h_N^N(p) < 0$, then we define recursively for $n = N-1, \cdots 1$ the demand functions $\{h_i^n\}_{i=1}^n$ as in (19). In particular, if there exists an $n < N$ such that $h_{n+1}^{n+1}(p_1, p_2, \cdots, p_n, p_{n+1}) < 0$ and $h_n^n(p_1, p_2, \cdots, p_n) \ge 0$, then we set

$$\widehat{h}_i(p) = \begin{cases} h_i^n(p_1, p_2, \cdots, p_n) & i = 1, \cdots, n \\ 0 & i = n+1, \cdots, N; \end{cases} \tag{20}$$

(iii) if there is no such n, then $\widehat{h}_i(p) = 0$ *for all* $i = 1, \cdots, N$.

We note that the actual demand function will always be non-negative, but for each price vector p, the number $\#\{i : \widehat{h}_i(p) > 0\} \leq N$, and could even be zero.

3.1 The Bertrand game and its Nash equilibrium

Besides the demand function, a key ingredient in the placement decision making process is the “waiting cost" for the time it takes for the limit order to be executed. We shall assume that each seller has her own waiting cost function $c_i^N \triangleq c_i^N(p_1, p_2, \cdots, p_N, Q)$, where Q is the total number of shares available in the LOB. Similar to the demand function, we shall assume the following assumptions for the waiting cost.

Assumption 3.2 *For each seller* $i \in \{1, \cdots, n\}$ *with* $n \in [1, N]$*, each* c_i^N *is smooth in all variables such that*

(i) (Monotonicity) $\frac{\partial c_i^N}{\partial p_i} > 0$*, and* $\frac{\partial c_i^N}{\partial p_j} < 0$*, for* $j \neq i$*;*

(ii) (Exchangeability) $c_i^N(p_1, \cdot\cdot, p_i, \cdot\cdot, p_i, \cdot\cdot, p_N) = c_j^N(p_1, \cdot\cdot, p_j, \cdot\cdot, p_i, \cdot\cdot, p_N)$*;*

(iii) $c_i^N(p)\big|_{p_i=0} = 0$*, and* $\frac{\partial c_i^N}{\partial p_i}\Big|_{p_i=0+} \in (0, 1)$*;*

(iv) $\lim_{p_i\to\infty} \frac{p_i}{c_i^N(p)} = 0$*,* $i = 1, \cdots, N$.

Remark 3.3 (a) Assumption 3.2-(i), (ii) ensure that the price ordering leads to the same ordering for waiting cost functions, similar to what we argued before for demand functions. In particular, the second part of Assumption 3.2-(i) is due to (15). That is, if other seller submits an order at a higher price, the demand for seller i increases, which would lead to faster execution, hence shorter waiting time and lower waiting cost.

(b) Consider the function $J_i(p, Q) = p_i - c_i^N(p, Q)$. Assumption 3.2 amounts to saying that $J_i(p, Q)\big|_{p_i=0} = 0$, $\frac{\partial J_i(p,Q)}{\partial p_i}\Big|_{p_i=0+} > 0$, and $\lim_{p_i\to\infty} J_i(p, Q) < 0$. Thus, there exists $p_i^0 = p_i^0(p_{-i}, Q) > 0$ such that $\frac{\partial J_i(p,Q)}{\partial p_i}\Big|_{p_i=p_i^0} = 0$, and $\frac{\partial J_i(p,Q)}{\partial p_i}\Big|_{p_i>p_i^0} < 0$.

(c) Since $J_i(0, Q) = 0$, and $\frac{\partial J_i(p,Q)}{\partial p_i}\Big|_{p_i=0+} > 0$, one can easily check that $J_i(p_i^0, Q) > 0$. This, together with Assumption 3.2-(iv), shows that there exists $\tilde{p}_i = \tilde{p}_i(p_{-i}, Q) > p_i^0$, such that $J_i(p_i, Q)\big|_{p_i=\tilde{p}_i} = 0$ (or, equivalently $c_i^N(p_1, \cdots, p_{i-1}, \tilde{p}_i, p_{i+1}, \cdots, p_N, Q) = \tilde{p}_i$). Furthermore, remark above implies that $J_i(p_i, Q) < 0$ for all $p_i > \tilde{p}_i(p_{-i}, Q)$. In other words, any selling price higher than $\tilde{p}_i(p_{-i}, Q)$ would yield a negative profit, and therefore should be prevented.

The Bertrand game among sellers can now be formally introduced: each seller chooses its price to maximize profit in a non-cooperative manner, and their decision will be based not only on her own price, but also on the actions of all other sellers. We denote the profit of each seller by

$$\Pi_i(p_1, p_2, \cdots, p_N, Q) := \widehat{h}_i(p_1, p_2, \cdots, p_N) \times \Big(p_i - c_i^N(p_1, p_2, \cdots, p_N, Q)\Big), \quad (21)$$

and each seller tries to maximize her profit Π. For each fixed Q, we are looking for a Nash equilibrium price vector $p^{*,N}(Q) = (p_1^{*,N}(Q), \cdots, p_N^{*,N}(Q))$. We note that in the case when $\widehat{h}_i(p^{*,N}) = 0$ for some i, the i-th seller will not participate in the game (with zero profit), so we shall modify the price

$$p_i^{*,N}(Q) \triangleq c_i^N(p_1^{*,N}, \cdots, p_N^{*,N}, Q) = c_i^N(p^{*,N}, Q), \quad (22)$$

and consider a subgame involving the $N-1$ sellers, and so on. That is, for a subgame with n sellers, they solve

$$p_i^{*,n} = \arg\max_{p \geq 0} \Pi_i^n(p_1^{*,n}, p_2^{*,n}, \cdots, p_{i-1}^{*,n}, p, p_{i+1}^{*,n}, \cdots, p_n^{*,n}, Q), \quad i = 1, \cdots, n \quad (23)$$

to get $p^{*,n} = (p_1^{*,n}, \cdots, p_n^{*,n}, c_{n+1}^{*,n+1}, \cdots, c_N^{*,N})$. More precisely, we define a Nash Equilibrium as follows.

Definition 3.4 *A vector of prices $p^* = p^*(Q) = (p_1^*, p_2^*, \cdots, p_N^*)$ is called a Nash equilibrium if*

$$p_i^* = \arg\max_{p \geq c_i} \Pi_i(p_1^*, p_2^*, \cdots, p_{i-1}^*, p, p_{i+1}^*, \cdots, p_N^*, Q), \quad (24)$$

and $p_i^* = c_i^{*,i}(p^*, Q)$ whenever $\widehat{h}_i(p^*) = 0$, $i = 1, 2, \cdots, N$.

We assume the following on a subgame for our discussion.

Assumption 3.5 *For $n = 1, \cdots, N$, we assume that there exists a unique solution to the system of maximization problems in equation (23).*

Remark 3.6 We observe from Definition of the Nash Equilibrium that, in equilibrium, a seller is actually participating in the Bertrand game only when her actual demand function is positive, and those with zero actual demand function will be ignored in the subsequent subgames. However, a participating seller does not necessarily have positive profit unless she sets the price higher than the waiting cost. In other words, it is possible that $\widehat{h}_i(p^*) > 0$, but $p_i^* = c_i(p^*, Q)$, so that $\Pi_i(p^*, Q) = 0$. We refer to such a case the *boundary* case, and denote the price to be $c_i^{*,b}$.

The following result details the procedure of finding the Nash equilibrium for the Bertrand competition. The idea is quite similar to that in [13], except for the general form of the waiting cost. We sketch the proof for completeness.

Proposition 3.7 *Assume that Assumption 3.2 is in force. Then there exists a Nash equilibrium to the Bertrand game (21) and (24).*

Moreover, the equilibrium point p^*, after modifications, should take the following form:

$$p^* = (p_1^*, \cdots, p_k^*, c_{k+1}^{*,b}, \cdots, c_n^{*,b}, c_{n+1}^*, \cdots, c_N^*), \tag{25}$$

from which we can immediately read: $\widehat{h}_i(p^*) > 0$ and $p_i^* > c_i^*$, $i = 1, \cdots, k$; $\widehat{h}_i(p^*) > 0$ but $p_i^* \le c_i^*$, $i = k+1, \cdots, n$; and $\widehat{h}_i(p^*) \le 0$, $i = n+1, \cdots, N$.

Proof We start with N sellers, and we shall drop the superscript N from all the notations, for simplicity. Let $p^* = (p_1^*, p_2^*, \cdots, p_N^*)$ be the candidate equilibrium prices (obtained by, for example, the first-order condition). By exchangeability, we can assume without loss of generality that the prices are ordered: $p_1^* \le p_2^* \le \cdots \le p_N^*$, and so are the corresponding cost functions $c_1^* \le c_2^* \le \cdots \le c_N^*$, where $c_i^* = c_i(p^*, Q)$ for $i = 1, \cdots, N$.

We first compare $p_N^{*,N}$ and $c_N^{*,N}$.

Case 1. $p_N^* > c_N^*$. We consider the following cases:

(a) If $h_N^N(p^*) > 0$, then by Definition 3.1 we have $\widehat{h}_i(p^*) = h_i^N(p^*) > 0$, for all i, and $p^* = (p_1^*, p_2^*, \cdots, p_N^*)$ is an equilibrium point.

(b) If $h_N^N(p^*) \le 0$, then in light of the definition of actual demand function (Definition 3.1), we have $\widehat{h}_N(p^*) = 0$. Thus, the N-th seller will have zero profit regardless where she sets the price. We shall require in this case that the N-th seller reduces her price to c_N^*, and we shall consider remaining $(N-1)$-sellers' candidate equilibrium prices $p^{*,N-1} = (p_1^{*,N-1}, \cdots, p_{N-1}^{*,N-1})$.

Case 2. $p_N^* \le c_N^*$. In this case the N-th seller would have a non-positive profit at the best. Thus, she sets $p_N^* = c_N^*$, and quits the game, and again the problem is reduced to a subgame with $(N-1)$ sellers, and to Case 1-(b). We should note that in the "boundary case" described in Remark 3.6, we will write $p_N^* = c_N^{*,b}$.

Repeating the same procedure for the subgames (for $n = N-1, \cdots, 2$), we see that eventually we will get a modified equilibrium point p^* of the form (25), proving the proposition. □

3.2 A linear mean-field case

In this subsection, we consider a special case, studied in [27], but with the modified waiting cost functions. More precisely, we assume that there are N sellers, each with demand function

$$h_i^N(p_1, \cdots, p_N) \triangleq A - Bp_i + C\bar{p}_i^N, \tag{26}$$

where $A, B, C > 0$, and $B > C$, and $\bar{p}_i^N = \frac{1}{N-1} \sum_{j \neq i}^{N} p_j$. We note that the structure of the demand function (26) obviously reflects a mean-field nature, and one can easily check that it satisfies all the assumptions mentioned in the previous subsection. Furthermore, as was shown in [27, Proposition 2.4], the actual demand function takes the form: for each $n \in \{1, \cdots, N-1\}$,

$$h_i^n(p_1, \cdots, p_n) = a_n - b_n p_i + c_n \bar{p}_i^n, \quad \text{for } i = 1, \cdots, n,$$

where $\bar{p}_i^n = \frac{1}{n-1} \sum_{j \neq i}^{n} p_j$, and the parameters (a_n, b_n, c_n) can be calculated recursively for $n = N, \cdots, 1$, with $a_N = A$, $b_N = B$ and $c_N = C$. We note that in these works the (waiting) costs are assumed to be constant.

Let us now assume further that the waiting cost is also linear. For example, for $n = 1, \cdots, N$,

$$c_i^n = c_i^n(p_i, \bar{p}_i^n, Q) \triangleq x_n(Q) p_i - y_n(Q) \bar{p}_i^n, \qquad x_n(Q), y_n(Q) > 0.$$

Note that the profit function for seller i is

$$\Pi_i(p_1, \cdots, p_n, Q) = (a_n - b_n p_i + c_n \bar{p}_i^n) \cdot \left(p_i - (x_n p_i - y_n \bar{p}_i^n)\right). \tag{27}$$

An easy calculation shows that the critical point for the maximizer is

$$p_i^{*,n} = \frac{a_n}{2b_n} + \left(\frac{c_n}{2b_n} - \frac{y_n}{2(1-x_n)}\right) \bar{p}_i^n, \tag{28}$$

which is the optimal choice of seller i if the other sellers set prices with average $\bar{p}_i^n = \frac{1}{n-1} \sum_{j \neq i}^n p_j^*$. Now, let us define

$$\bar{p}^n := \frac{1}{n} \sum_{i=1}^{n} p_i^* = \frac{a_n(1-x_n)}{2b_n(1-x_n) - c_n(1-x_n) + b_n y_n}. \tag{29}$$

Then, it is readily seen that $\bar{p}_i^n = \frac{n}{n-1} \bar{p}^n - \frac{1}{n-1} p_i^*$, which means (plugging back into (28))

$$p_i^{*,n} = \frac{a_n}{2b_n + \frac{c_n}{n-1} - \frac{1}{n-1} \frac{b_n y_n}{1-x_n}} + \frac{1}{\frac{n-1}{n} \frac{2b_n(1-x_n)}{c_n(1-x_n) - b_n y_n} + \frac{1}{n}} \bar{p}^n. \tag{30}$$

For the sake of argument, let us assume that the coefficients $(a_n, b_n, c_n, x_n(Q), y_n(Q))$ converge to $(a, b, c, x(Q), y(Q))$ as $n \to \infty$. Then, we see from (29) and (30) that

$$\begin{cases} \lim_{n\to\infty} \bar{p}^n = \dfrac{a(1-x)}{2b(1-x)-c(1-x)+by} =: \bar{p}; \\ \lim_{n\to\infty} p_i^{*,n} = \dfrac{a}{2b} + \dfrac{c(1-x)-by}{2b(1-x)} \lim_{n\to\infty} \bar{p}^n = \dfrac{a(1-x)}{(2b-c)(1-x)+by} =: p^*. \end{cases} \tag{31}$$

It is worth noting that if we assume that there is a "representative seller" who randomly sets prices $p = p_i$ with equal probability $\frac{1}{n}$, then we can randomize the profit function (27):

$$\Pi_n(p,\bar{p}) = (a_n - b_n p + c_n \bar{p})\Big(p - (x_n p - y_n \bar{p})\Big), \tag{32}$$

where p is a random variable taking value $\{p_i\}$ with equal probability, and $\bar{p} \sim \mathbb{E}[p]$, thanks to the Law of Large Numbers, when n is large enough. In particular, in the limiting case as $n \to \infty$, we can replace the randomized profit function Π_n in (32) by:

$$\Pi_\infty = \Pi(p,\mathbb{E}[p]) := (a - bp + c\mathbb{E}[p])\Big(p - (xp - y\mathbb{E}[p])\Big). \tag{33}$$

A similar calculation as (28) shows that $(p^*, \mathbb{E}[p^*]) \in \mathrm{argmax}\Pi(p,\mathbb{E}[p])$ will take the form

$$p^* = \frac{c(1-x)-by}{2b(1-x)}\mathbb{E}[p^*] + \frac{a}{2b} \qquad \text{and} \qquad \mathbb{E}[p^*] = \frac{a(1-x)}{2b(1-x)-c(1-x)+by}.$$

Consequently, we see that $p^* = \frac{a(1-x)}{(2b-c)(1-x)+by}$, as we see in (31).

Remark 3.8 The analysis above indicates the following facts: (i) If we consider the sellers in a "homogeneous" way, and as the number of sellers becomes large enough, all of them will actually choose the same strategy, as if there is a "representative seller" that sets the prices uniformly; (ii) The limit of equilibrium prices actually coincides with the optimal strategy of the representative seller under a limiting profit function. These facts are quite standard in mean-field theory, and will be used as the basis for our dynamic model for the (sell) LOB in the next section.

4 Mean-field type liquidity dynamics in continuous time

In this section we extend the idea of Bertrand game to the continuous time setting. To begin with, we assume that the contribution of each individual seller to the LOB is measured by the "liquidity" (i.e., the number of shares of the given asset) she provides, which is the function of the selling price she chooses, hence under the Bertrand game framework.

4.1 A general description

We begin by assuming that there are N sellers, and denote the liquidity that the i-th seller "adds" to the LOB at time t by Q^i_t. We shall assume that it is a pure jump Markov process, with the following generator: for any $f \in C([0,T]\times\mathbb{R}^N)$, and $(t,q)\in[0,T]\times\mathbb{R}^N$,

$$\mathscr{A}^{i,N}[f](t,q) := \int_{\mathbb{R}} \lambda^i(t,q,\theta)\Big(f(t,q_{-i}(q_i+h^i(t,\theta,z)) - f(t,q) - \langle \partial_{x_i} f, h^i(t,\theta,z)\rangle\Big)\nu^i(dz), \tag{34}$$

where $q \in \mathbb{R}^N$, and $q_{-i}(y) = (q_1,\cdots,q_{i-1},y,q_{i+1},\cdots,q_N)$. Furthermore, h^i denotes the demand function for the i-th seller, and $\theta \in \mathbb{R}^k$ is a certain market parameter which will be specified later. Roughly speaking, (34) indicates that the i-th seller would act (or "jump") at stopping times $\{\tau^i_j\}_{j=1}^\infty$ with the waiting times $\tau^i_{j+1}-\tau^i_j$ having exponential distribution with intensity $\lambda^i(\cdot)$, and jump size being determined by the demand function $h^i(\cdots)$. The total liquidity provided by all the sellers is then a pure jump process with the generator

$$\mathscr{A}^N[f](t,q,\theta) = \sum_{i=1}^N \mathscr{A}^{i,N}[f](t,q), \qquad q\in\mathbb{R}^N,\ N\in\mathbb{N}, \quad t\in[0,T]. \tag{35}$$

We now specify the functions λ^i and h^i further. Recalling the demand function introduced in the previous section, we assume that there are two functions λ and h, such that for each i, and for $(t,x,q,p)\in[0,T]\times\mathbb{R}\times\mathbb{R}^{2N}$,

$$\lambda^i(t,q,\theta) = \lambda(t,q^i,p^i,\mu^N), \quad h^i(t,\theta,z) = h(t,x,q^i,p^i,z), \tag{36}$$

where $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{p^i}$, x denotes the fundamental price at time t, and p^i is the sell price. We shall consider $p=(p^1,\cdots,p^N)$ as the control variable, as the Bertrand game suggests. Now, if we assume $\nu^i=\nu$ for all i, then we have a *pure jump Markov game of mean-field-type*, similar to the one considered in [6], in which each seller adds liquidity (in terms of number of shares) dynamically as a pure jump Markov process, denoted by Q^i_t, $t\geq 0$, with the kernel

$$\nu(t,q^i,\mu^N,p^i,dz) = \lambda(t,q^i,p^i,\mu^N)[\nu\circ h^{-1}(t,x,q^i,p^i,\cdot)](dz). \tag{37}$$

Furthermore, in light of the static case studied in the previous section, we shall assume that the seller's instantaneous profit at time $t>0$ takes the form $(p^i_t-c^i_t)\Delta Q^i_t$, where c^i_t is the "waiting cost" for i-th seller at time t. We observe that the submitted sell price p^i can be written as $p^i = x+l^i$, where x is the fundamental price and l^i is the distance from x that the i-th seller chooses to set. Now let us assume that there is an invertible relationship between the selling prices p and the corresponding number of shares q, e.g., $p=\varphi(q)$ (such a relation is often used to convert the Bertrand game to Cournot game, see, e.g., [27]), and consider l as the *control variable*. We can then rewrite the functions λ and h of (36) in the following form:

$$\lambda^i(t,q,\theta) = \lambda(t,q^i,l^i,\tilde{\mu}^N(\varphi(q))), \quad h^i(t,q,\theta) = h(t,x,q^i,l^i,z). \tag{38}$$

To simplify the presentation, in what follows, we shall assume that λ does not depend on the control variable l^i, and that both λ and h are time-homogeneous. In other words, we assume that each Q^i follows a pure jump SDE studied in §2:

$$Q^i_t = q^i + \int_0^t \int_{A\times\mathbb{R}_+} h(X_r, Q^i_{r-}, l^i_r, z)\mathbf{1}_{[0,\lambda(Q^i_{r-},\mu^N_{\varphi(\mathbf{Q}_r)})]}(y)\mathcal{N}^s(drdzdy), \tag{39}$$

where $\mathbf{Q}_t = (Q^1_t, \cdots, Q^N_t)$, $\mathcal{N}^s$ is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+$, and $\{X_t\}_{t\ge0}$ is the fundamental price process of the underlying asset which we assume to satisfy the SDE (cf. [32]):

$$X^{t,x}_s = x + \int_t^s b(X^{t,x}_r)dr + \int_t^s \sigma(X^{t,x}_r)dW_r, \tag{40}$$

where b and σ are deterministic functions satisfying some standard conditions. We shall assume that the i-th seller is aiming at maximizing the expected total accumulated profit:

$$\begin{aligned} &\mathbb{E}\Big\{\sum_{t\ge0}(p^i_t - c^i_t)\Delta Q^i_t\Big\} \\ &= \mathbb{E}\Big\{\int_0^\infty\int_A h(X_t, Q^i_t, l^i_t, z)(X_t + l^i_t - c^i_t)\lambda(Q^i_t, \mu^N_{\varphi(\mathbf{Q}_t)})\nu^s(dz)dt\Big\}. \end{aligned} \tag{41}$$

We remark that in (41) the time horizon is allowed to be infinity, which can be easily converted to finite horizon by setting $h(X_t,\cdots) = 0$ for $t \ge T$, for a given time horizon $T > 0$, which we do not want to specify at this point. Instead, our focus will be mainly on the limiting behavior of the equilibrium when $N \to \infty$. In fact, given the "symmetric" nature of the problem (i.e., all seller's having the same λ and h), as well as the results in the previous section, we envision a "representative seller" in a limiting mean-field type control problem whose optimal strategy coincides with the limit of N-seller Nash equilibrium as $N \to \infty$, just as the well-known continuous diffusion cases (see, e.g., [29] and [9, 12]). We should note such a result for pure jump cases has been substantiated in a recent work [6], in which it was shown that, under reasonable conditions, in the limit the total liquidity $Q_t = \sum_{i=1}^N Q^i_t$ will converge to a pure jump Markovian process with a mean-field type generator. Based on this result, as well as the individual optimization problem (39) and (41), it is reasonable to consider the following (limiting) mean-filed type pure jump stochastic control problem for a representative seller, whose total liquidity has a dynamics that can be characterized by the following mean-field type pure jump SDE:

$$Q_t = q + \int_0^t \int_{A\times\mathbb{R}_+} h(X_r, Q_{r-}, l_r, z)\mathbf{1}_{[0,\lambda(Q_{r-},\mathbb{P}_{Q_r})]}(y)\mathcal{N}^s(drdzdy), \quad (42)$$

where $\lambda(Q, \mathbb{P}_Q) := \lambda(Q, \mathbb{E}[\varphi(Q)])$ by a slight abuse of notation, and with the cost functional:

$$\Pi(q,l) = \mathbb{E}\Big\{\int_0^\infty \int_A h(X_t, Q_t, l_t, z)(X_t + l_t - c_t)\lambda(Q_t, \mathbb{P}_{Q_t})\nu^s(dz)dt\Big\}. \quad (43)$$

4.2 Problem formulation

With the general description in mind, we now give the formulation of our problem. First, we note that the liquidity of the LOB will not only be affected by the liquidity providers (i.e., the sellers), but also by liquidity consumer, that is, the market buy orders as well as the cancellations of sell orders (which we assume is free of charge). We shall describe its collective movement (in terms of number of shares) of all such consumptional orders as a compound Poisson process, denoted by $\beta_t = \sum_{i=1}^{N_t} \Lambda_t$, $t \geq 0$, where $\{N_t\}$ is a standard Poisson process with parameter λ, and $\{\Lambda_i\}$ is a sequence of i.i.d. random variables taking values in a set $B \subseteq \mathbb{R}$, with distribution ν. Without loss of generality, we assume that counting measure of β coincides with the canonical Poisson random measure $\mathcal{N}^b$, so that the Lévy measure is $\nu^b = \lambda\nu$. In other words, $\beta_t := \int_0^t \int_B z\, \widetilde{\mathcal{N}}^b(drdz)$, and the total liquidity satisfies the SDE:

$$Q_t^0 = q + \int_0^t \int_{A\times\mathbb{R}_+} h(X_r, Q_{r-}^0, l_r, z)\mathbf{1}_{[0,\lambda(Q_{r-}^0,\mathbb{P}_{Q_r^0})]}(y)\mathcal{N}^s(drdzdy) - \beta_t. \quad (44)$$

We remark that there are two technical issues for the dynamics (44). First, the presence of the buy order process β brings in the possibility that $Q_t^0 < 0$, which should never happen in reality. We shall therefore assume that the buy order has a natural upper limit: the total available liquidity Q_t^0. That is, if we denote $\mathcal{S}_\beta = \{t : \Delta\beta_t \neq 0\}$, then for all $t \in \mathcal{S}_\beta$, we have $Q_t^0 = (Q_{t-}^0 - \Delta\beta_t)^+$. Consequently, we can assume that there exists a process $K = \{K_t\}$, where K is a non-decreasing, pure jump process such that (i) $\mathcal{S}_K = \mathcal{S}_\beta$; (ii) $\Delta K_t := (Q_{t-}^0 - \Delta\beta_t)^-$, $t \in \mathcal{S}_K$; and (iii) the Q^0-dynamics (44) can be written as, for $t \geq 0$,

$$\begin{aligned} Q_t &= q + \int_0^t \int_{A\times\mathbb{R}_+} h(X_r, Q_{r-}, l_r, z)\mathbf{1}_{[0,\lambda(Q_{r-},\mathbb{P}_{Q_r})]}(y)\mathcal{N}^s(drdzdy) - \beta_t + K_t \\ &= q + \int_0^t \int_{A\times\mathbb{R}_+} h(X_r, Q_{r-}, l_r, z)\mathbf{1}_{[0,\lambda(Q_{r-},\mathbb{P}_{Q_r})]}(y)\widetilde{\mathcal{N}}^s(drdzdy) \qquad (45) \\ &\quad - \int_0^t \int_B z\, \widetilde{\mathcal{N}}^b(drdz) + \int_0^t \int_A h(X_r, Q_r, l_r, z)\lambda(Q_r, \mathbb{P}_{Q_r})\nu^s(dz)dr + K_t. \end{aligned}$$

where K is a "reflecting process", and $\widetilde{\mathcal{N}}^s(drdzdy)$ is the compensated Poisson martingale measure of $\mathcal{N}^s$. That is, (45) is a (pure jump) mean-field SDE with reflection as was studied in §2.

Now, in light of the discussion of MFSDEDR in §2, we shall consider the following two MFSDEDRs that are slightly more general than (45): for $\xi \in L^2(\mathcal{F}_t;\mathbb{R})$, $q \in \mathbb{R}$, and $0 \le s \le t$,

$$\begin{aligned} Q_s^{t,\xi} &= \xi + \int_t^s \int_{A\times\mathbb{R}_+} h(X_r^{t,x}, Q_{r-}^{t,\xi}, l_r, z)\, \mathbf{1}_{[0,\lambda(Q_{r-}^{t,\xi},\mathbb{P}_{Q_r^{t,\xi}})]}(y)\widetilde{\mathcal{N}}^s(drdzdy) \\ &\quad - \int_t^s \int_B z\, \widetilde{\mathcal{N}}^b(drdz) + \int_t^s a(X_r^{t,x}, Q_r^{t,\xi}, \mathbb{P}_{Q_r^{t,\xi}}, l_r)dr + K_s^{t,\xi}, \qquad (46) \\ Q_s^{t,q,\xi} &= q + \int_t^s \int_{A\times\mathbb{R}_+} h(X_r^{t,x}, Q_{r-}^{t,q,\xi}, l_r, z)\, \mathbf{1}_{[0,\lambda(Q_{r-}^{t,q,\xi},\mathbb{P}_{Q_r^{t,\xi}})]}(y)\widetilde{\mathcal{N}}^s(drdzdy) \\ &\quad - \int_t^s \int_B z\, \widetilde{\mathcal{N}}^b(drdz) + \int_t^s a(X_r^{t,x}, Q_r^{t,q,\xi}, \mathbb{P}_{Q_r^{t,\xi}}, l_r)dr + K_s^{t,q,\xi}, \qquad (47) \end{aligned}$$

where $l = \{l_s\}$ is the control process for the representative seller, and $Q_s = Q_s^{t,q,\xi}$, $s \ge t$, is the total liquidity of the sell-side LOB. We shall consider the following set of *admissible strategies*:

$$\mathscr{U}_{ad} := \{l \in L^1_{\mathbb{F}}([0,\infty);\mathbb{R}_+) : l \text{ is } \mathbb{F}\text{-predictable}\}. \qquad (48)$$

The objective of the seller is to solve the following mean-field stochastic control problem:

$$v(x,q,\mathbb{P}_\xi) = \sup_{l\in\mathscr{U}_{ad}} \Pi(x,q,\mathbb{P}_\xi,l) = \sup_{l\in\mathscr{U}_{ad}} \mathbb{E}\Big[\int_0^\infty e^{-\rho r} L(X_r^x, Q_r^{q,\xi}, \mathbb{P}_{Q_r^\xi}, l_r)dr\Big] \qquad (49)$$

where $L(x,q,\mu,l) := \int_A h(x,q,l,z)c(x,q,l)\lambda(q,\mu)\nu^s(dz)$, and $\mathscr{U}_{ad}$ is defined in (48). Here we denote $X^x := X^{0,x}$, $Q^{q,\xi} := Q^{0,q,\xi}$.

Remark 4.1 (i) In (46) and (47), we allow a slightly more general drift function a, which in particular could be $a(x,q,\mu,l) = \lambda(q,\mu)\int_A h(x,q,l,z)\nu^s(dz)$, as is in (45).

(ii) In (49), the pricing function $c(x,q,l)$ is a more general expression of the original form $x + l - c$ in (43), taking into account the possible dependence of the waiting cost c_t on the sell position l and the total liquidity q at time t.

(iii) Compared to (43), we see that a *discounting factor* $e^{-\rho t}$ is added to the cost functional $\Pi(\cdots)$ in (49), reflecting its nature as the "present value".

In the rest of the paper we shall assume that the market parameters b, σ, λ, h, the pricing function c in (46) – (49), and the discounting factor ρ satisfy the following assumptions.

Assumption 4.2 *All functions $b, \sigma \in C^0(\mathbb{R})$, $\lambda \in L^0(\mathbb{R} \times \mathscr{P}_2(\mathbb{R}); \mathbb{R}_+)$, $h \in L^0(\mathbb{R}^2 \times \mathbb{R}_+ \times A)$, and $c \in L^0(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$ are bounded, and satisfy the following conditions, respectively:*

(i) b and σ are uniformly Lipschitz continuous in x with Lipschitz constant $L > 0$;

(ii) $\sigma(0) = 0$ and $b(0) \geq 0$;

(iii) λ and h satisfy Assumption 2.1;

(iv) For $l \in \mathbb{R}_+$, $c(x, q, l)$ is Lipschitz continuous in (x, q), with Lipschitz constant $L > 0$;

(v) h is non-increasing, and c is non-decreasing in the variable l;

(vi) $\rho > L + \frac{1}{2}L^2$, where $L > 0$ is the Lipschitz constant in Assumption 2.1;

(vii) For $(x, \mu, l) \in \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}) \times \mathbb{R}_+$, $\Pi(x, q, \mu, l)$ is convex in q.

Remark 4.3 (i) The monotonicity assumptions in Assumption 4.2-(v) are inherited from §3. Specifically, they are the assumption (15) for h, and Assumption 3.1-(i) for c, respectively.

(ii) Under Assumption 4.2, one can easily check that the SDEs (40) as well as (46) and (47) all have pathwisely unique strong solutions in $L^2_{\mathbb{F}}(\mathbb{D}([0, T]))$, thanks to Theorem 2.3; and Assumption 4.2-(ii) implies that $X^{t,x}_s \geq 0$, $s \in [t, \infty)$, $\mathbb{P}$-a.s., whenever $x \geq 0$.

5 Dynamic programming principle

In this section we substantiate the *dynamic programming principle* (DPP) for the stochastic control problem (46)–(49). We begin by examining some basic properties of the value function.

Proposition 5.1 *Under the Assumptions 2.1 and 4.2, the value function $v(x, q, \mathbb{P}_\xi)$ is Lipschitz continuous in $(x, q, \mathbb{P}_\xi)$, non-decreasing in x, and decreasing in q.*

Proof We first check the Lipschitz property in x. For $x, x' \in \mathbb{R}$, denote $X^x = X^{0,x}$ and $X^{x'} = X^{0,x'}$ as the corresponding solutions to (40), respectively. Denote $\Delta X_t = X^x_t - X^{x'}_t$, and $\Delta x = x - x'$. Then, applying Itô's formula to $|\Delta X_t|^2$ and by some standard arguments, one has

$$|\Delta X_t|^2 = |\Delta x|^2 + \int_0^t (2\alpha_s + \beta_s^2)|\Delta X_s|^2 ds + \int_0^t 2\beta_s |\Delta X_s|^2 dW_s,$$

where α, β are two processes bounded by the Lipschitz constants L in Assumption 2.1, thanks to Assumption 4.2. Thus, one can easily check, by taking expectation and applying Burkholder-Davis-Gundy and Gronwall inequalities, that

$$\mathbb{E}[|\Delta X|^{*,2}_t] \leq |\Delta x|^2 e^{(2L+L^2)t}, \qquad t \geq 0. \tag{50}$$

Furthermore, it is clear that, under Assumption 4.2, the function $L(x, q, \mu, l)$ is uniformly Lipschitz in x, uniformly in (q, μ, l). That is, for some generic constant $C > 0$ we have

$$|\Pi(x, q, \mathbb{P}_\xi, l) - \Pi(x', q, \mathbb{P}_\xi, l)| \le C\mathbb{E}\Big[\int_0^\infty \int_A e^{-\rho t}|\Delta X_t|\nu^s(dz)dt\Big]$$
$$\le C\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\sqrt{\mathbb{E}[|\Delta X|_t^{*,2}]}dt\Big] \le C|\Delta x|\int_0^\infty e^{-\rho t}e^{(L+\frac{1}{2}L^2)t}dt \le C|x-x'|.$$

Here the last inequality is due to Assumption 4.2-(vi). Consequently, we obtain

$$|v(x, q, \mathbb{P}_\xi) - v(x', q, \mathbb{P}_\xi)| \le C|x - x'|, \quad \forall x, x' \in \mathbb{R}. \tag{51}$$

To check the Lipschitz properties for q and $\mathbb{P}_\xi$, we denote, for $(q, \mathbb{P}_\xi) \in \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R})$, $h_s^{q,\xi} \equiv h(X_s, Q_s^{t,q,\xi}, l_s, z)$, $\lambda_s^{q,\xi} \equiv \lambda(Q_s^{t,q,\xi}, \mathbb{P}_{Q_s^{t,\xi}})$, and $c_s^{q,\xi} \equiv c(X_s, Q_s^{t,q,\xi}, l_s)$, $s \ge t$. Furthermore, for $q, q' \in \mathbb{R}_+$ and $\mathbb{P}_\xi, \mathbb{P}_{\xi'} \in \mathscr{P}_2(\mathbb{R})$, we denote $\Delta\psi_r \equiv \psi_r^{q,\xi} - \psi_r^{q',\xi'}$ for $\psi = h, \lambda, c$. Now, by Assumptions 2.1 and 4.2, and following a similar argument of Theorem 2.3, one shows that

$$|\Pi(x, q, \mathbb{P}_\xi, l) - \Pi(x, q', \mathbb{P}_{\xi'}, l)|$$
$$\le \mathbb{E}\Big\{\int_0^\infty\int_A e^{-\rho r}\big(h_r^{q,\xi}c_r^{q,\xi}|\Delta\lambda_r| + c_r^{q,\xi}\lambda_r^{q',\xi'}|\Delta h_r| + h_r^{q',\xi'}\lambda_r^{q',\xi'}|\Delta c_r|\big)\nu^s(dz)dr\Big\}$$
$$\le \mathbb{E}\Big\{\int_0^\infty\int_A e^{-\rho r}|Q_r^{t,q,\xi} - Q_r^{t,q',\xi'}|\nu^s(dz)dr\Big\} \le C\left(|q-q'| + W_1(\mathbb{P}_\xi, \mathbb{P}_{\xi'})\right),$$

which implies that

$$|v(x, q, \mathbb{P}_\xi) - v(x, q', \mathbb{P}'_\xi)| \le C\left(|q - q'| + W_1(\mathbb{P}_\xi, \mathbb{P}_{\xi'})\right). \tag{52}$$

Finally, the respective monotonicity of the value function on x and q follows from the comparison theorem of the corresponding SDEs and Assumption 4.2. This completes the proof. □

We now turn our attention to the DPP. The argument will be very similar to that of [32], except for some adjustments to deal with the mean-field terms. But, by using the flow-property (5) we can carry out the argument without substantial difficulty.

Theorem 5.2 *Assume that Assumptions 2.1 and 4.2 are in force. Then, for any* $(x, q, \mathbb{P}_\xi) \in \mathbb{R}^2 \times \mathscr{P}_2(\mathbb{R})$ *and for any* $t \in (0, \infty)$,

$$v(x, q, \mathbb{P}_\xi) = \sup_{l\in\mathscr{U}_{ad}} \mathbb{E}\Big[\int_0^t e^{-\rho s}L(X_s^x, Q_s^{q,\xi;l}, \mathbb{P}_{Q_s^{\xi;l}}, l_s)ds$$
$$+ e^{-\rho t}v(X_t^x, Q_t^{q,\xi;l}, \mathbb{P}_{Q_t^{\xi;l}})\Big]. \tag{53}$$

Proof Let us denote the right side of (53) by $\tilde{v}(x, q, \mathbb{P}_\xi) = \sup_l \tilde{\Pi}(x, q, \mathbb{P}_\xi; l)$. We first note that X_r and $(Q_r^{t,\xi}, Q_r^{t,q,\xi})$ have the flow property. So, for any $l \in \mathscr{U}_{ad}$,

$$
\begin{aligned}
&\Pi(x,q,\mathbb{P}_\xi;l)=\mathbb{E}\Big[\int_0^\infty e^{-\rho s}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds\Big]\\
&=\mathbb{E}\Big[\int_0^t e^{-\rho s}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds\\
&\qquad+e^{-\rho t}\mathbb{E}\Big\{\int_t^\infty e^{-\rho(s-t)}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds\Big|\mathcal{F}_t\Big\}\Big] \qquad (54)\\
&=\mathbb{E}\Big[\int_0^t e^{-\rho s}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds+e^{-\rho t}\Pi(X_t^x,Q_t^{q,\xi;l},\mathbb{P}_{Q_t^{\xi;l}};l)\Big]\\
&\le\mathbb{E}\Big[\int_0^t e^{-\rho s}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds+e^{-\rho t}v(X_t^x,Q_t^{q,\xi;l},\mathbb{P}_{Q_t^{\xi;l}})\Big]\\
&=\tilde\Pi(x,q,\mathbb{P}_\xi;l).
\end{aligned}
$$

This implies that $v(x,q,\mathbb{P}_\xi)\le\tilde v(x,q,\mathbb{P}_\xi)$.

To prove the other direction, let us denote $\Gamma=\mathbb{R}_+\times\mathbb{R}\times\mathscr{P}_2(\mathbb{R})$, and consider, at each time $t\in(0,\infty)$, a countable partition $\{\Gamma_i\}_{i=1}^\infty$ of Γ and $(x_i,q_i,\mathbb{P}_{\xi_i})\in\Gamma_i$, $\xi_i\in L^2(\mathcal{F}_t)$, $i=1,2,\cdots$, such that for any $(x,q,\mu)\in\Gamma_i$ and for fixed $\varepsilon>0$, it holds $|x-x_i|\le\varepsilon$, $q_i-\varepsilon\le q\le q_i$, and $W_2(\mu,\mathbb{P}_{\xi_i})\le\varepsilon$. Now, for each i, choose an ε-optimal strategy $l^i\in\mathscr{U}_{ad}$, such that $v(t,x_i,q_i,\mathbb{P}_{\xi_i})\le\Pi(t,x_i,q_i,\mathbb{P}_{\xi_i};l^i)+\varepsilon$, where $\Pi(t,x_i,q_i,\mathbb{P}_{\xi_i};l^i):=\mathbb{E}[\int_t^\infty e^{-\rho(s-t)}L(X_s^{t,x_i},Q_s^{t,q_i,\xi_i},\mathbb{P}_{Q_s^{t,\xi_i}},l_s^i)ds]$ and $v(t,x_i,q_i,\mathbb{P}_{\xi_i})=\sup_{l^i\in\mathscr{U}_{ad}}\Pi(t,x_i,q_i,\mathbb{P}_{\xi_i};l^i)$.

Then, by definition of the value function and the Lipschitz properties (Proposition 5.1) with some constant $C>0$, for any $(x,q,\mu)\in\Gamma_i$, it holds that

$$
\begin{aligned}
\Pi(t,x,q,\mu;l^i)&\ge\Pi(t,x_i,q_i,\mathbb{P}_{\xi_i};l^i)-C\varepsilon\ge v(t,x_i,q_i,\mathbb{P}_{\xi_i})-(C+1)\varepsilon\\
&\ge v(t,x,q,\mu)-(2C+1)\varepsilon. \qquad (55)
\end{aligned}
$$

Now, for any $l\in\mathscr{U}_{ad}$, we define a new strategy $\tilde l$ as follows:

$$
\tilde l_s:=l_s\mathbf{1}_{[0,t]}(s)+\Big[\sum_i l_s^i\mathbf{1}_{\Gamma_i}(X_t^x,Q_t^{q,\xi;l},\mathbb{P}_{Q_t^{\xi;l}})\Big]\mathbf{1}_{(t,\infty)}(s). \qquad (56)
$$

Then, clearly $\tilde l\in\mathscr{U}_{ad}$. To simplify notation, let us denote

$$
I_1=\int_0^t e^{-\rho s}L(X_s^x,Q_s^{q,\xi;l},\mathbb{P}_{Q_s^{\xi;l}},l_s)ds. \qquad (57)
$$

By applying (55) and flow property, we have

$$
\begin{aligned}
&v(x,q,\mu) \geq \Pi(x,q,\mu;\tilde{l})\\
=&\mathbb{E}\Big[I_1 + e^{-\rho t}\mathbb{E}\Big\{\int_t^\infty e^{-\rho(s-t)}L(X_s^x, Q_s^{q,\xi;l}, \mathbb{P}_{Q_s^{\xi;l}}, l_s)ds\Big|\mathcal{F}_t\Big\}\Big]\\
=&\mathbb{E}\Big[I_1 + e^{-\rho t}\Pi(t, X_t^x, Q_t^{q,\xi}, \mathbb{P}_{Q_t^\xi}; \tilde{l})\Big]\\
=&\mathbb{E}\Big[I_1 + e^{-\rho t}\sum_i \Pi(t, X_t^x, Q_t^{q,\xi}, \mathbb{P}_{Q_t^\xi}; l^i)\mathbf{1}_{\Gamma_i}(X_t^x, Q_t^{q,\xi}, \mathbb{P}_{Q_t^\xi})\Big]\\
\geq&\mathbb{E}\Big[I_1 + e^{-\rho t}v(X_t^x, Q_t^{q,\xi}, \mathbb{P}_{Q_t^\xi})\Big] - (2C+1)\varepsilon = \tilde{\Pi}(x,q,\mathbb{P}_\xi;l) - (2C+1)\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we get $v(x,q,\mathbb{P}_\xi) \geq \tilde{v}(x,q,\mathbb{P}_\xi)$, proving (53). □

Remark 5.3 We should note that while it is difficult to specify all the boundary conditions for the value function, the case when $q = 0$ is relatively clear. Note that $q = 0$ means there is zero liquidity for the asset. Then by definition of the liquidity dynamics (45) we see that Q_t will stay at zero until the first positive jump happens. During that period of time there would be no trade, thus by DPP (53) we should have

$$v(x,0,\mu) \equiv 0. \tag{58}$$

Furthermore, since the value function v is non-increasing in q, thanks to Proposition 5.1, and is always non-negative, we can easily see that the following boundary condition is also natural

$$\partial_q v(x,0,\mu) \equiv 0. \tag{59}$$

We shall use (58) and (59) frequently in our future discussion.

6 HJB equation and its viscosity solutions

In this section, we shall formally derive the HJB equation associated to the stochastic control problem studied in the previous section, and show that the value function of the control problem is indeed a viscosity solution of the HJB equation.

To begin with, we first note that, given the DPP (53), as well as the boundary conditions (58) and (59), if the value function v is smooth, then by standard arguments with the help of the Itô's formula (14) and the fact that

$$\partial_q v(X_{t-}, Q_{t-}, \mathbb{P}_{X_{t-}})\mathbf{1}_{\{Q_{t-}=0\}}dK_t = \partial_q v(X_{t-}, 0, \mathbb{P}_{X_{t-}})\mathbf{1}_{\{Q_{t-}=0\}}dK_t \equiv 0,$$

it is not difficult to show that the value function should satisfy the following HJB equation: for $(x,q,\mu) \in \mathbb{R}\times\mathbb{R}_+\times\mathscr{P}_2(\mathbb{R})$,

$$\begin{cases} \rho v(x,q,\mu) = \sup_{l\in\mathbb{R}_+}[\mathscr{J}^l[v](x,q,\mu) + L(x,q,\mu,l)], \\ v(x,0,\mu) = 0, \quad \partial_q v(x,0,\mu) = 0, \end{cases} \tag{60}$$

where $\mathscr{J}^l$ is an integro-differential operator defined by, for any $\phi \in \mathbb{C}_b^{2,(2,1)}(\mathbb{R} \times \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R}))$,

$$\begin{aligned}
&\mathscr{J}^l[\phi](x,q,\mu) \triangleq \Big(b(x)\partial_x + \sigma^2(x)\frac{1}{2}\partial_{xx}^2 + a(x,q,\mu,l)\partial_q\Big)\phi(x,q,\mu)\\
&+\int_A \Big(\phi(x,q+h(x,q,l,z),\mu) - \phi(x,q,\mu) - \partial_q\phi(x,q,\mu)h(x,q,l,z)\Big)\lambda(q,\mu)\nu^s(dz)\\
&-\int_B \Big(\phi(x,q-z,\mu) - \phi(x,q,\mu) - \partial_q\phi(x,q,\mu)z\Big)\nu^b(dz)\\
&+\tilde{\mathbb{E}}\left[\partial_\mu\phi(x,q,\mu,\tilde{\xi})a(x,\tilde{\xi},\mu,l)\right] + \tilde{\mathbb{E}}\Big[\int_0^1\int_A \Big(\partial_\mu\phi(x,q,\mu,\tilde{\xi}+\gamma h(x,\tilde{\xi},l,z))\\
&\qquad -\partial_\mu\phi(x,q,\mu,\tilde{\xi})\Big)h(x,\tilde{\xi},l,z)\lambda(\tilde{\xi},\mu)\nu^s(dz)d\gamma\Big]\\
&-\tilde{\mathbb{E}}\Big[\int_0^1\int_B \Big(\partial_\mu\phi(x,q,\mu,\tilde{\xi}-\gamma z) - \partial_\mu\phi(x,q,\mu,\tilde{\xi})\Big)\times z\nu^b(dz)d\gamma\Big].
\end{aligned} \tag{61}$$

We note that in general, whether there exists smooth solutions to the HJB equation (60) is by no means clear. We therefore introduce the notion of *viscosity solution* for (60). To this end, write $\mathscr{D} := \mathbb{R} \times \mathbb{R}_+ \times \mathscr{P}_2(\mathbb{R})$, and for $(x,q,\mu) \in \mathscr{D}$, we denote

$$\begin{aligned}
\mathscr{U}(x,q,\mu) &:= \Big\{\varphi \in \mathbb{C}_b^{2,(2,1)}(\mathscr{D}) : v(x,q,\mu) = \varphi(x,q,\mu)\Big\};\\
\overline{\mathscr{U}}(x,q,\mu) &:= \Big\{\varphi \in \mathscr{U}(x,q,\mu) : v - \varphi \text{ has a strict maximum at } (x,q,\mu)\Big\};\\
\underline{\mathscr{U}}(x,q,\mu) &:= \Big\{\varphi \in \mathscr{U}(x,q,\mu) : v - \varphi \text{ has a strict minimum at } (x,q,\mu)\Big\}.
\end{aligned}$$

Definition 6.1 *We say a continuous function $v : \mathscr{D} \mapsto \mathbb{R}_+$ is a viscosity subsolution (supersolution, resp.) of (60) in $\mathscr{D}$ if*

$$\rho\varphi(x,q,\mu) - \sup_{l\in\mathbb{R}_+}[\mathscr{J}^l[\varphi](x,q,\mu) + L(x,q,\mu,l)] \le 0, (\textit{resp.} \ge 0) \tag{62}$$

for every $\varphi \in \overline{\mathscr{U}}(x,q,\mu)$ (resp. $\varphi \in \underline{\mathscr{U}}(x,q,\mu)$).

A function $v : \mathscr{D} \mapsto \mathbb{R}_+$ is called a viscosity solution of (60) on $\mathscr{D}$ if it is both a viscosity subsolution and a viscosity supersolution of (60) on $\mathscr{D}$.

Our main result of this section is the following theorem.

Theorem 6.2 *Assume that the Assumptions 2.1 and 4.2 are in force. Then, the value function v, defined by (49), is a viscosity solution of the HJB equation (60).*

Proof For a fixed $\bar{\mathbb{x}} := (\bar{x},\bar{q},\bar{\mu}) \in \mathscr{D}$ with $\bar{\mu} = \mathbb{P}_{\bar{\xi}}$ and $\bar{\xi} \in L^2(\mathcal{F};\mathbb{R})$, and any $\eta > 0$, consider the set $\mathscr{D}_{\bar{\mathbb{x}},\eta} := \{\mathbb{x} = (x,q,\mu) \in \mathscr{D} : \|\mathbb{x} - \bar{\mathbb{x}}\| < \eta\}$, where $\|\mathbb{x} - \bar{\mathbb{x}}\| := \Big(|x-\bar{x}|^2 + |q-\bar{q}|^2 + W_2(\mu,\bar{\mu})\Big)^{1/2}$, and $\mu = \mathbb{P}_\xi$ with $\xi \in L^2(\mathcal{F};\mathbb{R})$.

We first prove that the value function v is a *subsolution* to the HJB equation (60). We proceed by contradiction. Suppose not. Then there exist some $\varphi \in \overline{\mathscr{U}}(\mathbb{x})$ and $\varepsilon_0 > 0$ such that

$$\rho\varphi(\bar{\mathbb{x}}) - \sup_{l\in\mathbb{R}_+}[\mathscr{J}^l[\varphi](\bar{\mathbb{x}}) + L(\bar{\mathbb{x}}, l)] =: 2\varepsilon_0 > 0. \tag{63}$$

Since $A^l(\mathbb{x}) := \mathscr{J}^l[\varphi](\mathbb{x}) + L(\mathbb{x}, l)$ is uniformly continuous in $\mathbb{x}$, uniformly in l, thanks to Assumption 4.2, one shows that there exists $\eta > 0$ such that for any $\mathbb{x} \in \mathscr{D}_{\bar{\mathbb{x}},\eta}$, it holds that

$$\rho\varphi(\mathbb{x}) - \sup_{l\in\mathbb{R}_+}[\mathscr{J}^l[\varphi](\mathbb{x}) + L(\mathbb{x}, l)] \geq \varepsilon_0. \tag{64}$$

Furthermore, since $\varphi \in \overline{\mathscr{U}}(\mathbb{x})$, we assume without loss of generality that $0 = v(\bar{\mathbb{x}}) - \varphi(\bar{\mathbb{x}})$ is the strict maximum. Thus for the given $\eta > 0$, there exists $\delta > 0$, such that

$$\max\left\{v(\mathbb{x}) - \varphi(\mathbb{x}) : \mathbb{x} \notin \mathscr{D}_{\bar{\mathbb{x}},\eta}\right\} = -\delta < 0. \tag{65}$$

On the other hand, for a fixed $\varepsilon \in (0, \min(\varepsilon_0, \delta\rho))$, by the continuity of v we can assume, modifying $\eta > 0$ if necessary, that

$$|v(\mathbb{x}) - v(\bar{\mathbb{x}})| = |v(\mathbb{x}) - \varphi(\bar{\mathbb{x}})| < \varepsilon, \qquad \mathbb{x} \in \mathscr{D}_{\bar{\mathbb{x}},\eta}. \tag{66}$$

Next, for any $T > 0$ and any $l \in \mathscr{U}_{ad}$ we set $\tau^T := \inf\{t \geq 0 : \bar{\Theta}_t \notin \mathscr{D}_{\bar{\mathbb{x}},\eta}\} \wedge T$, where $\bar{\Theta}_t := (X_t^{\bar{x}}, Q_t^{\bar{q},\bar{\xi},l}, \mathbb{P}_{Q_t^{\bar{q},\bar{\xi}}})$. Applying Itô's formula (14) to $e^{-\rho t}\varphi(\bar{\Theta}_t)$ from 0 to τ^T and noting that $v(\bar{\mathbb{x}}) = \varphi(\bar{\mathbb{x}})$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{\tau^T} e^{-\rho t} L(\bar{\Theta}_t, l_t)dt + e^{-\rho\tau^T} v(\bar{\Theta}_{\tau^T})\Big]\\
&= \mathbb{E}\Big[\int_0^{\tau^T} e^{-\rho t} L(\bar{\Theta}_t, l_t)dt + e^{-\rho\tau^T}\varphi(\bar{\Theta}_{\tau^T}) + e^{-\rho\tau^T}[v-\varphi](\bar{\Theta}_{\tau^T})\Big] \qquad (67)\\
&= \mathbb{E}\Big[\int_0^{\tau^T} e^{-\rho t}\Big(L(\bar{\Theta}_t, l_t) + \mathscr{J}^l[\varphi](\bar{\Theta}_t) - \rho\varphi(\bar{\Theta}_t)\Big)dt + e^{-\rho\tau^T}[v-\varphi](\bar{\Theta}_{\tau^T})\Big] + v(\bar{\mathbb{x}})\\
&\leq \mathbb{E}\Big[-\frac{\varepsilon}{\rho}(1 - e^{-\rho\tau^T}) + e^{-\rho\tau^T}[v-\varphi](\bar{\Theta}_{\tau^T})\Big] + v(\bar{\mathbb{x}})\\
&= \mathbb{E}\Big[e^{-\rho\tau^T}\Big(\frac{\varepsilon}{\rho} + [v-\varphi](\bar{\Theta}_{\tau^T})\Big) : \tau^T < T\Big]\\
&\quad + \mathbb{E}\Big[e^{-\rho\tau^T}\Big(\frac{\varepsilon}{\rho} + [v-\varphi](\bar{\Theta}_{\tau^T})\Big) : \tau^T = T\Big] + v(\bar{\mathbb{x}}) - \frac{\varepsilon}{\rho}.
\end{aligned}$$

Now note that on the set $\{\tau^T < T\}$ we must have $\bar{\Theta}_{\tau^T} \notin \mathscr{D}_{\bar{\mathbb{x}},\eta}$, thus $[v - \varphi](\bar{\Theta}_{\tau^T}) \leq -\delta$, thanks to (65). On the other hand, on the set $\{\tau^T = T\}$ we have $\bar{\Theta}_{\tau^T} = \bar{\Theta}_T \in \mathscr{D}_{\bar{\mathbb{x}},\eta}$, and then (66) implies that $[v - \varphi](\bar{\Theta}_T) \leq v(\bar{\mathbb{x}}) - \varphi(\bar{\Theta}_T) + \varepsilon$. Plugging these facts in (67), we can easily obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{\tau^T} e^{-\rho t} L(\bar{\Theta}_t, l_t)dt + e^{-\rho\tau^T} v(\bar{\Theta}_{\tau^T})\Big]\\
&\le \Big(\frac{\varepsilon}{\rho} - \delta\Big)\mathbb{P}\{\tau^T < T\} + (\frac{\varepsilon}{\rho} + \varepsilon)e^{-\rho T} + v(\bar{\mathbb{x}}) - \frac{\varepsilon}{\rho}\\
&\le \Big(\frac{\varepsilon}{\rho} + \varepsilon\Big)e^{-\rho T} + v(\bar{\mathbb{x}}) - \frac{\varepsilon}{\rho}.
\end{aligned}$$

Here in the last inequality above we used the fact that $\varepsilon/\rho - \delta < 0$, by definition of ε. Letting $T \to \infty$ we have

$$\mathbb{E}\Big[\int_0^{\tau^T} e^{-\rho t} L(\bar{\Theta}_t, l_t)dt + e^{-\rho\tau^T} v(\bar{\Theta}_{\tau^T})\Big] \le v(\bar{\mathbb{x}}) - \frac{\varepsilon}{\rho}.$$

Since $l \in \mathscr{U}_{ad}$ is arbitrary, this contradicts the dynamic programming principle (53).

The proof that v is viscosity supersolution of (60) is more or less standard, again with the help of Itô's formula (14). We only give a sketch here.

Let $\bar{\mathbb{x}} \in \mathscr{D}$ and $\varphi \in \underline{\mathscr{U}}(\bar{\mathbb{x}})$. Without loss of generality we assume that $0 = v(\bar{\mathbb{x}}) - \varphi(\bar{\mathbb{x}})$ is a global minimum. That is, $v(\mathbb{x}) - \varphi(\mathbb{x}) \ge 0$ for all $\mathbb{x} \in \mathscr{D}$. For any $h > 0$ and $l \in \mathscr{U}_{ad}$, we apply DPP (53) to get

$$\begin{aligned}
0 &\ge \mathbb{E}\Big[\int_0^h e^{-\rho t} L(\Theta_t, l_t)dt + e^{-\rho h} v(\Theta_h)\Big] - v(\mathbb{x})\\
&\ge \mathbb{E}\Big[\int_0^h e^{-\rho t} L(\Theta_t, l_t)dt + e^{-\rho h} \varphi(\Theta_h)\Big] - \varphi(\mathbb{x}).
\end{aligned} \tag{68}$$

Applying Itô's formula to $e^{-\rho t}\varphi(\Theta_t)$ from 0 to h we have

$$0 \ge \mathbb{E}\Big[\int_0^h e^{-\rho t} \Big(L(\Theta_t, l_t) + \mathscr{J}^l[\varphi](\Theta_t) - \rho\varphi(\Theta_t)\Big) dt\Big]. \tag{69}$$

Dividing both sides by h and sending h to 0, we obtain $\rho\varphi(x, q, \mathbb{P}_\xi) \ge \mathscr{J}^l[\varphi](x, q, \mathbb{P}_\xi) + L(x, q, \mathbb{P}_\xi, l)$. By taking supremum over $l \in \mathscr{U}_{ad}$ on both sides, we conclude

$$\rho\varphi(x, q, \mathbb{P}_\xi) \ge \sup_{l \in \mathscr{U}_{ad}} [\mathscr{J}^l[\varphi](x, q, \mathbb{P}_\xi) + L(x, q, \mathbb{P}_\xi, l)].$$

The proof is now complete. □

Finally, we remark that, as the limiting case of a Bertrand-type of game for a large number of sellers, the value function $v(x, q, \mathbb{P}_\xi)$ in (49) can be thought of as the discounted lifelong expected utility of a representative seller, and thus can be considered as "equilibrium" discounted expected utility for all sellers. Moreover, as one can see in Proposition 5.1, the value function $v(x, q, \mathbb{P}_\xi)$ is uniformly Lipschitz continuous, non-decreasing in x, and decreasing in q. Also, by Assumption 4.2-(vii), the value function is convex in q. Consequently, we see that the value function $v(x, q, \mathbb{P}_\xi)$ resembles the *expected utility function* $U(x, q)$ in [32] which was defined by the following properties:

(i) the mapping $x \mapsto U(x, q)$ is non-decreasing, and $\frac{\partial U(x,q)}{\partial q} < 0$, $\frac{\partial^2 U(x,q)}{\partial q^2} > 0$;

(ii) the mapping $(x, q) \mapsto U(x, q)$ is uniformly Lipschitz continuous.

In particular, we may identify the two functions by setting $U(x, q) = v(x, q, \mathbb{P}_\xi)|_{\xi \equiv q}$, which amounts to saying that the equilibrium density function of a LOB is fully described by the value function of a control problem of the representative seller's Bertrand-type game. This would enhance the notion of "endogenous dynamic equilibrium LOB model" of [32] in a rather significant way.

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