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On non-Markovian forward–backward SDEs and backward stochastic PDEs

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Abstract

In this paper, we establish an equivalence relationship between the wellposedness of *forward–backward SDEs* (FBSDEs) with random coefficients and that of *backward stochastic PDEs* (BSPDEs). Using the notion of the "decoupling random field", originally observed in the well-known Four Step Scheme (Ma et al., 1994 [13]) and recently elaborated by Ma et al. (2010) [14], we show that, under certain conditions, the FBSDE is wellposed if and only if this random field is a Sobolev solution to a *degenerate* quasilinear BSPDE, extending the existing non-linear Feynman–Kac formula to the random coefficient case. Some further properties of the BSPDEs, such as comparison theorem and stability, will also be discussed.

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1. Introduction

A forward–backward SDE (FBSDE) is the following system of Itô-type of SDEs:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dB_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \end{cases} \tag{1.1}$$

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where B is a standard Brownian motion, the coefficients b, σ, f, g are measurable functions, and in general they could be random. The purpose is to seek the *adapted* solution to the FBSDE (1.1), namely a triple of adapted processes (X, Y, Z) with appropriate dimensions that satisfies (1.1) almost surely. It has been well-known that there are three solution schemes that are effective for the FBSDEs. (1) The contraction mapping approach (see, e.g. Antonelli [1], Pardoux and Tang [22]). This is the most straightforward approach, but works well only when the time duration $T > 0$ is small; (2) The Four Step Scheme, first initiated by Ma et al. [13], and later improved by Delarue [5]. This method allows arbitrary duration, but requires the Markovian structure, high regularity of the coefficients, and the non-degeneracy of the forward diffusion; (3) The method of continuation (see, e.g Hu and Peng [11], Peng and Wu [23], Yong [25]). This method allows non-Markovian structure, but requires the "monotonicity" conditions on the coefficients. It should be noted that these three approaches do not cover each other, and each has its own limitations.

During the past two decades tremendous effort has been made to understand the solvability of the FBSDEs over arbitrary durations, with minimum requirements on the coefficients, but most of the results were still within the paradigm of the three aforementioned methods. In the non-Markovian cases, the progress was even more limited. Following the decoupling strategy of [13,5], Zhang [26] studied the solvability of FSBDEs with random coefficients, under certain compatibility conditions. In a recent work Ma et al. [14] proposed a much more unified approach which extended all the existing approaches in the literature. The key observation of these works is that the solvability of FBSDE (1.1) depends rather heavily on the existence of a (possibly random) function $u(t, x)$, which is uniformly Lipschitz continuous in x, so that $Y_t = u(t, X_t)$ for all $t \in [0, T]$, almost surely. Such a random field, if exists, is called the *decoupling field* of the FBSDE (see [14]).

The main purpose of this paper is to characterize the decoupling random field u in terms of the so-called *backward stochastic PDEs* (BSPDEs), and consequently find the equivalence between the wellposedness of the two stochastic equations. We recall that in the Markovian case (that is, the coefficients are deterministic), the field u becomes deterministic, and in light of the Four Step Scheme (or simply the Feynman–Kac formula), the function u is known to satisfy the following quasilinear PDE with terminal condition (in one dimensional setting, cf. e.g. [9,13,21]):

$$\begin{cases} u_t + \frac{1}{2}u_{xx}\sigma^2(t, x, u) + u_x b(t, x, u, u_x\sigma(t, x, u)) + f(t, x, u, u_x\sigma(t, x, u)) = 0; \\ u(T, x) = g(x). \end{cases} \tag{1.2}$$

In the general case when the coefficients are random, the decoupling function u will naturally become a random field, and the corresponding PDE is expected to become a *backward stochastic PDE* (BSPDE). In the decoupled case when the generator of the backward SDE either is linear or depends only on Y, the BSPDEs and the associated "stochastic Feynman–Kac formula" were studied by Ma and Yong [15,16] and Hu et al. [10]. We note that in these cases the BSPDEs are either linear or semi-linear, but the main difficulty is that they are *degenerate* in the sense of SPDEs (cf. [12,15,16]).

In light of the general requirement of the decoupling random field in [14], in this paper we are to show that the FBSDE (1.1) is wellposed *if and only if* we can find a random field u that satisfies the following *quasilinear* BSPDE and that is *uniformly Lipschitz* in its spatial variable:

$$\begin{cases} du = -\Big\{\frac{1}{2}u_{xx}\sigma^2(t, x, u) + \beta_x\sigma(t, x, u) + u_x b(t, x, u, \beta + u_x\sigma(t, x, u)) \\ \quad + f(t, x, u, \beta + u_x\sigma(t, x, u))\Big\}dt + \beta dB_t; \\ u(T, x) = g(x). \end{cases} \tag{1.3}$$

We should note that a solution to the BSPDE is defined as the pair of progressively measurable random fields (u, β). Clearly, when the coefficients are deterministic, we must have $\beta = 0$ and the BSPDE (1.3) is reduced to the PDE (1.2). We would like to point out here that while it is well understood that in the Markovian case a classical solution to the PDE (1.2) clearly leads to a solution to the FBSDE, as we see in the Four Step Scheme, in many cases, the solvability of the FBSDE depends only on the Sobolev type weak solution to the PDE (1.2) (see, e.g. [2]). Thus, unlike the previous works [15,16,10] in this paper we shall focus on the *regular* weak solution to the BSPDE (1.3), that is, the weak solution in which the random field u is uniformly Lipschitz continuous in its spatial variable. We should point out that since BSPDE (1.3) is quasi-linear and it is always degenerate, combining with the wellposedness results in the FBSDE literature (e.g., [14]), a direct byproduct of this paper is the existence and uniqueness of weak solutions to the degenerate quasi-linear BSPDE. This, to the best of our knowledge is novel in the literature.

Finally, we remark that the well-posedness of the BSPDEs and some of solution properties such as the comparison principle, have been studied in different forms recently, either in the linear cases or in the semilinear but non-degenerate (super-parabolic) cases (cf. e.g., [6,7,18]). But these results and methodology do not seem to be applicable to the current situation due to both the non-Markovian and coupling nature of the FBSDE and the degeneracy of the BSPDE. In fact, we shall present new arguments for the comparison theorem and the stability results, taking advantage of the relationship between the BSPDE and FBSDE established in this paper, and some recent developments in the FBSDE theory.

The rest of the paper is organized as follows. In Section 2, we give the preliminaries. In Section 3, we introduce the notion of the decoupling random field, and establish the connection between the existence of the decoupling field and the wellposedness of the FBSDE. In Section 3, we prove the main theorem in decoupled case, and in Section 4, we prove the general case. Finally, in Section 5, we establish some further properties of the solutions to BSPDEs, including the comparison theorem and the stability result.

2. Preliminaries

Throughout this paper, we denote $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ to be a filtered probability space on which is defined a d-dimensional Brownian motion $B = (B_t)_{t\geq 0}$. We assume that $\mathbb{F} \triangleq \mathbb{F}^B \triangleq \{\mathcal{F}_t^B\}_{t\geq 0}$, the natural filtration generated by B, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. We consider the following FBSDE:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)dB_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \end{cases} \tag{2.1}$$

Here we assume that $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^m$, and $Z \in \mathbb{R}^{m\times d}$. In this paper, we shall always assume the coefficients b, σ, f, g could be random, and take values in $\mathbb{R}^n$, $\mathbb{R}^{n\times d}$, $\mathbb{R}^m$, and $\mathbb{R}^m$, respectively.

Throughout the paper, we shall make use of the following *Standing Assumptions*:

Assumption 2.1. (i) The coefficients b, σ, f are $\mathbb{F}$-progressively measurable for any fixed (x, y, z), and g is $\mathcal{F}_T$-measurable for any fixed x.

(ii) b, σ, f, g are uniformly Lipschitz continuous in (x, y, z).

(iii) b and σ are bounded, and

$$I_0^2 \triangleq E\left\{\int_0^T |f(t,0,0,0)|^2 dt + |g(0)|^2\right\} < \infty. \tag{2.2}$$

In what follows we write $\mathbb{E}$ (also $\mathbb{E}_1, \ldots$) for a generic Euclidean space, whose inner products and norms will be denoted as the same ones $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively and write $\mathbb{B}$ for a generic Banach space with norm $\|\cdot\|$. The following notations for the high dimensional operators will be frequently used in the sequel.

- For a matrix $A \in \mathbb{R}^{n\times m}$, we denote A^* to be its transpose, and $|A|^2 \triangleq \operatorname{tr}(AA^*)$.
- For a function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, $D\varphi \triangleq \left[\frac{\partial \varphi^i}{\partial x_j}\right]_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}$ denotes the derivative of φ.
- For a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, $D^2\varphi \triangleq \left[\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right]_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ denotes the Hessian of φ.
- For a function $\varphi : \mathbb{R}^n \to \mathbb{R}^{m\times n}$, $D\cdot\varphi \triangleq \sum_{i=1}^n \frac{\partial \varphi^i}{\partial x_i} \in \mathbb{R}^m$, where φ^i is the ith column of φ.

We shall use the standard notations for the spaces of continuously differentiable functions, and pth integrable functions such as $\mathbb{C}^p(\mathbb{E}_1;\mathbb{E}_2)$, $0 \le p \le \infty$ and $L^p(\mathbb{E}_1;\mathbb{E}_2)$, $L^p([0,T]\times\mathbb{E}_1;\mathbb{E}_2)$, $1 \le p \le \infty$, etc. In particular, if $\mathbb{E}_2 = \mathbb{R}$, we shall omit it (hence $\mathbb{C}^p(\mathbb{E})$, $L^p(\mathbb{E})$, etc.). Furthermore, the following differential rules are easy to verify:

$$\begin{cases} D^2\varphi = D(D\varphi)^*, \quad \varphi : \mathbb{R}^n \to \mathbb{R}; \\ \operatorname{tr}(D^2\varphi\psi) = D\cdot(D\varphi\psi) - D\varphi D\cdot\psi, \quad \varphi : \mathbb{R}^n \to \mathbb{R},\ \psi : \mathbb{R}^n \to \mathbb{R}^{n\times n}; \\ \operatorname{tr}(D\varphi\psi) = D\cdot(\varphi^*\psi^*) - \varphi^* D\cdot\psi^*, \quad \varphi : \mathbb{R}^n \to \mathbb{R}^m,\ \psi : \mathbb{R}^n \to \mathbb{R}^{n\times m}. \end{cases} \tag{2.3}$$

We next introduce the notion of the *weighted Sobolev space*. We begin by considering a function $\phi \in \mathbb{C}^\infty(\mathbb{R}^n)$ that satisfies the following conditions:

$$\begin{cases} 0 < \phi(x) \le 1, \quad \displaystyle\int_{\mathbb{R}^n} \phi(x)dx = 1; \\ \phi(x) = e^{-|x|} \quad \text{for } x \text{ large enough.} \end{cases} \tag{2.4}$$

We shall call such a smooth function ϕ the *weight function*. One can easily check that if ϕ is a weight function, then one has

$$K_\phi \triangleq \sup_{x\in\mathbb{R}^n} \frac{|D\phi(x)| + |D^2\phi(x)|}{\phi(x)} < \infty. \tag{2.5}$$

As we will see in Lemma 4.3 and Proposition 5.1 below, the constant K_ϕ is important for our estimates.

Now for a given weight function ϕ, we denote $H_\phi^0(\mathbb{R}^n;\mathbb{R}^{k\times l})$ to be the space of all Lebesgue measurable functions $h : \mathbb{R}^n \to \mathbb{R}^{k\times l}$ such that

$$\|h\|_0^2 \triangleq \int_{\mathbb{R}^n} |h(x)|^2 \phi(x)dx < \infty.$$

When the weight function and the dimension of the domain and range spaces are clear from the context, and there is no danger of confusion, we often drop the subscript ϕ and the spaces in the notation, and denote simply as H^0. Clearly H^0 is a Hilbert space equipped with the following inner product:

$$\langle h^1, h^2\rangle_0 \triangleq \int_{\mathbb{R}^n} \operatorname{tr}\left(h^1(x)(h^2(x))^*\right)\phi(x)dx. \tag{2.6}$$

We can now define the Weighted Sobolev spaces as usual. For example, we shall denote $H^1 = H^1_\phi \subset H^0$ to be the subspace of H^0 that consists of all those h such that its generalized derivative, still denoted as Dh, is also in H^0. Clearly, H^1 is a Hilbert space with the inner product

$$\langle h^1, h^2\rangle_1 \triangleq \langle h^1, h^2\rangle_0 + \langle Dh^1, Dh^2\rangle_0, \quad h^1, h^2 \in H^1.$$

By (2.3) one can easily prove the integration by parts formula: for any $h^1 \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ and $h^2 \in H^1(\mathbb{R}^n, \mathbb{R}^{m\times n})$,

$$\langle Dh^1, h^2\rangle_0 = -\langle h^1, D\cdot h^2\rangle_0 - \left\langle h^{1*}h^2, \frac{1}{\phi}D\phi\right\rangle_0. \tag{2.7}$$

Similarly, we denote $H^2 = H^2_\phi \subset H^1$ to be the subspace of H^1 that contains all $h \in H^0$ such that $Dh \in H^1$. Thus, H^2 is again a Hilbert space with inner product

$$\langle h^1, h^2\rangle_2 \triangleq \langle h^1, h^2\rangle_0 + \langle Dh^1, Dh^2\rangle_1, \quad h^1, h^2 \in H^2.$$

Moreover, let H^{-1} be the dual space of H^1, endowed with the dual product $\langle\cdot,\cdot\rangle_{-1}$. Then H^{-1} is equipped with the following norm:

$$\|h\|_{-1} \triangleq \sup\left\{\langle h, \varphi\rangle_{-1} : \varphi \in H^1, \|\varphi\|_1 = 1\right\}.$$

Clearly, $H^0 \subset H^{-1}$ in the sense that for any $\alpha \in H^0$, it holds that

$$\langle \alpha, \varphi\rangle_{-1} = \langle \alpha, \varphi\rangle_0, \quad \forall \varphi \in H^1. \tag{2.8}$$

Furthermore, for any $h \in H^0(\mathbb{R}^n, \mathbb{R})$, in light of (2.7), we have $Dh \in H^{-1}$ in the following sense: for any $\varphi \in H^1(\mathbb{R}^n, \mathbb{R}^{1\times n})$,

$$\langle Dh, \varphi\rangle_{-1} \triangleq -\langle h, D\cdot\varphi\rangle_0 - \left\langle h^*\varphi, \frac{1}{\phi}D\phi\right\rangle_0.$$

Remark 2.2. It is worth noting that

(i) For any two weight functions ϕ_1, ϕ_2 satisfying (2.4), there must exist constants $0 < c < C$ such that $c\phi_1 \le \phi_2 \le C\phi_1$. So the norms defined via ϕ_1 and ϕ_2 are equivalent, and therefore, *the spaces* H^i, $i = -1, 0, 1, 2$, *are independent of the choices of* ϕ.
(ii) It is readily seen that the weight function belongs to the class of the so-called Schwartz functions, and consequently *any functions with polynomial growth are in* H^0. □

We conclude this section by introducing some spaces of stochastic processes that will be useful for the study of the backward SPDEs. First, for any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}$, and $0 \le p \le \infty$, we denote $L^p(\mathcal{G})$ to be the spaces of all $\mathcal{G}$-measurable, L^p-integrable random variables. Next, for any generic Banach space $\mathbb{B}$, we denote $L^p_{\mathbb{F}}([0,T];\mathbb{B})$ to be all $\mathbb{B}$-valued, $\mathbb{F}$-progressively measurable random fields (or processes) $h : [0,T] \mapsto \mathbb{B}$ such that

$$\|h\|_{L^p_{\mathbb{F}}([0,T];\mathbb{B})} \triangleq \mathbb{E}\left\{\int_0^T \|h(t,\cdot)\|^p_{\mathbb{B}}dt\right\}^{1/p} < \infty. \tag{2.9}$$

In particular, if $\mathbb{B} = H^i_\phi$, where ϕ is a given weight function, we denote $\mathscr{H}^i_\phi = L^2([0,T]; H^i_\phi)$, $i = -1, 0, 1, 2$, respectively. Again, we often drop the subscript ϕ from the notations when the context is clear. Finally, the spaces of Banach-space-valued processes such as $\mathbb{C}^{\alpha}_{\mathbb{F}}([0,T];\mathbb{B})$, for $\alpha \geq 1$, are defined in the obvious way.

3. The decoupling random field

In this section, we introduce the notion of the "decoupling random field", and establish its relationship with the well-posedness of the FBSDE (1.1). We should note that the definition of the decoupling field here is slightly different from that in [14]. But one can easily check that a *regular* decoupling field in this paper is equivalent to the decoupling field in [14].

To begin with, we note that, in light of [14] (or the Four Step Scheme in the Markovian case), when FBSDE (2.1) is wellposed, one would expect that the relationship

$$Y_t = u(t, X_t), \quad \forall t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{3.1}$$

holds for some random field $u(\cdot,\cdot)$. Our decoupling field of FBSDE (2.1) is thus defined as follows. For any $0 \leq t_1 < t_2 \leq T$ and any random variable $\eta \in L^2(\mathcal{F}_{t_1})$, consider the following "localized" FBSDE:

$$\begin{cases} X_t^{t_1,\eta} = \eta + \int_{t_1}^t b(s, X_s^{t_1,\eta}, Y_s^{t_1,\eta}, Z_s^{t_1,\eta})ds + \int_{t_1}^t \sigma(s, X_s^{t_1,\eta}, Y_s^{t_1,\eta})dB_s; \\ Y_t^{t_1,\eta} = u(t_2, X_{t_2}^{t_1,\eta}) + \int_t^{t_2} f(s, X_s^{t_1,\eta}, Y_s^{t_1,\eta}, Z_s^{t_1,\eta})ds - \int_s^{t_2} Z_s^{t_1,\eta}dB_s, \\ \quad t \in [t_1, t_2]. \end{cases} \tag{3.2}$$

Definition 3.1. We say u is a decoupling field of FBSDE (2.1) if

(i) $u(T,x) = g(x)$ and
(ii) for any $0 \leq t_1 < t_2 \leq T$ and $\eta \in L^2(\mathcal{F}_{t_1})$, FBSDE (3.2) has a solution $(X^{t_1,\eta}, Y^{t_1,\eta}, Z^{t_1,\eta})$ satisfying

$$Y_t^{t_1,\eta} = u(t, X_t^{t_1,\eta}), \quad t \in [t_1, t_2]. \tag{3.3}$$

Moreover, we say a decoupling field u is regular if it is uniformly Lipschitz continuous in x.

Remark 3.2. The condition that u is uniformly Lipschitz continuous in x is crucial in this paper. In fact, from the general theory of FBSDEs (see, e.g., [14,26] and the references therein), we see that in many cases a well-posed FBSDE does possess a uniform Lipschitz decoupling field. One should note, however, that this is by no means a general statement. For example, the triplet $(X,Y,Z) = (W, W^2, 2W)$, where W is a standard Brownian motion, is obviously the unique solution to a trivial FBSDE, but in this case $u = x^2$ is not Lipschitz. □

The main result of this section concerns the wellposedness of the FBSDE and the existence of the regular decoupling field. An important starting point of our argument is the wellposedness of the FBSDE on a small duration, due to Antonelli [1]. We summarize it into the following lemma.

Lemma 3.3. *(Antonelli [1]) Assume Assumption 2.1 holds. Let K denote the Lipschitz constant of the terminal condition g. Then there exists a constant $\delta(K) > 0$, which depends only on the Lipschitz constants of b, σ, f, the dimensions, and the constant K, such that whenever $T \leq \delta(K)$, the FBSDE (2.1) has a unique solution.*

We now give the main result of this section.

Theorem 3.4. *Assume Assumption 2.1 holds. Then FBSDE (2.1) has at most one regular decoupling field u. Furthermore, if the regular decoupling field exists, then the FBSDE (2.1) is wellposed.*

Proof. We first show that the existence of a regular decoupling field implies the wellposedness of the FBSDE (2.1). Indeed, the existence of the solution to FBSDE (2.1) follows directly from the definition (by simply taking $t_1 = 0, t_2 = T$, and $\eta \equiv x$). We need only show that the solution is unique.

Let (X, Y, Z) be an arbitrary solution, and let u be a regular decoupling field and denote the solution associated to u by (X^0, Y^0, Z^0). We show that (X, Y, Z) must coincide with (X^0, Y^0, Z^0). To this end, let K be the Lipschitz constant of u and $0 = t_0 < \cdots < t_k = T$ be a partition of $[0, T]$ such that $\Delta t_i \le \delta(K), i = 1, \ldots, k$, where $\delta(K)$ is the constant in Lemma 3.3. Note that (X, Y, Z) satisfies the following FBSDE on $[t_{k-1}, t_k]$:

$$\begin{cases} X_t = X_{t_{k-1}} + \displaystyle\int_{t_{k-1}}^t b(s, X_s, Y_s, Z_s)ds + \int_{t_{k-1}}^t \sigma(s, X_s, Y_s)dB_s; \\ Y_t = g(X_T) + \displaystyle\int_t^{t_k} f(s, X_s, Y_s, Z_s)ds - \int_t^{t_k} Z_s dB_s. \end{cases}$$

Since $\Delta t_k \le \delta(K)$, the solution to the above FBSDE is unique, thanks to Lemma 3.3. Then by the definition of the decoupling field we must have $Y_t = u(t, X_t), t \in [t_{k-1}, t_k]$. Assume now that $Y_t = u(t, X_t)$ holds for $t \in [t_i, T]$. Then for $t \in [t_{i-1}, t_i]$, it holds that

$$\begin{cases} X_t = X_{t_{i-1}} + \displaystyle\int_{t_{i-1}}^t b(s, X_s, Y_s, Z_s)ds + \int_{t_{i-1}}^t \sigma(s, X_s, Y_s)dB_s; \\ Y_t = u(t_i, X_{t_i}) + \displaystyle\int_t^{t_i} f(s, X_s, Y_s, Z_s)ds - \int_t^{t_i} Z_s dB_s. \end{cases} \tag{3.4}$$

Again, by the wellposedness of the above FBSDE and the definition of the decoupling field, we see that $Y_t = u(t, X_t)$ holds for $t \in [t_{i-1}, t_i]$. Repeating this argument we conclude that $Y_t = u(t, X_t)$ holds for all $t \in [0, T]$.

Now note that $X_{t_0} = x$. Considering FBSDE (3.4) for $i = 1$, by the uniqueness (X, Y, Z) must coincide with (X^0, Y^0, Z^0) on $[t_0, t_1]$. In particular, $X_{t_1} = X^0_{t_1}$, a.s. Then considering the FBSDE (3.4) for $i = 2$, we see that (X, Y, Z) coincides with (X^0, Y^0, Z^0) on $[t_1, t_2]$. Repeating this argument forwardly finitely times we see that (X, Y, Z) coincides with (X^0, Y^0, Z^0) on the whole interval $[0, T]$, proving the uniqueness, whence the wellposedness of (1.1).

It remains to show that the regular decoupling field, if exists, must be unique. Indeed, assume that $\tilde{u}$ is another regular decoupling field. For any (t, x), the FBSDE (3.2) with $t_1 = t, t_2 = T, \eta = x$ has a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$, and by definition of the decoupling field we must have

$$\tilde{u}(t, x) = Y_t^{t,x} = u(t, x), \quad \mathbb{P}\text{-a.s.}$$

Since (t, x) is arbitrary, this implies that the regular decoupling field is unique. □

4. Decoupling random field via BSPDE

In this section, we study the regular decoupling random field u from the perspective of the Feynman–Kac formula, that is, we shall characterize u as a solution to some Backward SPDE.

Such a point of view was a natural extension of the original decoupling idea of the Four Step Scheme [13], and was studied in the simpler versions in [15,16,10].

Let us begin with the following heuristic argument. Assume that the decoupling field u takes the form of an Itô-type random field:

$$du(t,x) = \alpha(t,x)dt + \beta(t,x)dB_t, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{4.1}$$

where α, β are (smooth) progressively measurable random fields taking values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively. Assume also that FBSDE (1.1) is well-posed, with the solution (X,Y,Z), then by applying Itô–Ventzell's formula we have, for $i = 1,\dots,m$,

$$\begin{aligned} du^i(t,X_t) = \Bigg[&\alpha^i(t,X_t) + Du^i(t,X_t)b(t,X_t,Y_t,Z_t) \\ &+ \frac{1}{2}\mathrm{tr}\,[D^2u^i(t,X_t)\sigma\sigma^*(t,X_t,Y_t)] + \mathrm{tr}\,(D(\beta^i)^*(t,X_t)\sigma(t,X_t,Y_t))\Bigg]dt \\ &+ \Big[\beta^i(t,X_t) + Du^i(t,X_t)\sigma(t,X_t,Y_t)\Big]dB_t, \end{aligned} \tag{4.2}$$

where β^i is the ith row of β. Noting that u is the decoupling field and comparing (4.2) with the BSDE in (1.1) we must have

$$\begin{cases} Y_t = u(t,X_t), \\ Z_t = \beta(t,X_t) + Du(t,X_t)\sigma(t,X_t,u(t,X_t)), \end{cases} \quad t \in [0,T], \ \mathbb{P}\text{-a.s.} \tag{4.3}$$

Furthermore, for $i = 1,\dots,m$, we have

$$\begin{aligned} -\alpha^i(t,X_t) = \ & Du^i(t,X_t)b(t,X_t,Y_t,Z_t) + \frac{1}{2}\mathrm{tr}\,[D^2u^i(s,X_t)\sigma\sigma^*(t,X_t,Y_t)] \\ &+ \mathrm{tr}\,(D(\beta^i)^*(t,X_t)\sigma(t,X_t,Y_t)) + f^i(t,X_t,Y_t,Z_t). \end{aligned}$$

Consequently, if we define

$$\gamma(t,x) \triangleq \beta(t,x) + Du(t,x)\sigma(t,x,u(t,x)), \tag{4.4}$$

$$\begin{aligned} \alpha^i(t,x) \ \triangleq \ -\Bigg[&\frac{1}{2}\mathrm{tr}\,[D^2u^i\sigma\sigma^*(t,x,u)] + \mathrm{tr}\,(D(\beta^i)^*\sigma(t,x,u)) \\ &+ Du^i b(t,x,u,\gamma) + f^i(t,x,u,\gamma)\Bigg], \quad i = 1,\dots,m, \end{aligned} \tag{4.5}$$

then we have $Z_t = \gamma(t,X_t)$, $t \in [0,T]$, $\mathbb{P}$-a.s., and (4.1) becomes

$$\begin{cases} du^i(t,x) = -\Bigg[\frac{1}{2}\mathrm{tr}\,[D^2u^i\sigma\sigma^*(t,x,u)] + \mathrm{tr}\,(D(\beta^i)^*\sigma(t,x,u)) + Du^i b(t,x,u,\gamma) \\ \qquad + f^i(t,x,u,\gamma)\Bigg]dt + \beta^i(t,x)dB_t, \quad (t,x) \in [0,T]\times\mathbb{R}^n, \ i = 1,\dots,m; \\ u(T,x) = g(x). \end{cases} \tag{4.6}$$

In other words, the decoupling random field u must satisfy a (quasilinear) *backward stochastic PDE* (BSPDE), which we will call the *BSPDE associated to FBSDE* (2.1).

We are more interested in the converse, that is, the *Stochastic Feynman–Kac* formula. Suppose that the BSPDE (4.6) has a "classical" solution $(u, \beta) \in \mathbb{C}^0_{\mathbb{F}}([0, T]; \mathbb{C}^2(\mathbb{R})) \times L^2_{\mathbb{F}}([0, T]; \mathbb{C}^1(\mathbb{R}))$, such that all the derivatives involved are uniformly bounded. Then, we define

$$\hat{b}(t, x) \triangleq b(t, x, u(t, x), \gamma(t, x)), \qquad \hat{\sigma}(t, x) \triangleq \sigma(t, x, u(t, x)), \tag{4.7}$$

and consider the following SDE:

$$X_t = x + \int_0^t \hat{b}(s, X_s)ds + \int_0^t \hat{\sigma}(s, X_s)dB_s.$$

Since $\hat{b}, \hat{\sigma}$ are uniformly Lipschitz continuous in x, the above SDE has a unique solution X. Setting

$$Y_t \triangleq u(t, X_t), \qquad Z_t \triangleq \gamma(t, X_t),$$

and applying Itô–Ventzell's formula we can check (with the help of BSPDE (4.6)) that (X, Y, Z) satisfies the FBSDE (3.2). This, together with the fact that Du is bounded, shows that u is a regular decoupling field. In other words, combining with Theorem 3.4 and the "heuristic argument" at the beginning of this section we have actually proved the following version of the "stochastic Feynman–Kac formula".

Theorem 4.1. *Assume Assumption 2.1 holds. Let $(u, \beta) \in \mathbb{C}^0_{\mathbb{F}}([0, T]; \mathbb{C}^2(\mathbb{R})) \times L^2_{\mathbb{F}}([0, T]; \mathbb{C}^1(\mathbb{R}))$ be a pair of random fields, such that all the (spatial) derivatives involved are uniformly bounded. Then, (u, β) is a classical solution to BSPDE (4.6) if and only if FBSDE (2.1) is wellposed, and u is the regular decoupling field of FBSDE (2.1) such that (4.3) holds.* □

Remark 4.2. (i) When σ also depends on z, under additional technical conditions, one can still have the regular wellposedness of the FBSDE; see e.g. [14]. However, in this case the corresponding BSPDE will involve an implicit function and the technical arguments will become much more involved. Since the main focus of this paper is to establish the connection between FBSDEs and BSPDEs, rather than to explore the most general conditions for wellposedness of the systems, we content ourselves with the case $\sigma = \sigma(t, x, y)$.

(ii) It is well-known (cf. e.g., [20,13]) that if the FBSDE (2.1) is well-posed, then under the standard assumption the process Y is "Malliavin differentiable", and $Z_t = D_t Y_t$, where D is the Malliavin derivative. A similar relation also holds for the pair u and β. In fact, when BSPDE (4.6) is linear, it is shown in [15] that $Du(t, x) = \beta(t, x)$, with D being the Malliavin derivative. Such a relation can also be established using the newly developed notion of ω-derivatives in the sense of Dupire [8] (see also Cont and Fournié [3]), but we prefer not to pursue any further as this is not the main purpose of this paper. □

We should note that Theorem 4.1 may very well be an empty statement if BSPDE (4.6) does not have a classical solution. In the rest of the paper we shall focus on the Sobolev type weak solutions. We begin with the following important fact.

Lemma 4.3. *Assume Assumption 2.1 holds. Assume also that $(u, \beta) \in \mathscr{H}^1_\phi \times \mathscr{H}^0_\phi$ for some weight function ϕ, and α is defined by (4.5). If $D\hat{\sigma}$ is uniformly bounded, where $\hat{\sigma}$ is defined by (4.7), then $\alpha \in \mathscr{H}^{-1}_\phi$.*

Proof. For notational simplicity we shall drop the subscript ϕ from all the notations. We claim that, for any $(u,\beta) \in \mathscr{H}^2 \times \mathscr{H}^1$ and any $\varphi \in \mathscr{H}^1$,

$$\left|\langle \alpha^i, \varphi\rangle_{-1}\right| \le C[1+K]\,[1+I_0+\|u\|_1+\|\beta\|_0]\,\|\varphi\|_1, \tag{4.8}$$

where K is the bound of $D\hat{\sigma}$, and the constant C depends only on T, the dimensions, the bound of b, σ and the Lipschitz constant in Assumption 2.1, and the K_ϕ in (2.5). Then, for $(u,\beta) \in \mathscr{H}^1_\phi \times \mathscr{H}^0_\phi$, by standard approximating arguments we see that $\alpha \in \mathscr{H}^{-1}$ and

$$\|\alpha\|_{-1} \le C[1+K]\,[1+I_0+\|u\|_1+\|\beta\|_0]\,.$$

To prove (4.8), we note that in the case when $(u,\beta) \in \mathscr{H}^2 \times \mathscr{H}^1, \alpha \in \mathscr{H}^0$. Thus by (2.3), (2.7) and (2.8), we have (suppressing variables)

$$\begin{aligned}
\langle \alpha^i, \varphi\rangle_{-1} &= \langle \alpha^i, \varphi\rangle_0 = \int_{\mathbb{R}^n} \varphi(x)\alpha^i(t,x)\phi(x)dx \\
&= \int_{\mathbb{R}^n} \Bigg[[\phi D\varphi + \varphi D\phi]\sigma \left(\beta^i + \frac{1}{2}Du^i\sigma\right)^* + \varphi\phi[\beta^i + Du^i\sigma]D\cdot\sigma^* \\
&\quad - \varphi\phi Du^i b\big(t,x,u,\gamma\big) - \varphi\phi f^i\big(t,x,u,\gamma\big)\Bigg]dx \\
&= \left\langle \left[\beta^i + \frac{1}{2}Du^i\sigma\right]\sigma^*, D\varphi\right\rangle_0 + \langle \hat{\alpha}^i, \varphi\rangle_0,
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\hat{\alpha}^i &\triangleq \left(\beta^i + \frac{1}{2}Du^i\sigma\right)\sigma^*\frac{1}{\phi}(D\phi)^* + \gamma^i D\cdot\sigma^* - Du^i b\big(t,x,u,\gamma\big) \\
&\quad - f^i\big(t,x,u,\gamma\big).
\end{aligned} \tag{4.10}$$

By Assumption 2.1 and (2.5) we get

$$\begin{aligned}
&\left|\left[\beta^i + \frac{1}{2}Du^i\sigma\right]\sigma^*\right| \le C\left[1+|\beta^i|+|Du^i|\right]; \\
&|\hat{\alpha}^i| \le C\,[1+|\beta|+|Du|]\,[1+K] + C\,[|f(t,0,0,0)|+|x|+|u|]\,.
\end{aligned}$$

Consequently we obtain from (4.9) that

$$\begin{aligned}
\left|\langle \alpha^i, \varphi\rangle_0\right| &\le C\left[1+\|\beta^i\|_0+\|Du^i\|_0\right]\|D\varphi\|_0 \\
&\quad + C(1+K)\,[1+\|\beta\|_0+I_0+\|u\|_1]\,\|\varphi\|_0.
\end{aligned}$$

This implies (4.8) immediately. □

We now define the notion of Sobolev weak solutions to BSPDE (4.6).

Definition 4.4. We say that the pair of random fields $(u,\beta) \in \mathscr{H}^1 \times \mathscr{H}^0$ is a weak solution to BSPDE (4.6) if $D\hat{\sigma}$ is uniformly bounded and, for any $\varphi \in H^1$, it holds that

$$d\langle u^i(t,\cdot), \varphi\rangle_0 = \langle \alpha^i(t,\cdot), \varphi\rangle_{-1}dt + \langle \beta^i(t,\cdot)dB_t, \varphi\rangle_0, \quad \text{a.s.},\ i=1,\ldots,m. \tag{4.11}$$

We say that $(u,\beta) \in \mathscr{H}^1 \times \mathscr{H}^0$ is a regular weak solution to BSPDE (4.6) if (u,β) is a weak solution such that Du is uniformly bounded.

Remark 4.5. Under Assumption 2.1, two typical cases such that $D\hat{\sigma}$ is bounded are: (i) Du is uniformly bounded and (ii) σ does not depend on y. In particular, (4.11) is well defined for a regular weak solution. □

5. The decoupled case

In this and next sections, we shall extend the stochastic Feynman–Kac formula (Theorem 4.1) to the case when BSPDE only allows a regular weak solution. To this end, we first consider a simple but important case, that is, when the FBSDE is of the following decoupled form:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s. \end{cases} \tag{5.1}$$

Clearly, the associated BSPDE (4.6) will then take the following form:

$$\begin{cases} du^i(t,x) = -\left[\frac{1}{2}\mathrm{tr}\,[D^2u^i\sigma\sigma^*] + \mathrm{tr}\,(D(\beta^i)^*\sigma) + Du^ib + f^i(\cdot, u, \gamma)\right](t,x)dt \\ \qquad + \beta^i(t,x)dB_t, \quad (t,x) \in [0,T) \times \mathbb{R}^n,\ i = 1, \ldots, m; \\ u(T,x) = g(x) \end{cases} \tag{5.2}$$

where γ is defined by (4.4). We should point out that, to our best knowledge, even in this simple form the well-posedness of the BSPDE (5.2) is still open in the literature due to the dependence of f on γ. Therefore the discussion in this section is interesting in its own right.

We first note that if Assumption 2.1 holds, then the decoupled FBSDE (5.1) is always well-posed, and we can define the decoupling field as $u(t,x) \triangleq Y_t^{t,x}$, where $Y^{t,x}$ is the solution to FBSDE (3.2) with $t_1 = t, t_2 = T, \eta = x$. The main task of this section is to obtain some *a priori* estimates for the weak solutions of BSPDE (5.2) which will be useful for the proof of the main results in the next section. We begin with the following proposition.

Proposition 5.1. *Assume Assumption 2.1 holds. Let $(u, \beta) \in \mathscr{H}^1 \times \mathscr{H}^0$ be a weak solution to the BSPDE (5.2). Then there exists a constant $C > 0$, depending only on the bounds in Assumption 2.1, the dimensions, the duration T and the constant K_ϕ for the given weight function (2.5), such that*

$$\mathbb{E}\left\{\sup_{t\in[0,T]} \|u(t,\cdot)\|_0^2 + \int_0^T \|\gamma(t,\cdot)\|_0^2 dt\right\} \le C[1 + I_0^2], \tag{5.3}$$

where I_0 is defined by (2.2).

Proof. We first recall the following extended Itô's formula (see Pardoux [19, Theorem 1.2]). Let $u \in \mathscr{H}^1$ be an Itô-type random field:

$$du(t,x) = \alpha(t,x)dt + \beta(t,x)dB_t, \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $\alpha \in \mathscr{H}^{-1}$ and $\beta \in \mathscr{H}^0$. Then it holds that

$$d\|u(t,\cdot)\|_0^2 = \left[2\langle \alpha(t,\cdot), u(t,\cdot)\rangle_{-1} + \|\beta(t,\cdot)\|_0^2\right]dt + 2\langle u(t,\cdot), \beta(t,\cdot)dB_t\rangle_0, \\ t \in [0,T]. \tag{5.4}$$

Now let $(u, \beta) \in \mathscr{H}^1 \times \mathscr{H}^0$ be a weak solution to the BSPDE (5.2). Then

$$du^i(t,x) = \alpha^i(t,x)dt + \beta^i(t,x)dB_t, \quad i = 1, \ldots, m,$$

where α^i is defined by (4.5). Since $\alpha^i \in \mathscr{H}^{-1}$ by Lemma 4.3 and Remark 4.5, we can apply the extended Itô formula (5.4) and use the identity (4.9) to get

$$\begin{aligned} d\|u^i(t,\cdot)\|_0^2 &= 2\langle u^i, \beta^i dB_t\rangle_0 + \Big[\langle [2\beta^i + Du^i\sigma]\sigma^*, Du^i\rangle_0 + 2\langle \hat{\alpha}^i, u^i\rangle_0 + \|\beta^i\|_0^2\Big]dt \\ &= 2\langle u^i, \beta^i dB_t\rangle_0 + \Big[\|\gamma^i\|_0^2 + 2\langle \hat{\alpha}^i, u^i\rangle_0\Big]dt, \end{aligned} \tag{5.5}$$

where $\hat{\alpha}^i$ is defined by (4.10). Furthermore, by (4.10) we have

$$\begin{aligned} \langle \hat{\alpha}^i, u^i\rangle_0 &= \left\langle \left(\gamma^i - \frac{1}{2}Du^i\sigma\right)\sigma^*\frac{1}{\phi}(D\phi)^* + \gamma^i D\cdot\sigma^* - Du^i b - f^i(t,x,u,\gamma), u^i\right\rangle_0 \\ &= \left\langle \gamma^i \left[\sigma^*\frac{1}{\phi}(D\phi)^* + D\cdot\sigma^*\right]\right. \\ &\qquad \left. - Du^i\left[\frac{1}{2}\sigma\sigma^*\frac{1}{\phi}(D\phi)^* + b\right] - f^i(t,x,u,\gamma), u^i\right\rangle_0. \end{aligned} \tag{5.6}$$

It is easy to check that

$$\begin{cases} \left|\left\langle \gamma^i\left[\sigma^*\frac{1}{\phi}(D\phi)^* + D\cdot\sigma^*\right], u^i\right\rangle_0\right| \le C\|\gamma^i\|_0\|u^i\|_0; \\ \left|\langle f^i(t,x,u,\gamma), u^i\rangle_0\right| \le C[\|f(t,\cdot,0,0)\|_0 + \|u\|_0 + \|\gamma\|_0]\|u^i\|_0. \end{cases} \tag{5.7}$$

Moreover, by Assumption 2.1 and (2.5), we see that

$$\left|\frac{1}{2}\sigma\sigma^*\frac{1}{\phi}(D\phi)^* + b\right| + \left|D\left(\frac{1}{2}\sigma\sigma^*\frac{1}{\phi}(D\phi)^* + b\right)\right| \le C.$$

Thus by integrating by parts formula (2.7) we get

$$\begin{aligned} &\left|\left\langle Du^i\left[\frac{1}{2}\sigma\sigma^*\frac{1}{\phi}(D\phi)^* + b\right], u^i\right\rangle_0\right| \\ &\quad \le \frac{1}{2}\left|\left\langle u^i D\cdot\left[\frac{1}{2}\sigma\sigma^*\frac{1}{\phi}(D\phi)^* + b\right], u^i\right\rangle_0\right| \\ &\qquad + \frac{1}{2}\left|\left\langle |u^i|^2\left[\frac{1}{2}\frac{1}{\phi}(D\phi)\sigma\sigma^* + b^*\right], \frac{1}{\phi}(D\phi)\right\rangle_0\right| \\ &\quad \le C\|u^i\|_0^2. \end{aligned} \tag{5.8}$$

Now, combining (5.5)–(5.8) we deduce from (5.6) that

$$\begin{aligned} d\|u^i(t,\cdot)\|_0^2 &\ge 2\langle u^i, \beta^i dB_t\rangle_0 \\ &\qquad + \Big[\|\gamma^i\|_0^2 - C[\|\gamma\|_0\|u\|_0 + \|u\|_0^2 + \|f(t,\cdot,0,0)\|_0^2]\Big]dt \\ &\ge 2\langle u^i, \beta^i dB_t\rangle_0 + \left[\frac{1}{2}\|\gamma^i\|_0^2 - C[\|u\|_0^2 + \|f(t,\cdot,0,0)\|_0^2]\right]dt. \end{aligned} \tag{5.9}$$

Since $u(T,x)=g(x)$, we get

$$\begin{aligned}
&\mathbb{E}\left\{\|u^i(t,\cdot)\|_0^2+\frac{1}{2}\int_t^T\|\gamma^i(s,\cdot)\|_0^2ds\right\}\\
&\quad\le\mathbb{E}\left\{\|g\|_0^2+C\int_t^T[\|u(s,\cdot)\|_0^2+\|f(s,\cdot,0,0)\|_0^2]ds\right\}\\
&\quad\le C\mathbb{E}\left\{\int_t^T\|u(s,\cdot)\|_0^2ds\right\}+C[1+I_0^2],
\end{aligned}\tag{5.10}$$

where I_0 is defined by (2.2). Now summing over all i and applying Gronwall's inequality we obtain that

$$\sup_{0\le t\le T}\mathbb{E}\left\{\|u(t,\cdot)\|_0^2\right\}+\mathbb{E}\left\{\int_0^T\|\gamma(t,\cdot)\|_0^2dt\right\}\le C[1+I_0^2].$$

Thus (5.3) follows from the standard application of the Burkholder–Davis–Gundy inequality, proving the proposition. □

We next estimate the difference of regular weak solutions to two BSPDEs. Let (b,σ,f,g) and $(\tilde b,\tilde\sigma,\tilde f,\tilde g)$ be two sets of coefficients of the (decoupled) BSPDE (5.2), and denote the corresponding weak solutions by (u,β) and $(\tilde u,\tilde\beta)$, respectively. For notational simplicity, we denote, for $\xi=u,\beta,b,\sigma,\gamma$, and g, $\Delta\xi\triangleq\tilde\xi-\xi$, and denote $\Delta f\triangleq(\tilde f-f)(t,x,u,\gamma)$. We have the following estimates.

Proposition 5.2. *Let (b,σ,f,g) and $(\tilde b,\tilde\sigma,\tilde f,\tilde g)$ be two sets of coefficients of the (decoupled) BSPDE (5.2) satisfying Assumption 2.1, and denote the corresponding weak solutions by (u,β) and $(\tilde u,\tilde\beta)$, respectively. Then*

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{t\in[0,T]}\|\Delta u(t,\cdot)\|_0^2+\int_0^T\|\Delta\gamma(t,\cdot)\|_0^2dt\right\}\\
&\quad\le C\mathbb{E}\left\{\|\Delta g\|_0^2+\int_0^T[\|\Delta f\|_0^2+I_t]dt\right\},
\end{aligned}\tag{5.11}$$

where

$$\begin{aligned}
I\triangleq{}&\|Du\Delta\sigma\|_0^2+\|Du^i\|_0\|D\tilde u^i\tilde\sigma(\Delta\sigma)^*\|_0\\
&+\|Du^i\|_0\|D\tilde u^i\Delta\sigma\sigma^*\|_0+\|\gamma\|_0\|D(\Delta u)\Delta\sigma\|_0\\
&+\|\gamma\|_0\left\|\Delta u\frac{1}{\phi}D\phi\Delta\sigma\right\|_0+\|Du\|_0\left\|\Delta u\frac{1}{\phi}D\phi[\tilde\sigma\tilde\sigma^*-\sigma\sigma^*]\right\|_0\\
&+\|\gamma\|_0\|\Delta u(D\cdot\Delta\sigma)^*\|_0+\|Du\|_0\|\Delta u(\Delta b)^*\|_0.
\end{aligned}\tag{5.12}$$

In particular, if $\tilde b=b$, $\tilde\sigma=\sigma$, then $I=0$ and thus

$$\mathbb{E}\left\{\sup_{t\in[0,T]}\|\Delta u(t,\cdot)\|_0^2+\int_0^T\|\Delta\gamma(t,\cdot)\|_0^2dt\right\}\le C\mathbb{E}\left\{\|\Delta g\|_0^2+\int_0^T\|\Delta f\|_0^2dt\right\}.\tag{5.13}$$

Proof. The estimate (5.13) obviously follows from (5.11). We shall prove only (5.11). First we write

$$d(\Delta u) = [\tilde{\alpha} - \alpha]dt + \Delta\beta dB_t,$$

where $\tilde{\alpha}$ and α are defined by (4.5), with corresponding coefficients, respectively. We can apply the extended Itô's formula (5.4) again to get

$$d\|\Delta u^i(t,\cdot)\|_0^2 - 2\langle \Delta u^i(t,\cdot), \Delta\beta^i(t,\cdot)dB_t\rangle = \Big[2\langle \tilde{\alpha}^i - \alpha^i, \Delta u^i\rangle_{-1} + \|\Delta\beta^i\|_0^2\Big]dt. \quad (5.14)$$

To estimate the right hand side of (5.14) we first note that by (4.9) and (4.10) one has

$$\begin{aligned}
&2\langle \tilde{\alpha}^i - \alpha^i, \Delta u^i\rangle_{-1} + \|\Delta\beta^i\|_0^2 \\
&\quad = \|\Delta\beta^i\|_0^2 + \langle 2\tilde{\gamma}^i - D\tilde{u}^i\tilde{\sigma}, D\Delta u^i\tilde{\sigma}\rangle_0 - \langle 2\gamma^i - Du^i\sigma, D\Delta u^i\sigma\rangle_0 \\
&\qquad + \left\langle 2\tilde{\gamma}^i - D\tilde{u}^i\tilde{\sigma}, \Delta u^i\frac{1}{\phi}D\phi\tilde{\sigma}\right\rangle_0 - \left\langle 2\gamma^i - Du^i\sigma, \Delta u^i\frac{1}{\phi}D\phi\sigma\right\rangle_0 \\
&\qquad + 2\langle \tilde{\gamma}^i D\cdot(\tilde{\sigma})^* - D\tilde{u}^i\tilde{b} - \tilde{f}^i(\cdot,\tilde{u},\tilde{\gamma}), \Delta u^i\rangle_0 - 2\langle \gamma^i D\cdot(\sigma)^* - Du^i b \\
&\qquad - f^i(\cdot,u,\gamma), \Delta u^i\rangle_0,
\end{aligned} \quad (5.15)$$

where $\tilde{\gamma}^i$ and γ^i's are defined by (4.4) with corresponding coefficients. Thus the straightforward calculation shows that the right hand side above can be written as

$$\begin{aligned}
&\|\Delta\gamma^i\|_0^2 - 2\langle \Delta\gamma^i, Du^i\Delta\sigma\rangle_0 + 2\langle \gamma^i, D(\Delta u^i)\Delta\sigma\rangle_0 + \langle Du^i, D\tilde{u}^i(\tilde{\sigma}(\Delta\sigma)^* - \Delta\sigma\sigma^*)\rangle_0 \\
&\quad + 2\left\langle \Delta\gamma^i, \Delta u^i\frac{1}{\phi}D\phi\tilde{\sigma}\right\rangle_0 + 2\left\langle \gamma^i, \Delta u^i\frac{1}{\phi}D\phi\Delta\sigma\right\rangle_0 \\
&\quad - \left\langle D(\Delta u^i), \Delta u^i\frac{1}{\phi}D\phi\tilde{\sigma}\tilde{\sigma}^*\right\rangle_0 - \left\langle Du^i, \Delta u^i\frac{1}{\phi}D\phi[\tilde{\sigma}\tilde{\sigma}^* - \sigma\sigma^*]\right\rangle_0 \\
&\quad + 2\langle \Delta\gamma^i, \Delta u^i(D\cdot(\tilde{\sigma})^*)^*\rangle_0 + 2\langle \gamma^i, \Delta u^i(D\cdot(\Delta\sigma)^*)^*\rangle_0 - 2\langle D\Delta u^i, \Delta u^i(\tilde{b})^*\rangle_0 \\
&\quad - 2\langle Du^i, \Delta u^i(\Delta b)^*\rangle_0 - 2\langle \tilde{f}^i(\cdot,\tilde{u},\tilde{\gamma}) - \tilde{f}^i(\cdot,u,\gamma), \Delta u^i\rangle_0 - 2\langle \Delta f^i, \Delta u^i\rangle_0 \\
&= \|\Delta\gamma^i\|_0^2 - 2\langle \Delta\gamma^i, Du^i\Delta\sigma\rangle_0 + 2\langle \gamma^i, D(\Delta u^i)\Delta\sigma\rangle_0 \\
&\quad + \langle Du^i, D\tilde{u}^i(\tilde{\sigma}(\Delta\sigma)^* - \Delta\sigma\sigma^*)\rangle_0 \\
&\quad + 2\left\langle \Delta\gamma^i, \Delta u^i\frac{1}{\phi}D\phi\tilde{\sigma}\right\rangle_0 + 2\left\langle \gamma^i, \Delta u^i\frac{1}{\phi}D\phi\Delta\sigma\right\rangle_0 \\
&\quad + \frac{1}{2}\left\langle \Delta u^i, \Delta u^i\frac{1}{\phi}D\cdot(D\phi\tilde{\sigma}\tilde{\sigma}^*)\right\rangle_0 - \left\langle Du^i, \Delta u^i\frac{1}{\phi}D\phi[\tilde{\sigma}\tilde{\sigma}^* - \sigma\sigma^*]\right\rangle_0 \\
&\quad + 2\langle \Delta\gamma^i, \Delta u^i(D\cdot(\tilde{\sigma})^*)^*\rangle_0 + 2\langle \gamma^i, \Delta u^i(D\cdot(\Delta\sigma)^*)^*\rangle_0 + \langle \Delta u^i, \Delta u^i D\cdot(\tilde{b})^*\rangle_0 \\
&\quad - 2\langle Du^i, \Delta u^i(\Delta b)^*\rangle_0 - 2\langle \tilde{f}^i(\cdot,\tilde{u},\tilde{\gamma}) - \tilde{f}^i(\cdot,u,\gamma), \Delta u^i\rangle_0 - 2\langle \Delta f^i, \Delta u^i\rangle_0.
\end{aligned}$$

Applying the Cauchy–Schwartz inequality repeatedly we obtain that

$$\begin{aligned}
&2\langle \tilde{\alpha}^i - \alpha^i, \Delta u^i\rangle_{-1} + \|\Delta\beta^i\|_0^2 \\
&\quad \geq \frac{1}{2}\|\Delta\gamma^i\|_0^2 - C\Bigg[\|\Delta u\|_0^2 + \|Du\Delta\sigma\|_0^2 + \|Du^i\|_0\|D\tilde{u}^i\tilde{\sigma}(\Delta\sigma)^*\|_0 \\
&\qquad + \|Du^i\|_0\|D\tilde{u}^i\Delta\sigma\sigma^*\|_0 + \|\gamma\|_0\|D(\Delta u)\Delta\sigma\|_0
\end{aligned}$$

$$+\|\gamma\|_0 \left\| \Delta u \frac{1}{\phi} D\phi \Delta\sigma \right\|_0 + \|Du\|_0 \left\| \Delta u \frac{1}{\phi} D\phi [\tilde{\sigma}\tilde{\sigma}^* - \sigma\sigma^*] \right\|_0$$

$$\left. + \|\gamma\|_0 \|\Delta u (D \cdot \Delta\sigma)^*\|_0 + \|Du\|_0 \|\Delta u (\Delta b)^*\|_0 + \|\Delta f\|_0^2 \right]$$

$$= \frac{1}{2}\|\Delta\gamma^i\|_0^2 - C\left[\|\Delta u\|_0^2 + \|\Delta f\|_0^2 + I_t\right].$$

Now following the same arguments as in Proposition 5.1 one can easily prove the result. □

A direct consequence of Proposition 5.2 is the following uniqueness result.

Corollary 5.3. *Assume Assumption 2.1 holds. Then the decoupled BSPDE (5.2) has at most one weak solution.*

Another important consequence of Proposition 5.2 is the following "stability" result.

Proposition 5.4. *Let (b, σ, f, g) and $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)}), l = 1, 2, \ldots$ be a sequence of coefficients of the decoupled BSPDE (5.2) satisfying Assumption 2.1 uniformly. Assume that*

(i) $\lim_{l\to\infty}\left[\|b^{(l)} - b\|^2_{\mathscr{H}^0} + \|\sigma^{(l)} - \sigma\|^2_{\mathscr{H}^1} + \|g^{(l)} - g\|^2_{\mathscr{H}^0}\right] = 0$.

(ii) *For any fixed (x, y, z),*

$$\lim_{l\to\infty} \mathbb{E}\left\{\int_0^T |f^{(l)} - f|^2(t, x, y, z)dt\right\} = 0. \tag{5.16}$$

(iii) *For each l, BSPDE (5.2) with coefficients $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)})$ has a regular weak solution $(u^{(l)}, \beta^{(l)})$.*

(iv) *$Du^{(l)}$ are uniformly bounded, uniformly on l.*

(v) *There exists $u \in \mathscr{H}^0$ such that*

$$\lim_{l\to\infty} \|u^{(l)} - u\|^2_{\mathscr{H}^0} = 0. \tag{5.17}$$

Then $u \in \mathscr{H}^1$ and there exists $\beta \in \mathscr{H}^0$ such that (u, β) is a regular weak solution to the decoupled BSPDE (5.2) with coefficients (b, σ, f, g).

Proof. We first show that the limiting random field $u \in \mathscr{H}^1$ and that Du is bounded. In fact, by condition (iv) clearly $Du^{(l)}$ are bounded in $\mathcal{H}^0$. Then there exists $v \in \mathcal{H}^0$ such that $Du^{(l)} \to v$ weakly in $\mathscr{H}^0$. It is clear that v is bounded. Moreover, the differential operator D is a closed operator, that is, for any $h \in \mathscr{H}^1$,

$$\langle v, h\rangle_0 = \lim_{l\to\infty} \langle Du^{(l)}, h\rangle_0 = \lim_{l\to\infty} \langle u^{(l)}, Dh\rangle_0 = \langle u, Dh\rangle_0.$$

This implies that $Du = v$, and thus $u \in \mathscr{H}^1$.

Next, denote $\gamma^{(l)} \triangleq \beta^{(l)} + Du^{(l)}\sigma^{(l)}$. By (5.3), we get

$$\|\gamma^{(l)}\|^2_{\mathscr{H}^0} \le C \quad \text{and thus} \quad \|\beta^{(l)}\|^2_{\mathscr{H}^0} \le C.$$

Now by (5.11), our conditions imply that $\{\gamma^{(l)}, l \ge 1\}$ is a Cauchy sequence in $\mathscr{H}^0$. Then there exists $\gamma \in \mathscr{H}^0$ such that

$$\lim_{l\to\infty} \|\gamma^{(l)} - \gamma\|^2_{\mathscr{H}^0} = 0. \tag{5.18}$$

We now define $\beta \triangleq \gamma - Du\sigma$. Note that, for any $h \in \mathscr{H}^0$,

$$\begin{aligned}\langle \beta^{(l)}, h\rangle_{\mathcal{H}^0} &= \langle \gamma^{(l)} - Du^{(l)}\sigma^{(l)}, h\rangle_{\mathcal{H}^0} \\ &= \langle \gamma^{(l)}, h\rangle_{\mathcal{H}^0} - \langle Du^{(l)}\sigma, h\rangle_{\mathcal{H}^0} + \langle Du^{(l)}[\sigma - \sigma^{(l)}], h\rangle_{\mathcal{H}^0} \\ &\to \langle \gamma, h\rangle_{\mathcal{H}^0} - \langle Du\sigma, h\rangle_{\mathcal{H}^0} + 0 = \langle \beta, h\rangle_{\mathcal{H}^0}, \quad \text{as } l \to \infty,\end{aligned}$$

where the second convergence is due to the weak convergence of $Du^{(l)}$ and the boundedness of σ, and the third convergence is due to the uniform boundedness of $Du^{(l)}$. That is, $\beta^{(l)}$ converges to β weakly in $\mathscr{H}^0$.

It remains to show that (u, β) is a weak solution to BSPDE (5.2) with coefficients (b, σ, f, g). To simplify notations, in this part of the proof we assume $m = 1$ so that we can drop the superscript i, but all our arguments are still valid in high dimensional case.

It suffices to check (4.11). We fix $t_1 < t_2$ and a smooth function φ with compact support. For each l, since $(u^{(l)}, \beta^{(l)})$ is a weak solution to the corresponding BSPDE, by (4.9)–(4.11) we have

$$\begin{aligned}&\langle \varphi, u^{(l)}(t_2, \cdot) - u^{(l)}(t_1, \cdot)\rangle_0 \\ &= \int_{t_1}^{t_2} \langle \varphi, \beta^{(l)}(t, \cdot)dB_t\rangle_0 + \int_{t_1}^{t_2} \left\langle \left[\beta^{(l)} + \frac{1}{2}Du^{(l)}\sigma^{(l)}\right](\sigma^{(l)})^*, D\varphi\right\rangle_0 dt \\ &\quad + \int_{t_1}^{t_2} \left\langle \left(\beta^{(l)} + \frac{1}{2}Du^{(l)}\sigma^{(l)}\right)(\sigma^{(l)})^* \frac{1}{\phi}(D\phi)^* + \gamma^{(l)} D \cdot (\sigma^{(l)})^* \right. \\ &\quad \left. - Du^{(l)}b^{(l)}, \varphi\right\rangle_0 dt - \int_{t_1}^{t_2} \left\langle f^{(l)}\big(t, x, u^{(l)}, \gamma^{(l)}\big), \varphi\right\rangle_0 dt \\ &= \int_{t_1}^{t_2} \langle \varphi, \beta^{(l)}(t, \cdot)dB_t\rangle_0 + \int_{t_1}^{t_2} \left\langle \left[\gamma^{(l)} - \frac{1}{2}Du^{(l)}\sigma\right]\sigma^*, D\varphi\right\rangle_0 dt \\ &\quad + \int_{t_1}^{t_2} \left\langle \left(\gamma^{(l)} - \frac{1}{2}Du^{(l)}\sigma\right)\sigma^* \frac{1}{\phi}(D\phi)^* + \gamma^{(l)} D \cdot \sigma^* - Du^{(l)}b \right. \\ &\quad \left. - f^{(l)}\big(t, x, u, \gamma\big), \varphi\right\rangle_0 dt + \int_{t_1}^{t_2} [\langle I_t^{(l)}, D\varphi\rangle_0 + \langle J_t^{(l)}, \varphi\rangle_0]dt,\end{aligned}$$

where

$$\begin{aligned}I^{(l)} &\triangleq \gamma^{(l)}[(\sigma^{(l)})^* - \sigma^*] - \frac{1}{2}Du^{(l)}[\sigma^{(l)}(\sigma^{(l)})^* - \sigma\sigma^*]; \\ J^{(l)} &\triangleq I^{(l)}\frac{1}{\phi}(D\phi)^* + \gamma^{(l)} D \cdot [(\sigma^{(l)})^* - \sigma^*] - Du^{(l)}[b^{(l)} - b] \\ &\quad - \big[f^{(l)}\big(t, x, u^{(l)}, \gamma^{(l)}\big) - f^{(l)}\big(t, x, u, \gamma\big)\big].\end{aligned}$$

Sending $l \to \infty$ we obtain

$$\begin{aligned}&\langle \varphi, u(t_2, \cdot) - u(t_1, \cdot)\rangle_0 \\ &= \int_{t_1}^{t_2} \langle \varphi, \beta(t, \cdot)dB_t\rangle_0 + \int_{t_1}^{t_2} \left\langle \left[\gamma - \frac{1}{2}Du\sigma\right]\sigma^*, D\varphi\right\rangle_0 dt \\ &\quad + \int_{t_1}^{t_2} \left\langle \left(\gamma - \frac{1}{2}Du\sigma\right)\sigma^* \frac{1}{\phi}(D\phi)^* + \gamma D \cdot \sigma^* - Dub - f\big(t, x, u, \gamma\big), \varphi\right\rangle_0 dt \\ &\quad + \lim_{l\to\infty} \int_{t_1}^{t_2} [\langle I_t^{(l)}, D\varphi\rangle_0 + \langle J_t^{(l)}, \varphi\rangle_0]dt, \qquad (5.19)\end{aligned}$$

thanks to our assumptions on the coefficients. Furthermore, note that

$$
\begin{aligned}
|I^{(l)}| &\le C|\gamma^{(l)}-\gamma| + |\gamma||\sigma^{(l)}-\sigma| + C|\sigma^{(l)}-\sigma|;\\
|J^{(l)}| &\le C\left[|\gamma^{(l)}-\gamma| + |\sigma^{(l)}-\sigma| + |u^{(l)}-u|\right]\\
&\quad + |\gamma|\left[|\sigma^{(l)}-\sigma| + |D\cdot((\sigma^{(l)})^*-\sigma^*)|\right].
\end{aligned}
$$

Again, by our assumptions on the convergence of the coefficients we have

$$
\lim_{l\to\infty}\int_{t_1}^{t_2}[\langle I_t^{(l)}, D\varphi\rangle_0 + \langle J_t^{(l)}, \varphi\rangle_0]dt = 0.
$$

This, together with (5.19), implies that (4.11) holds for (u,β). That is, (u,β) is a weak solution. □

6. The main results

We are now ready to prove our main result of the paper. Note that except for the decoupled case, in general neither the well-posedness of the original coupled FBSDE nor that of the BSPDEs is known. Our main purpose here is to establish the equivalence of the solvability between the FBSDEs and BSPDEs.

We begin our discussion from the simple, decoupled case. In this case, the FBSDE (5.1) is always solvable, and its decoupling field always exists. On the other hand, by Theorem 4.1 we know that a classical solution (u,β) of the associate BSPDE (5.2) is always a regular decoupling field. However, the following result is by no means trivial.

Theorem 6.1. *Let Assumption 2.1 hold. Then, a random field $u \in \mathscr{H}^1$ is a regular decoupling field of the decoupled FBSDE (5.1) if and only if there exists a random field $\beta \in \mathscr{H}^0$ such that (u,β) is the (unique) regular weak solution to the BSPDE (5.2).*

Proof. We first note that in the decoupled case the solution to the BSPDE (5.2) has a unique weak solution, thanks to Corollary 5.3. Thus, since in this case the regular decoupling field u always exists, it suffices to find $\beta \in \mathscr{H}^0$ so that (u,β) is a weak solution to BSPDE (5.2).

Recall that the filtration $\mathbb{F}$ is Brownian, and the coefficients b, σ, and f are all $\mathbb{F}$-progressively measurable, we can assume without loss of generality that they take the form: for $\mathbb{P}$-a.e. ω, and all (x,y,z),

$$
\begin{cases}
\theta(t,\omega,x,y,z) = \theta(t,(B)_t(\omega),x,y,z), \quad \theta = b,\sigma,f;\\
g(\omega,x) = g(B_T(\omega),x).
\end{cases}
$$

Here $(B)_t \triangleq \{B_{s\wedge t}; s\in[0,T]\}$ denote the path of B up to time t. We now proceed in several steps, following the standard procedure of approximating non-Markovian processes with discrete Markovian ones.

Step 1. Assume that

$$
\begin{cases}
b(t,\omega,x) = \hat{b}(t,B_t(\omega),x), \qquad \sigma(t,\omega,x) = \hat{\sigma}(t,B_t(\omega),x),\\
f(t,\omega,x,y,z) = \hat{f}(t,B_t(\omega),x,y,z), \qquad g(\omega,x) = \hat{g}(B_T(\omega),x),
\end{cases}
\tag{6.1}
$$

where $\hat{b}, \hat{\sigma}, \hat{f}, \hat{g}$ are deterministic and smooth functions with bounded derivatives. Then, we can consider the following decoupled FBSDE on $[t, T]$:

$$\begin{cases} A_s = a + \int_t^s 1 dB_r; \\ X_s = x + \int_t^s \hat{b}(r, A_r, X_r)dr + \int_t^s \hat{\sigma}(r, A_r, X_r)dB_r; \\ Y_s = \hat{g}(A_T, X_T) + \int_s^T \hat{f}(r, A_r, X_r, Y_r, Z_r)dr - \int_t^T Z_r dB_r. \end{cases}$$

Denoting its solution by $(A, X, Y, Z) \triangleq (A^{t,a,x}, X^{t,a,x}, Y^{t,a,x}, Z^{t,a,x})$, by standard arguments in BSDE theory, we know that the (deterministic) function defined by $\hat{u}(t, a, x) \triangleq Y_t^{t,a,x}$ is smooth, and $Y_s = \hat{u}(s, A_s, X_s)$, $s \in [t, T]$, $\mathbb{P}$-a.s. Thus, applying Ito's formula, we have

$$\begin{aligned} dY_s^i &= \left[\hat{u}_s^i + D_x\hat{u}^i\hat{b} + \frac{1}{2}\mathrm{tr}\,(D_{aa}^2\hat{u}^i) + \mathrm{tr}\,(D_{ax}^2\hat{u}^i\hat{\sigma}) + \frac{1}{2}\mathrm{tr}\,(D_{xx}^2\hat{u}^i\hat{\sigma}\hat{\sigma}^*)\right](s, A_s, X_s)ds \\ &\quad + \left[D_a\hat{u}^i + D_x\hat{u}^i\hat{\sigma}\right](s, A_s, X_s)dB_s, \quad i = 1, \ldots, m, \end{aligned}$$

where D_a denotes the gradient with respect to a, and D_{xx} denotes the Hessian with respect to x. Other notations are defined in an obvious way. Compare this with the BSDE, we have

$$Z_s = \left[D_a\hat{u} + D_x\hat{u}\hat{\sigma}\right](s, A_s, X_s),$$

and $\hat{u}$ satisfies the following PDE:

$$\begin{cases} \hat{u}_s^i + D_x\hat{u}^i\hat{b} + \frac{1}{2}\mathrm{tr}\,(D_{aa}^2\hat{u}^i) + \mathrm{tr}\,(D_{ax}^2\hat{u}^i\hat{\sigma}) + \frac{1}{2}\mathrm{tr}\,(D_{xx}^2\hat{u}^i\hat{\sigma}\hat{\sigma}^*) \\ \qquad + \hat{f}^i(t, a, x, \hat{u}, D_a\hat{u} + D_x\hat{u}\hat{\sigma}) = 0; \\ \hat{u}(T, a, x) = \hat{g}(a, x). \end{cases} \tag{6.2}$$

Now consider the random fields

$$u(t, x) \triangleq \hat{u}(t, B_t, x) \quad \text{and set} \quad \beta(t, x) \triangleq (D_a\hat{u})(t, B_t, x).$$

For each fixed x we apply Itô's formula to $(t, B_t) \mapsto \hat{u}(t, B_t, x)$ and notice (6.2) we deduce that

$$\begin{aligned} du^i(t, x) &= \left[\hat{u}_t^i + \frac{1}{2}\mathrm{tr}\,(D_{aa}^2\hat{u}^i)\right](t, B_t, x)dt + D_a\hat{u}^i(t, B_t, x)dB_t \\ &= -\left[D_x u^i b + \mathrm{tr}\,(D_x(\beta^i)^*\sigma) + \frac{1}{2}\mathrm{tr}\,(D_{xx}^2 u^i\sigma\sigma^*) + f^i(t, x, u, \beta + D_x u\sigma)\right](t, x)dt \\ &\quad + \beta^i(t, x)dB_t, \quad i = 1, \ldots, m. \end{aligned}$$

That is, (u, β) is a classical solution to BSPDE (5.2).

Step 2. We now assume (6.1) again, but without requiring the smoothness of $\hat{b}, \hat{\sigma}, \hat{f}, \hat{g}$. Let $(\hat{b}^{(l)}, \hat{\sigma}^{(l)}, \hat{f}^{(l)}, \hat{g}^{(l)})$ be the standard smooth mollifiers of $(\hat{b}, \hat{\sigma}, \hat{f}, \hat{g})$, and consider the random fields

$$\theta^{(l)}(t, \omega, x, y, z) \triangleq \hat{\theta}^{(l)}(t, B_t, x, y, z), \quad \theta = b, \sigma, f;$$

and $g^{(l)}(\omega, x) \triangleq \hat{g}^{(l)}(B_T(\omega, x))$. Then $b^{(l)}, \sigma^{(l)}, f^{(l)}$, and $g^{(l)}$ obviously satisfy condition (i) and (ii) of Proposition 5.4. Also, the conditions (iii) and (iv) of Proposition 5.4 follow from Step 1 and the standard result in BSDE. Finally, the stability result for decoupled FBSDEs leads to condition (v) there. Therefore, applying Proposition 5.4 we conclude again that (u, β) is a regular weak solution to BSPDE (5.2).

Step 3. We now assume that all the coefficients are "discrete functionals" of B. That is, there exists a partition of $[0, T]\pi : 0 = t_0 < \cdots < t_l = T$ such that

$$\begin{cases} b(t, \cdot, x) = \hat{b}(t, B_{t_1\wedge t}, \ldots, B_{t_l\wedge t}, x), & \sigma(t, \cdot, x) = \hat{\sigma}(t, B_{t_1\wedge t}, \ldots, B_{t_l\wedge t}, x), \\ f(t, \cdot, x, y, z) = \hat{f}(t, B_{t_1\wedge t}, \ldots, B_{t_l\wedge t}, x, y, z), & g(\cdot, x) = \hat{g}(B_{t_1}, \ldots, B_{t_l}, x), \end{cases} \tag{6.3}$$

For $t \in [t_{l-1}, t_l]$, consider the decoupled FBSDE:

$$\begin{cases} A_s = a + \int_t^s I_{d\times d} dB_r; \\ X_s = x + \int_t^s \hat{b}(r, x_1, \ldots, x_{l-1}, A_r, X_r)dr + \int_t^s \hat{\sigma}(r, x_1, \ldots, x_{l-1}, A_r, X_r)dB_r; \\ Y_s = \hat{g}(x_1, \ldots, x_{l-1}, A_T, X_T) + \int_s^T \hat{f}(r, x_1, \ldots, x_{l-1}, A_r, X_r, Y_r, Z_r)dr \\ \qquad - \int_t^T Z_r dB_r. \end{cases}$$

Define $\hat{u}(t, a_1, \ldots, a_{l-1}, a, x) \triangleq Y_t^{t,a_1,\ldots,a_{l-1},a,x}$. Then, apply the result of Step 2, we see that $u(t, x) \triangleq \hat{u}(t, B_{t_1}, \ldots, B_{t_{l-1}}, B_t, x)$ is a regular weak solution to BSPDE (4.6) on $[t_{l-1}, t_l]$.

Now as a regular decoupling field (on $[t_{l-1}, t_l]$), $u(t_{l-1}, x) = \hat{u}(t_{l-1}, B_{t_1}, \ldots, B_{t_{l-1}}, B_{t_{l-1}}, x)$ is uniformly Lipschitz continuous in x. Thus we can consider the system on $[t_{l-2}, t_{l-1}]$ with terminal condition $u(t_{l-1}, x)$. Repeating this argument backwardly on each interval $[t_i, t_{i+1}]$, $i = l-1, l-2, \ldots, 0$ we obtain the result again.

Step 4. Finally we consider the general case. For each l, denote $t_i^{(l)} \triangleq \frac{iT}{2^l}$, $i = 0, \ldots, 2^l$. Let $\mathcal{F}_t^{(l)}$ be the σ-field generated by $(B_{t_1^{(l)}\wedge t}, B_{t_2^{(l)}\wedge t}, \ldots, B_{t_{2^l}^{(l)}\wedge t})$ and define

$$b^{(l)}(t, x) \triangleq \mathbb{E}\{b(t, x)|\mathcal{F}_t^{(l)}\}, \qquad \sigma^{(l)}(t, x) \triangleq \mathbb{E}\{\sigma(t, x)|\mathcal{F}_t^{(l)}\},$$
$$f^{(l)}(t, x, y, z) \triangleq \mathbb{E}\{f(t, x, y, z)|\mathcal{F}_t^{(l)}\}, \qquad g^{(l)}(x) \triangleq \mathbb{E}\{g(x)|\mathcal{F}_T^{(l)}\}.$$

We claim that $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)})$ satisfy all the conditions in Proposition 5.4. Indeed, first note that $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)})$ satisfies Assumption 2.1. Next, by the Dominated Convergence Theorem, we have

$$D\sigma^{(l)}(t, x) \triangleq \mathbb{E}\{D\sigma(t, x)|\mathcal{F}_t^{(l)}\}.$$

Note that $\mathcal{F}_t^{(l)}$ is increasing in l, and clearly $\mathcal{F}_t$ is generated by $\bigcup_l \mathcal{F}_t^{(l)}$. Then $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)})$ satisfy Proposition 5.4 (i) and (ii). To see (iii), note that for each l, $(b^{(l)}, \sigma^{(l)}, f^{(l)}, g^{(l)})$ take the form in Step 3 and thus applying the result in Step 3 we know that the corresponding BSPDE has a regular weak solution $u^{(l)}$. The conditions (iv) and (v) of Proposition 5.4 follow from the standard estimates as well as the stability result in the BSDE literature, respectively. Thus, applying Proposition 5.4 we obtain the result. □

We now extend the result to the general case. That is, the FBSDE (2.1) is truly coupled. However, for technical reasons, in the rest of the paper we consider only the case when the forward diffusion is independent of the backward component Z. In other words, we shall assume that the coefficient b also takes the form

$$b = b(t, \omega, x, y), \quad (t, \omega, x, y) \in [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m. \tag{6.4}$$

We should point out here that in this case the FBSDE (2.1) is non-Markovian, and strongly coupled, thus its well-posedness and the existence of the regular decoupling field is not known. We nevertheless have the following analogue of Theorem 6.1.

Theorem 6.2. *Assume Assumption 2.1 and (6.4) hold. Then a random field $u \in \mathscr{H}^1$ is a regular decoupling field of the FBSDE (2.1) if and only if there exists $\beta \in \mathscr{H}^0$ such that (u, β) is a regular weak solution to BSPDE (4.6).*

Furthermore, the regular weak solution to BSPDE (4.6), if exists, must be unique.

Proof. We first assume that FBSDE (2.1) has a regular decoupling random field u, then by Theorem 3.4 the FBSDE (2.1) has a (unique) solution (X, Y, Z). We shall find the random field $\beta \in \mathscr{H}^0$ such that (u, β) is a regular weak solution to BSPDE (4.6). Note that in this case (4.7) becomes

$$\hat{b}(t, x) \triangleq b(t, x, u(t, x)), \qquad \hat{\sigma}(t, x) \triangleq \sigma(t, x, u(t, x)). \tag{6.5}$$

Since u is uniformly Lipschitz continuous, we see that $\hat{b}$ and $\hat{\sigma}$ are still uniformly Lipschitz continuous. Now note that (X, Y, Z) satisfies the following decoupled FBSDE:

$$\begin{cases} \hat{X}_t = x + \int_0^t \hat{b}(s, \hat{X}_s)ds + \int_0^t \hat{\sigma}(s, \hat{X}_s)dB_s; \\ \hat{Y}_t = g(\hat{X}_T) + \int_t^T f(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s)ds - \int_t^T \hat{Z}_s dB_s; \end{cases} \tag{6.6}$$

Denote $\hat{u}(t, x) \triangleq \hat{Y}_x^{t,x}$. We can apply Theorem 6.1 to conclude that there exists $\beta \in \mathscr{H}^0$ such that $(\hat{u}, \beta)$ is a regular weak solution to the following (decoupled) BSPDE:

$$\begin{cases} d\hat{u}^i(t, x) = -\Bigg[\frac{1}{2}\mathrm{tr}\,[D^2\hat{u}^i\hat{\sigma}\hat{\sigma}^*(t, x)] + \mathrm{tr}\,(D(\hat{\beta}^i)^*\hat{\sigma}(t, x)) + D\hat{u}^i\hat{b}(t, x) \\ \qquad + f^i(t, x, \hat{u}, \hat{\gamma})\Bigg]dt + \hat{\beta}^i dB_t, \quad i = 1, \ldots, m; \\ \hat{u}(T, x) = g(x). \end{cases} \tag{6.7}$$

But the uniqueness of the decoupling field and that of the BSPDE then imply that $\hat{Y}_t^{t,x} = Y_t^{t,x}$ and thus $\hat{u} = u$. Then $(u, \hat{\beta})$ is a regular weak solution to BSPDE (4.6).

Conversely, if (u, β) is a regular weak solution to BSPDE (4.6). Define $\hat{b}, \hat{\sigma}$ by (6.5). Then (u, β) is the (unique) weak solution to BSPDE (6.7). Let (X, Y, Z) be the unique solution to the decoupled FBSDE (6.6). By Theorem 6.1, we see that $Y_t = u(t, X_t)$. Thus actually (X, Y, Z) satisfies FBSDE (2.1). We note that this argument applies to FBSDE (3.2) on any subinterval $[t_1, t_2]$. Thus u is a regular decoupling field.

To show the uniqueness, we note from the above argument that any regular weak solution to BSPDE (4.6) must correspond to a regular weak solution to the decoupled BSPDE (6.7), and thus is a regular decoupling field of FBSDE (2.1). Then the uniqueness follows from Theorem 3.4. □

Remark 6.3. From the proof of Theorem 6.2 the only reason that we require the assumption (6.4) is that the decoupled SDE can have Lipschitz coefficients. This condition can be relaxed if we raise the regularity of the solution (u, β) (e.g., requiring the function γ to be Lipschitz). But we do not intend to pursue such generality in this paper, and the interested reader can extend the result in a case by case basis. □

To conclude this section we would like to point out that the significance of Theorems 6.1 and 6.2 is that it gives the equivalence of solvability between the FBSDE (2.1) and BSPDE (4.6). This provides a new tool for solving BSPDEs that has not been established before, especially given the recent developments on the theory of non-Markovian FBSDEs. For example, an interesting by-product of Theorems 6.1 and 6.2 is the following well-posedness result for the degenerate quasilinear BSPDE (4.6) which, to our best knowledge, is novel in the literature.

Let us assume that all involved processes are 1-dimensional. For any t, any $x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2$, and $\varphi = b, \sigma, f$, let us define

$$\begin{aligned}
\varphi_1(t) &\triangleq \varphi_1(t, x_1, y_1, z_1, x_2, y_2, z_2) \triangleq \frac{\varphi(t, x_1, y_1, z_1) - \varphi(t, x_2, y_1, z_1)}{x_1 - x_2};\\
\varphi_2(t) &\triangleq \varphi_2(t, x_1, y_1, z_1, x_2, y_2, z_2) \triangleq \frac{\varphi(t, x_2, y_1, z_1) - \varphi(t, x_2, y_2, z_1)}{y_1 - y_2};\\
\varphi_3(t) &\triangleq \varphi_3(t, x_1, y_1, z_1, x_2, y_2, z_2) \triangleq \frac{\varphi(t, x_2, y_2, z_1) - \varphi(t, x_2, y_2, z_1)}{z_1 - z_2};\\
h &\triangleq h(x_1, x_2) \triangleq \frac{g(x_1) - g(x_2)}{x_1 - x_2};
\end{aligned}$$

and

$$F(t, y) \triangleq F(x_1, y_1, z_1, x_2, y_2, z_2; t, y) \triangleq f_1 + [f_2 + b_1 + f_3\sigma_1]y + [b_2 + f_3\sigma_2]y^2.$$

Theorem 6.4. *Assume Assumption 2.1 and (6.4) hold, and that all processes involved are 1-dimensional. Assume further that there exist a constant c and a constant $\varepsilon > 0$ small enough such that one of the following two conditions holds true: for any t and any $x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2$*

$$\begin{aligned}
&F(t, c) \geq 0, \quad c \leq h, \qquad b_2 + f_3\sigma_2 \leq \varepsilon;\\
&F(t, c) \leq 0, \quad c \geq h, \qquad b_2 + f_3\sigma_2 \geq -\varepsilon.
\end{aligned}$$

Then the FBSDE (2.1) has a regular decoupling field and thus the BSPDE (4.6) has a unique regular weak solution.

Proof. Applying the wellposedness result in [14] we know that under the given assumptions the FBSDE (2.1) is well-posed, and that the regular decoupling field exists. The result then follows from Theorem 6.2. □

7. Comparison theorem

In this section, we investigate the comparison theorem for the quasilinear BSPDE (4.6). We should note that while such result is more or less standard in BSDE theory (see, e.g., [15,16] for the case of linear BSPDEs), it is more significant because its connection to the fully coupled FBSDE (2.1), for which the comparison theorem generally fails (cf. e.g., [4,17]). Our argument follows that of [26, Theorem 7.1].

Theorem 7.1. *Assume that $(b, \sigma, f_i, g_i), i = 1, 2$ satisfy Assumption 2.1 and (6.4), and that the BSPDE (4.6) with coefficients (b, σ, f_i, g_i) has a regular weak solution $u_i, i = 1, 2$, respectively. Assume further that $m = 1$ and*

$$f_1(t, \cdot, x, y, z) \leq f_2(t, \cdot, x, y, z), \qquad g_1(\cdot, x) \leq g_2(\cdot, x), \quad \forall (t, x, y, z),\ \mathbb{P}\text{-a.s.},$$

then it holds that $u_1(t, x) \leq u_2(t, x), \forall (t, x), \mathbb{P}$-a.s.

Proof. Let K denote the common Lipschitz constant of u_1, u_2, and $\delta(K)$ be the constant in Lemma 3.3. We first claim that it suffices to prove the result for the case that

$$T \le \delta(K). \tag{7.1}$$

Indeed, if comparison holds whenever $T \le \delta(K)$, let $0 = t_0 < \cdots < t_k = T$ be a partition such that $\Delta t_i \le \delta(K)$. Then by applying our assumption on $[t_{k-1}, t_k]$ we get $u_1(t, x) \le u_2(t, x)$ for $t \in [t_{k-1}, t_k]$. In particular, $u_1(t_{k-1}, x) \le u_2(t_{k-1}, x)$. We can then repeat the argument on $[t_{k-2}, t_{k-1}]$ to get $u_1(t, x) \le u_2(t, x)$ for $t \in [t_{k-2}, t_{k-1}]$, and so on, to obtain the result.

Next, we note that by introducing the solution to FBSDE (2.1), say, (X^3, Y^3, Z^3), with coefficients b, σ, f_1, g_2, and then arguing that $Y_0^1 \le Y_0^3$ and $Y_0^3 \le Y_0^2$, respectively, we can split the arguments for the comparisons for f and for g separately.

We now assume (7.1) holds, and consider the following two cases.

Case 1. *Assume* $f_1 = f_2 = f$ *and* $g_1 \le g_2$.

Let u^i be the regular weak solution to BSPDE (4.6), and (X^i, Y^i, Z^i) the solution to FBSDEs (2.1), $i = 1, 2$, respectively. Without loss of generality, we shall only prove

$$u_1(0, x) \le u_2(0, x), \quad \text{or equivalently } Y_0^1 \le Y_0^2. \tag{7.2}$$

To this end, denote $X \triangleq X^2 - X^1$, $Y \triangleq Y^2 - Y^1$, $Z \triangleq Z^2 - Z^1$, and $\xi \triangleq g_2(X_T^1) - g_1(X_T^1)$. Then $\xi \ge 0$, and

$$\begin{cases} X_t = \displaystyle\int_0^t [\lambda_s^1 X_s + \theta_s^1 Y_s] ds + \int_0^t [\lambda_s^2 X_s + \theta_s^2 Y_s] dB_s; \\ Y_t = G X_T + \xi + \displaystyle\int_t^T \left[\lambda_s^3 X_s + \theta_s^3 Y_s + \kappa_s^3 Z_s\right] ds - \int_t^T Z_s dB_s, \end{cases} \tag{7.3}$$

for some appropriate bounded processes $\lambda^i, \theta^i, \kappa^i$, $i = 1, 2, 3$, and bounded random variable G (depending on the Lipschitz constants with respect to x, y, z of coefficients b, σ, f, and g, respectively).

We shall prove (7.2) by contradiction. Suppose not, then $Y_0 < 0$. Define

$$G_t \triangleq \mathbb{E}[G|\mathcal{F}_t], \quad \tau \triangleq \inf\{t : Y_t \ge G_t X_t\} \wedge T.$$

Since $Y_0 - G_0 X_0 = 0$, $Y_T - G_T X_T = \xi \ge 0$, and the process $Y - GX$ is continuous, we have $Y_\tau = G_\tau X_\tau$. Denote

$$\tilde{\zeta}_t \triangleq \xi_t \mathbf{1}_{[0,\tau]}(t) \quad \text{for } \zeta = \lambda^i, \theta^i, \kappa^i, \quad \text{and} \quad \tilde{\eta}_t \triangleq \eta_{\tau \wedge t} \quad \text{for } \eta = X, Y, Z.$$

One can easily check that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ satisfies:

$$\begin{cases} \tilde{X}_t = \displaystyle\int_0^t [\tilde{\lambda}_s^1 \tilde{X}_s + \tilde{\theta}_s^1 \tilde{Y}_s] ds + \int_0^t [\tilde{\lambda}_s^2 \tilde{X}_s + \tilde{\theta}_s^2 \tilde{Y}_s] dB_s; \\ \tilde{Y}_t = G_\tau \tilde{X}_T + \displaystyle\int_t^T \left[\tilde{\lambda}_s^3 \tilde{X}_s + \tilde{\theta}_s^3 \tilde{Y}_s + \tilde{\kappa}_s^3 (\tilde{Z}_s)^*\right] ds - \int_t^T \tilde{Z}_s dB_s. \end{cases}$$

Note that λ^i, θ^i and κ^i are all bounded, we assume that they are all bounded by K. Then, by our choice of $\delta(K)$ we see that the above FBSDE has a unique solution, and it must be the zero solution. Therefore $Y_0 = \tilde{Y}_0 = 0$, a contradiction. Thus (7.2) holds.

Case 2. $g_1 = g_2 = g$, *but* $f_1 \le f_2$. We still assume (7.1) holds, and make a further reduction. We claim that it suffices to prove the comparison for the case that

$$c_0 \le f_2 - f_1 \le c_0^{-1} \quad \text{for some } c_0 > 0. \tag{7.4}$$

Indeed, for general f_1, f_2, denote $f_2^\varepsilon \triangleq (f_2+\varepsilon)\wedge(f_1+\varepsilon^{-1})$. Then f_2^ε satisfies Assumption 2.1 uniformly, $f_2^\varepsilon \to f_2$, almost surely, as $\varepsilon \to 0$, and $\varepsilon \le f_2^\varepsilon - f_1 \le \varepsilon^{-1}$. Let $(X^{2,\varepsilon}, Y^{2,\varepsilon}, Z^{2,\varepsilon})$ denote the unique solution to FBSDE (2.1) with coefficients $(b, \sigma, f_2^\varepsilon, g)$. By our assumption, $Y_0^1 \le Y_0^{2,\varepsilon}$ for any $\varepsilon > 0$. By the stability of FBSDEs we get $\lim_{\varepsilon\to 0} Y_0^{2,\varepsilon} = Y_0^2$. Thus $Y_0^1 \le Y_0^2$.

In what follows we assume (7.1) and (7.4). Again, denote $X \triangleq X^2 - X^1$, $Y \triangleq Y^2 - Y^1$, $Z \triangleq Z^2 - Z^1$ and $\eta_t \triangleq (f_2 - f_1)(t, X_t^1, Y_t^1, Z_t^1)$. Then (7.4) implies that

$$c_0 \le \eta \le c_0^{-1},$$

and (7.3) is replaced by

$$\begin{cases} X_t = \int_0^t [\lambda_s^1 X_s + \theta_s^1 Y_s] ds + \int_0^t [\lambda_s^2 X_s + \theta_s^2 Y_s] dB_s; \\ Y_t = G X_T + \int_t^T \left[\lambda_s^3 X_s + \theta_s^3 Y_s + \kappa_s^3 Z_s + \eta_s\right] ds - \int_t^T Z_s dB_s, \end{cases} \tag{7.5}$$

for some appropriate bounded processes $\lambda^i, \theta^i, \kappa^i$ and bounded random variable G.

Following the arguments in Case 1, it suffices to prove $Y_0 \ge 0$ for $T \le \delta$, where $\delta \triangleq \delta(c) \le \delta(K)$ will be specified later. So we now assume $T \le \delta$. For any $\varepsilon > 0$, applying standard arguments on the BSDE in (7.5) we have

$$\mathbb{E}\left\{|Y_t|^2 + \frac{1}{2}\int_t^T |Z_s|^2 ds\right\} \le \mathbb{E}\left\{|X_T|^2 + C\varepsilon^{-1}\int_t^T [|X_t|^2 + |Y_t|^2]dt + \varepsilon \int_t^T |\eta_s|^2 ds\right\}.$$

Then

$$\sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\} \le [1 + C\varepsilon^{-1}T] \sup_{0\le t\le T} \mathbb{E}\{|X_t|^2\} + C\varepsilon^{-1}T \sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\} + \varepsilon T c_0^{-2}. \tag{7.6}$$

Moreover, from the FSDE in (7.5) we have

$$\sup_{0\le t\le T} \mathbb{E}\{|X_t|^2\} \le C\mathbb{E}\left\{\int_0^T |Y_t|^2 dt\right\} \le CT \sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\}. \tag{7.7}$$

Plugging (7.7) into (7.6) we get

$$\sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\} \le C_0 \varepsilon^{-1} T \sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\} + c_0^{-2}\varepsilon T.$$

Now we set $\varepsilon \triangleq 2C_0T$ in the above, and deduce that

$$\sup_{0\le t\le T} \mathbb{E}\{|Y_t|^2\} \le C c_0^{-2} T^2, \quad \text{and thus} \quad \sup_{0\le t\le T} \mathbb{E}\{|X_t|^2\} \le C c_0^{-2} T^3. \tag{7.8}$$

Now let M be the solution to the following linear SDE:

$$M_t = 1 + \int_0^t M_s \kappa_s^3 dB_s, \quad t \in [0, T]. \tag{7.9}$$

Then applying Itô's formula we have

$$Y_t M_t = G X_T M_T + \int_t^T [\lambda_s^3 X_s + \theta_s^3 Y_s + \eta_s] M_s ds - \int_t^T M_s [Z_s + Y_s \kappa_s^3] dB_s.$$

Note that M is a positive martingale, we have

$$\begin{aligned} Y_0 = Y_0 M_0 &= \mathbb{E}\left\{ G X_T M_T + \int_0^T [\lambda_t^3 X_t + \theta_t^3 Y_t + \eta_t] M_t dt \right\} \\ &\geq c_0 \mathbb{E}\left\{ \int_0^T M_t dt \right\} - \left| \mathbb{E}\left\{ G X_T M_T + \int_0^T [\lambda_t^3 X_t + \theta_t^3 Y_t] M_t dt \right\} \right| \\ &\geq c_0 T - C \sup_{0 \leq t \leq T} \mathbb{E}\{|X_t M_t|\} - CT \sup_{0 \leq t \leq T} \mathbb{E}\{|Y_t M_t|\}. \end{aligned}$$

Since $\mathbb{E}\{|M_t|^2\} \leq C$, it follows from (7.8) that

$$\mathbb{E}\{|X_t M_t|\} \leq \left[\mathbb{E}\{|X_t|^2\} \right]^{\frac{1}{2}} \left[\mathbb{E}\{|M_t|^2\} \right]^{\frac{1}{2}} \leq C c_0^{-1} T^{\frac{3}{2}},$$

and

$$\mathbb{E}\{|Y_t M_t|\} \leq \left[\mathbb{E}\{|Y_t|^2\} \right]^{\frac{1}{2}} \left[\mathbb{E}\{|M_t|^2\} \right]^{\frac{1}{2}} \leq C c_0^{-1} T.$$

Consequently, we have

$$Y_0 \geq c_0 T - C c_0^{-1} [T^{\frac{3}{2}} + T^2] \geq c_0^{-1} T \left[c_0^2 - C \delta^{\frac{1}{2}} \right].$$

Choosing $\delta \leq c_0^4 C^{-2}$ we obtain $Y_0 \geq 0$. This proves Case 2, whence the theorem. □

Remark 7.2. It should be noted that the comparison between the BSPDE only leads to the comparison between the decoupling field u of FBSDEs, and consequently the comparison for $Y_0^1 = u^1(0, x) \leq u^2(0, x) = Y_0^2$. However, since in general there is no comparison between $X_t^1 \neq X_t^2$, we cannot conclude $Y_t^1 \leq Y_t^2$ for $t > 0$, except for some special situation in which some monotone properties of the coefficients hold so that the decoupling field becomes monotone, and the comparison between Y_t, for all t becomes possible (see e.g. [24]). □

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