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Dynamic Programming for Multidimensional Stochastic Control Problems

Jin Ma[1)]
Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395, *U S A*
E-mail: majin@math.purdue.edu

Jiongmin Yong[2)]
Laboratory of Mathematics for Nonlinear Science, Department of Mathematics, and Institute of Mathematical Finance, Fudan University, Shanghai 200433, *P. R. China*
E-mail: jyong@fudan.edu.cn

Abstract In this paper we study a general multidimensional diffusion-type stochastic control problem. Our model contains the usual regular control problem, singular control problem and impulse control problem as special cases. Using a unified treatment of dynamic programming, we show that the value function of the problem is a viscosity solution of certain Hamilton-Jacobi-Bellman (HJB) quasi-variational inequality. The uniqueness of such a quasi-variational inequality is proved.

Keywords Stochastic control, Dynamic programming, Viscosity solutions, Singular control, Impulse control
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1 Introduction

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space on which is defined an m-dimensional Brownian motion $\{W_t : t \geq 0\}$. Consider the following controlled stochastic system:

$$X_t = x + \int_s^t b(r, X_r, u_r)dr + \int_s^t \sigma(r, X_r, u_r)dW_r + \xi_t, \qquad s \leq t \leq T, \tag{1.1}$$

where $s \geq 0$ is called the *initial time*; x the *initial state*; and the pair (u, ξ) the *control process.* We assume that the process u_t is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and takes values in some separable metric

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space, and the process ξ_t is $\mathbb{R}^n$-valued, càglàd, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and with locally bounded variation paths. We often refer to u as the *regular control*, and to ξ as the *singular control*, for obvious reasons.

For each pair of controls (u, ξ), we consider the following cost functional:

$$J_{s,x}(u,\xi) = E\left\{ \int_s^T h(r, X_r, u_r)dr + \int_s^T f^a(r)\|\dot{\xi}_r^{ac}\|_1 dr + \int_s^T f^s(r)|d\xi_r^{sc}| + \sum_{r\in S_\xi[s,T]} \ell(r, \Delta\xi_r) + g(X_T)\right\}. \tag{1.2}$$

Here, g, h, f^a, f^s and ℓ are some given functions, $\xi_r = \xi_r^{ac} + \xi_r^{sc} + \xi_r^d$ is the Lebesgue decomposition of the path ξ, in which ξ^{ac} denotes its absolutely continuous part, ξ^{sc} the singularly continuous part, and ξ^d the pure jump part. Throughout the paper we denote by $\|\cdot\|_1$ the L^1-norm in $\mathbb{R}^n$; $\Delta\xi_t = \xi_{t+} - \xi_t$, and $S_\xi[s,T] = \{t \in [s,T] \mid \Delta\xi_t \neq 0\}$ (see §2 for details). Our goal is to minimize the cost functional (1.2) over (u,ξ) chosen from a certain class of *admissible controls.*

It is easily seen that our model covers the classical stochastic *regular* control problem ($\xi \equiv 0$); the standard *singular* stochastic control problem (b, σ being independent of u, $\ell(t,\xi) = f(t)\|\Delta\xi_t\|_1$ with $f^a = f^s = f$); and the *impulse* control problem ($\xi^{ac} \equiv \xi^{sc} \equiv 0$). We should note that although all three problems have been studied extensively in separate forms (see for example, [1–12], to mention a few), the combined models were only studied under special forms (cf. e.g. [13–16]). The closest references to our problem are the works of Haussmann and Suo [17–18], but in their setting the cost functional is simpler, and it does not seem to be able to handle the impulse control problem. Moreover, the uniqueness of the viscosity solution of the associated HJB equation was not discussed there.

The main purpose of the paper is to derive the HJB equation of the control problem via dynamic programming; and more importantly, to prove the *uniqueness* of the viscosity solution to such an HJB equation, which, to the best of the best of our knowledge, is not covered by any existing literature. We should also note that by considering function $\ell(t,\theta) = \ell_0(t) + f^d(t)\|\theta\|_1^\nu$ and allowing $\ell_0(t) \geq c_0 \geq 0$ and $0 < \nu \leq 1$ in the cost functional, we can treat the impulse control problem ($c_0 > 0$) and the singular control problem ($l_0 \equiv 0$, $\nu = 1$) within a unified framework; such a treatment seems to be novel as well.

This paper is organized as follows. In Sec. 2 we give some necessary preliminaries and a formulation of the problem. In Sec. 3 we prove the continuity of the value functions. Sec. 4 is devoted to the study of HJB equations; and finally we prove that the value function is the *unique* viscosity solution of the associated HJB equation in Secs. 5 and 6.

2 Preliminaries and Problem Formulation

Throughout this paper we let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a filtered probability space satisfying the *usual conditions*, i.e. $\mathcal{F}_t$ is right-continuous in t and $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$. Let $K \subseteq \mathbb{R}^n$ be a convex and closed cone and let $K_0 \triangleq \{\theta \in K \mid \|\theta\|_1 = 1\}$, where $\|\cdot\|_1$ is the

L^1-norm of $\mathbb{R}^n$.

We shall make use of the following *Standing Assumptions* on the functions b, σ, f and g appearing in the control system (1.1) and the cost functional (1.2):

(A1) The functions b and σ are uniformly continuous from $[0,T] \times \mathbb{R}^n \times U$ to $\mathbb{R}^n$ and to $\mathbb{R}^{n\times m}$, respectively. Further, $\exists L > 0$ and $\nu \in (0,1]$, such that for $t \in [0,T]$, $x, \bar{x} \in \mathbb{R}^n$ and $u \in U$, there holds that

$$\begin{cases} |b(t,x,u) - b(t,\bar{x},u)| + |\sigma(t,x,u) - \sigma(t,\bar{x},u)| \le L|x - \bar{x}|, \\ |b(t,x,u)| + |\sigma(t,x,u)|^2 \le L(1 + |x|^\nu). \end{cases} \tag{2.1}$$

(A2) The functions $h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $f^a, f^s : [0,T] \to \mathbb{R}$, $\ell : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuous. Further, $\exists c_0 > 0$, $L, L_0 \ge 0$, $0 < \nu \le 1$ and $\mu \in (0,\nu]$, and a nondecreasing continuous function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$, such that for all $t, \bar{t} \in [0,T]$, $x, \bar{x} \in \mathbb{R}^n$ and $u \in U$, there holds that

$$\begin{cases} |h(t,x,u) - h(t,\bar{x},u)| \le L|x - \bar{x}| + \omega(|t - \bar{t}|); \\ 0 \le h(t,x,u), \ g(x) \le L(1 + |x|^\mu); \end{cases} \tag{2.2}$$

$$|g(x) - g(\bar{x})| \le L_0|x - \bar{x}|; \tag{2.3}$$

$$c_0 \le f^a(t), \ f^s(t) \le L, \qquad f^a(T) \wedge f^s(T) \triangleq \min\{f^a(T), f^s(T)\} \ge L_0; \tag{2.4}$$

$$\begin{cases} \ell(t, \theta + \bar{\theta}) \le \ell(t,\theta) + \ell(t,\bar{\theta}), \\ \ell(t, \alpha\theta) \le \ell(t,\theta), \end{cases} \qquad \forall t \in [0,T], \ \theta \in K, \ \alpha \in [0,1] : \tag{2.5}$$

$$0 \le \ell(t,\theta) - \ell(s,\theta) \le \omega(t-s)\|\theta\|_1^\nu, \qquad \forall 0 \le s \le t \le T, \ \theta \in K; \tag{2.6}$$

$$\ell(t,\theta) \ge c_0\|\theta\|_1^\nu, \qquad \forall t \in [0,T], \ \theta \in K; \tag{2.7}$$

$$g(x) \le \inf_{\xi \in K} \big[g(x + \xi) + \ell(T,\xi)\big], \qquad \forall x \in K. \tag{2.8}$$

In the case $\nu = 1$ and $\ell(t,0) = 0$ for some $t \in [0,T]$, we assume that for some $f^d \in C[0,T]$,

$$L_0 \le f^d(T) \le L; \qquad \ell(t,\theta) = f^d(t)\|\theta\|_1, \qquad \forall (t,\theta) \in [0,T] \times K. \tag{2.9}$$

Remark 2.1 Among all the conditions, (2.1), (2.2) are standard; (2.3), (2.4), (2.8) and (2.9) are *compatibility conditions* often seen in singular control problems, and seem to be indispensable (see, for example, [6,8,18]). Condition (2.5) is common in impulse control problems (cf. [2,9,10]); and (2.6) is slightly more general than that given in [9,10]. The coercivity condition (2.7) plays an important role in proving the continuity of the value function, as well as the uniqueness of the viscosity solutions.

For $0 \le s \le t \le T$ let us denote by $D([s,t];\mathbb{R}^n)$ the space of all functions $\zeta : [s,t] \mapsto \mathbb{R}^n$ that are càglàd (left continuous with right limits). For $\zeta \in D([s,t];\mathbb{R}^n)$, we denote $\Delta\zeta_r \triangleq \zeta_{r+} - \zeta_r$, $S_\zeta[s,t] \triangleq \{r \in [s,t] \mid \Delta\zeta_r \ne 0\}$; and define the total variation of ζ on $[s,t]$ by $|\zeta|_{[s,t]} \triangleq \int_{[s,t]} |d\zeta_r| = \sum_{i=1}^n |\zeta^i|_{[s,t]}$, where $|\zeta^i|_{[s,t]}$ is the total variation of the i-th component of ζ on $[s,t]$ in the usual sense. We shall denote $|\zeta|_t \triangleq |\zeta|_{[0,t]}$ for simplicity; and denote $BV([s,t];\mathbb{R}^n) = \{\zeta \in D([s,t];\mathbb{R}^n) \big| |\zeta|_{[s,t]} < \infty\}$.

Now for each $\zeta \in BV([0,T];\mathbb{R}^n)$, by the Lebesgue decomposition we can write $\zeta_t = \zeta_t^{ac} + \zeta_t^{sc} + \zeta_t^d$, $t \in [0,T]$, where $\zeta_t^{ac} \triangleq \int_0^t \dot{\zeta}_s^c ds$ ($\dot{\zeta}_s^c = \frac{d}{ds}\zeta_s^c$) is the *absolutely continuous* part; $\zeta^d \triangleq \sum_{0\le s<t} \Delta\zeta_s$ is the *pure jump* part; and $\zeta^{sc} \triangleq \zeta_t - \zeta^{ac} - \zeta_t^d$ is the *singularly continuous* part of ζ^-. Since $|\zeta| \in BV([0,T];\mathbb{R}^n)$ as well, the same decomposition holds: $|\zeta| = |\zeta|^{ac} + |\zeta|^{as} + |\zeta|^d$. An easy application of Radon-Nikodým Theorem leads to the following representation:

$$\zeta_t = \int_{[s,t)} \theta_r d|\zeta|_r, \qquad \forall t \in [s,T], \tag{2.10}$$

where θ is some measurable function such that $\|\theta\|_1 \equiv 1$. We denote this relation by $\zeta \sim (\theta, |\zeta|)$. It is fairly easy to show that, for any $0 \le s \le t \le T$,

$$\begin{cases} |\zeta|_{[s,t]} = |\zeta|^d_{[s,t]} + |\zeta|^{ac}_{[s,t]} + |\zeta|^{sc}_{[s,t]} = |\zeta^d|_{[s,t]} + |\zeta^{ac}|_{[s,t]} + |\zeta^{sc}|_{[s,t]}, \\ \zeta_t = \int_{[s,t)} \theta_r d|\zeta|_r = \int_s^t \theta_r d|\zeta|_r^{ac} + \int_s^t \theta_r d|\zeta|_r^{as} + \sum_{r\in S_{|\zeta|}[s,t)} \theta_r \Delta|\zeta|_r, \\ \zeta_t^{ac} = \int_s^t \theta_r d|\zeta|_r^{ac},\ \zeta_t^{sc} = \int_s^t \theta_r d|\zeta|_r^{sc},\ \Delta\zeta_t = \theta_t \Delta|\zeta|_t. \end{cases} \tag{2.11}$$

Finally, we denote by $BV^0([s,T];K)$ the subset of $BV([s,T];\mathbb{R}^n)$ consisting of all elements $\zeta \sim (\theta, |\zeta|)$ such that $\theta_r \in K_0, r \in [s,T]$.

To describe the singular control process, we denote by $L^{2m}_{\mathcal{F}}(\Omega; BV([s,r];\mathbb{R}^n))$ the space of all $\{\mathcal{F}_t\}$-adapted processes $\{\xi_t\}_{s\le t\le r}$ with paths in $BV([s,r];\mathbb{R}^n)$, such that

$$\|\xi\|_{[s,r],2m} \triangleq \left\{ E\Big(\int_{[s,r]} |d\xi_t| \Big)^{2m} \right\}^{\frac{1}{2m}} < \infty.$$

We denote $\|\xi\|_{[s,r]} = \|\xi\|_{[s,r],2}$ for simplicity. Since ξ is *left continuous*, it is predictable. By a stochastic version of the Radon-Nikodým Theorem (see, for example, Dellacherie-Meyer [19], VI), we still have the representation $\xi \sim (\theta, |\xi|)$, where both $|\xi|$ and θ are predictable. Further, by redefining θ on a set of $d|\xi| \otimes dP$-measure zero if necessary, we shall assume that $\|\theta_t\|_1 = 1$, $t \in [s,T]$, a.s. P.

We are now ready to formulate the control problem. Let U be a separable metric space and $K \subseteq \mathbb{R}^n$ be a convex and closed cone. Set

$$\begin{cases} \mathcal{U}[s,T] \triangleq \Big\{ u : [s,T]\times\Omega \to U \mid u \text{ is } \{\mathcal{F}_t\}\text{-progressively measurable} \Big\}; \\ \mathcal{K}[s,T] \triangleq \Big\{ \xi \in L^2_{\mathcal{F}}(\Omega; BV^0([s,T];K)) \mid \xi_s = 0, \text{ a.s.} \Big\}. \end{cases} \tag{2.12}$$

We call $(u,\xi) \in \mathcal{U}[0,T]\times\mathcal{K}[0,T]$ an *admissible control*, in which u is called the *regular part* and ξ is called the *singular part*. Our optimal control problem is then formulated as follows.

For any given $(s,x) \in [0,T)\times\mathbb{R}^n$, find a pair $(\bar{u},\bar{\xi}) \in \mathcal{U}[s,T]\times\mathcal{K}[s,T]$, such that

$$J_{s,x}(\bar{u},\bar{\xi}) = \inf_{(u,\xi)\in\mathcal{U}[s,T]\times\mathcal{K}[s,T]} J_{s,x}(u,\xi) \triangleq V(s,x). \tag{2.13}$$

The function $V(s,x)$ is called the *value function* of the control problem.

To conclude this section we note that under the assumption (A1), for any $s \in [0,T)$, $x \in \mathbb{R}^n$ and $(u,\xi) \in \mathcal{U}[s,T] \times \mathcal{K}[s,T]$, the state equation (1.1) admits a (pathwise) unique solution (the initial state x can actually be replaced by any square integrable $\mathcal{F}_s$-measurable random vector X_s). We shall denote such a solution by $X^{s,X_s,u,\xi}$, which will be simplified as $X^{s,X_s,\xi}$, or X^{s,X_s}, or even X, when the dependence on the missing factor(s) in the superscript is clear or not important. The following proposition gives some useful estimates for the solutions to (1.1).

Proposition 2.1 *There exists a constant $C>0$, such that for all $0 \le s \le t \le T$, $u \in \mathcal{U}[s,T]$, $\xi, \bar{\xi} \in \mathcal{K}[s,T]$ and $\mathcal{F}_s$-measurable random variables X_s and $\bar{X}_s$, there holds that*

$$E|X_t^{s,X_s,u,\xi}| \le CE\Big\{1 + |X_s| + |\xi^c|_{[s,t]} + |\xi^d|_{[s,t)}\Big\}, \tag{2.14}$$

$$E|X_t^{s,X_s,u,\xi} - X_t^{s,\bar{X}_s,u,\bar{\xi}}| \le CE\Big\{|X_s - \bar{X}_s| + |(\xi-\bar{\xi})^c|_{[s,t]} + |(\xi-\bar{\xi})^d|_{[s,t)}\Big\}, \tag{2.15}$$

$$\begin{aligned} &E\Big|\int_s^t b(r, X_r^{s,X_s,u,\xi}, u_r)dr\Big| + E\Big|\int_s^t \sigma(r, X_r^{s,X_s,u,\xi}, u_r)dW_r\Big| \\ &\quad \le C\Big\{\Big(1 + E|X_s|^\nu + E|\xi^c|_{[s,T]} + E|\xi^d|^\nu_{[s,T]}\Big)(t-s) \\ &\qquad + \Big(1 + E|X_s|^\nu + E|\xi^c|_{[s,T]} + E|\xi^d|^\nu_{[s,T]}\Big)^{1/2}(t-s)^{1/2}\Big\}. \end{aligned} \tag{2.16}$$

Proof Denote $X_t = X_t^{s,X_s,u,\xi}$, and $\langle x \rangle = (1+|x|^2)^{1/2}$. Applying Itô's formula to $\langle X_t \rangle$, and noting that $|x|^\nu, |x|^{\nu/2} \le 1 + |x|$, and $|\langle x \rangle - \langle y \rangle| \le |x-y|$, for all $x, y \in [0,\infty)$, we have

$$\begin{aligned} E\langle X_t \rangle = E\Big\{&\langle X_s \rangle + \int_s^t \langle X_r \rangle^{-1} \langle X_r, b(r, X_r, u_r)\rangle dr \\ &+ \frac{1}{2}\int_s^t \mathrm{tr}\Big[\big(-1\langle X_r \rangle^{-3} X_r X_r^T + \langle X_r \rangle^{-1} I\big)\sigma\sigma^T(r, X_r, u_r)\Big]dr \\ &+ \int_s^t \langle X_r \rangle^{-1}\langle X_r, d\xi_r^c\rangle + \sum_{r \in S_\xi[s,t)} \Big(\langle X_r + \Delta\xi_r \rangle - \langle X_r \rangle\Big)\Big\} \\ \le CE\Big\{&1 + \langle X_s \rangle + \int_s^t \langle X_r \rangle dr + \int_s^t |d\xi_r^c| + \sum_{r \in S_\xi[s,t)} |\Delta\xi_r|\Big\}. \end{aligned} \tag{2.17}$$

Applying Gronwall's inequality to (2.17), we get (2.14).

Next, we let $\langle x \rangle_\varepsilon = (\varepsilon + |x|^2)^{1/2}$, $\bar{X}_t = X_t^{s,\bar{X}_s,u,\bar{\xi}}$ and X_t as above. A similar argument to that before then leads to that

$$E\langle X_t - \bar{X}_t \rangle_\varepsilon \le CE\Big\{\langle X_s - \bar{X}_s \rangle_\varepsilon + \int_s^t \langle X_r - \bar{X}_r \rangle_\varepsilon dr + |(\xi-\bar{\xi})^c|_{[s,t]} + |(\xi-\bar{\xi})^d|_{[s,t)}\Big\}.$$

First applying the Gronwall inequality and then letting $\varepsilon \to 0$, we obtain (2.15). Finally, by (2.14) and (2.2), we obtain (2.16) immediately.

3 Properties of Value Functions

In this section we present some basic properties of the value function. We note that although these properties are more or less standard in form, special attention is still necessary because of the generality of the model that we are dealing with. To simplify the arguments, in what follows we denote by $C > 0$ a generic constant depending only on $T > 0$ and those constants appearing in (A1) and (A2), and we allow C to vary from line to line. We have the following result.

Theorem 3.1 *Let* (A1)-(A2) *hold. Then, there exists* $C > 0$, *such that for any* $s \in [0,T]$ *and* $x, \bar{x} \in \times\mathbb{R}^n$, *and the constant* μ *in* (2.2), *there holds that*

(i) $0 \le V(s,x) \le C(1+|x|^\mu)$;

(ii) $|V(s,x) - V(s,\bar{x})| \le C|x - \bar{x}|$;

(iii) *For any fixed* $x \in \mathbb{R}^n$, *the mapping* $s \mapsto V(s,x)$ *is continuous on* $[0,T]$.

Proof (i) The nonnegativity of $V(s,x)$ is obvious. Setting $\xi \equiv 0$ in (1.1) and (1.2), using Condition (2.2) and applying Proposition 2.1 we derive the second inequality in (i). The Lipschitz property (ii) is a direct consequence of Proposition 2.1 as well, thanks to (A1) and (A2).

The proof of (iii) is more complex. We first consider the following subset of $\mathcal{K}[s,T]$: for any $x \in \mathbb{R}^n$, let

$$\mathcal{K}^{|x|}[s,T] \triangleq \Big\{\xi \in \mathcal{K}[s,T] \;\Big|\; E|\xi^c|_{[s,T]} + E|\xi^d|^\nu_{[s,T]} \le \frac{1}{c_0}\left[1 + C(1+|x|^\mu)\right]\Big\}, \tag{3.1}$$

where C is the generic constant in (i) and (ii). Note that if $u \in \mathcal{U}[s,T]$ and $\xi \in \times\{\mathcal{K}[s,T] \setminus \mathcal{K}^{|x|}[s,T]\}$, then (recalling (2.4), (2.7) and part (i))

$$\begin{aligned} J_{s,x}(u,\xi) &\ge c_0 E|\xi^c|_{[s,T]} + c_0 E \sum_{r \in S_\xi[s,T]} |\Delta\xi_r|^\nu \\ &\ge c_0\Big(E|\xi^c|_{[s,t]} + E|\xi^d|^\nu_{[s,T]}\Big) \ge 1 + C(1+|x|^\mu) \ge 1 + V(s,x). \end{aligned}$$

From this one sees easily that $V(s,x) = \inf_{\mathcal{U}[s,T]\times\mathcal{K}^{|x|}[s,T]} J_{s,x}(u,\xi)$. Therefore, in the sequel we consider only the case when $(u,\xi) \in \mathcal{U}[s,T] \times \mathcal{K}^{|x|}[s,T]$.

To prove the continuity of V in s we let $s, \bar{s} \in [0,T]$. Without loss of generality we assume that $s < \bar{s}$. For any $(\bar{u},\bar{\xi}) \in \mathcal{U}[\bar{s},T] \times \mathcal{K}^{|x|}[\bar{s},T]$, we let $u \in \mathcal{U}[s,T]$ be such that $u\big|_{[\bar{s},T]} = \bar{u}$, and define $\xi_r \triangleq \bar{\xi}_{r\vee\bar{s}}$, $r \in [s,T]$. Clearly, $|\xi|_{[s,T]} = |\bar{\xi}|_{[\bar{s},T]}$, a.s. P, and $\xi \in \mathcal{K}^{|x|}[s,T]$. Applying Proposition 2.1 it is fairly easy to show that

$$\begin{aligned} J_{s,x}(u,\xi) &\le J_{\bar{s},x}(\bar{u},\bar{\xi}) + |J_{s,x}(u,\xi) - J_{\bar{s},x}(\bar{u},\bar{\xi})| \\ &\le J_{\bar{s},x}(\bar{u},\bar{\xi}) + C(1+|x|^\mu)(\bar{s}-s) + C(1+|x|)(\bar{s}-s)^{1/2}. \end{aligned} \tag{3.2}$$

Since $(\bar{u},\bar{\xi})$ is arbitrary and $\mu \le 1$ (whence $|x|^\mu \le 1+|x|$), we deduce that

$$V(s,x) \le V(\bar{s},x) + C(1+|x|)(\bar{s}-s)^{1/2}, \qquad \forall\, 0 \le s < \bar{s} \le T. \tag{3.3}$$

To obtain the other side of the inequality we consider the following two cases.

Case 1 $\bar{s} < T$. For any $0 < \varepsilon < T - \bar{s}$ we define a (deterministic) time change

$$\tau_\varepsilon(r) = \left\{ s + \frac{\bar{s} - s + \varepsilon}{\varepsilon}(r - \bar{s}) \right\} \mathbf{1}_{[\bar{s}, \bar{s}+\varepsilon]}(r) + r\mathbf{1}_{(\bar{s}+\varepsilon, T]}(r), \qquad r \in [\bar{s}, T].$$

Clearly, $\tau_\varepsilon(\cdot)$ is an increasing, continuous, piecewise linear function on $[\bar{s}, T]$ such that $\tau_\varepsilon(r) = r$ for $r \in [\bar{s} + \varepsilon, T]$, $\tau_\varepsilon(\bar{s}) = s$, and $\tau_\varepsilon(r) \le r$ for $r \in [\bar{s}, T]$ (whence $r \le \tau_\varepsilon^{-1}(r)$, for $r \in [s, T]$). Next, for any $(u, \xi) \in \mathcal{U}[s, T] \times \mathcal{K}[s, T]$, we let $\bar{u} = u\big|_{[\bar{s}, T]} \in \mathcal{U}[\bar{s}, T]$ and define $\xi^\varepsilon_r \triangleq \xi_{\tau_\varepsilon(r)}$. It is easy to check that ξ^ε is $\{\mathcal{F}_t\}$-adapted and $|\xi^\varepsilon|_{[\bar{s}, T]} = |\xi|_{[s, T]}$.

Using the estimates in Proposition 2.1, the nonnegativity of the functions h, f^a, f^s, as well as the corresponding assumptions in (A2), one shows that

$$\begin{aligned}
&J_{s,x}(u, \xi) - J_{\bar{s},x}(\bar{u}, \xi^\varepsilon) \\
\ge &E \int_{\bar{s}}^T \left[h(r, X_r^{s,x,\xi}, u_r) - h(r, X_r^{\bar{s},x,\xi^\varepsilon}, u_r) \right] dr + E \left[g(X_T^{s,x,\xi}) - g(X_T^{\bar{s},x,\xi^\varepsilon}) \right] \\
&+ E \sum_{r \in S_\xi[s, \bar{s}+\varepsilon]} \left[\ell(r, \Delta\xi_r) - \ell(\tau_\varepsilon^{-1}(r), \Delta\xi_r) \right] \\
\ge &- L \int_{\bar{s}}^{\bar{s}+\varepsilon} (E|X_r^{s,x,\xi}| + E|X_r^{\bar{s},x,\xi^\varepsilon}|) dr \\
&- (LT + L_0) C E|X_{\bar{s}+\varepsilon}^{s,x,\xi} - X_{\bar{s}+\varepsilon}^{\bar{s},x,\xi^\varepsilon}| - E \sum_{r \in S_\xi[s, \bar{s}+\varepsilon)} \omega(\tau_\varepsilon^{-1}(r) - r) \|\Delta\xi_r\|_1^\nu.
\end{aligned} \tag{3.4}$$

Since $0 \le \tau_\varepsilon^{-1}(r) - r = \frac{\bar{s}+\varepsilon-r}{\bar{s}+\varepsilon-s}(\bar{s} - s) \le \bar{s} - s$ for $r \in [s, \bar{s} + \varepsilon]$ and $J_{\bar{s},x}(\bar{u}, \xi^\varepsilon) \ge V(\bar{s}, x)$, for any $\varepsilon > 0$, we derive from (3.4) that

$$J_{s,x}(u, \xi) \ge -(LT + L_0) C \varlimsup_{\varepsilon \to 0} E|X_{\bar{s}+\varepsilon}^{s,x,\xi} - X_{\bar{s}+\varepsilon}^{\bar{s},x,\xi^\varepsilon}| - \omega(\bar{s} - s)\|\Delta\xi_s\|_1^\nu + V(\bar{s}, x).$$

Noting that $\xi_{\bar{s}+\varepsilon} = \bar{\xi}_{\bar{s}+\varepsilon}$ and using the estimates (2.15) and (2.16) one shows that

$$\begin{aligned}
\varlimsup_{\varepsilon \to 0} E|X_{\bar{s}+\varepsilon}^{s,x,\xi} - X_{\bar{s}+\varepsilon}^{\bar{s},x,\xi^\varepsilon}| \le &C \Big\{ \Big(1 + |x|^\nu + E|\xi^c|_{[s,T]} + E|\xi^d|_{[s,T]}^\nu \Big) (\bar{s} - s) \\
&+ \Big(1 + |x|^\nu + E|\xi^c|_{[s,T]} + E|\xi^d|_{[s,T]}^\nu \Big)^{1/2} (\bar{s} - s)^{1/2} \Big\}.
\end{aligned} \tag{3.5}$$

Consequently, putting (3.4) and (3.5) together, taking the infimum for $(u, \xi) \in \mathcal{U}[s, T] \times \mathcal{K}^{|x|}[s, T]$, and noting that $\nu \le 1$ and $\xi \in \mathcal{K}^{|x|}[s, T]$, we get

$$V(s, x) - V(\bar{s}, x) \ge -C(1 + |x|)[(\bar{s} - s) + (\bar{s} - s)^{\frac{1}{2}} + \omega(\bar{s} - s)], \tag{3.6}$$

for $0 \le s < \bar{s} < T$, which, combined with (3.3), leads to the continuity of $V(\cdot, x)$ for $s \in [0, T)$.

Case 2 $\bar{s} = T$. Without loss of generality, we may assume that the $\omega(\cdot)$ appearing in (A2) is also a modulus of continuity for $f^a \wedge f^s$. Again, for any $(u, \xi) \in \mathcal{U}[s, T] \times \mathcal{K}[s, T]$, we can go through a similar computation as before, and use the assumptions (2.4)–(2.8), as well as

Proposition 2.1 to get

$$\begin{aligned}J_{s,x}(u,\xi)-g(x) &\geq E\Big\{g(X_T^{s,x,\xi})-g(x)\\ &\quad+\int_s^T \big(f^s(T)\wedge f^s(T)-\omega(T-s)\big)|d\xi_r^c|+\ell(T,\xi_T^d)-\omega(T-s)|\xi^d|_{[s,T]}^\nu\Big\}\\ &\geq -C\Big(1+|x|^\nu+E|\xi^c|_{[s,T]}+E|\xi^d|_{[s,T]}^\nu\Big)(T-s)^{1/2}\\ &\quad-\omega(T-s)\Big(E|\xi^c|_{[s,T]}+E|\xi^d|_{[s,T]}^\nu\Big).\end{aligned} \tag{3.7}$$

Taking the infimum over $(u,\xi)\in\mathcal{U}[s,T]\times\mathcal{K}^{|x|}[s,T]$, we obtain

$$V(s,x)-g(x)\geq -C(1+|x|)\Big[(T-s)^{1/2}+\omega(T-s)\Big],\qquad 0\leq s\leq T.$$

This, together with (3.3), proves the continuity of $V(\cdot,x)$ at $s=T$.

4 Generalized Gradient Constraints

In this and the following sections we derive the HJB equation associated with our control problem. To begin with, let us state the Dynamic Programming Principle (Bellman Principle), whose proof (which we omit) can be carried out by combining some standard techniques as used in, e.g. [3,4,10,11]. Denote by $\Im_{s,T}$ the set of all $\{\mathcal{F}_t\}$-stopping times taking values in $[s,T]$. We have

Bellman Principle *Let* (A1)–(A2) *hold. Let* $(s,x)\in[0,T]\times\mathbb{R}^n$, $y>0$ *and* $\tau\in\Im_{s,T}$ *be given; denote* $X=X^{s,x,u,\xi}$ *for* $(u,\xi)\in\mathcal{U}[s,T]\times\mathcal{K}[s,T]$. *Then the following holds:*

$$\begin{aligned}V(s,x)=\inf_{\mathcal{U}[s,T]\times\mathcal{K}[s,T]}E\Big\{&\int_s^\tau h(r,X_r,u_r)dr+\int_s^\tau f^a(r)\|\dot{\xi}_r^{ac}\|_1dr\\ &+\int_s^\tau f^s(r)|d\xi_r^{sc}|+\sum_{r\in S_\xi[s,\tau)}\ell(r,\Delta\xi_r)+V(\tau,X_\tau)\Big\}.\end{aligned} \tag{4.1}$$

In light of the existing results involving singular and impulse controls, we know that our HJB equation will take the form of a quasi-variational inequality, and the essential part will be the determination of the obstacle(s). However, unlike the well-known gradient constraints as one often sees in the literature on singular control the obstacle appearing in our HJB equation is quite different due to the generality of our control problem.

To begin our investigation, let us argue heuristically. First, if we set $\tau=s+\varepsilon$, $\varepsilon>0$, in Eq.(4.1), and for any $\theta\in K\setminus\{0\}$, and $u^0\in U$ we define $u_t\equiv u^0$, $t\in[s,T]$, and $\xi_t\equiv\theta\mathbf{1}_{(s,\infty)}(t)$, then $(u,\xi)\in\mathcal{U}[s,T]\times\mathcal{K}[s,T]$, and as $\varepsilon\to 0$ one derives easily that $V(s,x)\leq\ell(s,\theta)+V(s,x+\theta)$. Consequently

$$\frac{V(s,x+\theta)-V(s,x)}{\ell(s,\theta)}+1\geq 0,\qquad\forall\,\theta\in\mathcal{K}\setminus\{0\}. \tag{4.2}$$

Next, let us assume further that $V \in C^{1,2}([0,T] \times \mathbb{R})$. Choosing $u \equiv u_0 \in U$ again, but $\xi_t = \alpha\theta(t-s)$, $t \geq s$, with any $\alpha > 0$, $\|\theta\|_1 = 1$, $\theta \in K$. Using (4.1) and Itô's formula (applied to $V(\tau, X_\tau)$, $\tau = s+\varepsilon$) we get

$$0 \leq E\left\{\int_s^{s+\varepsilon} H(r)dr + \alpha\left[\int_\varepsilon^{s+\varepsilon}\{\langle V_x, \theta\rangle + f^a(r)\}dr\right]\right\},$$

where

$$H(r) = V_r(r, X_r) + \langle V_x(r, X_r), b(r, X_r, u_r)\rangle + \frac{1}{2}\mathrm{tr}\left[V_{xx}(r, X_r)(\sigma\sigma^T)(r, X_r, u_r)\right].$$

Since $\alpha > 0$ and $\varepsilon > 0$ are arbitrary, we deduce easily from the above that

$$\langle V_x(s,x), \theta\rangle + f^a(s) \geq 0, \qquad \forall \theta \in K, \|\theta\|_1 = 1. \tag{4.3}$$

On the other hand, if we let $\zeta_t^\varepsilon \triangleq \varepsilon\zeta(\frac{1}{\varepsilon}(t-s))\mathbf{1}_{[s,s+\varepsilon]}(t) + \varepsilon\mathbf{1}_{[s+\varepsilon,T]}$, $t \in [s,T]$, where ζ is the Cantor function defined on [0,1], i.e. ζ is continuous, monotone increasing with $\zeta = \zeta^s$, and define for $\alpha > 0$, $\theta \in K$, $\|\theta\|_1 = 1$, $\xi_t = \alpha\theta\zeta(t)$, $t \in [s,T]$, then a similar argument to that before will show that (4.3) holds with f^a being replaced by f^s.

Combining the above, let us now define for any $\varphi \in C([0,T] \times \mathbb{R})$, and for all $(s,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, the following operators

$$\begin{cases} M[\varphi](s,x) \triangleq \inf_{\theta \in K\setminus\{0\}}\left\{\dfrac{\varphi(s,x+\theta) - \varphi(s,x)}{\ell(s,\theta)}\right\} + 1; \\ F(s,p) = \inf_{\theta \in K_0}\left\{f^a(s) \wedge f^s(s) + \langle p, \theta\rangle\right\}, \quad \forall (s,p) \in [0,T] \times \mathbb{R}^n, \end{cases} \tag{4.4}$$

where $K_0 \triangleq \{\theta \in K \mid \|\theta\|_1 = 1\}$ (thus $K = \{\alpha\theta : \theta \in K_0, \alpha \geq 0\}$). Then (4.2) implies that $M[V](s,x) \geq 0$, $\forall (s,x)$. Further, if $V \in C^{1,2}$, then $F(s, V_x(s,x)) \geq 0$, $\forall (s,x)$. Notice that if $K = \mathbb{R}^n$ and $\ell(s,\theta) = f^d(s)\|\theta\|_1$, then $F(s,p) = 1 - \|p\|_\infty$, where $\|p\|_\infty \equiv \max_{1\leq i\leq n}|p^i|$, $\forall p = (p^1, \cdots, p^n) \in \mathbb{R}^n$; and the inequality $M[V](s,x) \geq 0$ becomes $\|V_x(s,x)\|_\infty \leq f^d(s)$ while $F(s, V_x(s,x)) \geq 0$ becomes $\|V_x(s,x)\|_\infty \leq 1$, both being the gradient constraints as we often see in the literature on singular control. Therefore, in the sequel we shall call the equalities $M[V](s,x) \geq 0$ and $F(s, V_x(s,x)) \geq 0$ the *generalized gradient constraints.*

The main purpose of this section is to prove certain kind of continuity for the function $M[V](s,x)$, which will be essential in our future discussion. For notational convenience we define

$$F_1(s,p) \triangleq \inf_{\theta \in K_0}\{f^d(s) + \langle p, \theta\rangle\}, \qquad \forall (s,p). \tag{4.5}$$

Theorem 4.1 *Let* (A1)–(A2) *hold. Let* V *be the value function of the control problem* (1.1) *and* (1.2). *Then*

(i) $M[V]$ *is continuous on the set* $\Sigma_0 \triangleq \{s \in [0,T] | \ell(s,0) > 0\} \times \mathbb{R}^n$*;*

(ii) *If* $\nu < 1$, *then* $M[V]$ *is continuous on the set*

$$\Sigma \triangleq \{(s,x,y) \in (0,T) \times \mathbb{R}^n | M[V](s,x) < 1\}. \tag{4.6}$$

Further, if $(\bar{s}, \bar{x}) \notin \Sigma$, *then, for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $M[V](s,x) > 1 - \varepsilon$, *for all* $|s - \bar{s}|, |x - \bar{x}| < \delta$.

(iii) *If $\nu = 1$, $\inf_{t\in[0,T]} \ell(t,0) = 0$, and $V(s,x)$ is C^1 in x. Then $M[V]$ is continuous on $[0,T]\times\mathbb{R}^n$. Moreover, for all $(s,x)\in[0,T]\times\mathbb{R}^n$, there holds that*

$$\begin{cases} \{(s,x) \mid M[V](s,x) = 0\} = \{(s,x) \mid F_1(s,V_x(s,x)) = 0\}; \\ \{(s,x) \mid M[V](s,x) > 0\} = \{(s,x) \mid F_1(s,V_x(s,x)) > 0\}. \end{cases} \tag{4.7}$$

Proof (i) For any $(\bar s,\bar x)\in\Sigma_0$, one has $\ell(\bar s,0) > 0$. Thus there exists a $\delta > 0$, such that $\ell(s,0) \ge \delta > 0$, for all $|s-\bar s| \le \delta$. Since $V(\cdot,\cdot)$ is continuous thanks to Theorem 3.1, we show readily that the function $\frac{V(s,x+\theta)-V(s,x)}{\ell(s,\theta)}$ is continuous for $(s,x)\in[\bar s-\delta,\bar s+\delta]\times\mathbb{R}^n$, uniformly in $\theta\in K$. This clearly implies the continuity of $M[V]$ on Σ_0.

(ii) Since $M[V]$ is by definition the infimum of a family of continuous functions, it is upper semicontinuous. Thus, the set Σ defined by (4.6) is open. Consequently, for any $(\bar s,\bar x)\in\Sigma$, we can find a $\delta > 0$ and a neighborhood $\mathcal{N}$ of $(\bar s,\bar x)$, such that $M[V](s,x) \le 1-\delta$, for all $(s,x)\in\mathcal{N}$. Further, using Theorem 3.1(ii) and (2.7) we see that if $\theta\in K$ and $\theta\to 0$, then (recalling $\nu < 1$)

$$\left|\frac{V(s,x+\theta)-V(s,x)}{\ell(s,\theta)}\right| \le \frac{C}{c_0}|\theta|^{1-\nu} \to 0. \tag{4.8}$$

Therefore we can find an $\varepsilon > 0$, such that for all $(s,x)\in\mathcal{N}$

$$M[V](s,x) = \inf_{\theta\in K,\|\theta\|_1\ge\varepsilon}\left\{\frac{V(s,x+\theta)-V(s,x)}{\ell(t,\theta)} + 1\right\}. \tag{4.9}$$

Note again that the function $\frac{V(s,x+\theta)-V(s,x)}{\ell(s,\theta)}$ is continuous with respect to (s,x), uniformly in $\theta\in K$ with $\varepsilon \le \|\theta\|_1 \le C$, $M[V](s,x)$ is continuous on $\mathcal{N}$, whence at $(\bar s,\bar x)$, thus proving the first part of (ii).

To see the second part of (ii), let us assume that $(\bar s,\bar x)\notin\Sigma$, and our assertion is not true. To wit, for some $\varepsilon > 0$, there exists a sequence $(s_k,x_k)\to(\bar s,\bar x)$, such that

$$M[V](s_k,x_k) \le 1-\varepsilon, \qquad \forall k\ge 1. \tag{4.10}$$

Recall from the proof of Theorem 3.1 that for each fixed (s,x), there holds that $V(s,x) = \inf_{\mathcal{U}[s,T]\times\mathcal{K}^{|x|}[s,T]} J_{s,x}(u,\xi)$, where $\mathcal{K}^{|x|}[s,T]$ is defined by (3.1). Thus in the dynamic programming equation (4.1) we can actually replace the set $\mathcal{K}[s,T]$ by $\mathcal{K}^{|x|}[s,T]$, for each fixed (s,x). Consequently, we may assume that the infimum in the definition of $M[V]$ is taken over all $\theta\in K$, $\theta\ne 0$, such that $\|\theta\| \le C_x \triangleq C(1+|x|^\mu)$, for each fixed (s,x), with some generic constant $C > 0$ that is independent of (s,x) (see (3.1)).

Now by (4.10) we can choose for each k a $\theta_k\in K$, such that $\varepsilon < \|\theta_k\| \le C_{x_k}$ and

$$\frac{V(s_k,x_k+\theta_k)-V(s_k,x_k)}{\ell(s_k,\theta_k)} + 1 \le M[V](s_k,x_k) + \frac{\varepsilon}{2} \le 1-\frac{\varepsilon}{2}, \qquad k\ge 1. \tag{4.11}$$

Since $x_k\to\bar x$, the sequence $\{C_{x_k}\} = \{C(1+|x_k|^\mu)\}$ is bounded; thus $\{\theta_k\}$ is bounded as well. Extracting a subsequence if necessary, we may assume that $\theta_k\to\theta_0$ for some θ_0 satisfying

$\|\theta_0\| \ge \varepsilon$. Then

$$\begin{aligned} M[V](\bar{s},\bar{x}) &\le \frac{V(\bar{s},\bar{x}+\theta_k)-V(\bar{s},\bar{x})}{\ell(\bar{s},\theta_k)}+1 \\ &\le \frac{V(s_k,x_k+\theta_k)-V(s_k,x_k)}{\ell(s_k,\theta_k)}+1 \\ &\quad + \Big[\frac{V(\bar{s},\bar{x}+\theta_k)-V(\bar{s},\bar{x})}{\ell(\bar{s},\theta_k)} - \frac{V(s_k,x_k+\theta_k)-V(s_k,x_k)}{\ell(s_k,\theta_k)}\Big] \\ &\le 1-\frac{\varepsilon}{2}+\big[\cdots\big] \to 1-\frac{\varepsilon}{2}, \end{aligned}$$

a contradiction, which proves the second part of (ii).

(iii) Now assume that $\nu = 1$, and $\ell(s,\theta) = f^d(s)\|\theta\|_1$, where f^d is continuous and satisfies (2.9). Since V is C^1 in x by assumption, denoting by ω_V the modulus of continuity of V_x we have

$$\begin{aligned} &\Big|\frac{V(s,x+\theta)-V(s,x)}{\|\theta\|_1} - \frac{V(\bar{s},\bar{x}+\theta)-V(\bar{s},\bar{x})}{\|\theta\|_1}\Big| \\ &\le \int_0^1 \Big|\Big\langle V_x(s,x+\beta\theta)-V_x(\bar{s},\bar{x}+\beta\theta), \frac{\theta}{\|\theta\|_1}\Big\rangle\Big| d\beta \le \omega_V(|s-\bar{s}|+|x-\bar{x}|). \end{aligned}$$

Namely, the function $\frac{V(s,x+\theta)-V(s,x)}{\|\theta\|_1}$ is continuous in (s,x), uniformly in $\theta \ne 0$. Thus $M[V]$ is continuous on $[0,T]\times\mathbb{R}^n$. To prove the last assertion of (iii) we note that for any $\theta_0 \in K_0$ and $\delta > 0$, one has $\ell(s,\delta\theta_0) = f^d(s)\delta$, and that as $\delta \to 0$,

$$\begin{aligned} f^d(s)M[V](s,x) &\le f^d(s)\Big\{\frac{V(s,x+\delta\theta_0)-V(s,x)}{\ell(s,\delta\theta_0)}+1\Big\} \\ &= \int_0^1 \langle V_x(s,x+\beta\delta\theta_0),\theta_0\rangle d\beta + f^d(s) \to \langle V_x(s,x),\theta_0\rangle + f^d(s). \end{aligned} \tag{4.12}$$

Taking the infimum on the right side of (4.12), and noting that $M[V](s,x) \ge 0$, $\forall (s,x)$, as we proved before, we get $0 \le f^d(s)M[V](s,x) \le F_1(s,V_x(s,x))$, where F_1 is defined by (4.5). Consequently, we have

$$M[V](s,x) > 0 \Longrightarrow F_1(s,V_x(s,x)) > 0; \quad F_1(s,V_x(s,x)) = 0 \Longrightarrow M[V](s,x) = 0. \tag{4.13}$$

On the other hand, for any $\theta \in K$, one has

$$\begin{aligned} f^d(s)\Big\{\frac{V(s,x+\theta)-V(s,x)}{\ell(s,\theta)}+1\Big\} &= \int_0^1 \Big\{\Big\langle V_x(s,x+\beta\theta),\frac{\theta}{\|\theta\|_1}\Big\rangle + f^d(s)\Big\}d\beta \\ &\ge \int_0^1 F_1(s,V_x(s,x+\beta\theta))d\beta \ge \inf_{\theta\in K}\int_0^1 F_1(s,V_x(s,x+\beta\theta))d\beta, \end{aligned}$$

and hence

$$f^d(s)M[V](s,x) \ge \inf_{\theta\in K}\int_0^1 F_1(s,V_x(s,x+\beta\theta))d\beta, \qquad \forall (s,x). \tag{4.14}$$

Now if $M[V](s,x) = 0$, then we can find some $\theta \in K$ such that $\int_0^1 F_1(s,V_x(s,x+\beta\theta))d\beta = 0$. Thus $F_1(s,V_x(s,x+\beta\theta)) = 0$, for a.e. $\beta \in [0,1]$, since the integrand above is nonnegative.

Using the continuity of V_x, we then conclude that $F_1(s, V_x(s,x)) = 0$. Combining this with (4.13) we can now derive both equalities in (4.7) easily.

Remark 4.2 It is important to note that, in general, $M[V]$ is *not* necessarily continuous on $[0,T] \times \mathbb{R}^n \times [0,\infty)$. But, Theorem 4.1(ii) tells us the following: If $M[V]$ is positive at a certain point, it will be positive in a neighborhood of that point. This will be enough for the derivation of our result in the next section.

Remark 4.3 We should note that Theorem 4.1(iii) covers a very important case: $\nu = 1$ and $\ell(s,0) = 0$, namely, the standard singular control model. By (2.9), we always assume that $\ell(t,\theta) = f^d(t)\|\theta\|_1$. Theorem 4.1(iii) then tells us that in such a case, if the value function $V(s,x)$ is C^1 (in x), then the constraints $M[V] \geq 0$ and $F_1(s, V_x(s,x)) \geq 0$ are actually equivalent. Thus, denoting $G(s,p) \triangleq \inf_{\theta\in K_0} \{f^a(s) \wedge f^s(s) \wedge f^d(s) + \langle p, \theta\rangle\}$ we have $M[V] \wedge F(s, V_x) \geq 0 \Longleftrightarrow G(s, V_x) \geq 0$.

5 The HJB Equation

In this section we derive the HJB equation associated with our control problem. Let $\mathcal{S}^n$ be the set of all $(n \times n)$ symmetric matrices, and define

$$\begin{cases} H(s,x,p,A) = \inf\limits_{u\in U} \Big\{ h(s,x,u) + \langle p, b(s,x,u)\rangle + \dfrac{1}{2}\mathrm{tr}\big[A\sigma\sigma^*(s,x,u)\big] \Big\}, \\ \qquad\qquad\qquad\qquad \forall (s,x,p,A) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n. \end{cases} \tag{5.1}$$

Consider the following quasi-variational inequality:

$$\begin{cases} \min\{V_s + H(s,x,V_x,V_{xx}), F(s,V_x), M[V](s,x)\} = 0, \\ V\big|_{s=T} = g(x). \end{cases} \tag{5.2}$$

More precisely, we shall prove that the value function $V(s,x)$ is a viscosity solution of HJB variational inequality (5.2) whenever $\nu < 1$ or $\ell(s,0) > 0$. Further, in light of Remark 4.3, we will show that if $\nu = 1$ and $\ell(s,\theta) = f^d(s)\|\theta\|_1$, then V will be a viscosity solution to the variational inequality

$$\begin{cases} \min\{V_s + H(s,x,V_x,V_{xx}), G(s,V_x)\} = 0, \\ V\big|_{s=T} = g(x). \end{cases} \tag{5.3}$$

The uniqueness of such a viscosity solution will be proved in the next section.

First, let us recall the definition of a viscosity solution (see [4,11,20]).

Definition 5.1 *A continuous function $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity subsolution (resp. supersolution) of (5.2), if $v(T,x) \leq g(x)$, ($\geq g(x)$, resp.) for all $x \in \mathbb{R}^n$; and for any $\varphi \in C^\infty([0,T] \times \mathbb{R}^n)$ for which $v - \varphi$ attains a local maximum (resp. minimum) at $(\bar{s},\bar{x}) \in (0,T) \times \mathbb{R}^n$, one has*

$$\min\{\varphi_s + H(\bar{s},\bar{x},\varphi_x,\varphi_{xx}), F(\bar{s},\varphi_x), M[v](\bar{s},\bar{x})\} \geq 0, \quad (\leq 0, \ \textit{resp.}),$$

where φ_s, φ_x *and* φ_{xx} *are evaluated at* $(\bar{s},\bar{x})$. *If* v *is both viscosity subsolution and supersolution, then it is called a viscosity solution of* (5.2).

We remark here that since the operator $M[v]$ is "non-local", one cannot replace $M[v](\bar{s},\bar{x})$ in the definition above by $M[\varphi](\bar{s},\bar{x})$, even when $V(\bar{s},\bar{x}) = \varphi(\bar{s},\bar{x})$ is assumed! This is why our obstacle in the HJB equation is different from the standard gradient constraints, where such replacement is always possible.

We have the following theorem.

Theorem 5.2 *Assume* (A1)–(A2). *Then the value function* $V(s,x)$ *is a viscosity solution of* (5.2). *In particular, if* $\nu = 1$, $\ell(t,0) = 0$, *and* (2.9) *holds, then* $V(s,x)$ *is a viscosity solution of* (5.3).

Proof We prove only that V is a viscosity supersolution of (5.2), since that it is a subsolution is much easier. Let $(\bar{s},\bar{x}) \in (0,T)\times\mathbb{R}^n$ and let $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$ be such that $V-\varphi$ attains its local minimum at $(\bar{s},\bar{x})$. It is known that, one may assume without loss of generality that the minimum is $V(\bar{s},\bar{x}) = \varphi(\bar{s},\bar{x})$. We claim that one of the following inequalities must hold:

$$\begin{cases} \varphi_s(\bar{s},\bar{x}) + H(\bar{s},\bar{x},\varphi_x(\bar{s},\bar{x}),\varphi_{xx}(\bar{s},\bar{x})) \le 0; \\ F(\bar{s},\varphi_x(\bar{s},\bar{x})) \le 0; \quad M[V](\bar{s},\bar{x}) \le 0. \end{cases}$$

Suppose this is not the case. Then there exists a $\delta_0 > 0$ such that

$$\begin{cases} \varphi_s(\bar{s},\bar{x}) + H(\bar{s},\bar{x},\varphi_x(\bar{s},\bar{x}),\varphi_{xx}(\bar{s},\bar{x})) \ge \delta_0; \\ F(\bar{s},\varphi_x(\bar{s},\bar{x})) \ge \delta_0; \quad M[V](\bar{s},\bar{x}) \ge \delta_0. \end{cases} \tag{5.4}$$

Note that in Theorem 3.1 we have shown that $V(\bar{s},\bar{x}) = \inf_{\mathcal{U}[\bar{s},T]\times\mathcal{K}^{|\bar{x}|}[\bar{s},T]} J_{\bar{s},\bar{x}}(u,\xi)$, where $\mathcal{K}^{|\bar{x}|}[\bar{s},T]$ is defined by (3.1). For any $(u,\xi) \in \mathcal{U}[\bar{s},T]\times\mathcal{K}^{|\bar{x}|}[\bar{s},T]$, let $X_t = X^{\bar{s},\bar{x},u,\xi}$ be the corresponding trajectory; and $X^c_t = X_t - \xi^d_t$. For any $\tau \in \Im_{\bar{s},T}$ we apply Itô's formula to $\varphi(t,X^c_t)$ to get

$$\begin{aligned} V(\tau,X_\tau) - V(\bar{s},\bar{x}) &\ge \varphi(\tau,X^c_\tau) - \varphi(\bar{s},\bar{x}) + V(\tau,X^c_\tau+\xi^d_\tau) - V(\tau,X^c_\tau) \\ &= \int_{\bar{s}}^{\tau} \Big\{\varphi_r(r,X^c_r) + \langle\varphi_x(r,X^c_r), b(r,X_r,u_r)\rangle \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\varphi_{xx}(r,X^c_r)\sigma\sigma^*(r,X_r,u_r)\right)\Big\}dr + \int_{\bar{s}}^{\tau}\langle\varphi_x(r,X^c_r), d\xi^c_r\rangle \\ &\quad + \int_{\bar{s}}^{\tau}\langle\varphi_x(r,X^c_r), dW_r\rangle + V(\tau,X^c_\tau+\xi^d_\tau) - V(\tau,X^c_\tau). \end{aligned} \tag{5.5}$$

Using the definitions of $J_{\bar{x},\bar{x}}$, $M[V]$ and F, we derive from an easy computation that

$$\begin{aligned} J_{\bar{s},\bar{x}}(u,\xi) - V(\bar{s},\bar{x}) &\ge E\Big\{V(\tau,X_\tau) - V(\bar{s},\bar{x}) + \int_{\bar{s}}^{\tau} h(r,X_r,u_r)dr \\ &\quad + \int_{\bar{s}}^{\tau} f^a(r)|d\xi^{ac}_r| + \int_{\bar{s}}^{\tau} f^s(r)|d\xi^{sc}_r| + \sum_{r\in S_\xi[\bar{s},\tau)} \ell(r,\Delta\xi_r)\Big\} \\ &\ge E\Big\{\int_{\bar{s}}^{\tau}[\varphi_r(r,X^c_r) + H(r,X_r,\varphi_x(r,X^c_r),\varphi_{xx}(r,X^c_r))]dr \\ &\quad + \int_{\bar{s}}^{\tau} F(r,\varphi_x(r,X^c))|d\xi^c_r| + \ell(\tau,\xi^d_\tau)M[V](\tau,X^c_\tau) \end{aligned}$$

$$+\Big[\sum_{r\in S_\xi[\bar s,\tau)} \ell(r,\Delta\xi_r)-\ell(\tau,\xi^d_\tau)\Big]\Big\}. \tag{5.6}$$

Now, for any $\varepsilon>0$, we choose $(u^\varepsilon,\xi^\varepsilon)\in\mathcal{U}[\bar s,T]\times\mathcal{K}^{|\bar x|}[\bar s,T]$ so that $J_{\bar s,\bar x}(u^\varepsilon,\xi^\varepsilon)-V(\bar s,\bar x)<\varepsilon$, and denote by X^ε_t the corresponding trajectory. Note that applying Theorem 4.1(i) if $\ell(s,0)>0$, $\forall s$; or Theorem 4.1(ii) if $\nu<1$, we deduce from (5.4)–(5.6) that, for some $\eta>0$, there holds for all $|r-\bar s|,|x-\bar x|,|\hat x-\bar x|\le\eta$ that

$$\begin{cases}\varphi_r(r,x)+H(r,\hat x,\varphi_x(r,x),\varphi_{xx}(r,x))\ge\dfrac{\delta_0}{2},\\ F(r,\varphi_x(r,x))\ge\dfrac{\delta_0}{2},\quad M[V](r,x)\ge\dfrac{\delta_0}{2}.\end{cases} \tag{5.7}$$

Set $Z^\varepsilon_t\triangleq\max\{t-\bar s,|(X^\varepsilon)^c_t-\bar x|,|X^\varepsilon_t-\bar x|\}$. Then Z^ε is also càglàd and adapted. Thus if we define $\tau^\varepsilon_\eta=\inf\{t>\bar s\mid Z^\varepsilon_t>\frac{\eta}{2}\}\wedge T$, then $\tau^\varepsilon_\eta\in\Im_{\bar s,T}$. Replace now τ,u,ξ by $\tau^\varepsilon_\eta,u^\varepsilon,\xi^\varepsilon$ in (5.6). Note that the function ℓ is nonincreasing in t by (2.6) which, combined with the "triangle inequality" in (2.5), shows that the last [...] on the right side of (5.6) is nonnegative. Therefore, using the definition of $(u^\varepsilon,\xi^\varepsilon)$ and (5.7) we get

$$\varepsilon>\frac{\delta_0}{2}E(\tau^\varepsilon_\eta-\bar s)+\frac{\delta_0}{2}E|(\xi^\varepsilon)^c_{\tau^\varepsilon_\eta}|_{[\bar s,\tau^\varepsilon_\eta]}+\frac{\delta_0c_0}{2}E\|(\xi^\varepsilon)^d_{\tau^\varepsilon_\eta}\|^\nu_1. \tag{5.8}$$

We claim that (5.8) will lead to a contradiction, indeed, since (5.8) implies that, along a sequence if necessary, $\tau^\varepsilon_\eta\to\bar s$, a.s., as $\varepsilon\to0$, and that $E|(\xi^\varepsilon)^c|_{[\bar s,\tau^\varepsilon_\eta]}+E|(\xi^\varepsilon)^d_{\tau^\varepsilon_\eta}|^\nu=O(\varepsilon)$. Thus it follows from Proposition 2.1 that $E|X^\varepsilon_{\tau^\varepsilon_\eta}-\bar x|=O(\varepsilon)$; $E|(X^\varepsilon)^c_{\tau^\varepsilon_\eta}-\bar x|=O(\varepsilon)$. Note that

$$\Big\{\omega:\tau^\varepsilon_\eta(\omega)\ge\bar s+\frac{\eta}{2}\Big\}\supseteq\Big\{\omega:|X^\varepsilon_{\tau^\varepsilon_\eta}(\omega)-\bar x|<\frac{\eta}{2},|(X^\varepsilon)^c_{\tau^\varepsilon_\eta}(\omega)-\bar x|<\frac{\eta}{2}\Big\},$$

whence

$$\begin{aligned}P\Big(\tau^\varepsilon_\eta\ge\bar s+\frac{\eta}{2}\Big)&\ge1-\Big\{P\Big(|X^\varepsilon_{\tau^\varepsilon_\eta}(\omega)-\bar x|\ge\frac{\eta}{2}\Big)+P\Big(|(X^\varepsilon)^c_{\tau^\varepsilon_\eta}(\omega)-\bar x|\ge\frac{\eta}{2}\Big)\Big\}\\&\ge1-\frac{2}{\eta}\{E|X^\varepsilon_{\tau^\varepsilon_\eta}-\bar x|+E|(X^\varepsilon)^c_{\tau^\varepsilon_\eta}-\bar x|\}\to1,\quad\text{as }\varepsilon\to0.\end{aligned}$$

Therefore there exists a constant $\varepsilon_0\in(0,\delta_0\eta/8)$, such that for all $0<\varepsilon\le\varepsilon_0$, there holds that $P(\tau^\varepsilon_\eta\ge\bar s+\frac{\eta}{2})>\frac12$. Hence for such ε, (5.8) becomes

$$\varepsilon>\frac{\delta_0}{2}E(\tau^\varepsilon_\eta-\bar s)\ge\frac{\delta_0}{2}E\Big[(\tau^\varepsilon_\eta-\bar s)1_{\{\tau^\varepsilon_\eta-\bar s\ge\frac{\eta}{2}\}}\Big]\ge\frac{\delta_0}{2}\cdot\frac{\eta}{2}P\Big(\tau^\varepsilon_\eta-\bar s\ge\frac{\eta}{2}\Big)>\frac{\delta_0\eta}{8}>0,$$

a contradiction. This proves the first part of the theorem.

Now let us assume that $\nu=1$ and $\ell(s,\theta)=f^d(t)\|\theta\|_1$, and prove that V is a viscosity solution to (5.3). Again, we only show that $V(s,x)$ is a viscosity supersolution. In other words, for $(\bar s,\bar x)\in[0,T]\times\mathbb{R}^n$, and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^n)$ such that $V-\varphi$ attains its zero minimum at $(\bar s,\bar x)$, we want to show that one of the following inequalities must hold

$$\varphi_s(\bar s,\bar x)+H(\bar s,\bar x,\varphi_x(\bar s,\bar x),\varphi_{xx}(\bar s,\bar x))\le0;\quad G(\bar s,\varphi_x(\bar s,\bar x))\le0.$$

Suppose this is not the case. Then there exists a $\delta_0>0$, such that

$$\varphi_s(\bar s,\bar x)+H(\bar s,\bar x,\varphi_x(\bar s,\bar x),\varphi_{xx}(\bar s,\bar x))\ge\delta_0;\quad G(\bar s,\varphi_x(\bar s,\bar x))\ge\delta_0. \tag{5.9}$$

Similarly to (5.5), for any $(u,\xi) \in \mathcal{U}[\bar{s},T] \times \mathcal{K}^{|\bar{x}|}[\bar{s},T]$, denote $X_t = X^{\bar{s},\bar{x},u,\xi}$ and $X_t^c = X_t - \xi_t^d$. Let $\tau \in \Im_{\bar{s},T}$. This time we apply Itô's formula to $\varphi(t,X_t)$ (instead of $\varphi(t,X_t^c)$!) to get

$$\begin{aligned}
&V(\tau,X_\tau) - V(\bar{s},\bar{x}) \geq \varphi(\tau,X_\tau) - \varphi(\bar{s},\bar{x}) \\
&= \int_{\bar{s}}^{\tau} \Big\{ \varphi_r(r,X_r) + \langle \varphi_x(r,X_r), b(r,X_r,u_r)\rangle + \frac{1}{2}\mathrm{tr}\left([\varphi_{xx}\sigma\sigma^*](r,X_r,u_r)\right)\Big\} dr \\
&+ \int_{\bar{s}}^{\tau} \langle \varphi_x(r,X_r), d\xi_r^c\rangle + \int_{\bar{s}}^{\tau} \langle \varphi_x(r,X_r^c), dW_r\rangle \\
&+ \sum_{r\in[\bar{s},\tau)} \varphi(\tau, X_\tau^c + \xi_\tau^d) - \varphi(\tau,X_\tau^c).
\end{aligned} \tag{5.10}$$

On the other hand, notice that for (4.14), similarly to (5.6) we have

$$\begin{aligned}
J_{\bar{s},\bar{x}}(u,\xi) - V(\bar{s},\bar{x}) \geq E\Big\{ &\int_{\bar{s}}^{\tau} \Big[\varphi_r(r,X_r) + H(r,X_r,\varphi_x(r,X_r),\varphi_{xx}(r,X_r))\Big] dr \\
&+ \int_{\bar{s}}^{\tau} F(r,\varphi_x(r,X_r))|d\xi_r^c| + \|\xi_\tau^d\|_1 \cdot F_1^*\Big\},
\end{aligned} \tag{5.11}$$

where $F_1^* \triangleq \int_0^1 F_1(\tau,\varphi_x(\tau,X_\tau^c + \beta\xi_\tau^d))d\beta$ and F_1 is defined by (4.5). Then, by a line to line analogy of the proof of the first part, and using Theorem 4.1(iii) if necessary, we can also derive a contradiction from (5.10) and (5.11), which shows that $V(s,x)$ is a viscosity supersolution of (5.3). The proof is now complete.

6 Uniqueness of Viscosity Solutions

In this section we prove the uniqueness of the viscosity solutions of (5.2), which will complete the characterization of the value function $V(s,x)$.

Let us first recall the notion of (parabolic) super-(sub-)jets (cf. e.g. User's Guide [20]). For any $v : [0,T]\times\mathbb{R}^n \to [-\infty,+\infty]$, we define (recalling $\mathcal{S}^n$ in (5.1))

$$\begin{aligned}
\mathcal{P}^{2,+}v(s,x) &= \{(a,p,A) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mid v(t,z) \leq v(s,x) + a(t-s) + \langle p, z-x\rangle \\
&\qquad + \frac{1}{2}\langle A(z-x), z-x\rangle + o(|t-s| + |z-x|^2), t\downarrow s, z\to x\}, \\
\overline{\mathcal{P}}^{2,+}v(s,x) &= \{(a,p,A) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}^n \mid \exists (s_i,x_i) \in [0,T]\times\mathbb{R}^n, \\
&\qquad (a_i,p_i,A_i) \in \mathcal{P}^{2,+}v(s_i,x_i), (s_i,x_i,v(s_i,x_i),a_i,p_i,A_i) \to (s,x,v(s,x),a,p,A)\}.
\end{aligned}$$

Further, we let $\mathcal{P}^{2,-}v(s,x) = -\mathcal{P}^{2,+}(-v)(s,x)$, $\overline{\mathcal{P}}^{2,-}v(s,x) = -\overline{\mathcal{P}}^{2,+}(-v)(s,x)$. Then, following the standard techniques (cf. [4,11,20]) it is not hard to show that $V(s,x)$ is a viscosity subsolution (supersolution, resp.) of (5.2) if and only if $V(T,x) \leq g(x)$ (resp. $V(T,x) \geq g(x)$) such that for all $(s,x) \in [0,T)\times\mathbb{R}^n$,

$$\begin{aligned}
&\min\{a + H(s,x,p,A), F_1(s,p), M_1[V](s,x)\} \geq 0, \quad (a,p,A) \in \overline{\mathcal{P}}^{2,+}V(s,x) \\
&\Big(\text{resp. } \min\{a + H(s,x,p,A), F_1(s,p), M_1[V](s,x)\} \leq 0, \quad (a,p,A) \in \overline{\mathcal{P}}^{2,-}V(s,x)\Big).
\end{aligned} \tag{6.1}$$

Again, we shall prove the uniqueness in the following two cases separately:

Case 1 $0 < \mu < \nu^2$; either $\nu < 1$ or $\ell(s,0) \geq \ell_0 > 0$, for all $s \in [0,T]$. We note that this case contains the standard impulse control problems.

To simplify the expressions let us define a set of functions

$$\mathcal{V}_\mu = \{v \in C([0,T]\times\mathbb{R}^n) \mid \exists C > 0,\ 0 \leq v(s,x) \leq C(1+|x|^\mu),\\ |v(s,x)-v(s,z)| \leq C|x-z|,\ \ \forall s \in [0,T], x, z \in \mathbb{R}^n\ \}.$$

Clearly, the value function $V(s,x)$ belongs to $\mathcal{V}_\mu$, thanks to Theorem 3.1.

Theorem 6.1 *Assume* (A1)–(A2), *and that either* $\nu < 1$ *or* $\ell(s,0) \geq \ell_0 > 0$ *holds for all* $s \in [0,T]$. *Assume further that* $0 < \mu < \nu^2$ *holds. Then the value function* $V(s,x)$ *is the unique viscosity solution of* (5.2) *in* $\mathcal{V}_\mu$.

Proof Let V and $\hat{V}$ be two viscosity solutions of (5.2) in $\mathcal{V}_\mu$. We want to show that $V(t,x) = \hat{V}(t,x)$. Our line of attack is more or less standard by now (see, e.g. [9,15,17]), except for some necessary adjustments and technicalities that are needed to treat the special form of our HJB equation. To better demonstrate our proof, we proceed with several simpler steps.

1. Auxiliary functions φ and Φ.

Take $\alpha,\beta,\varepsilon,l,\rho \in (0,1)$, $m \in (\mu,\nu)$ with $\mu < m\nu$ (this is possible since $0 < \mu < \nu^2$), and $0 < \eta < T$. Define

$$\varphi(t,x,z) = -\beta t + \frac{l}{t-T+\eta} + \alpha(T+\eta-t)(\langle x\rangle^m + \langle z\rangle^m) + \frac{1}{2\varepsilon}|x-z|^2,\\ (t,x,z) \in (T-\eta,T]\times\mathbb{R}^n\times\mathbb{R}^n, \tag{6.2}$$

where $\langle x\rangle \triangleq (1+|x|^2)^{1/2}$; and define

$$\Phi(t,x,z) = \rho V(t,x) - \hat{V}(t,z) - \varphi(t,x,z), \quad (t,x,z) \in (T-\eta,T]\times\mathbb{R}^n\times\mathbb{R}^n. \tag{6.3}$$

Since V and $\hat{V}$ are all in $\mathcal{V}_\mu$ with $\mu < m$, also by the definition of φ, it is not hard to check that $\Phi(t,x,z) \to -\infty$ as $t \to T-\eta$, $x, z \to \infty$. Therefore there exists $(t_0,x_0,z_0) \in (T-\eta,T]\times\mathbb{R}^n\times\mathbb{R}^n$, which may depend on $\alpha,\beta,\varepsilon,l,\rho,m,\eta$, such that

$$\Phi(t_0,x_0,z_0) = \max_{(T-\eta,T]\times\mathbb{R}^n\times\mathbb{R}^n} \Phi(t,x,z).$$

Further, since $\Phi(t_0,x_0,z_0) \geq \Phi(T,0,0)$, plugging into (6.2) and (6.3) we derive

$$\alpha(T+\eta-t_0)(\langle x_0\rangle^m + \langle z_0\rangle^m) + \frac{1}{2\varepsilon}|x_0-z_0|^2\\ \leq \rho V(t_0,z_0) - \hat{V}(t_0,z_0) + (1-\rho)g(0) + 2\alpha\eta \leq C(1+|x_0|^\mu + |z_0|^\mu). \tag{6.4}$$

Noting that $\mu < m$ and $T+\eta-t_0 \geq \eta$, (6.4) implies that there exists a constant $C > 0$, independent of $\alpha,\beta,\varepsilon,l,\rho,m,\eta$, such that

$$|x_0|, |z_0| \leq C(\alpha\eta)^{-\frac{1}{m-\mu}}. \tag{6.5}$$

Now using the inequality $2\Phi(t_0, x_0, z_0) \geq \Phi(t_0, x_0, x_0) + \Phi(t_0, z_0, z_0)$ one shows that

$$\frac{|x_0 - z_0|^2}{\varepsilon} \leq \rho\Big(V(t_0, x_0) - V(t_0, z_0)\Big) + \hat{V}(t_0, x_0) - \hat{V}(t_0, z_0) \leq C|x_0 - z_0|.$$

This implies that $\frac{|x_0 - z_0|}{\sqrt{\varepsilon}} \leq \sqrt{\varepsilon}C$, and hence

$$\lim_{\varepsilon \to 0} \frac{|x_0 - z_0|^2}{\varepsilon} = 0. \tag{6.6}$$

2. Applying Theorem 8.3 of [20] to the functions ρV, $\hat{V}$ and φ , we conclude that there exist $a_1, a_2 \in \mathbb{R}$ and $A_1, A_2 \in \mathcal{S}^n$ such that

$$\begin{cases} \min\Big\{\frac{a_1}{\rho} + H\Big(t_0, x_0, \frac{1}{\rho}\varphi_x, \frac{1}{\rho}A_1\Big), F\Big(t_0, \frac{1}{\rho}\varphi_x\Big), M[V]\Big\} \geq 0, \\ \min\{-a_2 + H(t_0, z_0, -\varphi_z, -A_2), F(t_0, -\varphi_z), M[\hat{V}]\} \leq 0, \end{cases} \tag{6.7}$$

in which $a_1 + a_2 = \varphi_t(t_0, x_0, z_0)$, and, denoting $B(t,x,z) = \begin{pmatrix} \varphi_{xx}(t,x,z) & \varphi_{xz}(t,x,z) \\ \varphi_{xz}(t,x,z) & \varphi_{zz}(t,x,z) \end{pmatrix}$, there holds that

$$-\Big(\frac{1}{\varepsilon} + \|B\|\Big) \leq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \leq B + \varepsilon B^2. \tag{6.8}$$

Since $m - 2 < 0$ and $\eta \leq T + \eta - t \leq 2\eta$ (noting $t \in (T - \eta, T]$), we have

$$B \leq \frac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\alpha\eta \begin{pmatrix} \langle x \rangle^{m-2} I & 0 \\ 0 & \langle z \rangle^{m-2} I \end{pmatrix}.$$

Consequently,

$$\begin{aligned} B + \varepsilon B^2 \leq{}& \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 4\alpha^2\eta^2\varepsilon \begin{pmatrix} \langle x \rangle^{2m-4} I & 0 \\ 0 & \langle z \rangle^{2m-4} I \end{pmatrix} \\ &+ 2\alpha\eta \begin{pmatrix} 3\langle x \rangle^{m-2} I & -\big(\langle x \rangle^{m-2} + \langle z \rangle^{m-2}\big) I \\ -\big(\langle x \rangle^{m-2} + \langle z \rangle^{m-2}\big) I & 3\langle z \rangle^{m-2} I \end{pmatrix}. \end{aligned} \tag{6.9}$$

3. Show that for $\alpha > 0$ small enough, there holds that $M[\hat{V}](t_0, z_0) > 0$.

Since V and $\hat{V}$ are viscosity solutions of (5.2), one can easily show that $M[V](t_0, x_0) \geq 0$ and $M[\hat{V}](t_0, z_0) \geq 0$. Let us assume that $M[\hat{V}](t_0, z_0) = 0$. Then, there exists a sequence $\theta_k \in K \setminus \{0\}$, such that $\frac{\hat{V}(t_0, z_0 + \theta_k) - \hat{V}(t_0, z_0)}{\ell(t_0, \theta_k)} + 1 \leq \frac{1}{k}$, $\forall k \geq 1$. Therefore, by (2.7) and (6.5), we have (noting that $\hat{V}(t_0, z_0 + \theta_k) \geq 0$)

$$\begin{aligned} c_0 \|\theta_k\|_1^\nu \leq \ell(t_0, \theta_k) &\leq \frac{k}{k-1}\big(\hat{V}(t_0, z_0) - \hat{V}(t_0, z_0 + \theta_k)\big) \\ &\leq C(1 + |z_0|^\mu) \leq C\Big\{1 + (\alpha\eta)^{-\frac{\mu}{m-\mu}}\Big\}. \end{aligned} \tag{6.10}$$

On the other hand, since $\frac{V(t_0, x_0 + \theta_k) - V(t_0, x_0)}{\ell(t_0, \theta_k)} + 1 \geq M[V](t_0, x_0) \geq 0$, noting that $m < 1$ and $m\nu > \mu$, one has

$$|\langle x \rangle^m - \langle x + \theta_k \rangle^m| \leq |\theta_k| \leq \|\theta_k\|_1; \quad 1 - \frac{\mu(1-\nu)}{(m-\mu)\nu} > 0,$$

and one derives from the definitions of φ, Φ, and (6.10) that

$$\begin{aligned}
&\Phi(t_0,x_0+\theta_k,z_0+\theta_k)-\Phi(t_0,x_0,z_0)\\
&=\rho\Big(V(t_0,x_0+\theta_k)-V(t_0,x_0)\Big)+\Big(\hat V(t_0,z_0)-\hat V(t_0,z_0+\theta_k)\Big)\\
&\qquad+\alpha(T+\eta-t_0)\Big(\langle x_0\rangle^m-\langle x_0+\theta_k\rangle^m+\langle z_0\rangle^m-\langle z_0+\theta_k\rangle^m\Big)\\
&\geq\Big(1-\rho-\frac{1}{k}\Big)\ell(t_0,\theta_k)-4\alpha\eta|\theta_k|\geq\Big\{\Big(1-\rho-\frac{1}{k}\Big)c_0-4\alpha\eta\|\theta_k\|_1^{1-\nu}\Big\}\|\theta_k\|_1^\nu\\
&\geq\Big\{\Big(1-\rho-\frac{1}{k}\Big)c_0-4\alpha\eta C\Big(1+(\alpha\eta)^{-\frac{\mu(1-\nu)}{(m-\mu)\nu}}\Big)\Big\}\|\theta_k\|_1^\nu>0,
\end{aligned}\tag{6.11}$$

provided $\alpha>0$ is small enough (depending only on ρ) and k is large enough. Thus the point (t_0,x_0,z_0) cannot be a maximum point of Φ, a contradiction. Thus $M[\hat V](t_0,z_0)>0$.

4. Show that for $\alpha>0$ small, there holds that $F(t_0,-\varphi_z(t_0,x_0,z_0))>0$.

Since $m<1$, a direct computation shows that $|\varphi_x(t_0,x_0,z_0)+\varphi_z(t_0,x_0,z_0)|\leq 4\alpha T$. Thus (6.7) and the fact that $F(t_0,\frac{1}{\rho}\varphi_x(t_0,x_0,z_0))\geq 0$ lead to (recalling the (4.4) for the definition of F)the following:

$$\begin{aligned}
F(t_0,-\varphi_z(t_0,x_0,z_0))&=f^a(t_0)\wedge f^s(t_0)+\inf_{\theta_0\in K_0}\langle-\varphi_z(t_0,x_0,z_0),\theta_0\rangle\\
&\geq f^a(t_0)\wedge f^s(t_0)+\inf_{\theta_0\in K_0}\langle\varphi_x(t_0,x_0,z_0),\theta_0\rangle-4\alpha T\\
&=(1-\rho)f^a(t_0)\wedge f^s(t_0)+\rho\Big\{f^a(t_0)\wedge f^s(t_0)+\inf_{\theta_0\in K_0}\Big\langle\frac{1}{\rho}\varphi_x(t_0,x_0,z_0),\theta_0\Big\rangle\Big\}-4\alpha T\\
&=(1-\rho)f^a(t_0)\wedge f^s(t_0)+\rho F_1\Big(t_0,\frac{1}{\rho}\varphi_x(t_0,x_0,z_0)\Big)-4\alpha T\\
&\geq(1-\rho)f^a(t_0)\wedge f^s(t_0)-4\alpha T\geq(1-\rho)c_0-4\alpha T>0,
\end{aligned}$$

provided $\alpha>0$ is small enough (depending only on ρ).

5. Show that $t_0=T$.

Suppose $t_0<T$. Since the facts proved in Steps **3** & **4** above imply that (6.7) should actually read

$$\frac{a_1}{\rho}+H\Big(t_0,x_0,\frac{1}{\rho}\varphi_x,\frac{1}{\rho}A_1\Big)\geq 0,\qquad -a_2+H(t_0,z_0,-\varphi_z,-A_2)\leq 0,\tag{6.12}$$

and (6.5) and (6.6) enable us to assume without loss of generality that $x_0,z_0\to\bar x$, as $\varepsilon\to 0$ for some $\bar x\in\mathbb{R}^n$ satisfying (6.5), using the definitions of φ and H (see (5.1)), as well as (6.12) we have

$$\begin{aligned}
&\beta+\alpha(\langle x_0\rangle^m+\langle z_0\rangle^m)\leq-\varphi_t(t_0,x_0,z_0)=-a_1-a_2\\
&\leq\rho H\Big(t_0,x_0,\frac{1}{\rho}\varphi_x,\frac{1}{\rho}A_1\Big)-H(t_0,z_0,-\varphi_z,-A_1)\\
&\leq\sup_{u\in U}\Big\{\Big[\frac{1}{2}\mathrm{tr}\Big(\sigma^*(t_0,x_0,u)A_1\sigma(t_0,x_0,u)+\sigma^*(t_0,z_0,u)A_2\sigma(t_0,z_0,u)\Big)\Big]\\
&\qquad+\big[\langle\varphi_x(t_0,x_0,z_0),b(t_0,x_0,u)+b(t_0,z_0,u)\rangle\big]+\big[\rho h(t_0,x_0,u)-h(t_0,z_0,u)\big]\Big\}\\
&\triangleq\sup_{u\in U}\{I_1(u)+I_2(u)+I_3(u)\},
\end{aligned}\tag{6.13}$$

where $I_1(\cdot)$, $I_2(\cdot)$ and $I_3(\cdot)$ are three [...]'s inside the $\sup\{\cdots\}$ above, which we now estimate. First, using (6.8) and (6.9) we have, for each $u \in U$,

$$
\begin{aligned}
2I_1(u) &= \mathrm{tr}\left\{(\sigma^*(t_0,x_0,u),\sigma^*(t_0,z_0,u))\begin{pmatrix} A_1 & 0 \\ 0 & A_2\end{pmatrix}\begin{pmatrix}\sigma^*(t_0,x_0,u)\\ \sigma^*(t_0,z_0,u)\end{pmatrix}\right\}\\
&\le \mathrm{tr}\left\{(\sigma^*(t_0,x_0,u),\sigma^*(t_0,z_0,u))(B+\varepsilon B^2)\begin{pmatrix}\sigma^*(t_0,x_0,u)\\ \sigma^*(t_0,z_0,u)\end{pmatrix}\right\}\\
&\le \frac{3}{\varepsilon}|\sigma(t_0,x_0,u)-\sigma(t_0,x_0,u)|^2\\
&\quad + 4\alpha^2\eta^2\varepsilon\Big[\langle x_0\rangle^{m-4}|\sigma(t_0,x_0,u)|^2+\langle z_0\rangle^{m-4}|\sigma(t_0,z_0,u)|^2\Big]\\
&\quad + 2\alpha\eta\Big[3\langle x_0\rangle^{m-2}|\sigma(t_0,x_0,u)|^2+3\langle z_0\rangle^{m-2}|\sigma(t_0,z_0,u)|^2\\
&\quad + 2(\langle x_0\rangle^{m-2}+\langle z_0\rangle^{m-2})|\sigma(t_0,x_0,u)||\sigma(t_0,z_0,u)|\Big]\\
&\to 20\alpha\eta\langle\bar x\rangle^{m-2}|\sigma(\bar t,\bar x,u)|^2 \le 40\alpha\eta L^2\langle\bar x\rangle^m, \qquad \text{as } \varepsilon\to 0,
\end{aligned}
\tag{6.14}
$$

thanks to assumption (A1). Similarly, we have, as $\varepsilon \to 0$,

$$
\begin{aligned}
I_2(u) &= \Big\langle \alpha(T+\eta-t_0)m\langle x_0\rangle^{m-2}x_0+\frac{1}{\varepsilon}(x_0-z_0), b(t_0,x_0,u)\Big\rangle\\
&\quad + \Big\langle \alpha(T+\eta-t_0)m\langle z_0\rangle^{m-2}z_0-\frac{1}{\varepsilon}(x_0-z_0), b(t_0,z_0,u)\Big\rangle\\
&\to 2\langle \alpha(T+\eta-\bar t)m\langle\bar x\rangle^{m-2}\bar x, b(\bar t,\bar x,u)\rangle \le 4\sqrt{2}\alpha\eta L\langle\bar x\rangle^m;
\end{aligned}
\tag{6.15}
$$

and

$$
\begin{aligned}
I_3(u) &\le L|x_0-z_0|+(1-\rho)L(1+|x_0|^\mu)\\
&\to (1-\rho)L(1+|\bar x|^\mu)\le(1-\rho)L(1+\langle\bar x\rangle^m).
\end{aligned}
\tag{6.16}
$$

Consequently, if we let $\varepsilon \to 0$ on both sides of (6.13) and apply (6.14)–(6.16), then we obtain that

$$
\beta+2\alpha\langle\bar x\rangle^m \le \Big[(40L^2+4\sqrt{2}L)\eta+(1-\rho)L\Big]\alpha\langle\bar x\rangle^m+(1-\rho)L. \tag{6.17}
$$

Choosing η and ρ such that

$$
(40L^2+4\sqrt{2}L)\eta\le 1, \qquad (1-\rho)L\le\min\Big\{1,\frac{\beta}{2}\Big\}, \tag{6.18}
$$

then (6.17) leads to $\beta \le \frac{\beta}{2}$, a contradiction since $\beta > 0$. Therefore, we must have $t_0 = T$.

6. Complete the proof.

Since $t_0 = T$, for any $(t,x) \in (T-\eta, T] \times \mathbb{R}^n$, $\varphi(t,x,x) \le \varphi(T,x_0,z_0)$. Namely, by the definitions of φ, Φ and (t_0,x_0,z_0) and a little computation, we have

$$
\begin{aligned}
\varphi(t,x,x) &= \rho V(t,x)-\hat V(t,x)-2\alpha(T+\eta-t)\langle x\rangle^m+\beta t-\frac{l}{t-T+\eta}\\
&\le \rho g(x_0)-g(z_0)-\alpha\eta(\langle x_0\rangle^m+\langle z_0\rangle^m)-\frac{1}{2\varepsilon}|x_0-z_0|^2+\beta T-\frac{l}{\eta}\\
&\le \rho g(x_0)-g(z_0)+\beta T\to(\rho-1)g(x_0)+\beta T\le\beta T,
\end{aligned}
$$

as $\varepsilon \to 0$. In the last inequality, we have used $g(x_0) \geq 0$ (see (2.2)) and $\rho < 1$. We now let $\alpha \to 0$, $l \to 0$, $\rho \to 1$ and $\beta \to 0$ to get $V(t,x) \leq \hat{V}(t,x)$ on $(T-\eta,T] \times \mathbb{R}^n$. Changing the roles of V and $\hat{V}$, we obtain that $V(t,x) = \hat{V}(t,x)$ on $[T-\eta,T] \times \mathbb{R}^n$. Since $\eta > 0$ depends only on L (see (6.18)), by repeating the above argument finitely many times we complete the proof.

Case 2 $\ell(s,\theta) = f^d(s)\|\theta\|_1$. This case contains the standard singular control problems.

In this case the result is essentially the same, and the proof is very similar to that of Theorem 6.1. Thus we only briefly sketch it and point out the difference.

Theorem 6.2 *Let* (A1)–(A2) *and* (2.9) *hold. Then the value function* $V(s,x)$ *is the unique viscosity solution of* (5.3) *in* $\mathcal{V}_1$.

Sketch of the Proof We define φ and Φ as in (6.2) and (6.3) with $\alpha,\beta,\varepsilon,l,\rho \in (0,1)$, $0<\eta<T$ and $1<m<2$ (instead of $m<1$). Since V and $\hat{V}$ are in $\mathcal{V}_1$ and $m>1$, similarly to Step **1** in the proof of Theorem 6.1, we can find a maximum point $(t_0,x_0,z_0) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ for the function Φ. It follows that

$$\begin{aligned}&\alpha(T+\theta-t_0)(\langle x_0\rangle^m + \langle z_0\rangle^m) + \frac{1}{2\varepsilon}|x_0-z_0|^2 \\ &\quad \leq \rho V(t_0,x_0) - \hat{V}(t_0,z_0) + (1-\rho)g(0) + 2\alpha\eta \leq C(1+|x_0|+|z_0|),\end{aligned}$$

and we have (comparing with (6.5) and (6.6)) $|x_0|,|z_0| \leq C(\alpha\eta)^{-\frac{1}{m-1}}$ and $\frac{|x_0-z_0|^2}{\varepsilon} \to 0$, since $m<2$. Applying Theorem 8.3 of [3] to the function $\rho V(t,z) - \hat{V}(t,z) - \varphi(t,x,z)$, we can similarly to Steps **2**–**4** of Theorem 6.1, also derive (comparing with (6.12)) that

$$\frac{a_1}{\rho} + H\Big(t_0,x_0,\frac{1}{\rho}\varphi_x,\frac{1}{\rho}A_1\Big) \geq 0, \qquad -a_2 + H(t_0,z_0,-\varphi_z,-A_2) \leq 0.$$

and prove further that $t_0 = T$. The rest of the proof is essentially the same as that of Theorem 6.1.

7 Finite Fuel Problem

In this section we discuss the possible extension of our results to the "finite-fuel" setting, that is, the singular control is subject to a constraint on its total resources available. To begin with let us denote

$$\mathcal{K}_y[s,T] \triangleq \Big\{\xi \in \mathcal{K}[s,T] \;\Big|\; |\xi|_{[s,T]} \leq y, \text{ a.s.}\Big\}.$$

We call an element $(u,\xi) \in \mathcal{U}[0,T] \times \mathcal{K}[0,T]$ an *admissible control* of *finite-fuel* if $\xi \in \mathcal{K}_y[0,T]$ for some $y>0$, where the $y>0$ stands for the total initial fuel available for the singular control ξ. Naturally, ξ is said to be of *unlimited-fuel* if $y=\infty$.

The "Finite-Fuel" control problem is then formulated as follows.

Finite-Fuel Problem *For any given* $(s,x,y) \in [0,T) \times \mathbb{R}^n \times [0,\infty)$, *find a pair* $(\bar{u},\bar{\xi}) \in \mathcal{U}[s,T] \times \mathcal{K}_y[s,T]$, *such that*

$$J_{s,x}(\bar{u},\bar{\xi}) = \inf_{\mathcal{U}[s,T]\times\mathcal{K}_y[s,T]} J_{s,x}(u,\xi) \triangleq V(s,x,y).$$

We call $V(s,x,y)$ the value function of the Finite-Fuel problem. It is clear that we must also have $V(T,x,y) = g(x)$, $\forall(x,y) \in \mathbb{R}^n \times [0,\infty)$. Note that if $y = 0$, then the Finite-Fuel Problem is reduced to a regular (or classical) control problem, and one has

$$V_0(s,x) \triangleq \inf_{\mathcal{U}[s,T]} J_{s,x}(u,0) = V(s,x,0), \qquad \forall(s,x) \in [0,T] \times \mathbb{R}^n, \tag{7.1}$$

where $V_0(s,x)$ is the value function of the pure regular optimal control problem. It is easily seen that for fixed (s,x), $V(s,x,y)$ is nonincreasing in y. In fact we can actually show that $\lim_{y\to\infty} V(s,x,y) = V(s,x)$. To see this, note that for any fixed $(s,x) \in [0,T] \times \mathbb{R}^n$, and any $\varepsilon > 0$, there exists a $(u,\xi) \in \mathcal{U}[s,T] \times \mathcal{K}[s,T]$, such that $V(s,x) + \varepsilon \geq J_{s,x}(u,\xi)$. For each integer $N > 0$, we define a stopping time $\tau^N \triangleq \inf\{t \in [s,T] : |\xi|_{[s,t]} \geq N\} \wedge T$ and a stopped process $\xi_t^{(N)} \triangleq \xi_{t\wedge\tau^N}$. Denote $X_t = X_t^{s,x,u,\xi}$, $X_t^{(N)} = X_t^{s,x,u,\xi^{(N)}}$. It is easy to check that each $\xi^{(N)} \in \mathcal{K}_N[s,T] \subset \bigcup_{y>0} \mathcal{K}_y[s,T]$, and $\xi_t^{(N)} = \xi_{\tau^N}^{(N)}$ for $t > \tau^N$, P-a.s. Also, one can show that for P-a.s.$\omega \in \Omega$, $\tau^N(\omega)$ is nondecreasing and that $\tau^N(\omega) = T$ for N large enough. Thus by Proposition 2.1 with $\beta = 1$, we have

$$\begin{aligned} V(s,x) + \varepsilon \geq J_{s,x}(u,\xi) &\geq V(s,x,N) - E\Big\{ \int_{\tau^N}^T L|X_r - X_r^{(N)}|dr + L_0|X_T - X_T^{(N)}|\Big\} \\ &\geq V(s,x,N) - CE\Big\{|\xi^c|_{[\tau^N,T]} + |\xi^d|_{[\tau^N,T]}\Big\} = V(s,x,N) - o(1). \end{aligned}$$

That is, $\lim_{y\to\infty} V(s,x,y) = V(s,x)$.

Similarly to Theorem 3.1, one can show that $\begin{cases} V(s,x,y) \leq C(1+|x|^\mu), \qquad \forall(s,x,y); \\ |V(s,x,y) - V(s,\bar{x},\bar{y}) \leq C(|x-\bar{x}| - |y-\bar{y}|), \\ \qquad\qquad \forall(s,x,y),(s,\bar{x},\bar{y}). \end{cases}$

Moreover, it can be shown that the Bellman Principle for the Finite-Fuel Problem takes the form

$$\begin{aligned} V(s,x,y) = \inf_{\mathcal{U}[s,T]\times\mathcal{K}_y[s,T]} E\Big\{ \int_s^\tau h(r,X_r,u_r)dr + \int_s^\tau f^a(r)\|\dot{\xi}_r^{ac}\|_1 dr \\ + \int_s^\tau f^s(r)|d\xi_r^{sc}| + \sum_{r\in S_\xi[s,\tau)} \ell(r,\Delta\xi_r) + V(\tau,X_\tau,y-|\xi|_{[s,\tau]})\Big\}, \end{aligned} \tag{7.2}$$

and if we define

$$\begin{cases} M'[V](s,x,y) = \inf_{\theta\in K\setminus\{0\},\|\theta\|_1\leq y} \Big\{\dfrac{V(s,x+\theta,y-\|\theta\|_1) - V(s,x,y)}{\ell(s,\theta)}\Big\} + 1; \\ \qquad\qquad \forall(s,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}; \\ F'(s,p,q) = \inf_{\theta\in K_0} \{f^a(s)\wedge f^s(s) + \langle p,\theta\rangle - q\}, \quad \forall(s,p,q) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}. \end{cases} \tag{7.3}$$

then the HJB equation for the Finite-Fuel Problem reads as follows.

$$\begin{cases} \min\{V_s + H(s,x,V_x,V_{xx}), F'(s,V_x,V_y), M'[V](s,x,y)\} = 0, \\ V\,\big|_{s=T} = g(x), \qquad V\,\big|_{y=0} = V^0(s,x), \end{cases} \tag{7.4}$$

where $V^0(s,x)$ is defined by (7.1).

To conclude this section we state without proof a theorem which is an analogue of Theorems 5.1, 6.1 and 6.2 combined. The proof of this result is essentially parallel to that of the theorems we have proved, if not easier (noting that the bound posed on the total variation of the singular control ξ will make many arguments easier; for example no reduction to the set $\mathcal{K}^{|x|}[s,T]$, defined by (3.1), will be necessary in this case!). However, we should note that the uniqueness proof of the Finite-Fuel Problem is much lengthier than the unlimited fuel case, due to more complicated notations.

Let us introduce $\tilde{\mathcal{V}}_\mu = \{v \in C([0,T] \times \mathbb{R}^n \times [0,\infty)) \mid \exists C > 0,\ 0 \le v(s,x,y) \le C(1+|x|^\mu), |v(s,x,y) - v(s,\bar{x},\bar{y})| \le C(|x-\bar{x}| + |y-\bar{y}|), \forall s \in [0,T],\ x,\bar{x} \in \mathbb{R}^n,\ y,\bar{y} \in [0,\infty)\}$, and recall the two cases in Sec.6. We have the following result.

Theorem 7.1 *Let* (A1)–(A2) *hold. Then in Case* 1, *the value function* $V(s,x,y)$ *is the unique viscosity solution of* (7.4) *in* $\tilde{\mathcal{V}}_\mu$; *in Case* 2, *the value function* $V(s,x,y)$ *is the unique viscosity solution of* (7.4) *in* $\tilde{\mathcal{V}}_1$. *Further, if we define for* $(s,p,q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}$,

$$G'(s,p,q) = \inf_{\theta \in K_0} \{f^a(s) \wedge f^s(s) \wedge f^d(s) + \langle p, \theta \rangle - q\}, \tag{7.5}$$

then in Case 2, (7.4) *is equivalent to the following:*

$$\begin{cases} \min\{V_s + H(s,x,V_x,V_{xx}), G'(s,V_x,V_y)\} = 0, \\ V\big|_{s=T} = g(x); \qquad V\big|_{y=0} = V^0(s,x). \end{cases} \tag{7.6}$$

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