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Optimal Portfolio Selection Under Concave Price Impact

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Abstract In this paper we study an optimal portfolio selection problem under instantaneous price impact. Based on some empirical analysis in the literature, we model such impact as a concave function of the trading size when the trading size is small. The price impact can be thought of as either a liquidity cost or a transaction cost, but the concavity nature of the cost leads to some fundamental difference from those in the existing literature. We show that the problem can be reduced to an impulse control problem, but without fixed cost, and that the value function is a viscosity solution to a special type of Quasi-Variational Inequality (QVI). We also prove directly (without using the solution to the QVI) that the optimal strategy exists and more importantly, despite the absence of a fixed cost, it is still in a "piecewise constant" form, reflecting a more practical perspective.

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1 Introduction

Modeling of the liquidity risk has attracted strong attention in the recent years in the quantitative finance literature, and there have been numerous publications on the subject. Among others, one of the core issues is to understand the price impact of individual tradings. Motivated by empirical observations, Bouchaud, Farmer, and Lillo [2] (and the references therein) suggested a price impact model in which the trading size affects the price in a "concave" way, when the trading size is small. Such a (concavity) assumption apparently leads to some fundamental differences from many existing results (see more detailed discussion in Sect. 2), and this paper is an attempt to understand these differences in the context of an optimal portfolio selection problem. Roughly speaking, we shall argue that under such a concavity assumption, the optimization problem can be reduced to an impulse control problem without a fixed cost, but nevertheless the optimal strategy still exists and, somewhat surprisingly, it is in a piecewise constant form. One can then easily conclude that the liquidity cost does exist.

Our model is mainly motivated by the work of Cetin, Jarrow, and Protter [3], in which the liquidity cost was characterized by the so-called "supply curve". The main feature of the model (along with its subsequent work by Cetin, Jarrow, Protter, and Warachka [4]) is that the dependence of the supply curve on the trading size is essentially quadratic when the size is small. Furthermore, it is shown in [3] that the supply-curve-based liquidity cost could be eliminated if one is allowed to split any (large) order into many small pieces, and trade them infinitely frequently (this amounts to saying that the continuous trading is allowed). Such a point was later amplified by Bank and Baum [1], in which they proved that one can always approximate a trading strategy by those that have continuous and finite variation paths, consequently the liquidity cost could always be eliminated. But on the other hand, both empirical evidences and other theoretical studies indicate that the liquidity risk does exist, even in the continuous trading paradigm. For instance, by considering the Gamma constraint on the admissible (continuous!) portfolios and by using the so-called second order backward SDEs, Cetin, Soner, and Touzi [6] proved that the super-hedging price is in general higher than the Black-Scholes price, and thus the liquidity cost must exist. Also, to make the model more realistic, various constraints on the trading strategies have been added in order to avoid the vanishing liquidity cost. For example, Cetin and Rogers [5] assumed that any two consecutive transactions have to be one unit of time apart. In a different work, Ly Vath, Mnif, and Pham [14] assumed heavy liquidity cost if two transactions were made too closely. We should note, however, in the last two works the optimal strategy being piecewise constant is (essentially) assumed *exogenously*. The main message of our result is that the concavity assumption of the liquidity cost provides an *endogenous* structure, from which the optimal strategy becomes intrinsically "piecewise constant", even in the absence of a fixed cost.

It is worth noting that since all the liquidity costs mentioned above have instantaneous (or temporary) price impact, technically they are equivalent to a type of transaction costs. Consequently, our approach can be easily applied to problems with transaction costs, which has been studied extensively (see, e.g., [7, 8, 10, 13–16], and the references therein). Most results in the literature assume either fixed cost, or proportional cost, or the linear combination of them. To be more precise, if we denote $c(z)$ to be the price impact or the transaction cost when the trading size is z, and we assume $c(z) \sim |z|^\alpha$ when z is small, then the fixed cost case corresponds to $\alpha = 0$, proportional cost or linear price impact case corresponds to $\alpha = 1$, and the price impact in [3] corresponds to $\alpha = 2$. When $\alpha > 1$, the liquidity (or transaction) cost vanishes in approximate sense by allowing multiple instantaneous trading. When $\alpha = 1$, this is typically a singular control problem and the optimal strategy is continuous. When $\alpha = 0$, this is typically an impulse control problem and the optimal strategy is discrete. We essentially assume $0 < \alpha < 1$, which is consistent with the concavity of the price impact as observed in [2]. We show that our problem is essentially an impulse control problem, but possibly without fixed cost.

Our second goal in this paper is to prove the existence of the optimal strategy and argue that it must be piecewise constant. We note that unlike most of impulse control problems in the literature, we do not assume that the cost function is strictly positive (no fixed cost). Thus the HJB equation, being a quasilinear-variational inequality (QVI), does not have a smooth solution in general. Consequently, the construction of the optimal strategy, whence in many cases the existence of it, become problematic if one follows the standard verification theorem approach (cf., e.g. [15]). In this paper we shall attack the existence of optimal strategy directly. We first consider a sequence of approximating problems for which the strategies are restricted to a fixed number (say, n) of trades. We show that for each n the optimal strategy, denoted by Z^n, exists. The main technical part in this analysis turns out to be some uniform estimates on the number of jumps of Z^n. These estimates will enable us to study the regularity of the value function and to construct the optimal strategy. We should note that the regularity of the value function, which we need to construct the optimal strategy, is weaker than those that are commonly seen in the literature.

The rest of the paper is organized as follows. In Sect. 2 we formulate the problem and state the main result. In Sects. 3 and 4 we study the approximating value function V^n and its corresponding optimal strategy Z^n. In Sect. 5 we obtain uniform estimates of Z^n, which leads to the regularity of the value function V. In Sect. 6 we study the optimal strategy of the original problem. Finally in Sect. 7 we give some technical proofs.

2 Problem Formulation

2.1 The Model

Let $(\Omega, \mathcal{F}, P; \mathbf{F})$ be a complete filtered probability space on a finite time interval $[0, T]$ and W be a standard Brownian motion. We assume that the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by W, augmented by all the P-null sets as usual. The financial market consists of two assets, a bank account and a stock. For simplicity, we assume that the interest rate is 0. Let X denote the fundamental value of the stock which follows the stochastic differential equation:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{2.1}$$

In this paper we consider the liquidity cost in the following general form: if one buys z shares of the stock (sells $-z$ shares if $z < 0$) at time t, then the liquidity cost of the trade is $c(t, X_t, z)$, where c is a deterministic function satisfying $c(t, x, 0) = 0$; and

$$c \text{ is increasing in } z \text{ when } z > 0 \text{ and decreasing in } z \text{ when } z < 0. \tag{2.2}$$

We shall give more specific assumptions on the cost function c in the next subsection. But we remark here that if $c_0 := \inf_{(t,x),z\neq 0} c(t, x, z) > 0$, then c_0 represents a "fixed cost". The following example shows that such a positive lower bound usually does not exist in the context of liquidity cost.

Example 2.1 *Consider the "supply curve" $\mathcal{S}(t, X_t, z)$ defined in* [3], *in which X_t is the fundamental price and z is the trading size at time t. We can view $\mathcal{S}$ as the market price of the stock, satisfying*

$$\mathcal{S}(t, X_t, 0) = X_t, \ \textit{and } \mathcal{S} \textit{ is increasing in } z. \tag{2.3}$$

Thus the liquidity cost should naturally be defined by

$$c(t, X_t, z) := z\big[\mathcal{S}(t, X_t, z) - X_t\big]. \tag{2.4}$$

One can easily check that the c satisfies (2.2).

We remark that in Example 2.1, if $\mathcal{S}$ is smooth in z, then $c(t, X_t, z) \sim z^2$ when z is small. Namely $z \mapsto c(t, X_t, z)$ is *convex* for z small. In this paper, however, we are interested in the case where $c(t, X_t, z) \sim |z|^\alpha$ for some $0 < \alpha < 1$, as supported by [2]. Therefore it is fundamentally different from the case in [3].

We next consider admissible trading strategies Z. We assume Z is $\mathbf{F}$-adapted, càdlàg, and piecewise constant. Let $\tilde{Y}$ denote the total value invested in the riskless asset, and define $Y := \tilde{Y} + ZX$. Assuming that the interest rate is 0, then except for countably many $t \in D_Z := \{t \in [0, T] : \delta Z_t := Z_t - Z_{t-} \neq 0\}$, one has

$$d\tilde{Y}_t = 0 \text{ and thus } dY_t = Z_t dX_t. \tag{2.5}$$

Namely, Z is "self-financing". Furthermore, for $t \in D_Z$ (i.e., $\delta Z_t \neq 0$), we impose the standard self-financing constraint:

$$\delta Y_t + c(t, X_{t-}, \delta Z_t) = \delta Y_t + c(t, X_t, \delta Z_t) = 0. \tag{2.6}$$

We note that (2.6) simply means that no instantaneous profit can be made by changing the investment positions. In the case of supply-curve (Example 2.1), the equation (2.6) amounts to saying that (noting that X is continuous)

$$\delta\tilde{Y}_t + \delta Z_t X_t + c(t, X_t, \delta Z_t) = \delta\tilde{Y}_t + \delta Z_t \mathcal{S}(t, X_t, \delta Z_t) = 0.$$

This is exactly the standard idea of "budget constraint".

2.2 The Optimization Problem

We now introduce our optimization problem on a subinterval $[t, T]$. Let $X^{t,x}$ denote the solution to SDE (2.1) with initial value $X_t = x$, a.s. Given (x, y, z) and an admissible trading strategy Z, we shall set $Y_{t-} := y$ and $Z_{t-} := z$. Then by (2.5) and (2.6) we have

$$Y_T^{t,x,y,z,Z} := Y_T = y + \int_t^T Z_s dX_s^{t,x} - \sum_{t \le s \le T} c\big(s, X_s^{t,x}, \delta Z_s\big). \tag{2.7}$$

Let U be a terminal payoff function, then our optimization problem is:

$$V(t, x, y, z) := \sup_{Z \in \mathcal{Z}_t} E\big[U\big(Y_T^{t,x,y,z,Z}\big)\big]. \tag{2.8}$$

Here the set $\mathcal{Z}_t$ of the admissible strategies is defined rigorously at below:

Definition 2.2 Given $t \in [0, T]$, the set of admissible strategies, denoted by $\mathcal{Z}_t$, is the space of $\mathbf{F}$-adapted processes Z over $[t, T]$ such that, for a.s. ω,

(i) Z is càdlàg and piecewise constant with finitely many jumps;
(ii) $Z_T = 0$, and $|Z| \le M$.

It is worth noting that a key assumption in Definition 2.2 is that Z is piecewise constant and has only finitely many jumps. While this is obviously more desirable in practice, it is by no means clear that an optimal strategy can be found in such a form. Thus the main goal of this paper is to show that the concavity assumption on c, see (H4) below, implies the existence of an optimal strategy in $\mathcal{Z}_t$.

Remark 2.3 (i) We require Z to be càdlàg for notational convenience. One can easily change it to càglàd if necessary.

(ii) Due to the liquidity risk, if $Z_T \neq 0$, the payoff of Y_T is not clear. As in [3] and [6], we require $Z_T = 0$ so that $Y_T = \tilde{Y}_T$. An alternative way is to introduce a payoff function $U(\tilde{Y}_T, Z_T)$ on both accounts, see, e.g. [9] in the formulation of superhedging.

(iii) The assumption that Z is bounded is merely technical. This restriction can be removed, with some extra efforts on the estimates, by requiring that the cost function c satisfies certain growth condition, for example, $\lim_{|z|\to\infty} \inf_x |c(z, x)|/|z| = \infty$. We prefer not to pursue such complexity in this paper. In fact, we will impost some stronger technical assumptions in order not to distract our attention from the main focus of the paper.

Remark 2.4 Technically, the optimization problem (2.8) can be extended to the cases where admissible strategies are allowed to be general $\mathbf{F}$-adapted, càdlàg processes. But in that case we need to redefine the aggregate liquidity cost. For example, we can consider the aggregate cost in the following forms:

$$\sup_\pi \sum_{i=0}^\infty c(\tau_i, X_{\tau_i}, Z_{\tau_i} - Z_{\tau_{i-1}}), \quad \text{or} \quad \lim_{|\pi|\to 0} \sum_{i=0}^\infty c(\tau_i, X_{\tau_i}, Z_{\tau_i} - Z_{\tau_{i-1}}), \tag{2.9}$$

where the supreme is over all possible random partitions of $[t,T]$ π: $t=\tau_0<\tau_1<\cdots\le T$; and $|\pi|$ is the "mesh size" of the partition. Then, under our conditions in next subsection on the function c, one can show that the value function V would be the same as the one where the supreme is taken over only piecewise constant strategies. Namely, it suffices to consider only $\mathcal{Z}_t$, and thus the aggregate cost (2.9) is again reduced to that in (2.7).

However, for more general c, typically there is no optimal strategy in $\mathcal{Z}_t$ and then one has to extend the space to allow more complex strategies. The following two cases are worth noting.

(i) Assume that $c(t,x,z)=|z|$. Then $\sup_\pi \sum_{i=0}^\infty c(\tau_i, X_{\tau_i}, Z_{\tau_i}-Z_{\tau_{i-1}}) = \int_t^T |dZ_r|$, the total variation of the process Z. This problem then becomes a more or less standard singular (or impulse) stochastic control problem (cf. e.g., [10, 13], and [14]). In these cases the optimal controls are of bounded variation, but not necessarily piecewise constant.

(ii) Assume the supply curve $\mathcal{S}(t,x,z)$ is smooth, as proposed in [3] and [4]. Then $c(t,x,z)\sim z^2$ when z is small. For any (random) partition $\pi: t=\tau_0<\tau_1<\cdots\le T$ and any $\mathbf{F}$-adapted semimartingale Z satisfying $Z_T=0$, we have

$$\begin{aligned}
&\sum_{i=0}^\infty c(\tau_i, X_{\tau_i}, Z_{\tau_i}-Z_{\tau_{i-1}})\\
&\quad=\sum_{i=0}^\infty [Z_{\tau_i}-Z_{\tau_{i-1}}]\big[\mathcal{S}(\tau_i, X_{\tau_i}, Z_{\tau_i}-Z_{\tau_{i-1}})-X_{\tau_i}\big]\\
&\quad=\sum_{i=0}^\infty [Z_{\tau_i}-Z_{\tau_{i-1}}]\big[\mathcal{S}(\tau_i, X_{\tau_i}, Z_{\tau_i}-Z_{\tau_{i-1}})-\mathcal{S}(\tau_i, X_{\tau_i},0)\big]\\
&\quad\to \sum_{t\le s\le T}\delta Z_s\big[\mathcal{S}(s,X_s,\delta Z_s)-\mathcal{S}(s,X_s,0)\big]+\int_t^T \frac{\partial \mathcal{S}}{\partial z}(s,X_s,0)d[Z,Z]_s^c.
\end{aligned}$$

This recovers the liquidity cost in [3] and [4], and in this case it is natural to set the admissible strategies as semimartingales.

2.3 Technical Assumptions

We now present our technical conditions. As mentioned in Remark 2.4, our main focus is to show that the concavity assumption on c implies the existence of an optimal strategy in $\mathcal{Z}_t$. However, in order not to over complicate our estimates, we shall impose some stronger technical conditions, some of which might be more than necessary. We remark that our approach can be extended to more general cases.

We first assume that all processes in this paper are one dimensional and, as mentioned already, the interest rate is 0. Moreover, we shall make use of the following assumptions:

(H1) The coefficients b and σ in (2.1) are bounded and uniformly Lipschitz continuous in x, with a common Lipschitz constant $K>0$.

(H2) The terminal payoff function U is concave, increasing such that $0 < \lambda \le U' \le \Lambda$ on $(-\infty, \infty)$ for some constants $0 < \lambda < \Lambda$.

(H3) The cost function c depends only on the trading size z, and satisfies:

(i) $c(0) = 0$ and $c(z) > 0$ for all $z \neq 0$.

(ii) c is increasing in $[-2M, 0)$ and decreasing in $(0, 2M]$; moreover, in both intervals, c is uniformly continuous with the same modulus of continuity function ρ.

(iii) the following subadditive property holds:

$$c(z_1 + z_2) \le c(z_1) + c(z_2), \quad \text{for any } z_1, z_2 \\ \text{such that } |z_1|, |z_2|, |z_1 + z_2| \le 2M. \tag{2.10}$$

(H4) There exists a constant $\varepsilon_0 > 0$ such that

(i) c is concave in $(0, 2\varepsilon_0]$ and in $[-2\varepsilon_0, 0)$, and

$$\begin{aligned} \eta_\theta &:= \overline{\lim_{z\to 0}} \frac{c(\theta z)}{c(z)} < \theta, \quad \text{for } \theta = \frac{3}{2}, 2, 3, \quad \text{and} \\ \gamma &:= \overline{\lim_{z\to 0}} \frac{c(-2z) - c(-z)}{c(z)} < \infty. \end{aligned} \tag{2.11}$$

(ii) c is uniformly Lipschitz continuous in $[-2M, -\varepsilon_0] \cup [\varepsilon_0, 2M]$ with a Lipschitz constant L_0.

We conclude this subsection by several important remarks.

Remark 2.5 The assumption (H2) indicates that the terminal payoff U is essentially a "utility function", except that it violates the well-known *Inada* condition:

$$\lim_{y\to -\infty} U'(y) = \infty, \qquad \lim_{y\to\infty} U'(y) = 0. \tag{2.12}$$

This is mainly for technical simplifications. The following observations are worth noting.

(i) If there is a fixed cost, namely if the cost function c satisfies

$$c(z) \ge c_0 > 0 \quad \text{for all } z \neq 0, \tag{2.13}$$

then one can prove our main result Theorem 2.8 under Inada condition (2.12) (see also Remark 2.6-(iii) below). In fact, in this case the conditions on c can also be further relaxed.

(ii) In the case when $U(y) = -e^{-y}$, $c(z) = |z|^\alpha$ for some $0 < \alpha < 1$, and $b(t, x) = b_0$, $\sigma(t, x) = \sigma_0$, then the assumptions (H1), (H3), (H4), and (2.12) are all satisfied, one can easily check that $V(t, x, y, z) = -e^{-y}\mathcal{V}(t, z)$, where

$$\mathcal{V}(t, z) := \inf_{Z\in\mathscr{Z}_t} E\left[\exp\left(-b_0 \int_t^T Z_s ds - \sigma_0 \int_t^T Z_s dW_s + \sum_{t\le s\le T} |\delta Z_s|^\alpha\right)\right]. \tag{2.14}$$

Thus the optimization problems (2.8) and (2.14) are equivalent. By utilizing the structure of $\mathcal{V}$ and modifying our arguments slightly we can also prove our main result in this case.

We believe our results hold true under even more general conditions. However, since the main focus of this paper is the impact of the concave cost function c, we choose not to over-complicate this already lengthy paper, and content ourselves with the stronger condition (H2) instead.

Remark 2.6 (i) We require the concavity of c only around 0. Typically, c is convex when z is large, as in the standard literature of liquidity risk.

(ii) The typical case satisfying (H3) and (H4) is: $c(z) = c_0|z|^\alpha$, $0 < \alpha < 1$. The condition (2.11) captures the behavior of c around 0. We consider those three values of θ just for technical reasons. One can of course make the assumption more symmetric by strengthening the condition to $\eta(\theta) < 1$ for all $\theta > 1$. The assumption on γ is merely technical. However, one cannot remove (2.11) for free. For example, $c(z) = |z|$ violates (2.11) and we know in this case the optimization problem becomes a singular control problem, see Remark 2.4(i).

(iii) Another typical case is when there is a fixed cost, namely (2.13) holds. Since in this case (2.11) automatically holds, we do not need the concavity assumption in (H4) and our main results will still be valid. See Theorem 2.8 below.

(iv) Note that we allow $c(0+) > 0$ and/or $c(0-) > 0$ in (H4). Moreover, combing the arguments for the two cases in (i) and (ii), we can easily prove our results in the case that (H4) holds in $(0, 2\varepsilon_0]$ and $c(z) \geq c_0 > 0$ for $z < 0$, and the case that (H4) holds in $[-2\varepsilon_0, 0)$ and $c(z) \geq c_0 > 0$ for $z > 0$.

Remark 2.7 (i) In this remark we justify the subadditive property (2.10). Note that our goal is to solve (2.8). For general c, by possibly splitting a transaction into many small pieces, we define,

$$\tilde{c}(z) := \inf\big\{c(z_1) + \cdots + c(z_n) : |z_i| \leq 2M, z_1 + \cdots + z_n = z, \forall n\big\}.$$

Then it is easy to see that $\tilde{c} \leq c$ and $\tilde{c}$ satisfies (2.10). Replacing c by $\tilde{c}$ in (2.7) we have

$$\tilde{Y}_T := y + \int_t^T Z_s dX_s - \sum_{t \leq s \leq T} \tilde{c}(\delta Z_s); \qquad \tilde{V}(t,x,y,z) := \sup_{Z \in \mathcal{Z}_t} E\big[U(\tilde{Y}_T)\big].$$

Under the continuity of U, one can easily show that $\tilde{V} = V$. In other words, we can always replace the cost function c to one that satisfies (2.10).

(ii) If the cost function c satisfies $c(z) \leq C|z|^\alpha$ for some constants $C > 0$ and $\alpha > 1$ near $z = 0$, then the corresponding $\tilde{c}(z) \equiv 0$. To see this, note that for any z and large n we have

$$\tilde{c}(z) \leq \sum_{i=1}^n c\left(\frac{z}{n}\right) \leq C \sum_{i=1}^n \left|\frac{z}{n}\right|^\alpha \leq \frac{CM^\alpha}{n^{\alpha-1}} \to 0, \qquad \text{as } n \to \infty.$$

Thus the optimization problem is reduced to a standard one without liquidity cost. This is consistent with the result of [3], where $\alpha = 2$.

2.4 Main Result

For any $Z \in \mathcal{Z}_t$, we shall always denote

$$\tau_0 := t, \qquad \tau_i := \inf\{s > \tau_{i-1} : Z_s \neq Z_{\tau_{i-1}}\} \wedge T, \quad i = 1, \dots. \tag{2.15}$$

Then clearly $\tau_i < \tau_{i+1}$ whenever $\tau_i < T$, $\tau_i = T$ when i is large enough, and

$$Z_s = \sum_{i=1}^{\infty} Z_{\tau_{i-1}} \mathbf{1}_{[\tau_{i-1}, \tau_i)}(s), \qquad s \in [t, T]. \tag{2.16}$$

Recall that $Z_{t-} = z$. Let $N(Z)$ denote the number of jumps of Z, that is,

$$N(Z) := \sum_{t \le s \le T} \mathbf{1}_{\{\delta Z_s \neq 0\}} = \sum_{i=0}^{\infty} \mathbf{1}_{\{Z_{\tau_i} \neq Z_{\tau_{i-1}}\}}. \tag{2.17}$$

Our main result of the paper is:

Theorem 2.8 *Assume* (H1)–(H3), *and assume either* (2.13) *or* (H4) *is in force. Then for any* (t, x, y, z), *the optimization problem* (2.8) *admits an optimal strategy* $Z^* \in \mathcal{Z}_t$. *Moreover,* $E[N(Z^*)] < \infty$.

3 The Approximating Problems

In this section, we shall approximate the original optimization problem (2.7) and (2.8) by those with only fixed number of transactions, for which the optimal strategies are easier to find. To begin with, for any $n \ge 1$ we consider a reduced problem with at most n transactions:

$$V^n(t, x, y, z) := \sup_{Z \in \mathcal{Z}_t^n(z)} E\big\{U\big(Y_T^{t,x,y,z,Z}\big)\big\}$$

$$\text{where } \mathcal{Z}_t^n(z) := \big\{Z \in \mathcal{Z}_t : N(Z) \le n\big\}. \tag{3.1}$$

We note that, for $Z \in \mathcal{Z}_t^n(z)$, if $Z_t = z$, then $\tau_n = T$, and if $Z_t \neq z$, then $\tau_{n-1} = T$. Moreover, when $n = 1$, we have $\mathcal{Z}_t^1(z) = \{z\mathbf{1}_{[t,\tau)}\}$ for all stopping time τ, and

$$V^1(t, x, y, z) = \sup_{\tau \ge t} E\big\{U\big(y + z\big(X_\tau^{t,x} - x\big) - c(-z)\big)\big\}. \tag{3.2}$$

It is then readily seen, assuming (H1)–(H3), that

$$\big|V^1(t, x, y, z)\big| \le C\big[1 + |y|\big], \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^2 \times [-M, M]. \tag{3.3}$$

Here and in the sequel $C > 0$ is a generic constant depending only on T, M, λ, Λ, K, and $|U(0)|$ in (H1)–(H3), as well as $\sup_{|z| \le 2M} c(z)$, and it is allowed to vary from line to line.

Proposition 3.1 *Assume* (H1)–(H3). *Then* $V^n(t,x,y,z)\uparrow V(t,x,y,z)$, *as* $n\to\infty$; *and*

$$V^n(t,x,y,z)\le V(t,x,y,z)\le C[1+|y|],\quad (t,x,y,z)\in[0,T]\times\mathbb{R}^2\times[-M,M]. \tag{3.4}$$

Proof It is clear by definition that V^n is increasing and $V^n\le V$. We first show that (3.4) holds for V (whence for V^n as well). For any $Z\in\mathscr{Z}_t$, let us denote $X=X^{t,x}$ and $Y=Y^{t,x,y,z,Z}$ for simplicity. Since the liquidity cost is positive, we have

$$Y_T\le y+\int_t^T Z_s dX_s=y+\int_t^T Z_s b(s,X_s)ds+\int_t^T Z_s\sigma(s,X_s)dW_s.$$

Then, using the monotonicity of U and boundedness of b, σ and Z, we have

$$\begin{aligned} EU(Y_T)&\le E\left\{U\left(y+\int_t^T Z_s dX_s\right)\right\}\\ &\le \bigl|U(0)\bigr|+\Lambda\left\{|y|+E\left|\int_t^T Z_s dX_s\right|\right\}\le C\bigl[1+|y|\bigr]. \end{aligned}\tag{3.5}$$

Since Z is arbitrary, we prove (3.4).

We now show that $V^n\to V$, as $n\to\infty$. We first note that V^n is non-decreasing, and bounded from above, thanks to (3.4). Thus $V^\infty(t,x,y,z):=\lim_{n\to\infty}V^n(t,x,y,z)$ exists, and $V^\infty(t,x,y,z)\le V(t,x,y,z)$, for all (t,x,y,z). We need only show that $V^\infty\ge V$. To this end, for any $Z\in\mathscr{Z}_t$ we define $Z^n_s:=Z_s\mathbf{1}_{\{s<\tau_{n-1}\}}$, $s\in[t,T]$. Clearly, $Z^n\in\mathscr{Z}^n_t(z)$. Denote $Y^n:=Y^{t,x,y,z,Z^n}$. Then by the subadditivity assumption (2.10) we have

$$Y_T-Y^n_T=\int_{\tau_{n-1}}^T Z_s dX_s-\sum_{i\ge n}c(\delta Z_{\tau_i})+c(-Z_{\tau_{n-1}})\le\int_{\tau_{n-1}}^T Z_s dX_s. \tag{3.6}$$

Now, for any n, using (H2), (3.4), and (3.6) we have

$$\begin{aligned} E\bigl\{U(Y_T)\bigr\}&=E\bigl\{U\bigl(Y^n_T\bigr)\bigr\}+E\bigl\{U(Y_T)-U\bigl(Y^n_T\bigr)\bigr\}\\ &=E\bigl\{U\bigl(Y^n_T\bigr)\bigr\}+E\left\{\left[\int_0^1 U'\bigl(Y^n_T+\theta\bigl(Y_T-Y^n_T\bigr)\bigr)d\theta\right]\bigl[Y_T-Y^n_T\bigr]\right\}\\ &\le V^\infty(t,x,y,z)+\Lambda E\left\{\left|\int_{\tau_{n-1}}^T Z_s dX_s\right|\right\}. \end{aligned}\tag{3.7}$$

Next, Definition 2.2(iii) implies that $\lim_{n\to\infty}\{|\int_{\tau_n}^T Z_s dX^{t,x}_s|\}=0$, P-a.s. This enables us to let $n\to\infty$ in (3.7) and apply the Dominated Convergence Theorem to get $E\{U(Y_T)\}\le V^\infty(t,x,y,z)$. Since this is true for any $Z\in\mathscr{Z}_t$, we conclude that $V(t,x,y,z)\le V^\infty(t,x,y,z)$, proving the proposition. □

The next result concerns the *uniform regularity* of $\{V^n:n\ge1\}$.

Proposition 3.2 *Assume* (H1)–(H3). *Then, for any* n, *it holds that*

$$\big|V^n(t,x_1,y,z)-V^n(t,x_2,y,z)\big|\le C|\Delta x|; \tag{3.8}$$

$$\lambda\Delta y\le V^n(t,x,y_1,z)-V^n(t,x,y_2,z)\le \Lambda\Delta y,\quad \forall y_1\ge y_2; \tag{3.9}$$

$$\big|V^n(t_1,x,y,z)-V^n(t_2,x,y,z)\big|\le C|\Delta t|^{\frac{1}{2}}. \tag{3.10}$$

Here and in the sequel, $\Delta\xi:=\xi_1-\xi_2$, $\xi=t,x,y,z$, *respectively.*

Moreover, for $z_1>z_2>0$ *or* $z_1<z_2<0$, *we have*

$$\begin{aligned}-C\big[|\Delta z|+\rho\big(|\Delta z|\big)\big]&\le V^n(t,x,y,z_1)-V^n(t,x,y,z_2)\\ &\le C\big[|\Delta z|+\rho_n\big(|\Delta z|\big)\big],\end{aligned} \tag{3.11}$$

where ρ *is the modulus of continuity of* c *in* (H3) (iii), *and*

$$\begin{aligned}\rho_n(|\Delta z|)&:=\sup\left\{\sum_{i=1}^n\rho\big(\theta_i|\Delta z|\big):\theta_1,\dots,\theta_n\ge 0,\sum_{i=1}^n\theta_i=1\right\}\\ &\le n\rho\big(|\Delta z|\big).\end{aligned} \tag{3.12}$$

In this below, we present the proof of (3.8), (3.9), and (3.10) only. The proof of (3.11) is more involved and thus is relegated to Sect. 7.

Proof First let us denote $X^i:=X^{t,x_i}$, $i=1,2$, and $\Delta X:=X^1-X^2$. Then by the standard arguments in SDEs we know that

$$E\Big\{\sup_{s\in[t,T]}|\Delta X_s|^2\Big\}\le C|\Delta x|^2. \tag{3.13}$$

Next, for any $Z\in\mathcal{Z}_t^n(z)$, denote $Y^i:=Y^{t,x_i,y,z,Z}$, $i=1,2$, and $\Delta Y:=Y^1-Y^2$. Then

$$\begin{aligned}|\Delta Y_T|\le&\int_t^T|Z_s|\big|b\big(s,X_s^{t,x_1}\big)-b\big(s,X_s^{t,x_2}\big)\big|ds\\ &+\left|\int_t^T Z_s\big[\sigma\big(s,X_s^{t,x_1}\big)-\sigma\big(s,X_s^{t,x_2}\big)\big]dW_s\right|.\end{aligned}$$

Since b and σ are Lipschitz continuous and Z is bounded, (3.13) leads to that

$$\big|E\big\{U\big(Y_T^1\big)-U\big(Y_T^2\big)\big\}\big|^2\le CE\big\{|\Delta Y_T|^2\big\}\le CE\left\{\int_t^T|Z_s\Delta X_s|^2ds\right\}\le C|\Delta x|^2.$$

Since Z is arbitrary, (3.8) follows easily.

To prove (3.9) we denote, for any $Z\in\mathcal{Z}_t^n(z)$ and $y_1>y_2$, $Y^i:=Y^{t,x,y_i,z,Z}$, $i=1,2$, and $\Delta Y:=Y^1-Y^2$. Note that $\Delta Y_T=\Delta y$, we have

$$E\big\{U\big(Y_T^1\big)-U\big(Y_T^2\big)\big\}=E\left\{\left[\int_0^1U'\big(Y_T^1+\theta\Delta y\big)d\theta\right]\Delta y\right\}.$$

Thus (3.9) follows from (H2) immediately.

We next prove (3.10). Assume $t_1 < t_2$. It is then standard to show that

$$E\big\{\big|X_t^{t_1,x} - X_t^{t_2,x}\big|^2\big\} \le C|\Delta t|, \quad t \ge t_2 > t_1. \tag{3.14}$$

Now for any $Z \in \mathcal{Z}_{t_2}^n(z)$, define $\tilde{Z}_t := z1_{[t_1,t_2)}(t) + Z_t 1_{[t_2,T]}$. Then $\tilde{Z} \in \mathcal{Z}_{t_1}^n(z)$. Denote $X^i := X^{t_i,x}$, $i = 1, 2$, and $Y^2 := Y^{t_2,x,y,z,Z}$, $\tilde{Y}^1 = Y^{t_1,x,y,z,\tilde{Z}}$, then

$$\begin{aligned}
Y_T^2 - \tilde{Y}_T^1 &= \int_{t_2}^T Z_t dX_t^2 - \int_{t_1}^T \tilde{Z}_t dX_t^1 \\
&= -z\big[X_{t_2}^1 - x\big] + \int_{t_2}^T Z_t\big[b\big(t, X_t^2\big) - b\big(t, X_t^1\big)\big]dt \\
&\quad + \int_{t_2}^T Z_t\big[\sigma\big(t, X_t^2\big) - \sigma\big(t, X_t^1\big)\big]dW_t.
\end{aligned}$$

Now by standard arguments one can easily derive from (3.14) that

$$EU\big(Y_T^2\big) - V^n(t_1, x, y, z) \le E\big\{U\big(Y_T^2\big) - U\big(\tilde{Y}_T^1\big)\big\} \le CE\big\{\big|Y_T^2 - \tilde{Y}_T^1\big|\big\} \le C|\Delta t|^{\frac{1}{2}}.$$

Since $Z \in \mathcal{Z}_{t_2}^n(z)$ is arbitrary, we get

$$V^n(t_2, x, y, z) - V^n(t_1, x, y, z) \le C|\Delta t|^{\frac{1}{2}}. \tag{3.15}$$

On the other hand, for any $Z = \sum_{i=1}^n Z_{\tau_{i-1}} \mathbf{1}_{[\tau_{i-1},\tau_i)} \in \mathcal{Z}_{t_1}^n(z)$, it is obvious that $Z \in \mathcal{Z}_{t_2}^n(z)$. Denote $Y_i := Y^{t_i,x,y,z,Z}$ and assume $\tau_j \le t_2 < \tau_{j+1}$. Note that $Z_{t_2} = Z_{\tau_j}$. Then, by the subadditivity assumption (2.10),

$$\begin{aligned}
Y_T^1 - Y_T^2 &= \int_{t_1}^T Z_t dX_t^1 - \int_{t_2}^T Z_t dX_t^2 - \sum_{i=0}^j c(\delta Z_{\tau_i}) + c(Z_{\tau_j} - z) \\
&\le \int_{t_1}^{t_2} Z_t dX_t^1 + \int_{t_2}^T Z_t dX_t^1 - \int_{t_2}^T Z_t dX_t^2.
\end{aligned}$$

Since b, σ and Z are bounded, one can easily check that

$$E\left[\left|\int_{t_1}^{t_2} Z_t dX_t^1\right|\right] = E\left[\left|\int_{t_1}^{t_2} Z_t\big[b\big(t, X_t^1\big)dt + \sigma\big(t, X_t^1\big)dW_t\big]\right|\right] \le C|\Delta t|^{\frac{1}{2}}.$$

Moreover, note that $X_t^1 = X_t^{t_2, X_{t_2}^1}$ for $t \ge t_2$. Following the arguments for (3.8) we have

$$\begin{aligned}
&E\left[\left|\int_{t_2}^T Z_t dX_t^1 - \int_{t_2}^T Z_t dX_t^2\right|\right] \\
&\quad = E\left[\left|\int_{t_2}^T Z_t\big[\big[b\big(t, X_t^1\big) - b\big(t, X_t^2\big)\big]dt + \big[\sigma\big(t, X_t^1\big) - \sigma\big(t, X_t^2\big)\big]dW_t\big]\right|\right]
\end{aligned}$$

$$\leq CE\big[\big|X_{t_2}^{t_1,x} - x\big|\big] = CE\bigg[\bigg|\int_{t_1}^{t_2}\big[b\big(t, X_t^1\big)dt + \sigma\big(t, X_t^1\big)dW_t\big]\bigg|\bigg]$$
$$\leq C|\Delta t|^{\frac{1}{2}}.$$

Then, by the assumption (H2) on the payoff function U,

$$\begin{aligned}
&E\big\{U\big(Y_T^1\big)\big\} - V^n(t_2, x, y, z)\\
&\quad\leq E\big\{U\big(Y_T^1\big) - U\big(Y_T^2\big)\big\}\\
&\quad\leq CE\bigg\{\bigg|\int_{t_1}^{t_2} Z_t dX_t^1\bigg| + \bigg|\int_{t_2}^{T} Z_t dX_t^1 - \int_{t_2}^{T} Z_t dX_t^2\bigg|\bigg\} \leq C|\Delta t|^{\frac{1}{2}}.
\end{aligned}$$

Since $Z \in \mathcal{Z}_{t_1}^n(z)$ is arbitrary, we get $V^n(t_1, x, y, z) - V^n(t_2, x, y, z) \leq C|\Delta t|^{\frac{1}{2}}$, which, together with (3.15), implies (3.10). □

We will also need the following result in next section.

Proposition 3.3 *Assume* (H1)–(H3). *Then for any n and any* (t, x, y),

$$V^n(t, x, y, 0+) \leq V^n(t, x, y, 0); \quad V^n(t, x, y, 0-) \leq V^n(t, x, y, 0).$$

Proof First by (3.11) we know $V^n(t, x, y, 0+)$ and $V^n(t, x, y, 0-)$ exist.

For $z > 0$ and $Z^1 = \sum_{i=1}^n Z_{\tau_{i-1}}^1 \mathbf{1}_{[\tau_{i-1},\tau_i)} \in \mathcal{Z}_t^n(z)$, we define $Z^2 \in \mathcal{Z}_t^n(0)$ as follows. Let $k := \inf\{i : Z_{\tau_i}^1 \leq 0\}$. We note that $k \leq n$ since $Z_{\tau_n} = 0$. Define $Z_s^2 := [Z_s^1 - z] \vee 0$ for $s < \tau_k$ and $Z_s^2 := Z_s^1$ for $s \geq \tau_k$. Denote $\Delta Z := Z^1 - Z^2$. It is straightforward to check that

$$0 \leq \Delta Z_{\tau_i} \leq z \quad \text{and} \quad \delta Z_{\tau_i}^1 \delta Z_{\tau_i}^2 \geq 0, \quad i = 0, \dots, n.$$

Note that

$$Y_T^{t,x,y,z,Z^1} - Y_T^{t,x,y,0,Z^2} = \int_t^{\tau_k} \Delta Z_s dX_s^{t,x} + \sum_{i=0}^k \big[c\big(\delta Z_{\tau_i}^2\big) - c\big(\delta Z_{\tau_i}^1\big)\big].$$

Fix $i \leq k$. If $\delta Z_{\tau_i}^1 \delta Z_{\tau_i}^2 > 0$, then by Assumption (H3) (iii) we get

$$c\big(\delta Z_{\tau_i}^2\big) - c\big(\delta Z_{\tau_i}^1\big) \leq \rho\big(\big|\delta Z_{\tau_i}^2 - \delta Z_{\tau_i}^1\big|\big) = \rho\big(|\Delta Z_{\tau_{i-1}} - \Delta Z_{\tau_i}|\big) \leq \rho(z).$$

Now assume $\delta Z_{\tau_i}^1 \delta Z_{\tau_i}^2 = 0$. If $\delta Z_{\tau_i}^1 = 0$, by Definition 2.2 and (2.15) we must have $i = 0$ and $Z_{\tau_0}^1 = z$. This implies that $Z_{\tau_0}^2 = 0$ and thus $\delta Z_{\tau_0}^2 = 0$. If $\delta Z_{\tau_i}^1 \neq 0$, then again we have $\delta Z_{\tau_i}^2 = 0$, and thus

$$c\big(\delta Z_{\tau_i}^2\big) - c\big(\delta Z_{\tau_i}^1\big) = -c\big(\delta Z_{\tau_i}^1\big) \leq 0 \leq \rho(z).$$

Therefore, for some appropriately defined $\mathcal{F}_T$-measurable random variable ξ, we have

$$
\begin{aligned}
&E\big\{U\big(Y_T^{t,x,y,z,Z^1}\big)\big\} - V^n(t,x,y,0)\\
&\quad\le E\big\{U\big(Y_T^{t,x,y,z,Z^1}\big) - U\big(Y_T^{t,x,y,0,Z^2}\big)\big\}\\
&\quad= E\big\{U'(\xi)\big[Y_T^{t,x,y,z,Z^1} - Y_T^{t,x,y,0,Z^2}\big]\big\} \le E\bigg\{U'(\xi)\bigg[\int_t^{\tau_k} \Delta Z_s dX_s^{t,x} + k\rho(z)\bigg]\bigg\}\\
&\quad\le \Lambda E\bigg\{\bigg|\int_t^{\tau_k} \Delta Z_s dX_s^{t,x}\bigg| + n\rho(z)\bigg\} \le C\big[z + n\rho(z)\big].
\end{aligned}
$$

This implies that

$$V^n(t,x,y,z) - V^n(t,x,y,0) \le C\big[z + n\rho(z)\big].$$

Sending $z \downarrow 0$ we obtain $V^n(t,x,y,0+) \le V^n(t,x,y,0)$.

Similarly, we can prove $V^n(t,x,y,0-) \le V^n(t,x,y,0)$. The proof is now complete. □

4 The Approximating Optimal Strategies

In this section we construct the optimal strategy $Z^n \in \mathcal{Z}_t^n(z)$ for the approximating problem (3.1). We will provide the uniform estimate on Z^n's in next section.

We start with some auxiliary results. For any function $\varphi(t,x,y,z)$, define

$$
\begin{aligned}
\bar\varphi(t,x,y,z) &:= \sup_{\tilde z\in[-M,M]} \varphi\big(t,x,y-c(\tilde z - z),\tilde z\big);\\
\hat\varphi(t,x,y,z) &:= \sup_{\tau\ge t} E\big[\bar\varphi\big(\tau, X_\tau^{t,x}, y + z\big[X_\tau^{t,x} - x\big], z\big)\big],
\end{aligned}
\tag{4.1}
$$

where the supremum is taken over all stopping times $\tau \ge t$. It is clear that

$$\bar\varphi \le \hat\varphi \quad \text{and} \quad \hat\varphi(T,x,y,z) = \bar\varphi(T,x,y,z)$$

The following lemma is important for our construction of Z^n.

Lemma 4.1 *Assume* (H1)–(H3). *Suppose that a function* $\varphi : [0,T]\times\mathbb{R}^3 \mapsto \mathbb{R}$ *enjoys the following properties*:

(a) $|\varphi(t,x,y,z)| \le C[1+|y|]$;
(b) φ *is increasing in* y; *uniformly continuous in* (t,x,y); *and uniformly continuous in* z *in* $[-M,0)$ *and in* $(0,M]$;
(c) $\varphi(t,x,y,0+) \le \varphi(t,x,y,0)$, $\varphi(t,x,y,0-) \le \varphi(t,x,y,0)$.

Then

(i) $|\bar{\varphi}(t,x,y,z)| \le C[1+|y|]$ *and* $\bar{\varphi}$ *is also uniformly continuous in* (t,x,y). *Moreover, there exists a Borel measurable function* $\psi(t,x,y,z)$ *such that* $|\psi| \le M$ *and*

$$\bar{\varphi}(t,x,y,z) = \varphi\big(t,x,y-c\big(\psi(t,x,y,z)-z\big),\psi(t,x,y,z)\big). \tag{4.2}$$

(ii) *The optimal stopping problem* $\hat{\varphi}$ *admits an optimal stopping time* τ^*:

$$\begin{aligned}\tau^* := \inf\big\{s \ge t : \hat{\varphi}\big(s, X_s^{t,x}, y + z\big[X_s^{t,x}-x\big], z\big) \\ = \bar{\varphi}\big(s, X_s^{t,x}, y + z\big[X_s^{t,x}-x\big], z\big)\big\}.\end{aligned}$$

Proof First, assume (i) holds true, then (ii) is a standard result in optimal stopping theory, see e.g. [12, Appendix D]. To prove (i), note that

$$\begin{aligned}\big|\bar{\varphi}(t,x,y,z)\big| &\le C \sup_{\tilde{z}\in[-M,M]}\big[1+\big|y-c(\tilde{z}-z)\big|\big] \le C\Big[1+\sup_{\tilde{z}\in[-2M,2M]}\big|c(\tilde{z})\big|+|y|\Big] \\ &\le C\big[1+|y|\big].\end{aligned}$$

Moreover, by (H3) and the regularity of φ we see that $\varphi(t,x,y-c(\tilde{z}-z),\tilde{z})$ is uniformly continuous in (t,x,y), uniformly in $(z,\tilde{z})$. Thus $\bar{\varphi}$ is uniformly continuous in (t,x,y).

It remains to construct the function ψ. We shall apply the measurable selection theorem in Wagner [17]. For notational convenience, we define $\theta := (t,x,y,z) \in [0,\infty)^2 \times \mathbb{R} \times [-M,M]$, $g(\theta,\tilde{z}) := \varphi(t,x,y-c(\tilde{z}-z),\tilde{z})$, and $\bar{g}(\theta,\Gamma) := \sup_{\tilde{z}\in\Gamma} g(\theta,\tilde{z})$ for any Borel set $\Gamma \subset [-M,M]$ (by convention $\bar{g}(\theta,\emptyset) := -\infty$). Consider a set-valued function defined by

$$F(\theta) = \Big\{z' \in [-M,M] : g\big(\theta,z'\big) = \sup_{\tilde{z}\in[-M,M]} g(\theta,\tilde{z})\Big\}.$$

By our conditions, one may easily check that g is upper semicontinuous in $\tilde{z}$. Then $F(\theta)$ is a nonempty and closed set for any θ in the domain $[0,\infty)^2 \times \mathbb{R} \times [-M,M]$. In light of [17, Theorem 4.1], to obtain the measurable ψ it suffices to prove:

$$\text{for any open set } \Gamma \subset [-M,M],\ \big\{\theta : F(\theta)\cap\Gamma \ne \emptyset\big\} \subset R^4 \text{ is a Borel set.} \tag{4.3}$$

To see this, we first assume $c(\cdot)$ is continuous. Since $\varphi(t,x,y,z)$ is continuous in $[0,\infty)^2 \times \mathbb{R} \times [-M,0)$, $g(\theta,\tilde{z})$ is also continuous in $[0,\infty)^2 \times \mathbb{R} \times [-M,0) \times [-M,0)$. Therefore, if $\Gamma \subset [-M,0)$, then we can write, denoting the set of all rational numbers by $\mathbb{Q}$, that

$$\bar{\varphi}(\theta) := \bar{g}(\theta,\Gamma) = \sup_{\tilde{z}\in\Gamma} g(\theta,\tilde{z}) = \sup_{\tilde{z}\in\Gamma\cap\mathbb{Q}} g(\theta,\tilde{z}).$$

Thus, $\bar{g}(\cdot,\Gamma)$ is a Borel measurable function (in fact, it is a Baire function of Class 1) for $\Gamma \subset [-M,0)$. Similar argument shows that $\bar{g}(\cdot,\Gamma)$ is also Borel measurable if $\Gamma \subset (0,M]$. On the other hand, if $\Gamma = \{0\}$, then $\bar{g}(\theta,\Gamma) = g(\theta,0) = \varphi(t,x,y-c(-z),0)$ is obviously continuous. In general, if $\Gamma \subset [-M,M]$ is an open set, we can partition this set into $\Gamma = \cup_{i=1,2,3}\Gamma_i$, where $\Gamma_1 = \Gamma\cap[-M,0)$, $\Gamma_2 = \Gamma\cap(0,M]$, and $\Gamma_3 = \Gamma\cap\{0\}$. Then, we can see $\bar{g}(\cdot,\Gamma)$ is Borel measurable, since

$\bar{g}(\theta, \Gamma) = \max_{i=1,2,3} \bar{g}(\theta, \Gamma_i)$. Therefore, noting that $\bar{g}(\theta, \Gamma) \le \bar{g}(\theta, [-M, M])$ as $\Gamma \subset [M, M]$, we can conclude that the set

$$\{\theta : F(\theta) \cap \Gamma \neq \emptyset\} = \{\theta : \bar{g}(\theta, \Gamma) = \bar{g}(\theta, [-M, M])\},$$

whence a Borel set, and thus (4.3) holds when c is continuous at 0. In the general case, since c is lower semicontinuous at 0, one can prove (4.3) by repeating the above arguments but with the utilization of $\bar{g}(\theta, \Gamma) = \max\{\sup_{\tilde{z} \in \Gamma \cap \mathbb{Q}} g(\theta, \tilde{z}), g(\theta, z)\}$. □

We now give the main existence result of Z^n for this section.

Theorem 4.2 *Assume* (H1)–(H3). *Then, for each n and any fixed* (t, x, y, z),

$$V^n(t, x, y, z) = \hat{V}^{n-1}(t, x, y, z). \tag{4.4}$$

Moreover, there exists an optimal $Z^n \in \mathscr{Z}_t^n(z)$ *such that* $V^n(t, x, y, z) = E[U(Y_T^{t,x,y,z,Z^n})]$.

Proof We proceed in several steps.

Step 1. We first show that

$$V^n(t, x, y, z) \le \hat{V}^{n-1}(t, x, y, z). \tag{4.5}$$

Indeed, let $Z \in \mathscr{Z}_t^n(z)$. If $Z_{\tau_0} \neq z$, then $Z \in \mathscr{Z}_t^{n-1}(Z_{\tau_0})$, and

$$\begin{aligned} E\big[U\big(Y_T^{t,x,y,z,Z}\big)\big] &= E\big[U\big(Y_T^{t,x,y-c(Z_{\tau_0}-z),Z_{\tau_0},Z}\big)\big] \\ &\le V^{n-1}\big(t, x, y - c(Z_{\tau_0} - z), Z_{\tau_0}\big) \le \bar{V}^{n-1}(t, x, y, z) \\ &\le \hat{V}^{n-1}(t, x, y, z). \end{aligned}$$

If $Z_{\tau_0} = z$, then we denote

$$X_s := X_s^{t,x} \quad \text{and} \quad Y_s^0 := y + z[X_s - x], \quad s \in [t, T]. \tag{4.6}$$

Clearly we have $Z \in \mathscr{Z}_{\tau_1}^{n-1}(Z_{\tau_1})$ and

$$\begin{aligned} Y_T^{t,x,y,z,Z} &= y + z\big(X_{\tau_1}^{t,x} - x\big) - c(Z_{\tau_1} - z) + \int_{\tau_1}^T Z_s dX_s^{t,x} - \sum_{i=2}^\infty c(\delta Z_{\tau_i}) \\ &= Y_T^{\tau_1, X_{\tau_1}, Y_{\tau_1}^0 - c(Z_{\tau_1} - z), Z_{\tau_1}, Z}. \end{aligned}$$

This implies that

$$\begin{aligned} E\big[U\big(Y_T^{t,x,y,Z}\big)\big] &\le E\big[V^{n-1}\big(\tau_1, X_{\tau_1}, Y_{\tau_1}^0 - c(Z_{\tau_1} - z), Z_{\tau_1}\big)\big] \\ &\le E\big[\bar{V}^{n-1}\big(\tau_1, X_{\tau_1}, Y_{\tau_1}^0, z\big)\big] \le \hat{V}^{n-1}(t, x, y, z). \end{aligned}$$

Since Z is arbitrary, we obtain (4.5).

Step 2. We now construct Z^n. By the results in Sect. 3, we see that we may apply Lemma 4.1 on $\varphi := V^{n-1}$. Let ψ and $\tau_1^n := \tau^*$ be given as in Lemma 4.1(ii). Set $Z_s^n := z$, for $s \in [t, \tau_1^n)$, and $Z_{\tau_1^n}^n := \psi(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}^0, z)$. Then by Lemma 4.1 we get

$$\hat{V}^{n-1}(t,x,y,z) = E\big[V^{n-1}\big(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}^0 - c\big(Z_{\tau_1^n}^n - z\big), Z_{\tau_1^n}^n\big)\big]. \tag{4.7}$$

We remark that if $\tau_1^n = t$, then Z^n has a jump at t, and if $\tau_1^n > t$, then $Z_t^n = z$ and does not jump at t. Note that $Y_{\tau_1^n}^0 - c(Z_{\tau_1^n}^n - z) = Y_{\tau_1^n}^{t,x,y,z,Z^n}$. Then, by (4.5) we obtain

$$V^n(t,x,y,z) \le E\big[V^{n-1}\big(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}^{t,x,y,z,Z^n}, Z_{\tau_1^n}^n\big)\big]. \tag{4.8}$$

Repeating the above arguments, we define τ_i^n, $i = 2, \dots, n-1$ and Z^n on $[t, \tau_{n-1}^n]$ such that

$$\begin{aligned} &V^{n-i+1}\big(\tau_{i-1}^n, X_{\tau_{i-1}^n}, Y_{\tau_{i-1}^n}^{t,x,y,z,Z^n}, Z_{\tau_{i-1}^n}^n\big) \\ &\quad \le E_{\tau_{i-1}^n}\big[V^{n-i}\big(\tau_i^n, X_{\tau_i^n}, Y_{\tau_i^n}^{t,x,y,z,Z^n}, Z_{\tau_i^n}^n\big)\big]. \end{aligned} \tag{4.9}$$

Finally, for V^1, there exists $\tau_n^n \ge \tau_{n-1}^n$ such that, by setting $Z_s^n := Z_{\tau_{n-1}^n}^n$ for $s \in [\tau_{n-1}^n, \tau_n^n)$ and $Z_s^n := Z_{\tau_{n-1}^n}^n$ for $s \in [\tau_n^n, T]$,

$$\begin{aligned} V^1\big(\tau_{n-1}^n, X_{\tau_{n-1}^n}, Y_{\tau_{n-1}^n}^{t,x,y,z,Z^n}, Z_{\tau_{n-1}^n}^n\big) &= E_{\tau_{n-1}^n}\big[U\big(Y_{\tau_n^n}^{t,x,y,z,Z^n}\big)\big] \\ &= E_{\tau_{n-1}^n}\big[U\big(Y_T^{t,x,y,z,Z^n}\big)\big]. \end{aligned} \tag{4.10}$$

Now combining (4.8)–(4.10) we obtain

$$V^n(t,x,y,z) \le E\big[U\big(Y_T^{t,x,y,z,Z^n}\big)\big].$$

Since clearly $Z^n \in \mathscr{Z}_t^n(z)$, it is an optimal strategy for the optimization problem V^n.

Step 3. Since $V^n(t,x,y,z) = E[U(Y_T^{t,x,y,z,Z^n})]$. By Step 2 we see that (4.8) (and (4.9)) should hold with equality. This, together with (4.7), implies (4.4). □

5 Regularity of the Value Function

In this section we give some *uniform estimates* of the value function V. We should note that the regularity of V with respect to the variables (t,x,y) are clear, since the estimates (3.8), (3.9), and (3.10) in Proposition 3.2 are already uniform with respect to n. The estimate (3.11), however, depends heavily on n. In fact, in the case $|z| = |z|^\alpha$, $0 < \alpha < 1$, one can check that $\rho_n(|z|) = n^{1-\alpha}|z|^\alpha \to \infty$. Therefore the regularity of V with respect to z is by no means clear.

We first take a closer look at the approximating optimal strategies $\{Z^n\}_{n=1}^\infty$. Since our purpose is to construct the optimal piecewise constant control, it is thus conceivable that a uniform bound on $N(Z^n)$ would be extremely helpful.

We begin by considering the case where a fixed cost is present. For each (t,x,y,z), we denote Z^n to be the optimal portfolio for $V^n(t,x,y,z)$, when the context is clear.

Proposition 5.1 *Assume* (H1)–(H3), *and assume further that* $c(z) \geq c_0 > 0$ *for any* $z \neq 0$. *Then there exists a constant* $C > 0$ *such that*

$$E\{N(Z^n)\} \leq \frac{C}{\lambda c_0}, \quad \text{for all } n \text{ and all } (t,x,y,z). \tag{5.1}$$

Proof Denote $Z^0 := z\mathbf{1}_{[t,T)} \in \mathcal{Z}^1(z)$. Then

$$Y_T^{t,x,y,z,Z^0} - Y_T^{t,x,y,Z^n} = \sum_{i=0}^{n} c(\delta Z_{\tau_i}^n) + \int_t^T [z - Z_s^n] dX_s^{t,x} - c(-z).$$

Note that V^n's are non-decreasing in n. Then

$$\begin{aligned}
0 &\geq V^0(t,x,y,z) - V^n(t,x,y,z) \geq E\{U(Y_T^{t,x,y,z,Z^0})\} - E\{U(Y_T^{t,x,y,Z^n})\} \\
&\geq \lambda E\left\{\sum_{i=0}^{n} c(\delta Z_{\tau_i}^n)\right\} - \Lambda E\left\{\left|\int_t^T [z - Z_s^n] dX_s^{t,x}\right| + c(-z)\right\} \\
&\geq \lambda c_0 E\{N(Z^n)\} - C.
\end{aligned}$$

The result follows immediately. □

We next investigate the problem under (H4). We first have the following technical lemma.

Lemma 5.2 *Assume* (H1)–(H4) *hold. Denote*:

$$\alpha_1 := \frac{1-\eta_2}{\eta_2}, \qquad \beta_1 := \frac{1-\eta_2}{1+\gamma};$$

$$C_0 := \frac{\Lambda}{\lambda}\big[\|b\|_\infty T + \|\sigma\|_\infty \sqrt{T} + L_0\big] + 1; \qquad C_1 := C_0\left[2 + \Lambda\left(\frac{1}{\alpha_1} + \frac{1}{\beta_1}\right)\right], \tag{5.2}$$

There exists a constant $\varepsilon_1 \in (0, \varepsilon_0]$ *such that, for any* $0 < |z_1| < \varepsilon_1$,

(i) $c(z_1) \geq C_0|z_1|$.
(ii) *For any* $z_2 \geq z_1 > 0$ *or* $z_2 \leq z_1 < 0$, *we have*

$$\begin{aligned}
c(z_1) + c(z_2) - c(z_1+z_2) \geq {} & \big[\alpha_1\big[c(z_1+z_2) - c(z_2)\big]\big] \\
& \vee \big[\beta_1\big[c(-z_1-z_2) - c(-z_2)\big]\big].
\end{aligned}$$

(iii) *For any* $z_2 \geq \frac{1}{2}z_1 > 0$, *or* $z_2 \leq \frac{1}{2}z_1 < 0$, *or* $|z_2| > |z_1|$, *we have*

$$c(z_1) + c(z_2) - c(z_1+z_2) \geq C_1|z_1|.$$

Proof For $\theta = \frac{3}{2}, 2, 3$, set

$$\tilde{\eta}_\theta := \frac{1}{2}[\eta_\theta + 1], \quad \text{so that } \eta_\theta < \tilde{\eta}_\theta < 1.$$

(i) By (2.11), there exists $0 < \varepsilon \le \varepsilon_0$ such that $c(2z) \le 2\tilde{\eta}_2 c(z)$ for all $|z| \le \varepsilon$. By induction one can easily show that $\frac{c(2^{-n}\varepsilon)}{2^{-n}\varepsilon} \ge \frac{c(\varepsilon)}{\varepsilon \tilde{\eta}_2^n}$. Fix n_0 such that $\frac{c(\varepsilon)}{\varepsilon \tilde{\eta}_2^{n_0}} \ge 2C_0$, and set $\varepsilon_1 := 2^{1-n_0}\varepsilon$. For any $0 < |z| < \varepsilon_1$, there exists $n \ge n_0$ such that $2^{-n}\varepsilon < |z| \le 2^{1-n}\varepsilon$. Then

$$\frac{c(z)}{|z|} \ge \frac{c(2^{-n}\varepsilon)}{2^{1-n}\varepsilon} \ge \frac{1}{2}\frac{c(\varepsilon)}{\varepsilon \tilde{\eta}_2^n} \ge \frac{1}{2}\frac{c(\varepsilon)}{\varepsilon \tilde{\eta}_2^{n_2}} \ge C_0.$$

(ii) Without loss of generality, we assume $z_2 \ge z_1 > 0$. We may rewrite the required inequality as

$$\begin{aligned} &f(z_1, z_2) \le c(z_1) \quad \text{where} \\ f(z_1, z_2) := &\big[c(z_1 + z_2) - c(z_2)\big] + \big[\alpha_1\big[c(z_1 + z_2) - c(z_2)\big]\big] \\ &\vee \big[\beta_1\big[c(-z_1 - z_2) - c(-z_2)\big]\big]. \end{aligned}$$

If $z_2 \in [z_1, \varepsilon_0]$, by the concavity of c, $f(z_1, z_2)$ is decreasing in z_2, then

$$\begin{aligned} f(z_1, z_2) \le f(z_1, z_1) = &\big[c(2z_1) - c(z_1)\big] + \big[\alpha_1\big[c(2z_1) - c(z_1)\big]\big] \\ &\vee \big[\beta_1\big[c(-2z_1) - c(-z_1)\big]\big]. \end{aligned}$$

By choosing ε_1 small enough, we have

$$\begin{aligned} &c(2z_1) - c(z_1) \le [2\tilde{\eta}_2 - 1]c(z_1) = \eta_2 c(z_1) \quad \text{and} \\ &c(-2z_1) - c(-z_1) \le (1 + \gamma)c(z_1). \end{aligned}$$

Then

$$f(z_1, z_1) \le \big[\eta_2 + \big[(\alpha_1\eta_2) \vee \big(\beta_1(1+\gamma)\big)\big]\big]c(z_1) = c(z_1).$$

If $z_2 \in [\varepsilon_0, 2M]$, by (H4)-(ii) we have

$$f(z_1, z_2) \le L_0 z_1 + [\alpha_1 \vee \beta_1]L_0 z_1 = [1 + \alpha_1 \vee \beta_1]L_0 z_1.$$

By replacing C_0 with $[1 + \alpha_1 \vee \beta_1]L_0$ and setting ε_1 smaller if necessary, it follows from (i) that $f(z_1, z_2) \le c(z_1)$.

(iii) Without loss of generality, we assume $z_1 > 0$, and it suffices to show that

$$g(z_1, z_2) := c(z_1 + z_2) - c(z_2) + C_1|z_1| \le c(z_1).$$

If $z_2 \le -z_1$, then $z_2 < z_1 + z_2 \le 0$, and thus $g(z_1, z_2) \le C_1|z_1|$. By setting ε_1 smaller if necessary, the result follows from the proof of (i) by replacing C_0 with C_1.

If $z_2 \geq \varepsilon_0$, then $g(z_1, z_2) \leq [L_0 + C_1]|z_1|$. The result follows from the proof of (i) by replacing C_0 with $L_0 + C_1$. Finally, if $\frac{1}{2}z_1 \leq z_2 \leq \varepsilon_0$, then $g(z_1, z_2)$ is decreasing in z_2, and thus

$$g(z_1, z_2) \leq g\left(z_1, \frac{1}{2}z_1\right) = c\left(\frac{3z_1}{2}\right) - c\left(\frac{z_1}{2}\right) + C_1 z_1.$$

Then, by choosing ε_1 smaller if necessary, we have

$$\begin{aligned} c(z_1) - g(z_1, z_2) &\geq \left[c(z_1) - \frac{2}{3}c\left(\frac{3z_1}{2}\right)\right] + \left[c\left(\frac{z_1}{2}\right) - \frac{1}{3}c\left(\frac{3z_1}{2}\right)\right] - C_1 z_1 \\ &\geq [1 - \tilde{\eta}_{\frac{3}{2}}]c(z_1) + [1 - \tilde{\eta}_3]c\left(\frac{z_1}{2}\right) - C_1 z_1. \end{aligned}$$

Now the result follows from the proof of (i) by replacing C_0 with an appropriate larger constant. □

To extend Proposition 5.1 under (H4), we need an analysis on the number of the small jumps. For this purpose, we fix the constants ε_1, C_0, and C_1 given in Lemma 5.2. Define:

$$A_i^n := \left\{0 < \left|\delta Z_{\tau_i}^n\right| < \varepsilon_1\right\}, \quad B_i^n := \left\{\left|\delta Z_{\tau_i}^n\right| \geq \varepsilon_1\right\}, \quad i = 0, \dots, n;\ n > 0. \tag{5.3}$$

The following result is crucial.

Theorem 5.3 *Assume* (H1)–(H4). *Then for any fixed* m,

$$P\left(\sum_{i=0}^{n} \mathbf{1}_{A_i^n} \geq m\right) \leq \frac{1}{2^m}, \quad \forall n \geq m. \tag{5.4}$$

The proof of Theorem 5.3 depends heavily on the following technical result, whose proof is quite lengthy and will be deferred to Sect. 7 in order not to distract the discussion.

Proposition 5.4 *Assume* (H1)–(H4). *Then, for any* n *and* i, P*-a.s. in* A_i^n *one has*: (i) $P\{B_{i+1}^n | \mathcal{F}_{\tau_i}\} \leq \frac{C_0}{C_1} < \frac{1}{2}$ *for the constants* C_0 *and* C_1 *defined in* (5.2), *and* (ii) $Z_{\tau_i}^n = 0$.

Proof of Theorem 5.3 Define $k_{-1} := -1$, and

$$k_j := \inf\left\{i > k_{j-1} : 0 < \left|\delta Z_{\tau_i}^n\right| < \varepsilon_1\right\} \wedge (n+1), \quad j = 0, 1, \dots, n.$$

Then $P(\sum_{i=0}^n \mathbf{1}_{A_i^n} \geq m) = P(k_m \leq n)$. We claim that, for each $0 \leq j < n$,

$$\{k_{j+1} \leq n\} \subseteq A_{k_j}^n \cap B_{k_j+1}^n, \quad P\text{-a.s.} \tag{5.5}$$

(It is important to note here that the left side contains k_{j+1} while the superscript of B on the right side is $k_j + 1$!)

Indeed, we first note that $\{k_{j+1} \le n\} \subset \{k_j \le n\} \subseteq A^n_{k_j}$, and consider the set $A^n_{k_j} \setminus B^n_{k_j+1}$. Suppose that $Z^n_{\tau_{k_j+1}} \neq Z^n_{\tau_{k_j}}$ on $A^n_{k_j} \setminus B^n_{k_j+1}$. Then $0 < |Z^n_{\tau_{k_j+1}} - Z^n_{\tau_{k_j}}| < \varepsilon_1$, and by Proposition 5.4(ii) we must have both $Z^n_{\tau_{k_j}} = 0$ and $Z^n_{\tau_{k_j+1}} = 0$, P-a.s., a contradiction. Thus we must have $Z^n_{\tau_{k_j+1}} = Z^n_{\tau_{k_j}}$ on $A^n_{k_j} \setminus B^n_{k_j+1}$. Then by the definition of τ_i in (2.15) we know $\tau_{k_j+1} = T$ and thus $Z^n_{\tau_{k_j}} = Z^n_{\tau_{k_j+1}} = \cdots = Z^n_{\tau_n} = 0$. Namely $k_{j+1} = n+1$. In other words, $A^n_{k_j} \setminus B^n_{k_j+1} \subseteq \{k_{j+1} = n+1\}$. Note that $\{k_{j+1} \le n\} \subseteq A^n_{k_j} \setminus \{k_{j+1} = n+1\}$, (5.5) follows.

Next, applying Proposition 5.4(i) we derive from (5.5) that

$$
\begin{aligned}
P\left(\sum_i \mathbf{1}_{A^n_i} \ge m\right) &= P(k_m \le n) \le P\left(\bigcap_{j=0}^{m-1} \left[A^n_{k_j} \cap B^n_{k_j+1}\right]\right) \\
&= E\left\{\prod_{j=0}^{m-1} [\mathbf{1}_{A^n_{k_j}} \mathbf{1}_{B^n_{k_j+1}}]\right\} \\
&= E\left\{\left[\prod_{j=0}^{m-2} [\mathbf{1}_{A^n_{k_j}} \mathbf{1}_{B^n_{k_j+1}}]\right] \mathbf{1}_{A^n_{k_{m-1}}} E\{\mathbf{1}_{B^n_{k_{m-1}+1}} | \mathcal{F}_{\tau_{k_{m-1}}}\}\right\} \\
&\le E\left\{\left[\prod_{j=0}^{m-2} [\mathbf{1}_{A^n_{k_j}} \mathbf{1}_{B^n_{k_j+1}}]\right] \mathbf{1}_{A^n_{k_{m-1}}} \frac{1}{2}\right\} \le \frac{1}{2} E\left\{\prod_{j=0}^{m-2} [\mathbf{1}_{A^n_{k_j}} \mathbf{1}_{B^n_{k_j+1}}]\right\}.
\end{aligned}
$$

Repeating the argument $m-1$ more times we prove the theorem. □

The following theorem is a generalized version of Proposition 5.1.

Theorem 5.5 *Assume Assumptions* (H1)–(H4). *Then it holds that*

$$
E\{N(Z^n)\} \le C\left[1 + \frac{1}{c(\varepsilon_1) \wedge c(-\varepsilon_1)}\right] < \infty, \quad \forall n.
$$

Proof Denote

$$
N_1(Z^n) := \sum_{i=0}^n \mathbf{1}_{A^n_i}, \qquad N_2(Z^n) := \sum_{i=0}^n \mathbf{1}_{B^n_i}.
$$

Then $E\{N(Z^n)\} = E\{N_1(Z^n)\} + E\{N_2(Z^n)\}$. First, Theorem 5.3 implies that

$$
E\{N_1(Z^n)\} = \sum_{m=0}^n P(N_1(Z^n) \ge m) \le \sum_{m=0}^n \frac{1}{2^m} \le 2. \tag{5.6}
$$

Next, one can estimate $E\{N_2(Z^n)\}$ along the lines as Proposition 5.1. Indeed, note that

$$
E\left\{U\left(y + \int_t^T Z^n_s dX_s\right)\right\} - V^n(t,x,y,z)
$$

$$= E\Big\{U\Big(y+\int_t^T Z_s^n dX_s\Big) - U\Big(y+\int_t^T Z_s^n dX_s - \sum_i c\big(\delta Z_{\tau_i}^n\big)\Big)\Big\}$$

$$\geq \lambda E\Big\{\sum_i c\big(\delta Z_{\tau_i}^n\big)\Big\} \geq \lambda E\Big\{\sum_i \big(c(\varepsilon_1)\wedge c(-\varepsilon_1)\big)\mathbf{1}_{B_i^n}\Big\}$$

$$= \lambda\big[c(\varepsilon_1)\wedge c(-\varepsilon_1)\big]E\big\{N_2\big(Z^n\big)\big\}.$$

On the other hand, recalling (3.2) we have

$$E\Big\{U\Big(y+\int_t^T Z_s^n dX_s\Big)\Big\} - V^n(t,x,y,z)$$

$$\leq E\Big\{U\Big(y+\int_t^T Z_s^n dX_s\Big)\Big\} - V^1(t,x,y,z)$$

$$= \sup_{\tau\geq t}\Big|E\Big\{U\Big(y+\int_t^T Z_s^n dX_s\Big) - U\Big(y+\int_t^\tau z dX_s - c(-z)\Big)\Big\}\Big|$$

$$\leq \Lambda E\Big\{\Big|\int_t^\tau \big(Z_s^n - z\big)dX_s\Big| + \Big|\int_\tau^T Z_s^n dX_s\Big| + c(-z)\Big\} \leq C\Lambda.$$

Then $E\{N_2(Z^n)\} \leq \frac{C\Lambda}{\lambda[c(\varepsilon_1)\wedge c(-\varepsilon_1)]}$. This, together with (5.6), proves the theorem. □

As a consequence Theorem 5.5, we have the second main result of this section, which improves (3.11) and whose proof is also postponed to Sect. 7.

Theorem 5.6 *Assume* (H1)–(H3). *Assume further that either* $c(z)\geq c_0>0$, *for all* $z\neq 0$ *or* (H4) *holds. Then there exists a generic constant* $C>0$, *such that for any* z_1, z_2 *with the same sign, and for all* n, *it holds that*

$$\big|V^n(t,x,y,z_1) - V^n(t,x,y,z_2)\big| \leq C\big[|\Delta z| + \rho\big(|\Delta z|\big)\big]; \tag{5.7}$$

$$V(t,x,y,z) - V^n(t,x,y,z) \leq \frac{C}{n}. \tag{5.8}$$

As the direct consequences of Propositions 3.2 and 3.3, and Theorem 5.6 we have

Theorem 5.7 *Assume*(H1)–(H3), *and assume either* $c(z)\geq c_0$, $z\neq 0$ *or* (H4). *Then*

(i) $|V(t,x_1,y,z) - V(t,x_2,y,z)| \leq C|\Delta x|$.
(ii) $\lambda\Delta y \leq V(t,x,y_1,z) - V(t,x,y_2,z) \leq \Lambda\Delta y, \forall\Delta y := y_1 - y_2 \geq 0$.
(iii) $|V(t_1,x,y,z) - V(t_2,x,y,z)| \leq C|\Delta t|^{\frac{1}{2}}$.
(iv) $|V(t,x,y,z_1) - V(t,x,y,z_2)| \leq C[|\Delta z| + \rho(|\Delta z|)], \forall z_1, z_2$ *with the same sign*.
(v) $V(t,x,y,0+) \leq V(t,x,y,0)$, $V(t,x,y,0-) \leq V(t,x,y,0)$.

6 The Optimal Strategy Z^*

In this section we construct the optimal controls for the original problem (2.8). We should note that by virtue of Proposition 5.1 and Theorem 5.5, one can easily show that under our assumptions Z^n should converge in distribution. But this does not seem to be helpful for our construction of the optimal strategy. In fact, in general we will have to extend the probability space, and it is not clear whether the limit process will have the desired adaptedness that is essential in our application. We thus construct the optimal portfolio Z^* for (2.8) directly.

In light of the construction of the optimal strategy Z^n, we know that the function $\bar{V} = V$ should play the role of an "obstacle" that will trigger the jumps, as it is usually the case in impulse control literature. To this end let us consider the following set

$$\begin{aligned} \mathcal{O}(z) &:= \big\{(t,x,y) : V(t,x,y,z) > V\big(t,x,y-c(\tilde{z}-z),\tilde{z}\big), \forall \tilde{z} \neq z\big\}, \\ \mathcal{O} &:= \bigcup_z \mathcal{O}(z). \end{aligned} \tag{6.1}$$

Intuitively, the set $\mathcal{O}(z)$ should define an "inaction region", since a change of position (on z) would decrease the value function. Furthermore, following the standard impulse control theory one would expect that $\mathcal{O}(z)$ is an open set and the trade will take place when $(t,x,y) \in \partial\mathcal{O}(z)$. This is indeed the case when $c(z) \geq c_0 > 0$ for $z \neq 0$. However, unfortunately in our more general case we only have the following result.

Lemma 6.1 *Assume* (H1)–(H4). *Define*

$$\mathcal{O}_n(z) := \left\{(t,x,y) : V(t,x,y,z) > V\big(t,x,y-c(\tilde{z}-z),\tilde{z}\big), \forall |\tilde{z}-z| \geq \frac{1}{n}\right\}. \tag{6.2}$$

Then $\mathcal{O}_n(z)$ is open, for all n, and $\mathcal{O}(z) = \bigcap_n \mathcal{O}_n(z)$.

Proof Denote

$$\bar{V}_n(t,x,y,z) := \sup_{|\tilde{z}-z|\geq \frac{1}{n}} V\big(t,x,y-c(\tilde{z}-z),\tilde{z}\big). \tag{6.3}$$

Apply Theorem 5.7 and follow the proof of Lemma 4.1, we know $\bar{V}_n$ is continuous in (t,x,y) and there exists a Borel measurable function ψ_n such that $|\psi_n(t,x,y,z) - z| \geq \frac{1}{n}$ and

$$V\big(t,x,y-c\big(\psi_n(t,x,y,z)-z\big),\psi_n(t,x,y,z)\big) = \bar{V}_n(t,x,y,z).$$

This implies that

$$\mathcal{O}_n(z) = \big\{(t,x,y) : V(t,x,y,z) > \bar{V}_n(t,x,y,z)\big\}$$

and thus $\mathcal{O}_n(z)$ is open. That $\mathcal{O}(z) = \bigcap_{n=1}^\infty \mathcal{O}_n(z)$ is obvious. The proof is complete. □

We remark that Lemma 6.1 does not imply that the set $\mathcal{O}(z)$ is an open set. Therefore, if we follow the scheme in the previous sections to define, for given $(t,x,y,z)\in\mathcal{O}$ and recalling (4.6),

$$\tau := \inf\big\{s\ge t : \big(s, X_s, Y_s^0\big)\notin \mathcal{O}(z)\big\}\wedge T. \tag{6.4}$$

Then intuitively it is possible that $P\{\tau = t\}>0$ and/or $P\{(\tau, X_\tau, Y_\tau^0)\in\mathcal{O}(z)\}>0$. In either case the construction procedure will fail. The following Theorem, which excludes the above cases, is therefore essential.

Theorem 6.2 *Assume* (H1)–(H4). *Define, for each* $(t,x,y,z)\in\mathcal{O}$ *and* $n>0$,

$$\tau^n := \inf\big\{s\ge t : \big(s, X_s, Y_s^0\big)\notin \mathcal{O}_n(z)\big\}\wedge T, \tag{6.5}$$

and let τ *be defined by* (6.4). *Then*

- (i) τ^n *are decreasing stopping times and* $(\tau^n, X_{\tau^n}, Y^0_{\tau^n})\notin\mathcal{O}_n(z)$ *whenever* $\tau^n<T$.
- (ii) $\tau^n\downarrow\tau$ *and thus* τ *is also a stopping time.*
- (iii) $P(\tau^n>\tau, \forall n)=0$ *and thus,* P*-a.s.,* $(\tau, X_\tau, Y_\tau^0)\notin\mathcal{O}(z)$ *when* $\tau<T$. *In particular, this implies that* $\tau>t$.
- (iv) $V(t,x,y,z)=E\{V(\tau, X_\tau, Y_\tau^0, z)\}$.

The proof of Theorem 6.2 will depend heavily on an important, albeit technical, lemma that characterizes the possible behavior of the small jumps under our basic assumptions on the liquidity/transaction cost function. The proof of this lemma is again rather tedious, and we defer it to Sect. 7.

Lemma 6.3 *Assume* (H1)–(H4) *and let* ε_1 *be that in Lemma* 5.2. *Suppose that for given* (t,x,y,z), $\tilde z$ *is such that* $0<|\tilde z - z|<\varepsilon_1$ *and* $V(t,x,y,z)=V(t,x,y-c(\tilde z - z),\tilde z)$, *then* $\tilde z = 0$.

Proof of Theorem 6.2 (i) That τ^n's are decreasing stopping times is obvious by definition. Also, since each $\mathcal{O}_n(z)$ is an open set, thanks to Lemma 6.1, it follows immediately that $(\tau^n, X_{\tau^n}, Y^0_{\tau^n})\notin\mathcal{O}_n(z)$, whenever $\tau^n<T$.

(ii) Denote $\tau^\infty := \lim_{n\to\infty}\tau^n$. Since $\mathcal{O}_n\supseteq\mathcal{O}$, we have $\tau^n\ge\tau$ for any n and thus $\tau^\infty\ge\tau$, P-a.s. The claim is trivial when $\tau=T$. Now assume $\tau(\omega)<T$. Then for any $\varepsilon>0$, there exists $s<\tau(\omega)+\varepsilon$ such that $(s, X_s, Y_s^0)\notin\mathcal{O}(z)$. Since $\mathcal{O}(z)=\bigcap_n\mathcal{O}_n(z)$, there exists $n:=n(\omega)$ such that $(s, X_s(\omega), Y_s^0(\omega))\notin\mathcal{O}_n(z)$. Thus $\tau^n(\omega)\le s<\tau(\omega)+\varepsilon$ and therefore $\tau^\infty(\omega)<\tau(\omega)+\varepsilon$. Since ε is arbitrary, we get $\tau^\infty\le\tau$, and hence $\tau^\infty=\tau$.

(iii) Choose n_0 such that $n_0>\max\{\frac{1}{\varepsilon_1}, \frac{1}{|z|}\mathbf{1}_{\{z\neq 0\}}\}$, and note that $\{\tau^n>\tau,\forall n\}\subset\{\tau<T\}$. On $\{\tau<T\}$, for $n\ge n_0$ large enough, by (ii) we have $\tau^n<T$ and thus there exists Z_{τ^n} such that $|Z_{\tau^n}-z|\ge\frac{1}{n}$ and $V(\tau^n, X_{\tau^n}, Y^0_{\tau^n}, z)=V(\tau^n, X_{\tau^n}, Y^0_{\tau^n}-c(Z_{\tau^n}-z), Z_{\tau^n})$. By Lemma 6.3, either $Z_{\tau^n}=0$ or $|Z_{\tau^n}-z|\ge\varepsilon_1$. If $z=0$, then $Z_{\tau^n}\neq 0$ and thus $|Z_{\tau^n}-z|\ge\varepsilon_1\ge\frac{1}{n_0}$. If $z\neq 0$, then either $|Z_{\tau^n}-z|=|z|\ge\frac{1}{n_0}$ or $|Z_{\tau^n}-z|\ge\varepsilon_1\ge\frac{1}{n_0}$. So in all the cases we have $|Z_{\tau^n}-z|\ge\frac{1}{n_0}$. This implies that $\tau^n=\tau^{n_0}$ for all n large enough. Therefore, $\tau=\tau^{n_0}$ and thus $(\tau, X_\tau, Y_\tau^0)\notin\mathcal{O}(z)$.

(iv) We first note that, taking τ as the first trading time, we should have

$$E\{V(\tau, X_\tau, Y_\tau^0, z)\} = \sup\{E\{U(Y_T^{t,x,y,z,Z})\} : Z \in \mathcal{Z}_t, Z_s = z \text{ for } \forall s < \tau\}.$$

It then follows that $E\{V(\tau, X_\tau, Y_\tau^0, z)\} \le V(t,x,y,z)$.

On the other hand, note that $\mathbf{F}$ is quasi-left continuous, we can choose a sequence of stopping times $\tau_m \uparrow \tau$ such that $\tau_m < \tau$ whenever $\tau > t$. We claim that

$$V(t,x,y,z) \le E\{V(\tau_m, X_{\tau_m}, Y_{\tau_m}^0, z)\}. \tag{6.6}$$

Then by sending $m \to \infty$ we prove the theorem.

To prove (6.6), we recall (6.3) and choose n_0 as in (iii). On the set $\{\tau > t\}$ and for $t \le s < \tau$, denote

$$I_s := V(s, X_s, Y_s^0, z) - \bar{V}_{n_0}(s, X_s, Y_s^0, z).$$

By the proof of Lemma 6.1 we have $I_s > 0$. Since I is continuous in s, we get

$$I^m := \inf_{s \le \tau_m} I_s > 0. \tag{6.7}$$

For any $n \ge n_0$, let Z^n be the optimal portfolio of $V^n(t,x,y,z)$. If $Z_t^n \ne z$, by Proposition 5.4(ii) and following similar arguments as in (iii), we have $|Z_t^n - z| \ge \frac{1}{n_0}$. Then

$$\begin{aligned} V^n(t,x,y,z) &= V^{n-1}(t,x,y-c(Z_t^n - z), Z_t^n) \le V(t,x,y-c(Z_t^n - z), Z_t^n) \\ &\le \bar{V}_{n_0}(t,x,y,z). \end{aligned}$$

Thus, by (5.8),

$$V(t,x,y,z) \le V^n(t,x,y,z) + \frac{C}{n} \le \bar{V}_{n_0}(t,x,y,z) + \frac{C}{n},$$

and therefore $Z_t^n = z$ for $n \ge n_1 := \frac{C}{V(t,x,y,z) - \bar{V}_{n_0}(t,x,y,z)} \vee n_0$. Now assume $n \ge n_1$, and let $\tau_1^n > t$ be the first jump time of Z^n. Again by Proposition 5.4(ii) and following similar arguments as in (iii), we have $|Z_{\tau_1^n}^n - z| \ge \frac{1}{n_0}$ on $\{\tau_1^n < T\}$. Then, for any m, on $\{\tau_1^n < \tau_m\} \subset \{\tau_1^n < T\}$, using (5.8) we have

$$\begin{aligned} I^m \le I_{\tau_1^n} &= V(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) - \bar{V}_{n_0}(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) \\ &\le V(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) - V(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n} - c(Z_{\tau_1^n}^n - z), Z_{\tau_1^n}^n) \\ &\le V(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) - V^{n-1}(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n} - c(Z_{\tau_1^n}^n - z), Z_{\tau_1^n}^n) \\ &= V(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) - V^n(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n}, z) \le \frac{C}{n}. \end{aligned}$$

This, together with (6.7), implies that

$$\lim_{n\to\infty} P(\tau_1^n < \tau_m) = 0. \tag{6.8}$$

Next, recall from the proof of Theorem 4.2 that τ_1^n is a solution to an optimal stopping problem, and thus (cf. e.g., [11]), $V^n(s, X_s, Y_s, z)$ is a martingale for $t \le \tau_1^n$. Therefore

$$\begin{aligned}
V^n(t,x,y,z) &= E\big\{V^n\big(\tau_1^n \wedge \tau_m, X_{\tau_1^n\wedge\tau_m}, Y_{\tau_1^n\wedge\tau_m}, z\big)\big\} \\
&= E\big\{V^n(\tau_m, X_{\tau_m}, Y_{\tau_m}, z)\mathbf{1}_{\{\tau_m \le \tau_1^n\}} \\
&\quad + V^{n-1}\big(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n} - c\big(Z^n_{\tau_1^n} - z\big), Z^n_{\tau_1^n}\big)\mathbf{1}_{\{\tau_1^n < \tau_m\}}\big\} \\
&\le E\big\{V(\tau_m, X_{\tau_m}, Y_{\tau_m}, z)\mathbf{1}_{\{\tau_m \le \tau_1^n\}} \\
&\quad + V\big(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n} - c\big(Z^n_{\tau_1^n} - z\big), Z^n_{\tau_1^n}\big)\mathbf{1}_{\{\tau_1^n < \tau_m\}}\big\} \\
&= E\big\{V(\tau_m, X_{\tau_m}, Y_{\tau_m}, z)\big\} \\
&\quad + E\big\{\big[V\big(\tau_1^n, X_{\tau_1^n}, Y_{\tau_1^n} - c\big(Z^n_{\tau_1^n} - z\big), Z^n_{\tau_1^n}\big) \\
&\quad - V(\tau_m, X_{\tau_m}, Y_{\tau_m}, z)\big]\mathbf{1}_{\{\tau_1^n < \tau_m\}}\big\}.
\end{aligned}$$

Applying Proposition 3.1 we then have

$$V^n(t,x,y,z) \le E\Big\{V(\tau_m, X_{\tau_m}, Y_{\tau_m}, z) + C\Big[1 + \sup_{t\le s\le T}|Y_s|\Big]\mathbf{1}_{\{\tau_1^n < \tau_m\}}\Big\}.$$

Sending $n \to \infty$ and by (6.8) we obtain (6.6), whence the theorem. □

To construct the optimal strategy, we also need

Lemma 6.4 *Assume* (H1)–(H4). *If* $(t,x,y) \notin \mathcal{O}(z)$, *then there exists* $\tilde z$ *such that*

$$V(t,x,y,z) = V\big(t,x,y - c(\tilde z - z), \tilde z\big) \quad \textit{and} \quad \big(t,x,y - c(\tilde z - z)\big) \in \mathcal{O}(\tilde z).$$

Proof Assume the result is not true. Since $(t,x,y) \notin \mathcal{O}(z)$, there exists $z_1 \ne z$ such that $V(t,x,y,z) = V(t,x,y - c(z_1 - z), z_1)$. By our assumption, $(t,x,y - c(z_1 - z)) \notin \mathcal{O}(z_1)$. Then there exists $z_2 \ne z_1$ such that $V(t,x,y - c(z_1 - z), z_1) = V(t,x,y - c(z_1 - z) - c(z_2 - z_1), z_2)$. Note that $c(z_1 - z) + c(z_2 - z_1) \ge c(z_2 - z)$. By the optimality of V we must have $c(z_1 - z) + c(z_2 - z_1) = c(z_2 - z)$ and

$$V(t,x,y,z) = V\big(t,x,y - c(z_1 - z), z\big) = V\big(t,x,y - c(z_2 - z), z_2\big).$$

This also implies that $z_2 \ne z$. By our assumption again, $(t,x,y - c(z_2 - z)) \notin \mathcal{O}(z_2)$. Repeating this argument yields the different $z_1, z_2, \dots$ such that $c(z_i - z) + c(z_{i+1} - z_i) = c(z_{i+1} - z)$, $i = 1, 2, \dots$, and

$$V(t,x,y,z) = V\big(t,x,y - c(z_1 - z), z_1\big) = \cdots = v\big(t,x,y - c(z_{i+1} - z), z_{i+1}\big).$$

Note that since z_i's are all different, there is at most one z_i equal to 0. Thus, by Lemma 6.3, except for one i, we have $|z_{i+1} - z_i| \ge \varepsilon_1$. This implies that $c(z_i - z) \ge (i-1)[c(\varepsilon_1) \wedge c(-\varepsilon_1)]$ for all i. This contradicts with the fact that $c(z_i - z)$ is bounded. □

We are now ready to construct the optimal strategy Z^*. Let (t, x, y, z) be given and denote $X_s := X_s^{t,x}$.

First, set $\tau_0^* := t$; if $(t, x, y) \in \mathcal{O}(z)$, set $Z^*_{\tau_0^*} := z$ and $Y^*_{\tau_0^*} := y$; if $(t, x, y) \notin \mathcal{O}(z)$, applying Lemma 6.4 we may find Z_0^* such that $V(t, x, y, z) = V(t, x, y - c(Z_0^* - z), Z_0^*)$ and $(t, x, y - c(Z_0^* - z)) \in \mathcal{O}(Z_0^*)$. In this case, set $Y^*_{\tau_0^*} := y - c(Z_0^* - z)$. So in both cases we have $(\tau_0^*, X_{\tau_0^*}, Y^*_{\tau_0^*}) \in \mathcal{O}(Z^*_{\tau_0^*})$.

Assume we have defined τ_i^* and (Y^*, Z^*) on $[t, \tau_i^*]$ such that $(\tau_i^*, X_{\tau_i^*}, Y^*_{\tau_i^*}) \in \mathcal{O}(Z^*_{\tau_i^*})$. Denote $Y_s^i := Y^*_{\tau_i^*} + Z^*_{\tau_i^*}[X_s - X_{\tau_i^*}]$, $s \geq \tau_i^*$, and define

$$\tau_{i+1}^* := \inf\{s \geq \tau_i^* : \big(s, X_s, Y_s^i\big) \notin \mathcal{O}\big(Z_i^*\big)\} \wedge T.$$

By Theorem 6.2, τ_{i+1}^* is a stopping time and $\tau_{i+1}^* > \tau_i^*$ whenever $\tau_i^* < T$. Set $Z_s^* := Z^*_{\tau_i^*}$ and $Y_s^* := Y_s^i$ for $s \in [\tau_i^*, \tau_{i+1}^*)$. If $\tau_{i+1}^* = T$, then we set $Z^*_{\tau_{i+1}^*} := 0$ and $Y^*_{\tau_{i+1}^*} := Y^i_{\tau_{i+1}^*} - c(-Z^*_{\tau_i^*})$. If $\tau_{i+1}^* < T$, by Theorem 6.2 again we know $(\tau_{i+1}^*, X_{\tau_{i+1}^*}, Y^i_{\tau_{i+1}^*}) \notin \mathcal{O}(Z^*_{\tau_i^*})$. Applying Lemma 6.4 we may find $Z^*_{\tau_{i+1}^*}$ such that, by defining $Y^*_{\tau_{i+1}^*} := Y^i_{\tau_{i+1}^*} - c(Z^*_{\tau_{i+1}^*} - Z^*_{\tau_i^*})$,

$$V\big(\tau_{i+1}^*, X_{\tau_{i+1}^*}, Y^i_{\tau_{i+1}^*}, Z^*_{\tau_i^*}\big) = V\big(\tau_{i+1}^*, X_{\tau_{i+1}^*}, Y^*_{\tau_{i+1}^*}, Z^*_{\tau_{i+1}^*}\big), \quad \text{and}$$
$$\big(\tau_{i+1}^*, X_{\tau_{i+1}^*}, Y^*_{\tau_{i+1}^*}\big) \in \mathcal{O}\big(Z^*_{\tau_{i+1}^*}\big).$$

Repeat the procedure we obtain τ_i^* for $i = 0, 1, \ldots$ and (Y^*, Z^*).

We should point out that at this point we do not know if the above construction will stop after finitely many times. We shall prove this and our main result Theorem 2.8 in Sect. 7.

7 Some Technical Proofs

In this section we provide the technical proofs we miss in the previous sections. We note that these results are instrumental in the construction of the piecewise constant optimal strategy, and some of these results are of interest in their own right. As a matter of fact, many of these results can be considered as the necessary conditions of the optimality.

7.1 Proofs of (3.11) and Theorem 5.6

To prove the regularity of the V^n's with respect to z, we first introduce the following notion of "domination" of strategies. Assume $Z^j \in \mathcal{Z}_t^n(z_j)$, $j = 1, 2$, where either $z_1 > z_2 > 0$, or $z_1 < z_2 < 0$. Denote $\Delta Z := Z^1 - Z^2$, as usual. We say that Z^1 dominates Z^2 if Z^1 and Z^2 have the same jump times τ_i's, and

$$\begin{aligned} &\Delta z = \Delta Z_{\tau_{-1}} \geq \Delta Z_{\tau_0} \geq \cdots \geq \Delta Z_{\tau_n} = 0 \quad \text{or} \\ &\Delta z = \Delta Z_{\tau_{-1}} \leq \Delta Z_{\tau_0} \leq \cdots \leq \Delta Z_{\tau_n} = 0, \end{aligned} \tag{7.1}$$

and, by denoting $\mathrm{sgn}(0) := 0$ and $\delta Z^j_{\tau_i} := Z^j_{\tau_i} - Z^j_{\tau_{i-1}}$,

$$\mathrm{sgn}\big(\delta Z^1_{\tau_i}\big) = \mathrm{sgn}\big(\delta Z^2_{\tau_i}\big). \tag{7.2}$$

Remark 7.1 We remark that the requirements (7.1) and (7.2) guarantee not only that Z^1 and Z^2 stay close, but that they are on the same side of the origin. This is mainly due to the fact that the cost function c is allowed to behave differently on the two sides of the origin (i.e., $c(0+) \neq c(0-)$).

Recall (3.12). Note that if $z^1 > z^2 > 0$ and Z^1 dominates Z^2, then, denoting $Y^i := Y^{t,x,y,z_i,Z^i}$, $i = 1, 2$, and $X = X^{t,x}$, by induction one can easily check that

$$\begin{aligned} \big|E\big\{U\big(Y^1_T\big)\big\} - E\big\{U\big(Y^2_T\big)\big\}\big| &\le CE\left\{\left|\int_t^T \Delta Z_s dX_s\right| + \left|\sum_{i=0}^n \big[c\big(\delta Z^1_{\tau_i}\big) - c\big(\delta Z^2_{\tau_i}\big)\big]\right|\right\} \\ &\le C|\Delta z| + CE\left\{\sum_{i=0}^n \rho\big(\big|\delta Z^1_{\tau_i} - \delta Z^2_{\tau_i}\big|\big)\right\} \\ &= C|\Delta z| + CE\left\{\sum_{i=0}^n \rho(\Delta Z_{\tau_{i-1}} - \Delta Z_{\tau_i})\right\} \\ &\le C|\Delta z| + C\rho_n\big(|\Delta z|\big). \end{aligned} \tag{7.3}$$

Proof of (3.11) By the definitions one can easily check that

$$V^n(T, x, y, z) = V(T, z, y, z) = U\big(y - c(-z)\big). \tag{7.4}$$

Then the estimate is obvious for $t = T$. So we may assume $t < T$. Without loss of generality assume $z_1 > z_2 > 0$.

We first prove the right inequality. In light of the estimate (7.3), we need only prove the following claim: For any $Z^1 \in \mathcal{Z}^n_t(z_1)$, there exists $Z^2 \in \mathcal{Z}^n_t(z_2)$ dominated by Z^1. Indeed, for any $\varepsilon > 0$, we can find $Z^{1,\varepsilon} \in \mathcal{Z}^n_t(z_1)$ such that $E\{U(Y_T^{t,x,y,z,Z^{1,\varepsilon}})\} > V^n(t, x, y, z_1) - \varepsilon$. If the claim is true, then (7.3) leads to that

$$V^n(t, x, y, z_1) \le C\big[|\Delta z| + \rho_n(|\Delta z|)\big] + V^n(t, x, y, z_2) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the right inequality.

Now let $Z^1 = \sum_{i=0}^{n-1} Z^1_{\tau_i} \mathbf{1}_{[\tau_i,\tau_{i+1})} \in \mathcal{Z}^n_t(z_1)$ be given. We construct $Z^2 \in \mathcal{Z}^n_t(z_2)$ as follows. We begin by choosing the same jump times τ_i's. Define

$$Z^2_{\tau_0} := \begin{cases} z_2, & \text{if } Z^1_{\tau_0} = z_1; \\ Z^1_{\tau_0}, & \text{if } Z^1_{\tau_0} > z_1 \text{ or } Z^1_{\tau_0} < z_2; \\ z_2 - \frac{1}{2}[(z_1 - Z^1_{\tau_0}) \wedge z_2], & \text{if } z_2 \le Z^1_{\tau_0} < z_1. \end{cases}$$

Suppose that we have defined $Z^2_{\tau_i}$ such that either $Z^2_{\tau_i} = Z^1_{\tau_i}$ or $0 < Z^2_{\tau_i} < Z^1_{\tau_i}$, we then define $Z^2_{\tau_{i+1}}$ in the following way: if $\tau_{i+1} = T$ or $Z^2_{\tau_i} = Z^1_{\tau_i}$, then simply set

$Z^2_{\tau_{i+1}} := Z^1_{\tau_{i+1}}$. Assume $\tau_{i+1} < T$ and $0 < Z^2_{\tau_i} < Z^1_{\tau_i}$. Note that in this case, by (2.15) we have $Z^1_{\tau_{i+1}} \neq Z^1_{\tau_i}$. If $Z^1_{\tau_{i+1}} > Z^1_{\tau_i}$ or $Z^1_{\tau_{i+1}} < Z^2_{\tau_i}$, define $Z^2_{\tau_{i+1}} := Z^1_{\tau_{i+1}}$. Otherwise, we have $Z^1_{\tau_i} > Z^1_{\tau_{i+1}} \geq Z^2_{\tau_i} > 0$, then define $Z^2_{\tau_{i+1}} := Z^2_{\tau_i} - \frac{1}{2}[(Z^1_{\tau_i} - Z^1_{\tau_{i+1}}) \wedge Z^2_{\tau_i}]$. Note that we still have either $Z^2_{\tau_{i+1}} = Z^1_{\tau_{i+1}}$ or $0 < Z^2_{\tau_{i+1}} < Z^1_{\tau_{i+1}}$, so we may continue to define Z^2. One can check directly that Z^2 constructed in such a way satisfies both (7.1) and (7.2), hence Z^1 dominates Z^2.

It remains to prove the left inequality. To this end, let $Z^2 = \sum_{i=0}^{n-1} Z^2_{\tau_i} \mathbf{1}_{[\tau_i,\tau_{i+1})} \in \mathscr{Z}^n_t(z_2)$ be arbitrarily chosen. We define $Z^1 \in \mathscr{Z}^n_t(z_1)$ recursively as follows. First, define

$$Z^1_{\tau_0} := \begin{cases} z_1, & \text{if } Z^2_{\tau_0} = z_2; \\ Z^2_{\tau_0}, & \text{if } Z^2_{\tau_0} > z_1 \text{ or } Z^2_{\tau_0} < z_2; \\ z_1 + [Z^2_{\tau_0} - z_2], & \text{if } z_2 < Z^1_{\tau_0} \leq z_1. \end{cases}$$

Assume we have defined $Z^1_{\tau_i}$ such that either $Z^1_{\tau_i} = Z^2_{\tau_i}$ or $0 < Z^2_{\tau_i} < Z^1_{\tau_i}$. If $\tau_{i+1} = T$ or $Z^1_{\tau_i} = Z^2_{\tau_i}$, define $Z^1_{\tau_{i+1}} := Z^2_{\tau_{i+1}}$. Now assume $\tau_{i+1} < T$ and $0 < Z^2_{\tau_i} < Z^1_{\tau_i}$. Note that in this case $Z^2_{\tau_{i+1}} \neq Z^2_{\tau_i}$. If $Z^2_{\tau_{i+1}} < Z^2_{\tau_i}$ or $Z^2_{\tau_{i+1}} > Z^1_{\tau_i}$, define $Z^1_{\tau_{i+1}} := Z^2_{\tau_{i+1}}$. Otherwise, we have $Z^1_{\tau_i} \geq Z^2_{\tau_{i+1}} > Z^2_{\tau_i} > 0$, then define $Z^1_{\tau_{i+1}} := Z^1_{\tau_i} + [Z^2_{\tau_{i+1}} - Z^2_{\tau_i}]$. Note that we still have either $Z^1_{\tau_{i+1}} = Z^2_{\tau_{i+1}}$ or $0 < Z^2_{\tau_{i+1}} < Z^1_{\tau_{i+1}}$, so we may continue to define Z^1. One may check that (7.2) still holds, and for each ω, there exists k such that

$$\Delta Z_{\tau_0} = \cdots = \Delta Z_{\tau_k} = \Delta z \quad \text{and} \quad \Delta Z_{\tau_{k+1}} = \cdots = \Delta Z_{\tau_n} = 0. \tag{7.5}$$

Then, similar to (7.3), we have

$$\begin{aligned} &E\{U(Y^{t,x,y,z_2,Z^2})\} - V^n(t,x,y,z_1) \\ &\leq E\{U(Y^{t,x,y,z_2,Z^2})\} - E\{U(Y^{t,x,y,z_1,Z^1})\} \\ &\leq C|\Delta z| + CE\left\{\sum_{i=0}^{n} \rho(\Delta Z_{\tau_{i-1}} - \Delta Z_{\tau_i})\right\} = C\big[|\Delta z| + \rho(|\Delta z|)\big]. \end{aligned}$$

Since Z^2 is arbitrary, we prove the left inequality in (3.11). □

Proof of Theorem 5.6 Without loss of generality, assume $z_1 > z_2 > 0$. We first recall the left inequality in (3.11). So we need only check the other half of the inequality. To this end, let Z^1 be the optimal strategy of $V^n(t,x,y,z_1)$, and as in the proof of (3.11) we define $Z^2 \in \mathscr{Z}^n_t(z_2)$ that is "dominated" by Z^1. We note that, for $i > N(Z^1)$, $Z^1_{\tau_i} = Z^1_{\tau_{i-1}}$, which implies that $Z^2_{\tau_i} = Z^2_{\tau_{i-1}}$. Then, following (7.3) we have

$$\begin{aligned} &V^n(t,x,y,z_1) - V^n(t,x,y,z_2) \\ &\leq E\{U(Y_T^{t,x,y,z_1,Z^1})\} - E\{U(Y_T^{t,x,y,z_2,Z^2})\} \end{aligned}$$

$$
\begin{aligned}
&\le C|\Delta z| + CE\left\{\sum_{i=0}^{N(Z^1)} \rho\big(|\delta Z^1_{\tau_i} - \delta Z^2_{\tau_i}|\big)\right\}\\
&\le C|\Delta z| + C\rho\big(|\Delta z|\big)E\big\{N\big(Z^1\big)\big\} \le C\big[|\Delta z| + \rho\big(|\Delta z|\big)\big],
\end{aligned}
$$

where the last inequality is due to Theorems 5.1 and 5.5. This proves (5.7).

To prove (5.8),we denote, for any $m > n$, $Z^m = \sum_{i=1}^m Z^m_{\tau_{i-1}} \mathbf{1}_{[\tau_{i-1},\tau_i)}$ be the optimal strategy of $V^m(t,x,y,z)$. Define $Z^{n,m}_s := Z^m_s \mathbf{1}_{\{s<\tau_n\}}$. Then $Z^{n,m} \in \mathscr{Z}^{n+1}_t(z)$, and

$$
\begin{aligned}
&Y_T^{t,x,y,z,Z^m} - Y_T^{t,x,y,z,Z^{n,m}}\\
&\quad = \left[\int_{\tau_n}^T Z^m_s dX_s + c\big(-Z^m_{\tau_{n-1}}\big) - \sum_{i=n}^m c\big(\delta Z^m_{\tau_i}\big)\right]\mathbf{1}_{\{\tau_n<T\}}\\
&\quad \le \left[\int_{\tau_n}^T Z^m_s dX_s + c\big(-Z^m_{\tau_{n-1}}\big)\right]\mathbf{1}_{\{\tau_n<T\}}.
\end{aligned}
$$

Note that $\{\tau_n < T\} = \{N(Z^m) > n\}$, it follows that

$$
\begin{aligned}
&V^m(t,x,y,z) - V^{n+1}(t,x,y,z)\\
&\quad \le E\big\{U\big(Y_T^{t,x,y,z,Z^m}\big) - U\big(Y_T^{t,x,y,z,Z^{n,m}}\big)\big\}\\
&\quad \le CE\left\{\left[E_{\tau_n}\left\{\left|\int_{\tau_n}^T Z^m_s dX_s\right|\right\} + 1\right]\mathbf{1}_{\{\tau_n<T\}}\right\}\\
&\quad \le CP\{\tau_n < T\} = CP\big\{N\big(Z^m\big) > n\big\} \le \frac{C}{n} E\big\{N\big(Z^m\big)\big\} \le \frac{C}{n}.
\end{aligned}
$$

Sending $m \to \infty$ and applying Proposition 3.1, we obtain the result. □

7.2 Proof of Proposition 5.4

We split the proof into several lemmas. To begin with, we fix (t_0, x_0, y_0, z_0) and n, and let Z^n be the optimal strategy of $V^n(t_0, x_0, y_0, z_0)$. Recall (5.3) and for notational simplicity we suppress the superscript "n" and denote them as A_i and B_i. Throughout this subsection we assume that (H1)–(H4) are all in force. Keep in mind that our purpose is to show that on the set of small jumps (the set A_i's) the jump will only happen when it jumps to 0.

Lemma 7.2 *P-a.s. on A_i, either* $0 \vee Z^n_{\tau_i} \le Z^n_{\tau_{i-1}}$ *or* $Z^n_{\tau_{i-1}} \le Z^n_{\tau_i} \wedge 0$.

Proof Suppose that the lemma is not true. Then we may assume without loss of generality that $P(D_{i_0}) > 0$ for some $i_0 \ge 0$, where $D_{i_0} := \{Z^n_{\tau_{i_0}} > Z^n_{i_0-1} \ge 0\} \cap A_{i_0}$. Our goal is to construct some $\tilde{Z}^n \in \mathscr{Z}^n_{t_0}(z_0)$ such that

$$
\begin{aligned}
&E\big\{U\big(Y_T^{\tilde{Z}^n}\big)\big\} - E\big\{U\big(Y_T^{Z^n}\big)\big\} > 0,\\
&\quad \text{where } Y^{\tilde{Z}^n} := Y^{t_0,x_0,y_0,z_0,\tilde{Z}^n},\ Y^{Z^n} := Y^{t_0,x_0,y_0,z_0,Z^n}.
\end{aligned} \tag{7.6}
$$

This leads to $E\{U(Y_T^{\tilde{Z}^n})\} > V(t_0, x_0, y_0, z_0)$, an obvious contradiction.

We now define $\tilde{Z}^n$ as follows. First, let $k := \inf\{i \ge i_0 : Z^n_{\tau_i} \le 0\}$. Since $Z^n_{\tau_n} = 0$, we have $k \le n$. Now, set

$$\tilde{Z}^n_{\tau_i} := \begin{cases} Z^n_{\tau_i}, & i < i_0 \text{ or } i \ge k; \\ Z^n_{\tau_{i_0}-1}\mathbf{1}_{D_{i_0}} + Z^n_{\tau_{i_0}}\mathbf{1}_{D^c_{i_0}}, & i = i_0; \\ \{[Z^n_{\tau_i} - Z^n_{\tau_{i_0}} + Z^n_{\tau_{i_0-1}}] \vee 0\}\mathbf{1}_{D_{i_0}} + Z^n_{\tau_i}\mathbf{1}_{D^c_{i_0}}, & i_0 + 1 \le i < k. \end{cases}$$

Then $\tilde{Z}^n \in \mathcal{Z}^n_{\tau_0}(z_0)$. To prove (7.6), we denote $\Delta Z^n := \tilde{Z}^n - Z^n$. Then,

$$\Delta Y^n_T := Y_T^{\tilde{Z}^n} - Y_T^{Z^n} = \int_{\tau_{i_0}}^T \Delta Z^n_s dX_s + \sum_{i=i_0}^n \big[c\big(\delta Z^n_{\tau_i}\big) - c\big(\delta \tilde{Z}^n_{\tau_i}\big)\big].$$

By definition of $\tilde{Z}^n$ it is clear that $\Delta Y^n_T = 0$ on $D^c_{i_0}$. On D_{i_0}, first note that $|\Delta Z^n_{\tau_i}| \le \delta Z^n_{\tau_{i_0}}$ for all i. Further, for $i > k$, one has $\delta \tilde{Z}^n_{\tau_i} = \delta Z^n_{\tau_i}$; and for $i \le k$, one can check that either $0 \le \delta \tilde{Z}^n_{\tau_i} \le \delta Z^n_{\tau_i}$ or $\delta Z^n_{\tau_i} \le \delta \tilde{Z}^n_{\tau_i} \le 0$. It then follows from the monotonicity assumption in (H3)-(ii) that $c(\delta Z^n_{\tau_i}) \ge c(\delta \tilde{Z}^n_{\tau_i})$. Moreover, note that when $i = i_0$,

$$c\big(\delta Z^n_{\tau_{i_0}}\big) - c\big(\delta \tilde{Z}^n_{\tau_{i_0}}\big) = c\big(\delta Z^n_{\tau_{i_0}}\big) > C_0\big|\delta Z^n_{\tau_{i_0}}\big|,$$

thanks to Lemma 5.2(i). Thus, on D_{i_0},

$$\Delta Y^n_T \ge \int_{\tau_{i_0}}^T \Delta Z^n_s dX_s + c\big(\delta Z^n_{\tau_{i_0}}\big) > \int_{\tau_{i_0}}^T \Delta Z^n_s dX_s + C_0\big|\delta Z^n_{\tau_{i_0}}\big|;$$

and

$$\begin{aligned} E_{\tau_{i_0}}\left\{\left|\int_{\tau_{i_0}}^T \Delta Z^n_s dX_s\right|\right\} &\le E_{\tau_{i_0}}\left\{\int_{\tau_{i_0}}^T \big|\Delta Z^n_s b(s, X_s)\big| ds + \left|\int_{\tau_{i_0}}^T \Delta Z^n_s \sigma(s, X_s) dW_s\right|\right\} \\ &\le E_{\tau_{i_0}}\left\{\int_{\tau_{i_0}}^T \big|\Delta Z^n_s b(s, X_s)\big| ds\right\} \\ &\quad + \Lambda E_{\tau_{i_0}}\left\{\int_{\tau_{i_0}}^T \big|\Delta Z^n_s \sigma(s, X_s)\big|^2 ds\right\}^{\frac{1}{2}} \\ &\le \big[\|b\|_\infty T + \|\sigma\|_\infty \sqrt{T}\big]\big|\delta Z^n_{\tau_{i_0}}\big| = \frac{\lambda}{\Lambda}(C_0 - 1)\big|\delta Z^n_{\tau_{i_0}}\big|. \end{aligned}$$

Therefore, for some appropriately defined $\mathcal{F}_T$-measurable random variable ξ, we have

$$\begin{aligned} E\big\{U\big(Y_T^{\tilde{Z}^n}\big) - U\big(Y_T^{Z^n}\big)\big\} &= E\big\{U'(\xi)\Delta Y^n_T\big\} = E\big\{U'(\xi)\Delta Y^n_T \mathbf{1}_{D_{i_0}}\big\} \\ &\ge E\left\{\left[\lambda C_0\big|\delta Z^n_{\tau_{i_0}}\big| - \Lambda\left|\int_{\tau_{i_0}}^T \Delta Z^n_s dX_s\right|\right]\mathbf{1}_{D_{i_0}}\right\} \end{aligned}$$

$$= E\left\{\left[\lambda C_0\left|\delta Z^n_{\tau_{i_0}}\right| - \Lambda E_{\tau_{i_0}}\left\{\left|\int_{\tau_{i_0}}^T \Delta Z^n_s dX_s\right|\right\}\right]\mathbf{1}_{D_{i_0}}\right\}$$
$$\geq \lambda E\left\{\left|\delta Z^n_{\tau_{i_0}}\right|\mathbf{1}_{D_{i_0}}\right\} > 0.$$

This proves (7.6) and hence the lemma. □

Lemma 7.3 *For any $\tilde{A}_i \subset A_i$, if $P(\tilde{A}_i) > 0$, then $P(\tilde{D}_{i+1}) > 0$, where*

$$\tilde{D}_{i+1} := \left\{-1 \leq \frac{\delta Z^n_{\tau_{i+1}}}{\delta Z^n_{\tau_i}} \leq \frac{1}{2}\right\} \cap \tilde{A}_i. \tag{7.7}$$

Consequently, P-a.s. in A_i, it holds that $|Z^n_{\tau_i}| \leq |\delta Z^n_{\tau_i}|$.

Proof To simplify the presentation we prove the lemma only for $i = 1$. The general case can be proved in a line by line analogy. We will prove by contradiction, and without loss of generality, we assume $Z^n_{\tau_0} \geq 0$. Then by Lemma 7.2, we have $Z^n_{\tau_1} < Z^n_{\tau_0}$ in $\tilde{A}_1 \subset A_1$. Suppose that the result is not true, namely $P(\tilde{D}_2) = 0$. Then, with possibly an exception of a null set, one has

$$\tilde{A}_1 \subseteq \tilde{D}_{21} \cup \tilde{D}_{22} := \left(\left\{\delta Z^n_{\tau_2} > -\delta Z^n_{\tau_1}\right\} \cap A_1\right) \cup \left(\left\{\delta Z^n_{\tau_2} < \frac{1}{2}\delta Z^n_{\tau_1}\right\} \cap A_1\right).$$

Slightly different from the previous lemma, we now define $\tilde{Z}^n_{\tau_0} := Z^n_{\tau_0}$; $\tilde{Z}^n_{\tau_1} := Z^n_{\tau_1}\mathbf{1}_{\tilde{A}_1^c} + z_0\mathbf{1}_{\tilde{A}_1}$; and $\tilde{Z}^n_{\tau_i} := Z^n_{\tau_i}$, for $i \geq 2$. Then $\tilde{Z}^n \in \mathscr{Z}^n_{t_0}(z_0)$, and

$$\Delta Y^n_T = \left[-\delta Z^n_{\tau_1}[X_{\tau_2} - X_{\tau_1}] + c\left(\delta Z^n_{\tau_1}\right) + c\left(\delta Z^n_{\tau_2}\right) - c\left(Z^n_{\tau_2} - Z^n_{\tau_0}\right)\right]\mathbf{1}_{\tilde{A}_1}.$$

Note that, on $\tilde{D}_{21}$, $Z^n_{\tau_2} > Z^n_{\tau_0} > Z^n_{\tau_1}$. Then (H3)-(ii) and Lemma 5.2(i) yield that

$$c\left(\delta Z^n_{\tau_1}\right) + c\left(\delta Z^n_{\tau_2}\right) - c\left(Z^n_{\tau_2} - Z^n_{\tau_0}\right) \geq c\left(\delta Z^n_{\tau_1}\right) \geq C_0\left|\delta Z^n_{\tau_1}\right|.$$

On the set $\tilde{D}_{22}$, however, one has $\delta Z^n_{\tau_2} < \frac{1}{2}\delta Z^n_{\tau_1} < 0$. Then by Lemma 5.2(iii) we have

$$c\left(\delta Z^n_{\tau_1}\right) + c\left(\delta Z^n_{\tau_2}\right) - c\left(Z^n_{\tau_2} - Z^n_{\tau_0}\right) \geq C_1\left|\delta Z^n_{\tau_1}\right| \geq C_0\left|\delta Z^n_{\tau_1}\right|.$$

So, P-a.s. in $\tilde{A}_1$,

$$\Delta Y^n_T \geq -\delta Z^n_{\tau_1}[X_{\tau_2} - X_{\tau_1}] + C_0\left|\delta Z^n_{\tau_1}\right|.$$

Thus, following similar arguments as in Lemma 7.2, we have

$$\begin{aligned} E\left\{U\left(Y^{\tilde{Z}^n}_T\right) - U\left(Y^{Z^n}_T\right)\right\} &\geq E\left\{\left[\lambda C_0\left|\delta Z^n_{\tau_1}\right| - \Lambda\left|\delta Z^n_{\tau_1}\right|\left|X_{\tau_2} - X_{\tau_1}\right|\right]\mathbf{1}_{\tilde{A}_1}\right\} \\ &\geq \lambda E\left\{\left|\delta Z^n_{\tau_1}\right|\mathbf{1}_{\tilde{A}_1}\right\} > 0, \end{aligned} \tag{7.8}$$

a contradiction. Hence $P(\tilde{D}_2) > 0$ must hold.

To prove the last assertion we again assume $i = 1$ and $Z^n_{\tau_0} \geq 0$, and that the result is not true. That is, denoting $\widehat{D}_1 := \{|Z^n_{\tau_1}| > |\delta Z^n_{\tau_1}|\} \cap A_1$, one has $P(\widehat{D}_1) > 0$. Now, denote

$$\widehat{D}_{i+1} := \left\{-1 \leq \frac{\delta Z^n_{\tau_{i+1}}}{\delta Z^n_{\tau_i}} \leq \frac{1}{2}\right\} \cap \widehat{D}_i, \quad i = 1, \dots, n-1.$$

We shall prove by induction that $\widehat{D}_i \subset A_i$ and $Z^n_{\tau_{i-1}} \geq Z^n_{\tau_i} > \frac{1}{2} Z^n_{\tau_{i-1}}$ on $\widehat{D}_i$, for $i = 1, \dots, n$. Indeed, for $i = 1$, by definition $\widehat{D}_1 \subset A_1$. Moreover, Lemma 7.2 tells us that $Z^n_{\tau_1} < Z^n_{\tau_0}$ on $\widehat{D}_1$. If $Z^n_{\tau_1} \leq 0$, then obviously $|Z^n_{\tau_1}| \leq |\delta Z^n_{\tau_1}|$. If $Z^n_{\tau_1} > 0$ in $\widehat{D}_1$, then $Z^n_{\tau_1} > -\delta Z^n_{\tau_1}$ and hence $Z^n_{\tau_1} > \frac{1}{2} Z^n_{\tau_0}$ on $\widehat{D}_1$. Namely the claim holds for $i = 1$.

Assume now that for all $i \leq j$, the claim holds. In particular, this implies that $Z^n_{\tau_j} > \frac{1}{2^j} Z^n_{\tau_0} \geq 0$ on $\widehat{D}_j$, we show that the claim is true for $i = j + 1$. Note that on $\widehat{D}_{j+1}$, one has $|\delta Z^n_{\tau_{j+1}}| \leq |\delta Z^n_{\tau_j}| < \varepsilon_1$. Since $Z^n_{\tau_j} \neq 0$ on $\widehat{D}_{j+1} \subset \widehat{D}_j$, by (2.15) we know $Z^n_{\tau_{j+1}} \neq Z^n_{\tau_j}$. Thus $\widehat{D}_{j+1} \subset A_{j+1}$. Moreover, since $\delta Z^n_{\tau_j} < 0$, we have $\delta Z^n_{\tau_{j+1}} \geq \frac{1}{2} \delta Z^n_{\tau_j}$ on $\widehat{D}_{j+1}$. Thus by inductional hypothesis we have

$$Z^n_{\tau_{j+1}} \geq \frac{3}{2} Z^n_{\tau_j} - \frac{1}{2} Z^n_{\tau_{j-1}} > \frac{1}{2} Z^n_{\tau_j}, \quad \text{on } \widehat{D}_{j+1}.$$

That is, the claim is true for $i = j + 1$, and hence it is true for all i.

Finally, by applying the same argument repeatedly we have $P(\widehat{D}_n) > 0$. But the claim tells us that $Z^n_{\tau_n} > \frac{1}{2^n} Z^n_{\tau_0} \geq 0$ on $\widehat{D}_n$. This is impossible since $Z^n_{\tau_n} = 0$ must hold almost surely by definition of $\mathcal{Z}^n_t(z_0)$. The proof is now complete. □

Proof Proof of Proposition 5.4(i) We follow the arguments in Lemma 7.3. Again for simplicity we assume $i = 1$, $Z^n_{\tau_0} \geq 0$, and that the result is not true. Then $P(D_1) > 0$, where

$$D_1 := \left\{P\{B_2 | \mathcal{F}_{\tau_1}\} > \frac{C_0}{C_1}\right\} \cap A_1.$$

As before, we define $\tilde{Z}^n_{\tau_0} := Z^n_{\tau_0}$; $\tilde{Z}^n_{\tau_1} := Z^n_{\tau_1} \mathbf{1}_{D_1^c} + Z^n_{\tau_0} \mathbf{1}_{D_1}$, and $\tilde{Z}^n_{\tau_i} := Z^n_{\tau_i}$, for $i \geq 2$. Then $\tilde{Z}^n \in \mathcal{Z}^n_{t_0}(z_0)$, and

$$\Delta Y^n_T = \big[-\delta Z^n_{\tau_1}[X_{\tau_2} - X_{\tau_1}] + c\big(\delta Z^n_{\tau_1}\big) + c\big(\delta Z^n_{\tau_2}\big) - c\big(Z^n_{\tau_2} - Z^n_{\tau_0}\big)\big] \mathbf{1}_{D_1}.$$

On $D_1 \cap B_2^c$, we use (2.10) to get $c(\delta Z^n_{\tau_1}) + c(\delta Z^n_{\tau_2}) - c(Z^n_{\tau_2} - Z^n_{\tau_0}) \geq 0$. On $D_1 \cap B_2$, we have $|\delta Z^n_{\tau_2}| \geq \varepsilon_1 \geq |\delta Z^n_{\tau_1}|$. Thus Lemma 5.2(iii) tells us that

$$c\big(\delta Z^n_{\tau_1}\big) + c\big(\delta Z^n_{\tau_2}\big) - c\big(Z^n_{\tau_2} - Z^n_{\tau_0}\big) \geq C_1 \big|\delta Z^n_{\tau_1}\big|.$$

Combining above we conclude that

$$\begin{aligned} &E\big\{U\big(Y_T^{\tilde{Z}^n}\big) - U\big(Y_T^{Z^n}\big)\big\} \\ &\quad \geq E\big\{\lambda C_1 \big|\delta Z^n_{\tau_1}\big| \mathbf{1}_{D_1 \bigcap B_2} - \Lambda \big|\delta Z^n_{\tau_1}\big| E_{\tau_1}\big\{|X_{\tau_2} - X_{\tau_1}|\big\} \mathbf{1}_{D_1}\big\} \end{aligned}$$

$$= E\{\lambda C_1|\delta Z^n_{\tau_1}|E_{\tau_1}\{\mathbf{1}_{B_2}\}\mathbf{1}_{D_1} - \lambda(C_0-1)|\delta Z^n_{\tau_1}|\mathbf{1}_{D_1}\}$$

$$\geq \lambda E\left\{\left[C_1|\delta Z^n_{\tau_1}|\frac{C_0}{C_1} - (C_0-1)|\delta Z^n_{\tau_1}|\right]\mathbf{1}_{D_1}\right\} = \lambda E\{|\delta Z^n_{\tau_1}|\mathbf{1}_{D_1}\} > 0. \quad (7.9)$$

This is a contradiction and thus proves the part (i).

We shall prove part (ii) by backward induction on i. Since $Z^n_{\tau_n} = 0$, the result is true for $i = n$. Without loss of generality we assume it is true for $i = 2$ and will prove it for $i = 1$. Assume $Z^n_{\tau_0} \geq 0$. If it is not true for $i = 1$, then $P(\tilde{D}_1) > 0$ where

$$\tilde{D}_1 := \{Z^n_{\tau_1} \neq 0\} \cap A_1.$$

We now define $\tilde{Z}^n_{\tau_0} := Z^n_{\tau_0}$; $\tilde{Z}^n_{\tau_1} := Z^n_{\tau_1}\mathbf{1}_{\tilde{D}_1^c}$; and $\tilde{Z}^n_{\tau_i} := Z^n_{\tau_i}$, for $i \geq 2$. Then $\tilde{Z}^n \in \mathcal{Z}^n_{t_0}(z_0)$, and

$$\Delta Y^n_T = [-Z^n_{\tau_1}[X_{\tau_2} - X_{\tau_1}] + c(\delta Z^n_{\tau_1}) + c(\delta Z^n_{\tau_2}) - c(-Z^n_{\tau_0}) - c(Z^n_{\tau_2})]1_{\tilde{D}_1}.$$

We claim that:

$$\left.\begin{aligned} c(-Z^n_{\tau_0}) - c(\delta Z^n_{\tau_1}) &\leq \frac{1}{\alpha_1}[c(-Z^n_{\tau_1}) + c(\delta Z^n_{\tau_1}) - c(-Z^n_{\tau_0})]; \\ c(Z^n_{\tau_2}) - c(\delta Z^n_{\tau_2}) &\leq \frac{1}{\beta_1}[c(-Z^n_{\tau_1}) + c(\delta Z^n_{\tau_1}) - c(-Z^n_{\tau_0})] + L_0|Z^n_{\tau_1}|; \end{aligned}\right\} \text{on } \tilde{D}_1 \cap B_2. \quad (7.10)$$

Indeed, without loss of generality, we assume $Z^n_{\tau_0} > 0$. Then, by Lemmas 7.2 and 7.3, we have $0 \leq Z^n_{\tau_1} \leq -\delta Z^n_{\tau_1} < \varepsilon_1$ on $\tilde{D}_1 \subset A_1$. Thus the first inequality of (7.10) follows from Lemma 5.2(ii). To show the second inequality, note that $|\delta Z^n_{\tau_2}| \geq \varepsilon_1$ on $\tilde{D}_1 \cap B_2$. If $\delta Z^n_{\tau_2} \leq -\varepsilon_1$, then $\delta Z^n_{\tau_2} \leq Z^n_{\tau_2} < 0$, and thus $c(Z^n_{\tau_2}) - c(\delta Z^n_{\tau_2}) \leq 0$. If $\varepsilon_1 \leq \delta Z^n_{\tau_2} \leq \varepsilon_0$, note that $c(Z^n_{\tau_2}) - c(\delta Z^n_{\tau_2}) = c(\delta Z^n_{\tau_2} + Z^n_{\tau_1}) - c(\delta Z^n_{\tau_2})$ is decreasing in $\delta Z^n_{\tau_2}$. Then

$$\begin{aligned} c(Z^n_{\tau_2}) - c(\delta Z^n_{\tau_2}) &\leq c(-\delta Z^n_{\tau_1} + Z^n_{\tau_1}) - c(-\delta Z^n_{\tau_1}) \\ &\leq \frac{1}{\beta_1}[c(-Z^n_{\tau_1}) + c(\delta Z^n_{\tau_1}) - c(-Z^n_{\tau_0})], \end{aligned}$$

thanks again to Lemma 5.2(ii). Finally, if $\delta Z^n_{\tau_2} \geq \varepsilon_0$, then $c(Z^n_{\tau_2}) - c(\delta Z^n_{\tau_2}) \leq L_0|Z^n_{\tau_1}|$. This completes the proof of Claim (7.10).

Note that, by inductional hypothesis we have $Z^n_{\tau_2} = 0$ on $\tilde{D}_1 \cap B_2^c$. Then, for some appropriately defined $\mathcal{F}_T$-measurable random variable ξ, by (2.10) we have,

$$\begin{aligned} &E\{U(Y_T^{\tilde{Z}^n}) - U(Y_T^{Z^n})\} \\ &\quad = E\{U'(\xi)[Y_T^{\tilde{Z}^n} - Y_T^{Z^n}]\} \\ &\quad = E\{U'(\xi)[-Z^n_{\tau_1}[X_{\tau_2} - X_{\tau_1}]\mathbf{1}_{\tilde{D}_1} + [c(-Z^n_{\tau_1}) + c(\delta Z^n_{\tau_1}) - c(-Z^n_{\tau_0})]\mathbf{1}_{\tilde{D}_1 B_2^c} \\ &\qquad + [c(\delta Z^n_{\tau_1}) - c(-Z^n_{\tau_0}) + c(\delta Z^n_{\tau_2}) - c(Z^n_{\tau_2})]\mathbf{1}_{\tilde{D}_1 B_2}]\} \end{aligned}$$

$$\geq E\bigg\{U'(\xi)\bigg[-Z^n_{\tau_1}[X_{\tau_2}-X_{\tau_1}]\mathbf{1}_{\tilde{D}_1}-L_0|Z^n_{\tau_1}|\mathbf{1}_{\tilde{D}_1B_2}$$

$$+\big[c\big(-Z^n_{\tau_1}\big)+c\big(\delta Z^n_{\tau_1}\big)-c\big(-Z^n_{\tau_0}\big)\big]\bigg[\mathbf{1}_{\tilde{D}_1B_2^c}-\bigg(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\bigg)\mathbf{1}_{\tilde{D}_1B_2}\bigg]\bigg]\bigg\}$$

$$\geq E\bigg\{-\Lambda|Z^n_{\tau_1}|\big[|X_{\tau_2}-X_{\tau_1}|+L_0\big]\mathbf{1}_{\tilde{D}_1}$$

$$+\big[c\big(-Z^n_{\tau_1}\big)+c\big(\delta Z^n_{\tau_1}\big)-c\big(-Z^n_{\tau_0}\big)\big]\bigg[\lambda\mathbf{1}_{\tilde{D}_1B_2^c}-\Lambda\bigg(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\bigg)\mathbf{1}_{\tilde{D}_1B_2}\bigg]\bigg\}$$

$$=E\bigg\{\Big[-\Lambda|Z^n_{\tau_1}|\big[E_{\tau_1}\{|X_{\tau_2}-X_{\tau_1}|\}+L_0\big]+\big[c\big(-Z^n_{\tau_1}\big)+c\big(\delta Z^n_{\tau_1}\big)-c\big(-Z^n_{\tau_0}\big)\big]$$

$$\times\bigg[\lambda E_{\tau_1}\{\mathbf{1}_{B_2^c}\}-\Lambda\bigg(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\bigg)E_{\tau_1}\{\mathbf{1}_{B_2}\}\bigg]\Big]\mathbf{1}_{\tilde{D}_1}\bigg\}.$$

One can easily check that

$$\Lambda\big[E_{\tau_1}\big\{|X_{\tau_2}-X_{\tau_1}|\big\}+L_0\big]\leq\lambda[C_0-1].$$

Moreover, by part (i) we know that P-a.s. on $\tilde{D}_1\subset A_1$, $P\{B_2|\mathcal{F}_{\tau_1}\}\leq\frac{C_0}{C_1}$ and thus $P\{B_2^c|\mathcal{F}_{\tau_1}\}\geq 1-\frac{C_0}{C_1}$. Then

$$\lambda E_{\tau_1}\{\mathbf{1}_{B_2^c}\}-\frac{\Lambda}{\alpha_1}E_{\tau_1}\{\mathbf{1}_{B_2}\}\geq\lambda\bigg[1-\frac{C_0}{C_1}-\Lambda\bigg(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\bigg)\frac{C_0}{C_1}\bigg]=\frac{\lambda C_0}{C_1}.$$

Note that, on $\tilde{D}_1\subset A_1$, by Lemmas 7.2 and 7.3, we have $0\leq Z^n_{\tau_1}\leq\delta Z^n_{\tau_1}<\varepsilon_1$ or $0>\delta Z^n_{\tau_1}\geq Z^n_{\tau_1}>-\varepsilon_1$. Then it follows from Lemma 5.2(ii) that

$$E\big\{U\big(Y_T^{\tilde{Z}^n}\big)-U\big(Y^{Z^n_T}\big)\big\}$$

$$\geq E\bigg\{\bigg[-\lambda(C_0-1)|Z^n_{\tau_1}|+\frac{\lambda C_0}{C_1}C_1|Z^n_{\tau_1}|\bigg]\mathbf{1}_{\tilde{D}_1}\bigg\}=\lambda E\big\{|Z^n_{\tau_1}|\mathbf{1}_{\tilde{D}_1}\big\}>0, \tag{7.11}$$

a contradiction. □

7.3 Proofs of Lemma 6.3 and Theorem 2.8

Proof of Lemma 6.3 We follow the proof of Proposition 5.4. For each n, let $Z^n\in\mathcal{Z}^n_t(\tilde{z})$ be the optimal portfolio of $V^n(t,x,y-c(\tilde{z}-z),\tilde{z})$. We first prove several claims by contradiction. In each case, we show that if the claim is not true, then we can construct some $\tilde{Z}^n\in\mathcal{Z}^{n+1}_t(z)$ such that, by denoting $Y^{\tilde{Z}^n}:=Y^{t,x,y,z,\tilde{Z}^n}$, $Y^{Z^n}:=Y^{t,x,y-c(\tilde{z}-z),\tilde{z},Z^n}$,

$$E\big\{U\big(Y_T^{\tilde{Z}^n}\big)-U\big(Y_T^{Z^n}\big)\big\}\geq c(z,\tilde{z})>0 \tag{7.12}$$

where $c(z,\tilde{z})$ is some constant independent of n. This implies that

$$V(t,x,y,z)-V_n\big(t,x,y-c(\tilde{z}-z),\tilde{z}\big)\geq c(z,\tilde{z})>0.$$

Sending $n \to \infty$ and applying Proposition 3.1, we obtain the contradiction.

Without loss of generality we assume $z \ge 0$. The key observation is that we may also view $Y_T^{Z^n}$ as the wealth of the portfolio Z^n starting from (t,x,y,z), with two initial jumps first from z to $\tilde z$ and then from $\tilde z$ to $Z^n_{\tau_0}$.

Claim 1 *$\tilde z < z$. Indeed, if $\tilde z > z$, for fixed n, let $k := \inf\{i \ge 0 : Z^n_{\tau_i} \le 0\}$, and define $\tilde Z^n_{\tau_i} := [Z^n_{\tau_i} - \tilde z + z] \vee 0$ for $i < k$, and $\tilde Z^n_{\tau_i} := Z^n_{\tau_i}$ for $i \ge k$. Then $\tilde Z^n \in \mathcal{Z}_t^{n+1}(z)$. Following exactly the same arguments as in the proof of Lemma 7.2, we prove (7.12) with $c(z,\tilde z) = \lambda(\tilde z - z) > 0$ and thus obtain a contradiction.*

Claim 2 *$-1 \le \frac{Z^n_{\tau_0} - \tilde z}{\tilde z - z} \le \frac{1}{2}$, and if $Z^n_{\tau_0} = \tilde z$ then $P(-1 \le \frac{Z^n_{\tau_1} - \tilde z}{\tilde z - z} \le \frac12) > 0$. Indeed, assume the result is not true. Define $\tilde Z^n_{\tau_0} = z$, $\tilde Z^n_{\tau_i} = Z^n_{\tau_i}$, for all $i \ge 1$. Then $\tilde Z^n \in \mathcal{Z}_t^{n+1}(z)$, and similar to (7.8) we prove (7.12) with $c(z,\tilde z) = \lambda(z - \tilde z) > 0$.*

Claim 3 *$|\tilde z| \le z - \tilde z$. Indeed, if $Z^n_{\tau_0} \ne \tilde z$, then by Claim 2 we have $0 < |Z^n_{\tau_0} - \tilde z| \le z - \tilde z < \varepsilon_1$. Applying Proposition 5.4 we get $Z^n_{\tau_0} = 0$ and thus proving the claim. If $Z^n_{\tau_0} = \tilde z$, then Claim 2 leads to $P(-1 \le \frac{Z^n_{\tau_1} - \tilde z}{\tilde z - z} \le \frac12) > 0$. On $\{-1 \le \frac{Z^n_{\tau_1} - \tilde z}{\tilde z - z} \le \frac12\}$, we have $|Z^n_{\tau_1} - \tilde z| \le |\tilde z - z| < \varepsilon_1$. If $Z^n_{\tau_1} = \tilde z$, by (2.15) we get $\tau_1 = T$ and thus $\tilde z = 0$. If $Z^n_{\tau_1} \ne \tilde z$, by Proposition 5.4 again we get $Z^n_{\tau_1} = 0$. Then $-1 \le \frac{-\tilde z}{\tilde z - z} \le \frac12$ and thus the claim holds.*

Claim 4 *If $Z^n_{\tau_0} = \tilde z$, then $P(|Z^n_{\tau_1} - \tilde z| \ge \varepsilon_1) \le \frac{C_0}{C_1}$. Indeed, if $P(|Z^n_{\tau_1} - \tilde z| \ge \varepsilon_1) > \frac{C_0}{C_1}$, then we define $\tilde Z^n_{\tau_0} := z$, and $\tilde Z^n_{\tau_i} := Z^n_{\tau_i}$, for $i \ge 1$. Similar to (7.9) we prove (7.12) with $c(z,\tilde z) = \lambda(z - \tilde z) > 0$.*

We now prove the lemma. Define $\tilde Z^n_{\tau_0} := 0$ and $\tilde Z^n_{\tau_i} := Z^n_{\tau_i}$ for $i \ge 1$. Then

$$\begin{aligned}\Delta Y_T &:= Y_T^{\tilde Z^n} - Y_T^{Z^n}\\ &= c(\tilde z - z) + c\big(Z^n_{\tau_0} - \tilde z\big) + C\big(Z^n_{\tau_1} - Z^n_{\tau_0}\big) - c(-z) - c\big(Z^n_{\tau_1}\big) - Z^n_{\tau_0}[X_{\tau_1} - x].\end{aligned}$$

If $Z^n_{\tau_0} \ne \tilde z$, by the proof of Claim 3, we have $Z^n_{\tau_0} = 0$. Then

$$\Delta Y_T = c(\tilde z - z) + c(-\tilde z) - c(-z) \ge C_1|\tilde z|,$$

thanks to Lemma 5.2(iii) and Claims 1 and 3. If $Z^n_{\tau_0} = \tilde z$, then

$$\Delta Y_T = c(\tilde z - z) + c\big(Z^n_{\tau_1} - \tilde z\big) - c(-z) - c\big(Z^n_{\tau_1}\big) - \tilde z[X_{\tau_1} - x].$$

Similar to (7.11) we can prove

$$V(t,x,y,z) - V^n\big(t,x,y - c(\tilde z - z),\tilde z\big) \ge E\big\{U\big(Y_T^{\tilde Z^n}\big) - U\big(Y_T^{Z^n}\big)\big\} \ge \lambda|\tilde z|.$$

Send $n \to \infty$ and noting that $V(t,x,y,z) = V(t,x,y - c(\tilde z - z),\tilde z)$, we must have $\tilde z = 0$. □

Proof of Theorem 2.8 (i) If $c(z) \geq c_0 > 0$ for all $z \neq 0$, then following the arguments in Theorem 5.1 one can easily prove that $E\{N(Z^*)\} \leq \frac{C}{c_0}$.

Now assume (H4) holds. Following the proof of Theorem 6.3 it is readily seen that, for any $i \geq 1$ and P-a.s. on $\{0 < |Z^*_{\tau^*_i} - Z^*_{\tau^*_{i-1}}| < \varepsilon_1\}$, it holds that

$$Z^*_{\tau^*_i} = 0 \quad \text{and} \quad P\left(\left|Z^*_{\tau^*_{i+1}} - Z^*_{\tau^*_i}\right| \geq \varepsilon_1 | \mathcal{F}_{\tau^*_i}\right) \leq \frac{C_0}{C_1}.$$

Then following the proof of Theorem 5.3 we get

$$P\left(\sum_{i=0}^{n} \mathbf{1}_{\{0<|Z^*_{\tau^*_i} - Z^*_{\tau^*_{i-1}}|<\varepsilon_1\}} \geq m\right) \leq \frac{1}{2^m}, \qquad \forall n \geq m.$$

Similar to Theorem 5.5 one can then prove that $E\{\sum_{i=0}^{\infty} \mathbf{1}_{\{|Z^*_{\tau^*_i} - Z^*_{\tau^*_{i-1}}|>0\}}\} < \infty$. This implies that $P(\tau^*_i < T, \forall i) = 0$ and $E(N(Z^*)) < \infty$.

(ii) Applying Lemma 6.2 repeatedly we have

$$V(t, x, y, z) = E\left\{V\left(\tau^*_n, X_{\tau^*_n}, Y^*_{\tau^*_n}, Z^*_{\tau^*_n}\right)\right\}, \quad \forall n.$$

Now by (i), we conclude that $\tau^*_n = T$ and $Z^*_{\tau^*_n} = 0$ for n large enough. Sending $n \to \infty$ we obtain that $V(t, x, y, z) = E\{U(Y^*_T)\}$. This means that Z^* is an optimal portfolio for $V(t, x, y, z)$. □

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