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On semi-linear degenerate backward stochastic partial differential equations

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Abstract. In this paper we study a class of one-dimensional, degenerate, semilinear backward stochastic partial differential equations (BSPDEs, for short) of parabolic type. By establishing some new a priori estimates for both linear and semilinear BSPDEs, we show that the regularity and *uniform boundedness* of the adapted solution to the semilinear BSPDE can be determined by those of the coefficients, a special feature that one usually does not expect from a stochastic differential equation. The proof follows the idea of the so-called *bootstrap* method, which enables us to analyze each of the derivatives of the solution under consideration. Some related results, including some comparison theorems of the adapted solutions for semilinear BSPDEs, as well as a nonlinear stochastic Feynman-Kac formula, are also given.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which is defined a one-dimensional standard Brownian motion $W = \{W(t) : t \in [0, T]\}$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration generated by W, augmented by all the P-null sets in $\mathcal{F}$. We are interested in the following (one-dimensional) *terminal value problem* for a semilinear stochastic partial differential equation:

$$\begin{cases} du = -\{\mathcal{L}u + \mathcal{M}q + f(t, x, u)\}dt + qdW(t), & (t, x) \in [0, T] \times \mathbb{R}, \\ u\big|_{t=T} = g. \end{cases} \tag{1.1}$$

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Here and throughout the paper, we denote

$$\begin{cases} \mathcal{L}u \overset{\Delta}{=} \frac{1}{2}a(t,x)^2 u_{xx} + b(t,x)u_x + c(t,x)u \\ \mathcal{M}q \overset{\Delta}{=} \alpha(t,x)q_x + \gamma(t,x)q, \end{cases} \tag{1.2}$$

where $a, b, c, \alpha, \gamma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$, and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are real-valued random fields satisfying appropriate measurability and regularity conditions. As usual, equation (1.1) is understood as the following integral form:

$$\begin{aligned} u(t,x) = g(x) \ &+ \int_t^T \{\mathcal{L}u(s,x) + \mathcal{M}q(s,x) + f(s,x,u(s,x))\}ds \\ &- \int_t^T q(s,x)dW(s), \qquad (t,x) \in [0,T] \times \mathbb{R}, \end{aligned} \tag{1.3}$$

and throughout the paper we will not distinguish (1.1) and (1.3). Our purpose is to find a pair of random fields $(u,q) : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}^2$, called an *adapted solution* (more precise notions will be given shortly), such that for each fixed $x \in \mathbb{R}$, $\{u(\cdot,x,\cdot), q(\cdot,x,\cdot)\}$ is a pair of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes that satisfies (1.3) in a certain sense, under the following *parabolicity* condition:

$$a(t,x)^2 - \alpha(t,x)^2 \geq 0, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}, \quad \text{a.s.} \tag{1.4}$$

Equation (1.1) (or equivalently, (1.3)) is called a *backward stochastic partial differential equation* (BSPDE, for short). Such kind of equation, especially in the linear case (i.e., $f(t,x,u) \equiv h(t,x)$) has been studied by many authors, mainly within the topics of stochastic control and nonlinear filtering theory (see, for examples, Bensoussan [1,2], Nagasa–Nisio [12], Pardoux [13], Hu–Peng [6], Peng [16] and Zhou [17,18]). Such BSPDEs have also proved to be useful in mathematical finance as it provides a generalized version of the celebrated Black–Scholes formula in the case where the market parameters (such as interest rate, volatility, etc.) are allowed to be random (see Ma–Yong [9,10]). However, not until [9] and its subsequent version [10], all early studies of such BSPDEs required the so-called *super-parabolicity* condition, which translated into the present case is that there exists a constant $\delta > 0$, such that (1.4) is replaced by

$$a(t,x)^2 - \alpha(t,x)^2 \geq \delta, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}, \quad \text{a.s. } \omega \in \Omega. \tag{1.5}$$

We note that the super-parabolicity condition is removed in [9] and [10] for a class of *linear* BSPDEs under certain "compatibility" conditions on the coefficients, which in particular contains the one-dimensional version (1.3) with $f(t,x,u) \equiv h(t,x)$.

A fundamental result of this paper is a uniform boundedness estimate for the adapted solutions to BSPDEs of form (1.1). Such a result, slightly surprising from the stochastic differential equation perspective because of the presence of the stochastic integral, represents a special feature of a *backward* SDE. To our best knowledge, this result is new in the stochastic PDE literature. The well-posedness and the regularity of the adapted solutions to the semilinear BSPDE then follow from a combination of some *a priori* estimates for both linear and semilinear BSPDEs and a so-called *bootstrap* method. We should point out, however, that due to some essential technicalities in deriving the *a priori* estimate, in this paper we have to restrict ourselves to the one-dimensional case although almost all other aspects of the paper can be extended to the higher-dimensional case without substantial difficulty. We hope to be able to address the higher-dimensional case in our future publications. As applications we shall prove some comparison results for adapted solutions to semilinear BSPDEs; and finally we establish a nonlinear Feynman-Kac formula which generalizes the (deterministic) one developed by Pardoux and Peng [15], when degeneracy is possible.

This paper is organized as follows. In Section 2 we introduce all the notations and give the definitions of the adapted solutions. In Section 3 we prove some important a priori estimates, including the uniform boundedness estimate for semilinear BSPDEs. In Section 4 we study the well-posedness of the semilinear BSPDE (1.3) under super-parabolicity condition; and in Section 5 we extend the results to the possible degenerate cases. In Sections 6 we present the comparison theorems and the nonlinear Feynman-Kac formula.

2. Preliminaries

Throughout this paper we denote by $C^k(\mathbb{R})$ the set of functions that are continuously differentiable up to order k; by $C_b^k(\mathbb{R})$ the set of functions in $C^k(\mathbb{R})$ whose derivatives up to order k are uniformly bounded; by $C_0^k(\mathbb{R})$ the set of functions in $C^k(\mathbb{R})$ with compact supports. We denote ∂_x^k (resp. ∂_u^k) to be the k-th order partial derivative with respect to x (resp. to u). When the context is clear we often use the notations $\partial^k = \partial_x^k$, $\partial_x\varphi = \varphi_x$, and $\partial_x^2\varphi = \varphi_{xx}$ for the purpose of easier presentation. Next, we define some other spaces that will be used in the paper. Let $k \geq 0$ be an integer, $1 \leq p, r \leq \infty$ be any real numbers, X be any Banach space, and $\mathcal{G} \subset \mathcal{F}$ be any sub-σ-field. We denote

• by $W^{k,p}(\mathbb{R})$ the usual Sobolev space, $k = 0, 1, 2, \cdots$, $1 \leq p \leq \infty$, and $H^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$ (with $H^0(\mathbb{R}) = L^2(\mathbb{R})$) which is a Hilbert space with the usual inner product

$$(\varphi, \psi)_{H^k} \triangleq \int_{\mathbb{R}} \sum_{i=0}^{k} (\partial^i \varphi(x))(\partial^i \psi(x))dx, \quad \forall \varphi, \psi \in H^k(\mathbb{R});$$

• by $L^p_{\mathcal{G}}(\Omega; X)$ the set of all X-valued, $\mathcal{G}$-measurable random variable ξ such that $E\|\xi\|_X^p < \infty$;

• by $L^p_{\mathcal{F}}(0, T; L^r(\Omega; X))$ the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-predictable X-valued processes $\varphi(t, \omega) : [0, T] \times \Omega \to X$ such that $\|\varphi\|_{L^p_{\mathcal{F}}(0,T;L^r(\Omega;X))} \triangleq \left\{ \int_0^T \left[E\|\varphi(t)\|_X^r\right]^{p/r} dt\right\}^{1/p} < \infty$.

• by $C_{\mathcal{F}}([0, T]; L^r(\Omega; X))$ the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-predictable X-valued processes $\varphi(t, \omega) : [0, T]\times \Omega \to X$ such that

$$\|\varphi\|_{C_{\mathcal{F}}([0,T];L^r(\Omega;X))} \triangleq \sup_{t\in[0,T]} \left[E\|\varphi(t)\|_X^r\right]^{1/r} < \infty.$$

If $r = p$, we shall denote $L^p_{\mathcal{F}}(0,T;L^p(\Omega;X))$ by $L^p_{\mathcal{F}}(0,T;X)$ for simplicity. Also, we often simply denote $C^k(\mathbb{R}) = C^k$, $C^k_b(\mathbb{R}) = C^k_b$, $C^k_0(\mathbb{R}) = C^k_0$, $W^{k,m}(\mathbb{R}) = W^{k,m}$, $H^k(\mathbb{R}) = H^k, \dots$, etc., if there is no danger of confusion. For any $R > 0$, with $B_R \stackrel{\Delta}{=} [-R,R]$, we can similarly define $C^k(B_R)$, $H^k(B_R)$, etc. Finally, in this paper, we will use $C > 0$ to represent a generic constant which can be different at different places.

Now, let us give some assumptions. Let $\kappa \geq 0$ be an integer and $1 \leq p \leq \infty$.

($\mathbf{A}^\kappa$) The random fields a, b, c, α, γ satisfy the following:

$$a, \alpha \in L^\infty_{\mathcal{F}}(0,T;W^{\kappa+1,\infty}(\mathbb{R})), \qquad b, c, \gamma \in L^\infty_{\mathcal{F}}(0,T;W^{\kappa,\infty}(\mathbb{R})). \tag{2.1}$$

($\mathbf{F}^\kappa$) The random field f satisfies the following:

$$f \in L^\infty_{\mathcal{F}}(0,T;W^{\kappa,\infty}(\mathbb{R}^2)). \tag{2.2}$$

($\mathbf{G}^{\kappa,p}$) The random field g satisfies the following:

$$g \in L^2_{\mathcal{F}_T}(\Omega;W^{\kappa,\infty}) \bigcap L^p_{\mathcal{F}_T}(\Omega;W^{\kappa,\infty}). \tag{2.3}$$

We should point out that the assumptions above are slightly weaker than the corresponding ones in [10] or [11]. We will show that these assumptions are already sufficient for the well-posedness of strong and weak adapted solutions (to be defined below). However, in order to discuss the classical adapted solutions, the following stronger assumptions, which are the same as those in [10,11], will have to be in force.

($\mathbf{A}^\kappa_c$) The random fields a, b, c, α, γ satisfy the following:

$$a, \alpha \in L^\infty_{\mathcal{F}}(0,T;C^{\kappa+1}_b(\mathbb{R})), \qquad b, c, \gamma \in L^\infty_{\mathcal{F}}(0,T;C^\kappa_b(\mathbb{R})). \tag{2.1$'$}$$

($\mathbf{F}^\kappa_c$) The random field f satisfies the following:

$$f \in L^\infty_{\mathcal{F}}(0,T;C^\kappa_b(\mathbb{R}^2)). \tag{2.2$'$}$$

($\mathbf{G}^{\kappa,p}_c$) The random field g satisfies the following:

$$g \in L^2_{\mathcal{F}_T}(\Omega;C^\kappa_b) \bigcap L^p_{\mathcal{F}_T}(\Omega;C^\kappa_b). \tag{2.3$'$}$$

At times we need the following useful assumption ($\widetilde{\mathbf{A}}^1$) which is weaker than ($\mathbf{A}^1$) but stronger than ($\mathbf{A}^0$).

($\widetilde{\mathbf{A}}^1$) Random fields a, b, c, α, γ satisfy the following:

$$\begin{cases} a, \alpha \in L^\infty_{\mathcal{F}}(0,T;W^{2,\infty}(\mathbb{R})), \\ b, \gamma \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(\mathbb{R})), \\ c \in L^\infty_{\mathcal{F}}(0,T;L^\infty(\mathbb{R})). \end{cases} \tag{2.4}$$

Finally, we introduce the bounds K_κ and K' that will be used frequently in the sequel:

$$\begin{cases} K_\kappa \stackrel{\Delta}{=} \|a\|_{L^\infty_{\mathcal{F}}(0,T;W^{\kappa+1,\infty})} + \|\alpha\|_{L^\infty_{\mathcal{F}}(0,T;W^{\kappa+1,\infty})} + \|b\|_{L^\infty_{\mathcal{F}}(0,T;W^{\kappa,\infty})} \\ \qquad\quad +\|\gamma\|_{L^\infty_{\mathcal{F}}(0,T;W^{\kappa,\infty})} + \|c\|_{L^\infty_{\mathcal{F}}(0,T;W^{\kappa,\infty})}, \\ K' \stackrel{\Delta}{=} \|a\|_{L^\infty_{\mathcal{F}}(0,T;W^{2,\infty})} + \|\alpha\|_{L^\infty_{\mathcal{F}}(0,T;W^{2,\infty})} + \|b\|_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty})} \\ \qquad\quad +\|\gamma\|_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty})} + \|c\|_{L^\infty_{\mathcal{F}}(0,T;L^\infty)}. \end{cases} \tag{2.5}$$

Before giving definitions of solutions to (1.3), let us make the following observation: If a and α are differentiable, then

$$a^2u_{xx} = (a^2u_x)_x - 2aa_xu_x, \qquad \alpha q_x = (\alpha q)_x - \alpha_x q, \tag{2.6}$$

and hence, BSPDE (1.3) can be written as the following *divergence* form:

$$\begin{aligned} u(t,x) = \;& g(x) + \int_t^T \Big\{\frac{1}{2}(a^2u_x)_x + \widetilde{b}u_x + cu + (\alpha q)_x + \widetilde{\gamma}q + f(s,x,u)\Big\}ds \\ & - \int_t^T q\,dW(s), \qquad (t,x) \in [0,T]\times\mathbb{R}, \end{aligned} \tag{2.7}$$

where

$$\widetilde{b} = b - aa_x; \qquad \widetilde{\gamma} = \gamma - \alpha_x. \tag{2.8}$$

Thus, under ($\mathbf{A}^0$), (1.3) and (2.7) are equivalent. Consequently, all the results for (1.3) can be automatically carried over to (2.7) and vice versa. Therefore, in the rest of the paper, we will use both (1.3) and (2.7), whichever is convenient, for our discussion without further explanation.

The following definitions of *adapted solutions* to the BSPDE (1.3) are inherited from [10] (see [11] also), with slight modifications. To simplify notations, let us further introduce some spaces of random fields that will be frequently used in the sequel. For each integer $k \ge 1$ we define

$$\mathcal{H}^k \stackrel{\Delta}{=} \{(u,q) \mid u \in L^2_{\mathcal{F}}(0,T;H^k),\; q \in L^2_{\mathcal{F}}(0,T;H^{k-1})\}. \tag{2.9}$$

We note that if $u \in L^2_{\mathcal{F}}(0,T;H^k)$ and it is also continuous in t with respect to the norm $\{E\|\cdot\|^2_{H^k}\}^{1/2}$, then $u \in C_{\mathcal{F}}([0,T];L^2(\Omega;H^k))$.

Definition 2.1. *A pair of random fields* $(u,q) : [0,T]\times\mathbb{R}\times\Omega \mapsto \mathbb{R}^2$ *is called*

(i) a *classical adapted solution* of (2.7), if

$$\begin{cases} u \in C_{\mathcal{F}}(0,T;L^2(\Omega;C^2(\mathbb{R}))), \\ q \in L^2_{\mathcal{F}}(0,T;C^1(\mathbb{R})), \end{cases} \tag{2.10}$$

such that

$$\begin{aligned} u(t,x) = \;& g(x) + \int_t^T \{\frac{1}{2}[a^2u_x]_x + \widetilde{b}u_x + cu + [\alpha q]_x + \widetilde{\gamma}q + f(s,x,u)\}ds \\ & - \int_t^T q\,dW(s), \qquad (t,x) \in [0,T]\times\mathbb{R}, \text{ P-a.s.} \end{aligned} \tag{2.11}$$

(ii) a *strong adapted solution* of (2.7), if

$$\begin{cases} u \in \bigcap_{R>0} C_{\mathcal{F}}([0,T]; L^2(\Omega; H^2(B_R))), \\ q \in \bigcap_{R>0} L^2_{\mathcal{F}}(0,T; H^1(B_R)), \end{cases} \tag{2.12}$$

such that (2.11) holds.

(iii) a *weak adapted solution* of (2.7) if

$$\begin{cases} u \in \bigcap_{R>0} C_{\mathcal{F}}([0,T]; L^2(\Omega; H^1(B_R))), \\ q \in \bigcap_{R>0} L^2_{\mathcal{F}}(0,T; L^2(B_R)), \end{cases} \tag{2.13}$$

such that

$$\begin{aligned} &\int_{\mathbb{R}} u(t,x)\varphi(x)dx - \int_{\mathbb{R}} g(x)\varphi(x)dx \\ &= \int_t^T \int_{\mathbb{R}} \Big\{ -\frac{1}{2}a^2 u_x \varphi_x + \widetilde{b} u_x \varphi + cu\varphi - \alpha q \varphi_x + \widetilde{\gamma} q \varphi + f(s,x,u)\varphi \Big\} dxds \\ &\quad - \int_t^T \int_{\mathbb{R}} q\varphi dx dW(s), \qquad \forall \varphi \in C_0^\infty,\ t \in [0,T],\ \text{P-a.s.} \end{aligned} \tag{2.14}$$

We should point out that if $\widetilde{b}$ and $\widetilde{\gamma}$ in (2.7) are given by (2.8), then Definition 2.1 defines the weak, strong, and classical adapted solutions for BSPDE (1.3), respectively.

It is clear from the definition that a classical adapted solution is a strong adapted solution; and a strong adapted solution is a weak adapted solution. Conversely, using integration by parts it can be easily shown that if $(\mathbf{A}^0)$, $(\mathbf{F}^0)$ and $(\mathbf{G}^{0,2})$ hold, then a weak adapted solution of (2.7) satisfying (2.12) is a strong adapted solution. Furthermore, if $(\mathbf{A}^0_c)$, $(\mathbf{F}^0_c)$ and $(\mathbf{G}^{0,2}_c)$ hold, and (2.10) is true, then (u,q) is actually a classical adapted solution (see [10] and [11] for detailed arguments).

The following result is one of the fundamental lemmas in [10], rephrased to suit the current situation.

Lemma 2.2. (i) *Let the parabolicity condition (1.4) hold. Let* $(\mathbf{A}^\kappa)$ *hold for some* $\kappa \ge 1$*. Then there exists a constant* $C_0 > 0$*, depending only on* κ*,* T*, and the bound* K_κ *given in (2.5), such that for any* $u \in C_0^\infty(\mathbb{R})$ *and* $q \in C_0^\infty(\mathbb{R})$*, it holds, for* $k = 1, 2, \cdots, \kappa$*,* $dP \times dt$*-a.e. on* $[0,T] \times \Omega$ *that*

$$\begin{aligned} &\int_{\mathbb{R}} \Big\{ (a^2 - \alpha^2)(\partial^{k+1} u)^2 + \frac{1}{2} |\partial^k (\alpha u_x + q)|^2 \Big\} dx \\ &\le \int_{\mathbb{R}} \big\{ -2(\partial^k u)\partial^k(\mathcal{L}u + \mathcal{M}q) + \sum_{i=0}^k |\partial^i q|^2 + C_0 \sum_{i=0}^k |\partial^i u|^2 \big\} dx. \end{aligned} \tag{2.15}$$

For $k = 0$*, (2.15) remains true under* $(\widetilde{\mathbf{A}}^1)$*, with the constant* C_0 *depending only on* T *and the bound* K' *given in (2.5).*

(ii) *Let the super-parabolicity condition (1.5) hold for some* $\delta > 0$*, and let* $(\mathbf{A}^\kappa)$ *hold for some* $\kappa \ge 0$*. Then there exists a constant* $C > 0$*, depending only on* κ*,* T*,* *and the bound K_κ given in (2.5), such that for any $u \in C_0^\infty(\mathbb{R})$ and $q \in C_0^\infty(\mathbb{R})$, it holds, for $k = 0, 1, \cdots, \kappa$, $dP \times dt$-a.e. on $[0, T] \times \Omega$ that*

$$\delta \|u\|_{H^{k+1}}^2 \le \int_{\mathbb{R}} \sum_{j=0}^{k} \Big\{ -2(\partial^j u)\partial^j(\mathcal{L}u + \mathcal{M}q) \Big\} dx + \|q\|_{H^k}^2 + C\|u\|_{H^k}^2. \quad (2.16)$$

In particular, (2.16) holds for $k = 0$ under $(\mathbf{A}^0)$ only.

Proof. We prove the case $k = 0$ first to get the main idea. Note that in this case, one has, by using integration by parts,

$$\begin{aligned}
&\int_{\mathbb{R}} \{-2u(\mathcal{L}u + \mathcal{M}q)\} dx \\
&\quad = \int_{\mathbb{R}} \{-2u[\tfrac{1}{2}a^2 u_{xx} + bu_x + cu + \alpha q_x + \gamma q]\} dx \\
&\quad = \int_{\mathbb{R}} \{a^2|u_x|^2 + 2(aa_x - b)uu_x - 2cu^2 - 2\alpha q_x u - 2\gamma q u\} dx \\
&\quad = \int_{\mathbb{R}} \{a^2|u_x|^2 + (aa_x - b)(u^2)_x - 2cu^2 + 2[\alpha u_x + (\alpha_x - \gamma)u]q\} dx \\
&\quad = \int_{\mathbb{R}} \{a^2|u_x|^2 - [(aa_x - b)_x + 2c]u^2 + |\alpha u_x + (\alpha_x - \gamma)u + q|^2 \\
&\qquad - |\alpha u_x + (\alpha_x - \gamma)u|^2 - q^2\} dx \\
&\quad \ge \int_{\mathbb{R}} \{a^2|u_x|^2 - [(aa_x - b)_x + 2c]u^2 + \frac{1}{2}|\alpha u_x + q|^2 - (\alpha_x - \gamma)^2 u^2 \\
&\qquad - \alpha^2|u_x|^2 - (\alpha_x - \gamma)^2 u^2 - 2\alpha(\alpha_x - \gamma)uu_x - q^2\} dx \\
&\quad = \int_{\mathbb{R}} \{(a^2 - \alpha^2)|u_x|^2 - [(aa_x - b)_x + 2c + 2(\alpha_x - \gamma)^2]u^2 + \frac{1}{2}|\alpha u_x + q|^2 \\
&\qquad - \alpha(\alpha_x - \gamma)(u^2)_x - q^2\} dx \\
&\quad = \int_{\mathbb{R}} \Big\{(a^2 - \alpha^2)|u_x|^2 + \frac{1}{2}|\alpha u_x + q|^2 - q^2 \\
&\qquad - \{(aa_x - b)_x + 2c + 2(\alpha_x - \gamma)^2 + [\alpha(\alpha_x - \gamma)]_x\}u^2 \Big\} dx.
\end{aligned}$$

In the above, we have used the inequality $|a+b|^2 \ge \frac{1}{2}|a|^2 - |b|^2$ (for any $a, b \in \mathbb{R}$). Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}} \{(a^2 - \alpha^2)|u_x|^2 + \frac{1}{2}|\alpha u_x + q|^2 \\
&\quad \le \int_{\mathbb{R}} \Big\{ -2u(\mathcal{L}u + \mathcal{M}q) + q^2 \\
&\qquad + \{(aa_x - b)_x + 2c + 2(\alpha_x - \gamma)^2 + [\alpha(\alpha_x - \gamma)]_x\}u^2 \Big\} dx \\
&\quad \le \int_{\mathbb{R}} \Big\{ -2u(\mathcal{L}u + \mathcal{M}q) + q^2 + Cu^2 \Big\} dx.
\end{aligned}$$

This gives (2.15) for $k = 0$, with only $(\widetilde{\mathbf{A}}^1)$ needed. The constant C only depends on the bound of $a, a_x, a_{xx}, \alpha, \alpha_x, \alpha_{xx}, b_x, \gamma, \gamma_x, c$. The corresponding case in (2.16) can also be proved similarly to the above arguments.

Next, we look at the general case, with more careful calculations. We shall give a detailed argument for the case $k \geq 3$, since the cases $k = 1, 2$ can be proved along the same lines, only easier. First, using integration by parts, and denoting $C_k^i = \binom{k}{i}$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \{ -2(\partial^k u)\partial^k(\mathcal{L}u + \mathcal{M}q)\} dx \\
&= \int_{\mathbb{R}} \Big\{ -2(\partial^k u)\partial^k(\frac{1}{2}a^2 u_{xx} + bu_x + cu + \alpha q_x + \gamma q)\Big\} dx \\
&= \int_{\mathbb{R}} \Big\{(\partial^{k+1}u)\partial^{k-1}(a^2 u_{xx} + 2\alpha q_x) - 2(\partial^k u)\partial^k(bu_x + cu + \gamma q)\Big\} dx \\
&= \int_{\mathbb{R}} \Big\{(\partial^{k+1}u)\sum_{i=0}^{k-1} C_{k-1}^i\big[(\partial^{k-1-i}(a^2))(\partial^{i+2}u) + 2(\partial^{k-1-i}\alpha)(\partial^{i+1}q)\big] \\
&\quad -2(\partial^k u)\sum_{i=0}^{k} C_k^i\big[(\partial^{k-i}b)(\partial^{i+1}u) + (\partial^{k-i}c)(\partial^i u) + (\partial^{k-i}\gamma)(\partial^i q)\big]\Big\} dx \\
&= \int_{\mathbb{R}} \Big\{a^2(\partial^{k+1}u)^2 + (k-1)(\partial(a^2))(\partial^{k+1}u)(\partial^k u) \\
&\quad +(\partial^{k+1}u)\sum_{i=0}^{k-3} C_{k-1}^i(\partial^{k-1-i}(a^2))(\partial^{i+2}u) \\
&\quad +2\alpha(\partial^{k+1}u)(\partial^k q) + 2(\partial^{k+1}u)\sum_{i=0}^{k-2} C_{k-1}^i(\partial^{k-1-i}\alpha)(\partial^{i+1}q) \\
&\quad -2b(\partial^k u)(\partial^{k+1}u) - 2(\partial^k u)\sum_{i=0}^{k-1} C_k^i(\partial^{k-i}b)(\partial^{i+1}u) - 2(\partial^k u)\sum_{i=0}^{k} C_k^i(\partial^{k-i}c)(\partial^i u) \\
&\quad -2\gamma(\partial^k u)(\partial^k q) - 2(\partial^k u)\sum_{i=0}^{k-1} C_k^i(\partial^{k-i}\gamma)(\partial^i q)\Big\} dx \\
&= \int_{\mathbb{R}} \Big\{a^2(\partial^{k+1}u)^2 + 2\{\alpha(\partial^{k+1}u) - [\gamma + (k-1)(\partial\alpha)](\partial^k u)\}(\partial^k q) \\
&\quad +\big[\frac{k-1}{2}(\partial(a^2)) - b\big]\partial[(\partial^k u)^2] \\
&\quad -(\partial^k u)\sum_{i=0}^{k-3} C_{k-1}^i\big[(\partial^{k-i}(a^2))(\partial^{i+2}u) + (\partial^{k-1-i}(a^2))(\partial^{i+3}u)\big] \\
&\quad -2(\partial^k u)\big\{\sum_{i=0}^{k-2} C_{k-1}^i(\partial^{k-i}\alpha)(\partial^{i+1}q) + \sum_{i=0}^{k-3} C_{k-1}^i(\partial^{k-1-i}\alpha)(\partial^{i+2}q)\big\} \\
&\quad -2(\partial^k u)\Big[\sum_{i=0}^{k-1} C_k^i(\partial^{k-i}b)(\partial^{i+1}u) + \sum_{i=0}^{k} C_k^i(\partial^{k-i}c)(\partial^i u) + \sum_{i=0}^{k-1} C_k^i(\partial^{k-i}\gamma)(\partial^i q)\Big]\Big\} dx.
\end{aligned}
$$

Thus, by "completing squares" and using the inequality $|a+b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ as before, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \big\{ -2(\partial^k u)\partial^k(\mathcal{L}u + \mathcal{M}q)\big\} dx \\
&= \int_{\mathbb{R}} \Big\{ a^2(\partial^{k+1}u)^2 + |\alpha(\partial^{k+1}u) - [\gamma + (k-1)(\partial\alpha)](\partial^k u) + (\partial^k q)|^2 \\
&\quad -|\alpha(\partial^{k+1}u)|^2 - |[\gamma + (k-1)(\partial\alpha)](\partial^k u)|^2 - |(\partial^k q)|^2 \\
&\quad -\alpha[\gamma + (k-1)(\partial\alpha)]\partial[(\partial^k u)^2] - \partial\Big[\frac{k-1}{2}(\partial(a^2)) - b\Big](\partial^k u)^2 \\
&\quad -(\partial^k u)\sum_{i=0}^{k-3} C_{k-1}^i \big[(\partial^{k-i}(a^2))(\partial^{i+2}u) + (\partial^{k-1-i}(a^2))(\partial^{i+3}u)\big] \\
&\quad -2(\partial^k u)\Big\{ \sum_{i=0}^{k-2} C_{k-1}^i (\partial^{k-i}\alpha)(\partial^{i+1}q) + \sum_{i=0}^{k-3} C_{k-1}^i (\partial^{k-1-i}\alpha)(\partial^{i+2}q)\Big\} \\
&\quad -2(\partial^k u)\Big[\sum_{i=0}^{k-1} C_k^i (\partial^{k-i}b)(\partial^{i+1}u) + \sum_{i=0}^{k} C_k^i (\partial^{k-i}c)(\partial^i u) \\
&\qquad + \sum_{i=0}^{k-1} C_k^i (\partial^{k-i}\gamma)(\partial^i q)\Big]\Big\} dx \\
&\geq \int_{\mathbb{R}} \Big\{ (a^2 - \alpha^2)(\partial^{k+1}u)^2 + \frac{1}{2}|\partial^k(\alpha u_x + q)|^2 - \sum_{i=0}^{k} |\partial^i q|^2 - C_0 \sum_{i=0}^{k} |\partial^i u|^2 \Big\} dx,
\end{aligned}
$$

which leads further to

$$
\begin{aligned}
&\int_{\mathbb{R}} \Big\{ (a^2 - \alpha^2)(\partial^{k+1}u)^2 + \frac{1}{2}|\partial^k(\alpha u_x + q)|^2 \Big\} dx \\
&\quad \leq \int_{\mathbb{R}} \Big\{ -2(\partial^k u)\partial^k(\mathcal{L}u + \mathcal{M}q)\big\} dx + \sum_{i=0}^{k} |\partial^i q|^2 + C_0 \sum_{i=0}^{k} |\partial^i u|^2 \Big\} dx
\end{aligned}
$$

This gives (2.15) for the case $k \geq 3$. The proof of (2.16) can now be carried out easily. □

3. Some a priori estimates

In this section we prove some a priori estimates that will be useful in the sequel. We shall consider both the linear and semilinear cases.

3.1. Linear case

We begin by considering the special case of (1.3) with $f(t, x, u) = h(t, x)$, that is, (1.3) is a *linear* BSPDE:

$$\begin{aligned} u(t,x) &= g(x) + \int_t^T \{\frac{1}{2}a^2 u_{xx} + bu_x + cu + \alpha q_x + \gamma q + h\}ds - \int_t^T q dW(s) \\ &= g(x) + \int_t^T \{\mathcal{L}u + \mathcal{M}q + h\}ds - \int_t^T q dW(s). \end{aligned} \tag{3.1}$$

The following lemma is a slight variation and extension of Theorem 3.1 of [10].

Theorem 3.1. *Let the parabolicity condition (1.4) hold. Let* $(\mathbf{A}^\kappa)$ *hold for some* $\kappa \geq 1$. *Then for any random fields* h *and* g *satisfying*

$$h \in L^2_{\mathcal{F}}(0,T;H^\kappa), \quad g \in L^2_{\mathcal{F}_T}(\Omega;H^\kappa), \tag{3.2}$$

BSPDE (3.1) admits a unique weak adapted solution $(u,q) \in \mathcal{H}^\kappa$. *Moreover,* (u,q) *satisfies the following estimate:*

$$\begin{aligned} &\max_{t\in[0,T]} E\|u(t)\|^2_{H^k} + E\int_0^T \Big\{ \int_{\mathbb{R}} \sum_{i=0}^k (a^2-\alpha^2)(\partial_x^{i+1}u)^2 dx \\ &\quad + \|\alpha u_x + q\|^2_{H^k} + \|q\|^2_{H^{k-1}} \Big\} dt \\ &\leq C\Big\{ E\|g\|^2_{H^k} + E\int_0^T \|h(t)\|^2_{H^k} dt \Big\}, \qquad k = 0,1,\cdots,\kappa. \end{aligned} \tag{3.3}$$

where $\|q\|^2_{H^{-1}} \stackrel{\Delta}{=} 0$. *Estimate (3.3) remains true for* $k=0$ *under* $(\widetilde{\mathbf{A}}^1)$.

Furthermore, if the super-parabolicity condition (1.5) holds for some $\delta > 0$, *then the condition (3.2) can be relaxed to*

$$h \in L^2_{\mathcal{F}}(0,T;H^{\kappa-1}), \quad g \in L^2_{\mathcal{F}_T}(\Omega;H^\kappa), \tag{3.4}$$

and the weak adapted solution $(u,q) \in \mathcal{H}^{\kappa+1}$. *Moreover, there exists a constant* $C>0$ *depending only on* κ, T, K_κ, *and* δ *such that*

$$\begin{aligned} &\max_{t\in[0,T]} E\|u(t)\|^2_{H^k} + E\int_0^T \{\|u(s)\|^2_{H^{k+1}} + \|q(s)\|^2_{H^k}\}ds \\ &\qquad \leq CE\Big\{ \int_0^T \|h(t)\|^2_{H^{(k-1)\vee 0}} dt + \|g\|^2_{H^k} \Big\}, \qquad k = 0,\cdots,\kappa. \end{aligned} \tag{3.5}$$

Proof. We first assume that (3.2) holds for some $\kappa \geq 1$. In this case, the existence and uniqueness of the weak adapted solution follows from [10, Theorem 3.1]. We need only establish the estimate (3.3).

Define the following equivalent inner product for H^κ:

$$(\varphi,\psi)_\kappa \equiv \int_{\mathbb{R}} (\partial^\kappa \varphi)(\partial^\kappa \psi) dx, \quad \forall \varphi,\psi \in H^\kappa. \tag{3.6}$$

Denote $|\varphi|_\kappa = (\varphi, \varphi)_\kappa^{1/2}$. Next, let $\{\varphi_i\}_{i\ge 1} \subseteq C_0^\infty(\mathbb{R})$ be an orthonormal basis for H^κ (with respect to the inner product (3.6)). Let $\ell \ge 1$ be fixed. Consider the following linear BSDE (instead of a BSPDE):

$$\begin{cases} du^{\ell j} = -\Big\{ \sum_{i=1}^{\ell} \big[(\mathcal{L}\varphi_i, \varphi_j)_\kappa u^{\ell i} + (\mathcal{M}\varphi_i, \varphi_j)_\kappa q^{\ell i}\big] + (h, \varphi_j)_\kappa \Big\} dt + q^{\ell j} dW(t), \\ u^{\ell j}(T) = (g, \varphi_j)_\kappa, \qquad 1 \le j \le \ell. \end{cases} \tag{3.7}$$

It is by now standard ([14]) that there exists a unique adapted solution $(u^{\ell j}, q^{\ell j}) \in C_{\mathcal{F}}([0,T];\mathbb{R}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R})$, $1 \le j \le k$. Let

$$\begin{cases} u^\ell(t,x,\omega) = \sum_{j=1}^{\ell} u^{\ell j}(t,\omega)\varphi_j(x), \\ q^\ell(t,x,\omega) = \sum_{j=1}^{\ell} q^{\ell j}(t,\omega)\varphi_j(x), \end{cases} \qquad (t,x,\omega) \in [0,T] \times \mathbb{R} \times \Omega.$$

Then we see that for any fixed $(t,\omega) \in [0,T] \times \Omega$, $u^\ell(t,\cdot,\omega), q^\ell(t,\cdot,\omega) \in C_0^\infty(\mathbb{R})$.

Further, if we denote $P^\ell : H^\kappa \to \operatorname{span}\{\varphi_1, \cdots, \varphi_\ell\} \stackrel{\Delta}{=} H^\kappa_\ell$ to be the orthogonal projection from H^κ onto H^κ_ℓ, then obviously we have $(P^\ell)^* = P^\ell$, and $P^\ell u^\ell = u^\ell$, $\ell \ge 1$, since u^ℓ (as well as q^ℓ) are H^κ_ℓ-valued processes. Now let $h^\ell = P^\ell h$ and $g^\ell = P^\ell g$. Then clearly the following holds:

$$\begin{cases} du^\ell = \big\{ - P^\ell[\mathcal{L}u^\ell + \mathcal{M}q^\ell] - h^\ell \big\} dt + q^\ell dW(t), \\ u^\ell\big|_{t=T} = g^\ell. \end{cases} \tag{3.8}$$

Applying the generalized Itô formula (cf. [10,11]) to $|u^\ell(t)|_k^2$ over $[t,T]$, for $k = 0, 1, \cdots, \kappa$, and using (3.8) and Lemma 2.2, we have

$$\begin{aligned} &E|g^\ell|_k^2 - E|u^\ell(t)|_k^2 \\ &= E\int_t^T \Big\{ -2(u^\ell, \mathcal{L}u^\ell + \mathcal{M}q^\ell + h^\ell)_k + |q^\ell|_k^2 \Big\} ds \\ &\ge E\int_t^T \Big\{ \int_{\mathbb{R}} (a^2 - \alpha^2)(\partial^{k+1}u^\ell)^2 dx + |\alpha u^\ell_x + q^\ell|_k^2 \Big\} ds \\ &\quad - E\int_t^T \big[C_0 \|u^\ell\|_{H^k}^2 + \|q^\ell\|_{H^{k-1}}^2 + |u^\ell|_k^2 + |h^\ell|_k^2 \big] ds. \end{aligned} \tag{3.9}$$

In the above, if $k = 0$, $\|q\|_{H^{k-1}}^2 \stackrel{\Delta}{=} 0$. We remark here that for the first equality in the above we assumed that the stochastic integral appearing in the Itô's formula is a true martingale with mean zero. In general one can use the standard localization arguments, together with the Dominated Convergence Theorem, to obtain the result; we omit the details here. Moreover, we have used the fact that $(P^\ell)^* = P^\ell$ and $P^\ell u^\ell = u^\ell$. One can rewrite (3.9) as follows:

$$\begin{aligned} &E|u^\ell(t)|_k^2 + E\int_t^T \Big\{ \int_{\mathbb{R}} (a^2 - \alpha^2)(\partial^{k+1}u^\ell)^2 dx + |\alpha u^\ell_x + q^\ell|_k^2 \Big\} ds \\ &\le E|g^\ell|_k^2 + E\int_t^T \big[C_0 \|u^\ell\|_{H^k}^2 + \|q^\ell\|_{H^{k-1}}^2 + |u^\ell|_k^2 + |h^\ell|_k^2 \big] ds. \end{aligned} \tag{3.10}$$

Now, we first take $k = 0$. Then (3.10) becomes

$$\begin{aligned} &E|u^\ell(t)|_0^2 + E\int_t^T \Big\{\int_{\mathbb{R}} (a^2-\alpha^2)(u_x^\ell)^2 dx + |\alpha u_x^\ell + q^\ell|_0^2\Big\} ds \\ &\quad \le E|g^\ell|_0^2 + E\int_t^T \big[(C_0+1)|u^\ell|_0^2 + |h^\ell|_0^2\big] ds. \end{aligned} \tag{3.11}$$

By Gronwall's inequality, one has

$$\begin{aligned} &\max_{t\in[0,T]} E|u^\ell(t)|_0^2 + E\int_0^T \Big\{\int_{\mathbb{R}} (a^2-\alpha^2)(u_x^\ell)^2 dx + |\alpha u_x^\ell + q^\ell|_0^2\Big\} ds \\ &\quad \le C\big\{E|g^\ell|_0^2 + E\int_0^T |h^\ell|_0^2 ds\big\}, \end{aligned} \tag{3.12}$$

with the constant C depending on C_0. Next, we take $k = 1$. Then (3.10) implies that (noting (3.12))

$$\begin{aligned} &E|u^\ell(t)|_1^2 + E\int_t^T \Big\{\int_{\mathbb{R}} (a^2-\alpha^2)(\partial^2 u^\ell)^2 dx + |\alpha u_x^\ell + q^\ell|_1^2\Big\} ds \\ &\le E|g^\ell|_1^2 + E\int_t^T \big[C_0(|u^\ell|_0^2 + |u^\ell|_1^2) + |q^\ell|_0^2 + |u^\ell|_1^2 + |h^\ell|_1^2\big] ds \\ &\le E|g^\ell|_1^2 + E\int_t^T \big[C_0|u^\ell|_0^2 + (C_0+1)|u^\ell|_1^2 + 2|\alpha u_x^\ell + q^\ell|_0^2 + 2|\alpha u_x^\ell|_0^2 + |h^\ell|_1^2\big] ds \\ &\le E|g^\ell|_1^2 + E\int_t^T \big[(C_0+1+2\|\alpha\|_\infty^2)|u^\ell|_1^2 + |h^\ell|_1^2\big] ds \\ &\qquad\qquad + C\big\{E|g^\ell|_0^2 + E\int_t^T |h^\ell|_0^2 ds\big\}. \end{aligned} \tag{3.13}$$

By Gronwall's inequality again, we obtain

$$\begin{aligned} &\max_{t\in[0,T]} E|u^\ell(t)|_1^2 + E\int_0^T \Big\{\int_{\mathbb{R}} (a^2-\alpha^2)(\partial^2 u^\ell)^2 dx + |\alpha u_x^\ell + q^\ell|_1^2\Big\} ds \\ &\le C\big\{E\|g^\ell\|_1^2 + E\int_0^T \|h^\ell\|_1^2 ds\big\}. \end{aligned} \tag{3.14}$$

Moreover, combining (3.12) and (3.14), we have

$$\begin{aligned} E\int_0^T |q^\ell|_0^2 ds &\le E\int_0^T \big[2|\alpha u_x^\ell|_0^2 + 2|\alpha u_x^\ell + q^\ell|_0^2\big] ds \\ &\le C\big\{E\|g^\ell\|_1^2 + E\int_0^T \|h^\ell\|_1^2 ds\big\}. \end{aligned} \tag{3.15}$$

Then, combining (3.12), (3.14) and (3.15), we obtain

$$\begin{aligned} &\max_{t\in[0,T]} E\|u^\ell(t)\|_{H^1}^2 + E\int_0^T \Big\{\int_{\mathbb{R}} \sum_{i=0}^{1} (a^2-\alpha^2)(\partial^2 u^\ell)^2 dx \\ &\qquad + \|\alpha u_x^\ell + q^\ell\|_{H^1}^2 + \|q^\ell\|_{H^0}^2\Big\} ds \\ &\le C\big\{E\|g^\ell\|_{H^1}^2 + E\int_0^T \|h^\ell\|_{H^1}^2 ds\big\}. \end{aligned} \tag{3.16}$$

Following the same argument, we are able to prove the following:

$$\max_{t\in[0,T]} E\|u^\ell(t)\|_{H^k}^2 + E\int_0^T \Big\{\int_{\mathbb{R}}\sum_{i=0}^k (a^2-\alpha^2)(\partial^{i+1}u^\ell)^2dx + \|\alpha u_x^\ell + q^\ell\|_{H^k}^2 + \|q^\ell\|_{H^{k-1}}^2\Big\}ds \le C\Big\{E\|g^\ell\|_{H^k}^2 + E\int_0^T \|h^\ell\|_{H^k}^2 ds\Big\}, \tag{3.17}$$

for $k=0,1,\cdots,\kappa$. Letting $\ell\to\infty$, we obtain (3.3).

Recall that $C>0$ is a generic constant depending only on κ, T and K, which can vary from line to line. Also, we note that by Lemma 2.2, when we prove (3.3) for $k=0$ only ($\widetilde{\mathbf{A}}^1$) is needed.

Now we assume that the super-parabolicity condition (1.5) holds. Note that

$$2\int_t^T\int_{\mathbb{R}}(\partial^k u^\ell)(\partial^k h^\ell)dxdt \le \varepsilon\int_t^T |u^\ell|_{k+1}^2dt + \frac{1}{\varepsilon}\int_t^T |h^\ell|_{k-1}^2dt.$$

Thus, (3.9) and (3.10) can be refined as follows:

$$E|u^\ell(t)|_k^2 + E\int_t^T \Big\{\delta|u^\ell|_{k+1}^2 + |\alpha u_x^\ell + q^\ell|_k^2\Big\}ds \le E|g^\ell|_k^2 + E\int_t^T \big[C_0\|u^\ell\|_{H^k}^2 + \|q^\ell\|_{H^{k-1}}^2 + \varepsilon|u^\ell|_{k+1}^2 + \frac{1}{\varepsilon}|h^\ell|_{k-1}^2\big]ds.$$

Thus, by choosing $\varepsilon>0$ small enough, we have

$$E|u^\ell(t)|_k^2 + E\int_t^T \Big\{\frac{\delta}{2}|u^\ell|_{k+1}^2 + |\alpha u_x^\ell + q^\ell|_k^2\Big\}ds \le E|g^\ell|_k^2 + E\int_t^T \big[C_0\|u^\ell\|_{H^k}^2 + \|q^\ell\|_{H^{k-1}}^2 + \frac{1}{\varepsilon}|h^\ell|_{k-1}^2\big]ds.$$

Next, using the same argument in (3.11)–(3.17) (starting from $k=0$, then $k=1,2,\cdots$, etc.), we are able to prove (3.5) for the case where (3.2) holds. In the case where only (3.4) holds, we can use a usual approximation argument since $L^2_{\mathcal{F}}(0,T;H^\kappa)$ is dense in $L^2_{\mathcal{F}}(0,T;H^{\kappa-1})$.

Finally, from estimate (3.5), we see that under super-parabolicity condition (1.5) and (3.4), the weak adapted solution (u,q) must be in $\mathcal{H}^{\kappa+1}$, again by an approximation argument. The proof is now complete. □

3.2. Semi-linear case

We now turn to the semilinear BSPDE (2.7). We shall make use of some further assumptions.

($\widehat{\mathbf{A}}$) The random fields α and γ are independent of x, that is, $\alpha,\gamma\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R})$.

($\widehat{\mathbf{F}}$) There exists a constant $L > 0$ and a random field $F \in L^2_{\mathcal{F}}(0, T; L^2(\mathbb{R}))$ such that

$$|f(t,x,u)| \le |F(t,x)| + L|u|, \qquad \forall (t,x,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \text{ P-a.s.} \tag{3.18}$$

Note that ($\widehat{\mathbf{F}}$) is weaker than ($\mathbf{F}^1$) when f is independent of u. The following estimate is fundamental for studying semi-linear BSPDEs.

Theorem 3.2. *Let the parabolicity condition (1.4) hold. Let ($\widetilde{\mathbf{A}}^1$), ($\widehat{\mathbf{A}}$), ($\widehat{\mathbf{F}}$) and ($\mathbf{G}^{1,p}$) hold for some $p \ge 2$. Let (u,q) be a strong adapted solution to the semi-linear BSPDE (2.7). Then, for each $m = 1, 2, \cdots, [\frac{p}{2}]$, it holds that*

$$\sup_{0\le t\le T} E\|u(t)\|^{2m}_{L^{2m}} + E\int_{\mathbb{R}_0} |u(s,x)|^{2m-2}[\alpha(s)u_x(s,x) + q(s,x)]^2 dxds$$
$$\le C\Big\{E\|g\|^{2m}_{L^{2m}} + E\int_0^T \|F(s)\|^{2m}_{L^{2m}} ds\Big\}. \tag{3.19}$$

where $\mathbb{R}_t = (t,T)\times\mathbb{R}$, for $t \in [0,T)$, and $C = e^{Bm}$ for some $B > 0$ depending only on T, K' and L.

Moreover, if ($\mathbf{G}^{1,\infty}$) holds, and the random field F in ($\widehat{\mathbf{F}}$) is uniformly bounded, (in particular if ($\mathbf{F}^1$) holds), then one has

$$|u(t,x)| \le C, \qquad \text{a.e. } (t,x) \in \mathbb{R}_0, \quad \text{a.s.} \tag{3.20}$$

Proof. For any $m = 1, 2, \cdots, [\frac{p}{2}]$, we apply Itô's formula to u^{2m} to get

$$\int_{\mathbb{R}} g(x)^{2m} dx - \int_{\mathbb{R}} u(t,x)^{2m} dx$$
$$= \int_{\mathbb{R}_t} \Big\{2mu^{2m-1}\big[-\frac{1}{2}(a^2u_x)_x - \widetilde{b}u_x - cu - \alpha q_x - \gamma q - f\big]$$
$$+ m(2m-1)u^{2m-2}q^2\Big\}dxds + \int_{\mathbb{R}_t} 2mu^{2m-1}q\,dxdW(s), \tag{3.21}$$

where $\widetilde{b}$ is defined by (2.8). (Note that $\widetilde{\gamma} = \gamma$ by ($\widehat{\mathbf{A}}$).) Using integration by parts we see that the first integral on the right hand side of (3.21) can be written as

$$\int_{\mathbb{R}_t} \Big\{-mu^{2m-1}(a^2u_x)_x - 2mu^{2m-1}\alpha q_x - 2mu^{2m-1}[\widetilde{b}u_x + \gamma q]$$
$$-2mcu^{2m} - 2mu^{2m-1}f + m(2m-1)u^{2m-2}q^2\Big\}dxds$$
$$= \int_{\mathbb{R}_t} \Big\{m(2m-1)u^{2m-2}(au_x)^2 + 2m(2m-1)u^{2m-2}(\alpha u_x)q - 2mcu^{2m}$$
$$-2mu^{2m-1}[\widetilde{b}u_x + \gamma q] + m(2m-1)u^{2m-2}q^2 - 2mu^{2m-1}f\Big\}dxds$$
$$= \int_{\mathbb{R}_t} \Big\{m(2m-1)u^{2m-2}(a^2 - \alpha^2)(u_x)^2 + m(2m-1)u^{2m-2}(\alpha u_x + q)^2$$
$$-2mu^{2m-1}\big[\widetilde{b}u_x + \gamma q\big] - 2mcu^{2m} - 2mu^{2m-1}f\Big\}dxds. \tag{3.22}$$

In the above, we have used the assumption that α is independent of x as well. Now note that

$$\begin{aligned}
&-\int_{\mathbb{R}_t} 2mu^{2m-1}(\widetilde{b}u_x+\gamma q)dxds\\
&\quad=-\int_{\mathbb{R}_t} 2mu^{2m-1}\big[(\widetilde{b}-\gamma\alpha)u_x+\gamma(\alpha u_x+q)\big]dxds\\
&\quad=-\int_{\mathbb{R}_t} \big[(\widetilde{b}-\gamma\alpha)(u^{2m})_x+2mu^{2m-1}\gamma(\alpha u_x+q)\big]dxds\\
&\quad=\int_{\mathbb{R}_t} \big[(\widetilde{b}-\gamma\alpha)_x u^{2m}-2mu^{2m-1}\gamma(\alpha u_x+q)\big]dxds, \qquad (3.23)
\end{aligned}$$

and that

$$2|u||\gamma||\alpha u_x+q|\le \frac{(2m-1)}{2}(\alpha u_x+q)^2+\frac{2}{2m-1}u^2\gamma^2. \qquad (3.24)$$

Further, making use of the assumption ($\widehat{\mathbf{F}}$) and Young's inequality one has

$$|u|^{2m-1}|f|\le |u|^{2m-1}(|F|+L|u|)\le \frac{(2m-1)}{2m}u^{2m}+\frac{1}{2m}|F|^{2m}+L|u|^{2m}. \qquad (3.25)$$

Hence, the right hand side of (3.22) is no less than

$$\begin{aligned}
&\int_{\mathbb{R}_t}\Big\{\frac{m(2m-1)}{2}u^{2m-2}(\alpha u_x+q)^2-|F|^{2m}\\
&\qquad+\Big((\widetilde{b}-\gamma\alpha)_x-2cm-\frac{2|\gamma|^2m}{(2m-1)}-(2m-1)-2Lm\Big)u^{2m}\Big\}dxds. \qquad (3.26)
\end{aligned}$$

Combining (3.22)–(3.26), we derive from (3.21) that

$$\begin{aligned}
&\int_{\mathbb{R}} g(x)^{2m}dx-\int_{\mathbb{R}} u(t,x)^{2m}dx\\
&\quad\ge\int_{\mathbb{R}_t}\Big\{\frac{m(2m-1)}{2}u^{2m-2}(\alpha u_x+q)^2-|F|^{2m}-Cmu^{2m}\Big\}dxds\\
&\qquad+\int_{\mathbb{R}_t} 2mu^{2m-1}q\,dxdW(s), \qquad (3.27)
\end{aligned}$$

where $C>0$ is some generic constant depending only on the constants K' in (2.5) and L in ($\widehat{\mathbf{F}}$). Taking expectation and applying the Gronwall inequality to (3.27) we obtain

$$\begin{aligned}
&E\int_{\mathbb{R}} u(x,t)^{2m}dx+\frac{m(2m-1)}{2}E\int_{\mathbb{R}_t} u(x,s)^{2m-2}[\alpha u_x+q]^2(s,x)dxds\\
&\quad\le e^{CmT}\Big\{E\int_{\mathbb{R}} g(x)^{2m}dx+\int_{\mathbb{R}_0}|F(s,x)|^{2m}dxds\Big\},\quad \forall 1\le m\le M,\ t\in[0,T], \qquad (3.28)
\end{aligned}$$

which gives (3.19).

Now, in the case $(\mathbf{G}^{1,\infty})$ holds, we drop the second term on the left side of (3.28), take $2m$-th root on both sides and send $m \to \infty$ to get

$$|u(x,t)| \le e^{\frac{CT}{2}}(\|g\|_\infty + \|F\|_\infty), \quad \forall t \in [0,T], \text{ a.e. } x \in \mathbb{R}, \text{ a.s.} \tag{3.29}$$

This completes the proof. □

We emphasize here that Theorem 3.2 holds only under the parabolicity condition (1.4), instead of the super-parabolicity condition (1.5).

4. Well-posedness: Super-parabolic Case

In this section we shall prove the well-posedness of the semi-linear BSPDE (1.3) (or (2.7)) under super-parabolicity condition (1.5). We note that even for this simple case, the well-posedness of the semi-linear BSPDE has not been fully explored so far. Therefore we shall give a detailed study here.

Let us first consider the existence and uniqueness of the weak solution.

Theorem 4.1. *Let the super-parabolicity condition (1.5) hold, and let* $(\widetilde{\mathbf{A}}^1)$, $(\mathbf{F}^1)$, *and* $(\mathbf{G}^{1,2})$ *hold. Then, the semi-linear BSPDE (1.3) admits a unique weak adapted solution.*

Proof. Let us first apply the Picard approximation to prove the existence.

We define

$$(u^0, q^0) = (0,0) \in \mathcal{H}^1, \tag{4.1}$$

and let $(u^n, q^n) \in \mathcal{H}^1$ be given. Since $f(\cdot,\cdot,u^n(\cdot,\cdot)) \in L^2_{\mathcal{F}}(0,T;H^0)$, and we have assumed the super-parabolicity condition (1.5), applying Theorem 3.1 with $\kappa = 1$, we see that the (linear) BSPDE

$$\begin{cases} du^{n+1} = -\{\frac{1}{2}a^2 u^{n+1}_{xx} + bu^{n+1}_x + cu^{n+1} + \alpha q^{n+1}_x + \gamma q^{n+1} + f(t,x,u^n)\}dt \\ \qquad\qquad + q^{n+1} dW(t), \\ u^{n+1}(T,x) = g(x) \end{cases} \tag{4.2}$$

admits a unique weak adapted solution $(u^{n+1}, q^{n+1}) \in \mathcal{H}^2 \subseteq \mathcal{H}^1$. We shall prove that $(u^n, q^n)_{n \ge 0}$ is a Cauchy sequence in $\mathcal{H}^1$.

Applying Theorem 3.1 ((3.3) with $k = 0$ to be more precise) to $u^{n+1} - u^n$, we get:

$$\begin{aligned} E\|u^{n+1}(t) - u^n(t)\|^2_{H^0} &\le CE\int_t^T \|f(s,\cdot,u^n(s,\cdot)) - f(s,\cdot,u^{n-1}(s,\cdot))\|^2_{H^0} ds \\ &\le CK^2 \int_t^T E\|u^n(s,\cdot) - u^{n-1}(s,\cdot)\|^2_{H^0} ds, \end{aligned} \tag{4.3}$$

where the last inequality is due to assumption $(\mathbf{F}^1)$.

By a simple iteration we derive from (4.3) that

$$E\|u^{n+1}(t)-u^n(t)\|_{H^0}^2 \le \frac{(CK^2)^n}{n!}(T-t)^n E\|u^1(t)-u^0(t)\|_{H^0}^2, \tag{4.4}$$

which implies that $\{u^n\}_{n\ge 0}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(0,T;H^0)$.

Now we apply Theorem 3.1 again ((3.5) with $k=0$ to be more precise) to get:

$$\begin{aligned} &E\int_0^T \{\|u^n(s)-u^m(s)\|_{H^1}^2 + \|q^n(s)-q^m(s)\|_{H^0}^2\}ds \le CK^2 \\ &E\int_0^T \|u^{n-1}(s)-u^{m-1}(s)\|_{H^0}^2 ds, \end{aligned} \tag{4.5}$$

which implies that $(u^n,q^n)_{n\ge 0}$ is also a Cauchy sequence in $\mathcal{H}^1$ and we denote its limit by $(u,q)\in\mathcal{H}^1$. It is easy to check that (u,q) is a weak adapted solution to (1.3).

The uniqueness follows easily from Theorem 3.1, and we omit it. □

A direct consequence of the a priori estimates in Theorems 3.1 and 3.2, given the existence of the weak adapted solution, is the following result which is simple but a slightly surprising.

Corollary 4.2. *Suppose that the assumptions of Theorem 4.1 are all in force. Then the weak adapted solution (u,q) of the semi-linear BSPDE (1.3) is indeed a strong adapted solution. Moreover, if* $(\widehat{\mathbf{A}})$ *and* $(\mathbf{G}^{1,\infty})$ *also hold, then u is uniformly bounded.*

Proof. Let $(u,q)\in\mathcal{H}^1$ be the weak adapted solution of (1.3). The super-parabolicity and Theorem 3.1 then tell us that this weak solution must belong to $\mathcal{H}^2$, thanks to the estimate (3.5). Then, as in [10] one shows that (u,q) is indeed a strong adapted solution of the linear BSPDE (3.1) with $h(t,x)=f(t,x,u(t,x))$, which amounts to saying that (u,q) is a strong adapted solution of the semi-linear BSPDE (1.3).

Finally, we apply Theorem 3.2 to conclude that when $(\widehat{\mathbf{A}})$ and $(\mathbf{G}^{1,\infty})$ hold, the component u of the strong adapted solution (u,q) is actually uniformly bounded. □

One of the interesting perspective in Corollary 4.2 is that, under the super-parabolicity condition, the *regularity* of the component u of the solution (u,q) is raised automatically by one. On the other hand, by assuming $(\widehat{\mathbf{A}})$, we derive the uniform boundedness of u. This enables us to use the following "boot-strap" method to obtain the further regularity of the adapted solution of (1.3) (or (2.7)).

Theorem 4.3. *Let the super-parabolicity condition (1.5) hold, and let* $(\mathbf{A}^\kappa_c)$, $(\widehat{\mathbf{A}})$, $(\mathbf{F}^\kappa_c)$, *and* $(\mathbf{G}^{\kappa,\infty}_c)$ *hold for $\kappa\ge 3$. Then, the semi-linear BSPDE (1.3) admits a classical adapted solution $(u,q)\in\mathcal{H}^{\kappa+1}$. Furthermore, there exists a constant $C>0$, such that*

$$|\partial_x^k u(t,x)|\le C,\quad \forall (t,x)\in[0,T]\times\mathbb{R},\ \text{a.s.},\ 0\le k\le\kappa-1. \tag{4.6}$$

Proof. By Corollary 4.2, we may let $(u, q) \in \mathcal{H}^2$ be the strong adapted solution of the BSPDE (1.3) such that u is bounded. We define, for each $\mu \in (0, 1]$, the *difference quotient*

$$U^\mu(t,x) \stackrel{\Delta}{=} \frac{u(t,x+\mu)-u(t,x)}{\mu}, \quad Q^\mu(t,x) \stackrel{\Delta}{=} \frac{q(t,x+\mu)-q(t,x)}{\mu}. \quad (4.7)$$

Clearly, $(U^\mu, Q^\mu) \in \mathcal{H}^2$ is the adapted strong solution of the following linear BSPDE:

$$\begin{aligned} U^\mu(t,x) &= g^\mu(x) + \int_t^T \{\mathcal{L}U^\mu(s,x) + \mathcal{M}Q^\mu(s,x) \\ &\quad + \frac{1}{\mu}[f(s,x+h,u(s,x+\mu)) - f(s,x,u(s,x))] \\ &\quad + \frac{1}{2}(a^2)^\mu(s,x)u_{xx}(s,x+\mu) + b^\mu(s,x)u_x(s,x+\mu) \\ &\quad + c^\mu(s,x)u(s,x+\mu)\}ds - \int_t^T Q^\mu(s,x)dW(s), \end{aligned} \quad (4.8)$$

where

$$\begin{cases} g^\mu(x) = \dfrac{g(x+\mu)-g(x)}{\mu}, \\ (a^2)^\mu(s,x) = \dfrac{a(s,x+\mu)^2 - a(s,x)^2}{\mu}, \\ b^\mu(s,x) = \dfrac{b(s,x+\mu)-b(s,x)}{\mu}, \\ c^\mu(s,x) = \dfrac{c(s,x+\mu)-c(s,x)}{\mu}. \end{cases} \quad (4.9)$$

Denote

$$\begin{cases} F_u^\mu(t,x) \stackrel{\Delta}{=} \displaystyle\int_0^1 f_u(t,x+\mu,u(t,x)+\theta(u(s,x+\mu)-u(s,x)))d\theta; \\ F_x^\mu(t,x) \stackrel{\Delta}{=} \displaystyle\int_0^1 f_x(t,x+\theta\mu,u(t,x))d\theta. \end{cases} \quad (4.10)$$

Then we have

$$\begin{aligned} &\frac{1}{\mu}[f(t,x+\mu,u(t,x+\mu)) - f(t,x,u(t,x))] \\ &= \frac{1}{\mu}[f(t,x+\mu,u(t,x+\mu)) - f(t,x+\mu,u(t,x))] \\ &\qquad + \frac{1}{\mu}[f(t,x+\mu,u(t,x)) - f(t,x,u(t,x))] \\ &= F_u^\mu(t,x)U^\mu(t,x) + F_x^\mu(t,x). \end{aligned} \quad (4.11)$$

Thus, (4.8) becomes

$$
\begin{aligned}
U^{\mu}(t,x) &= g^{\mu}(x) + \int_t^T \{\mathcal{L}U^{\mu}(s,x) + \mathcal{M}Q^{\mu}(s,x) + F_u^{\mu}(s,x)U^{\mu}(s,x) \\
&\quad + F_x^{\mu}(s,x)] \\
&\quad + \frac{1}{2}(a^2)^{\mu}(s,x)u_{xx}(s,x+\mu) + b^{\mu}(s,x)u_x(s,x+\mu) \\
&\quad + c^{\mu}(s,x)u(s,x+\mu)\}ds - \int_t^T Q^{\mu}(s,x)dW(s).
\end{aligned} \tag{4.12}
$$

Also, we formally differentiate (1.3) with respect to x, and denote (formally) $u_x = U$, $q_x = Q$. Then (U, Q) should satisfy the following (linear) BSPDE:

$$
\begin{aligned}
U(t,x) &= g_x(x) + \int_t^T \{\mathcal{L}U(s,x) + \mathcal{M}Q(s,x) + f_u(s,x,u)U(s,x) \\
&\quad + f_x(s,x,u(s,x)) + a(s,x)a_x(s,x)u_{xx}(s,x) \\
&\quad + b_x(s,x)u_x(s,x) + c_x(s,x)u(s,x)]\}ds \\
&\quad - \int_t^T Q(s,x)dW(s).
\end{aligned} \tag{4.13}
$$

For the given strong adapted solution $(u,q) \in \mathcal{H}^2$ of (1.3), we know that

$$
\widetilde{h} \stackrel{\Delta}{=} aa_x u_{xx} + b_x u_x + c_x u + f_x(\cdot,\cdot,u) \in L^2_{\mathcal{F}}(0,T;H^0). \tag{4.14}
$$

Since we have assumed the super-parabolicity condition (1.5), by second part of Theorem 3.1 we see that the linear BSPDE (4.13) admits a (unique) adapted strong solution $(U,Q) \in \mathcal{H}^2$. Denote $\Delta^{\mu}U(t,x) \stackrel{\Delta}{=} U^{\mu}(t,x) - U(t,x)$ and $\Delta^{\mu}Q(t,x) \stackrel{\Delta}{=} Q^{\mu}(t,x) - Q(t,x)$, then $(\Delta^{\mu}U, \Delta^{\mu}Q) \in \mathcal{H}^2$ is the strong adapted solution of the following linear BSPDE:

$$
\begin{aligned}
\Delta^{\mu}U(t,x) &= [g^{\mu}(x) - g_x(x)] + \int_t^T \Big\{\mathcal{L}\Delta^{\mu}U(s,x) + \mathcal{M}\Delta^{\mu}Q(s,x) \\
&\quad + F_u^{\mu}(s,x)\Delta^{\mu}U(s,x) + [F_u^{\mu}(s,x) - f_u(s,x,u(s,x))]U(s,x) \\
&\quad + (F_x^{\mu}(s,x) - f_x(s,x,u(s,x))) \\
&\quad + [\frac{1}{2}(a^2)^{\mu}(s,x) - a(s,x)a_x(s,x)]u_{xx}(s,x+\mu) \\
&\quad + a(s,x)a_x(s,x)[u_{xx}(s,x+\mu) - u_{xx}(s,x)] \\
&\quad + [b^{\mu}(s,x) - b_x(s,x)]u_x(s,x+\mu) \\
&\qquad + b_x(s,x)[u_x(s,x+\mu) - u_x(s,x)] \\
&\quad + [c^{\mu}(s,x) - c_x(s,x)]u(s,x+\mu) \\
&\qquad + c_x(s,x)[u(s,x+\mu) - u(s,x)]\Big\}ds \\
&\quad - \int_t^T \Delta Q^{\mu}(s,x)dW(s).
\end{aligned} \tag{4.15}
$$

Note that ($\widetilde{\mathbf{A}}^1$) holds for equation (4.15), applying Theorem 3.1 (with $k = 0$) we have

$$\begin{aligned}
&\max_{t\in[0,T]} E\|\Delta U^\mu(t)\|^2_{H^0} + E\int_0^T \{\|\Delta U^\mu(s)\|^2_{H^1} + \|\Delta Q^\mu(s)\|^2_{H^0}\}dt \\
&\le CE\Big\{\|g^\mu - g_x\|^2_{H^0} + \int_0^T \Big[\|[F^\mu_u(s,\cdot) - f_u(s,\cdot,u(s,\cdot))]U(s)\|^2_{H^0} \\
&\quad +\|(F^\mu_x(s,\cdot) - f_x(s,\cdot,u(s,\cdot)))\|^2_{H^0} + \|[\frac{1}{2}(a^2)^\mu - aa_x]u_{xx}(s,\cdot+\mu)\|^2_{H^0} \\
&\quad +\|aa_x[u_{xx}(s,\cdot+\mu) - u_{xx}(s,\cdot)]\|^2_{H^0} + \|[b^\mu - b_x]u_x(s,\cdot+\mu)\|^2_{H^0} \\
&\quad +\|b_x[u_x(s,\cdot+\mu) - u_x(s,\cdot)]\|^2_{H^0} + \|[c^\mu - c_x]u(s,\cdot+\mu)\|^2_{H^0} \\
&\quad +\|c_x[u(s,\cdot+\mu) - u(s,\cdot)]\|^2_{H^0}\Big]ds\Big\}. \qquad (4.16)
\end{aligned}$$

By our assumption, we can easily prove that the right side above tends to 0 as $\mu \to 0$. Thus $\lim_{\mu\to 0}(\Delta^\mu U, \Delta^\mu Q) = 0$ in $\mathcal{H}^1$, which implies $U = u_x$ and $Q = q_x$. Consequently, since $(U, Q) \in \mathcal{H}^2$, we see that $(u, q) \in \mathcal{H}^3$.

On the other hand, we can rewrite (4.13) as follows:

$$\begin{aligned}
U(t,x) &= g_x(x) + \int_t^T \{\mathcal{L}U(s,x) + \mathcal{M}Q(s,x) + a(s,x)a_x(s,x)U_x(s,x) \\
&\quad +[b_x(s,x) + f_u(s,x,u(s,x))]U(s,x) \\
&\quad +c_x(s,x)u(s,x) + f_x(s,x,u)\}ds - \int_t^T Q(s,x)dW(s). \qquad (4.17)
\end{aligned}$$

Then, by Theorem 3.2, we see that $U = u_x$ is bounded.

The higher regularity can also be obtained by repeating the above procedure. But since it is almost identical to the first part, we only point out the differences. Note that formally differentiating (4.13) and denoting $(\widehat{U}, \widehat{Q})$ to be the weak adapted solution of the resulting BSPDE, we have:

$$\begin{aligned}
\widehat{U}(t,x) &= g_{xx}(x) + \int_t^T \Big\{\mathcal{L}\widehat{U}(s,x) + \mathcal{M}\widehat{Q}(s,x) + 2a(s,x)a_x(s,x)\widehat{U}_x(s,x) \\
&\quad +[a(s,x)a_{xx}(s,x) + a_x(s,x)^2 + 2b_x(s,x) + f_u(s,x,u(s,x))]\widehat{U}(s,x) \\
&\quad +\big\{f_{uu}(s,x,u(s,x))u_x(s,x)^2 \\
&\quad +[b_{xx}(s,x) + 2c_x(s,x) + 2f_{xu}(s,x,u(s,x))]u_x(s,x) \\
&\quad +c_{xx}(s,x)u(s,x) + f_{xx}(s,x,u(s,x))\Big\}ds - \int_t^T \widehat{Q}(s,x)dW(s). \qquad (4.18)
\end{aligned}$$

Now, thanks to the assumption ($\mathbf{F}^\kappa_c$) and the uniform boundedness of u_x, we know from Theorems 3.1 and 3.2 that (4.18) has a unique strong adapted solution $(\widehat{U}, \widehat{Q}) \in \mathcal{H}^2$, and that $\widehat{U}$ is continuous and uniformly bounded. Repeating the same argument as in the first part we can then show that $\widehat{U} = u_{xx}$ and $\widehat{Q} = q_{xx}$. In other words, we now have $(u, q) \in \mathcal{H}^4$ and u_{xx} is bounded. Continuing this procedure we obtain that $(u, q) \in \mathcal{H}^{\kappa+1}$ and that (4.6) holds true.

Finally, by the equation (4.18) (which is the equation for (u_{xx}, q_{xx})) and the Sobolev Embedding Theorem, we see that $u \in C_{\mathcal{F}}([0, T]; C^2(B_R))$, and $q \in L^2_{\mathcal{F}}(0, T; C^1(B_R))$, for any $R > 0$. Hence, (u, q) is actually a classical adapted solution of (1.3). □

We remark here that the last conclusion of Theorem 4.3 takes the advantage that our problem is one-dimensional (hence the Sobolev embedding applies). In the higher dimensional case the situation will be more complicated, but the "bootstrap" method that we have applied should still be useful. Also, we note that the super-parabolicity condition (1.5) plays an important role here so that we only need condition $(\mathbf{A}^\kappa)$ (instead of $(\mathbf{A}^{\kappa+1})$ for the coefficients of the differential operator to obtain $(u, q) \in \mathcal{H}^{\kappa+1}$ (see Theorem 3.1 for a similar situation).

5. The parabolic case

In this section we consider the possible degenerate case, that is, we assume only the parabolicity condition (1.4) instead of the super-parabolicity condition (1.5). The idea here is that some of the previous a priori estimates, especially the uniform boundedness results in Theorems 3.2 and 4.3, were proved without using the super-parabolicity condition. Therefore we can reach our goal by the standard approach of "vanishing viscosity". We proceed as follows.

Define, for each $\varepsilon > 0$, $\mathcal{L}^\varepsilon$ to be the perturbed differential operator:

$$(\mathcal{L}^\varepsilon \varphi)(t, x) \stackrel{\Delta}{=} \frac{1}{2}\big[a(t, x)^2 + \varepsilon\big]\varphi_{xx}(t, x) + b(t, x)\varphi_x(t, x) + c(t, x)\varphi(t, x), \quad \forall \varphi \in C^2. \tag{5.1}$$

Now assume that $(\mathbf{A}^1)$, $(\mathbf{F}^1)$ and $(\mathbf{G}^{1,\infty})$ hold. Then from Corollary 4.2 we know that the semi-linear BSPDE

$$u(t, x) = g(x) + \int_t^T \{\mathcal{L}^\varepsilon u(s, x) + \mathcal{M}q(s, x) + f(s, x, u)\}ds - \int_t^T q\,dW(s) \tag{5.2}$$

admits a unique strong adapted solution, denoted by $(u^\varepsilon, q^\varepsilon)$. We shall prove that, under some slightly stronger conditions, the family $\{(u^\varepsilon, q^\varepsilon)\}$ has a limit point in $\mathcal{H}^1$ as $\varepsilon \to 0$, which will turn out to be the weak adapted solution to (1.3).

We have the following result.

Theorem 5.1. *Let the parabolicity condition (1.4) hold; and assume* $(\mathbf{A}^2_c)$, $(\mathbf{F}^2_c)$, $(\mathbf{G}^{2,\infty}_c)$ *and* $(\widehat{\mathbf{A}})$. *Then, the semi-linear BSPDE (1.3) admits a unique strong adapted solution* $(u, q) \in \mathcal{H}^2$, *with* u *and* u_x *being bounded.*

If in addition $(\mathbf{A}^3_c)$, $(\mathbf{F}^3_c)$ *and* $(\mathbf{G}^{3,\infty}_c)$ *hold, then this strong solution becomes a classical adapted solution, with* u, u_x *and* u_{xx} *all being bounded.*

Proof. We first assume that $(\mathbf{A}^3_c)$, $(\mathbf{F}^3_c)$, $(\mathbf{G}^{3,\infty}_c)$ and $(\widehat{\mathbf{A}})$ hold. For each $\varepsilon > 0$, by Theorem 4.3, we let $(u^\varepsilon, q^\varepsilon)$ be the unique classical adapted solution of the perturbed BSPDE $(5.1)_\varepsilon$. Let $\{\varepsilon_n\}$ be such that $\varepsilon_n \downarrow 0$, as $n \to \infty$, and denote $(u^n, q^n) = (u^{\varepsilon_n}, q^{\varepsilon_n})$. Applying Theorem 4.3, we see that u^n, u^n_x and u^n_{xx} are uniformly bounded, uniformly in $n \geq 1$. Next, applying Theorem 3.1 (with $k = 3$ there), we have

$$\begin{aligned} E\|u^n(t)\|^2_{H^3} &+ E\int_t^T \Big\{\|u^n(s)\|^2_{H^3} + \|q^n\|^2_{H^2}\Big\}ds \\ &\leq C\Big\{E\int_t^T \|f(s,\cdot,u^n)\|^2_{H^3}ds + E\|g\|^2_{H^3}\Big\} \\ &\leq C\Big\{E\int_t^T [1 + \|u^n(s)\|^2_{H^3}]ds + E\|g\|^2_{H^3}\Big\}. \end{aligned} \tag{5.3}$$

Thus, by Gronwall's inequality we obtain

$$E\|u^n(t)\|^2_{H^3} + E\int_t^T \Big\{\|u^n(s)\|^2_{H^3} + \|q^n(s)\|^2_{H^2}\Big\}ds \leq C\big\{1 + E\|g\|^2_{H^3}\big\}. \tag{5.4}$$

Define $(u^{n,m}, q^{n,m}) = (u^n - u^m, q^n - q^m)$. Then $(u^{n,m}, q^{n,m})$ is the strong adapted solution to the following (linear) BSPDE:

$$\begin{aligned} u^{n,m}(t,x) &= \int_t^T \Big\{\mathcal{L}u^{n,m}(s,x) + \mathcal{M}q^{n,m}(s,x) + (\varepsilon_n u^n_{xx}(s,x) - \varepsilon_m u^m_{xx}(s,x)) \\ &\qquad + (f(s,x,u^n) - f(s,x,u^m)\Big\}ds - \int_t^T q^{n,m}(s,x)dW(s) \\ &= \int_t^T \Big\{\mathcal{L}^{\varepsilon_n}u^{n,m}(s,x) + \mathcal{M}q^{n,m}(s,x) + (\varepsilon_n - \varepsilon_m)u^m_{xx}(s,x)) \\ &\qquad + (f(s,x,u^n) - f(s,x,u^m)\Big\}ds - \int_t^T q^{n,m}(s,x)dW(s). \end{aligned} \tag{5.5}$$

Now by (3.3), we see that the following estimate holds true:

$$\begin{aligned} E\|u^{n,m}(t)\|^2_{H^1} &+ E\int_t^T \|q^{n,m}(s)\|^2_{H^0}ds \\ &\leq CE\int_t^T \{\|(\varepsilon_n - \varepsilon_m)u^m_{xx}\|^2_{H^1} + \|f(s,\cdot,u^n) - f(s,\cdot,u^m)\|^2_{H^1}\}ds \\ &\leq E\int_t^T \{C|\varepsilon_n - \varepsilon_m|^2\|u^m\|^2_{H^3} + K^2\|u^{n,m}\|^2_{H^1}\}ds. \end{aligned} \tag{5.6}$$

Therefore, by Gronwall's inequality again we have

$$E\|u^{n,m}(t)\|^2_{H^1} + E\int_t^T \|q^{n,m}\|^2_{H^0}ds \leq C|\varepsilon_n - \varepsilon_m|^2 E\int_t^T \|u^m\|^2_{H^3}ds, \tag{5.7}$$

with slightly modified constant $C > 0$. Consequently, we obtain that (u^n, q^n) is Cauchy in $\mathcal{H}^1$. Let (u, q) be the limit of $\{(u^n, q^n)\}$, it is then obviously a weak solution of (1.3).

On the other hand, (5.4) yields the weak convergence of some subsequence of (u^n, q^n) to (u, q) in $\mathcal{H}^3$. Hence, by the uniqueness of the limit, it is necessary that $(u, q) \in \mathcal{H}^3$. This implies that (u, q) must be at least a strong solution of (1.3). Also, we must have the boundedness of u, u_x and u_{xx}. Now, using the equation (4.18) for u_{xx}, together with the Sobolev Embedding Theorem, we see that $u \in C_{\mathcal{F}}([0, T]; C^2(B_R))$ and $q \in L^2_{\mathcal{F}}(0, T; C^1(B_R))$ for any $R > 0$. In other words, (u, q) is actually a classical adapted solution to (1.3).

Next, we assume that $(\mathbf{A}^2_c)$, $(\mathbf{F}^2_c)$, $(\mathbf{G}^{2,\infty}_c)$ and $(\widehat{\mathbf{A}})$ hold. We approximate all the coefficients by functions which are smooth in x. To be more precise, we let $\psi \in C^\infty_0(\mathbb{R})$ be a nonnegative function, supported on $[-1, 1]$ such that $\int_{\mathbb{R}} \psi(x)dx = 1$. Define

$$a^n(t, x, \omega) = n \int_{\mathbb{R}} a(t, x - y, \omega)\psi(ny)dy, \quad (t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega. \tag{5.8}$$

Then $a^n \in L^\infty_{\mathcal{F}}(0, T; C^\infty)$ and since $a \in L^\infty_{\mathcal{F}}(0, T; C^3_b)$, we have

$$\lim_{n\to\infty} \|a^n(t, \cdot, \omega) - a(t, \cdot, \omega)\|_{L^\infty_{\mathcal{F}}(0,T;C^3_b)} = 0. \tag{5.9}$$

Similarly, we can construct b^n, c^n, f^n, and g^n which are smooth in x such that

$$\begin{cases} b^n \to b, & \text{in } L^\infty_{\mathcal{F}}(0, T; C^2_b), \\ c^n \to c, & \text{in } L^\infty_{\mathcal{F}}(0, T; C^2_b), \\ f^n \to f, & \text{in } L^\infty_{\mathcal{F}}(0, T; C^2_b(\mathbb{R}^2)), \\ g^n \to g, & \text{in } L^\infty_{\mathcal{F}_T}(\Omega; C^2_b). \end{cases} \tag{5.10}$$

(We should point out here that there is no need to approximate α and γ, thanks to $(\widehat{\mathbf{A}})$.) Denote $\mathcal{L}^n$ to be the corresponding differential operator, and (u^n, q^n) be the classical adapted solution of the corresponding BSPDE:

$$\begin{aligned} u^n(t, x) = \ & g^n(x) + \int_t^T \left\{\mathcal{L}^n u^n(s, x) + \mathcal{M}q^n(s, x) + f^n(s, x, u^n)\right\}ds \\ & - \int_t^T q^n(s, x)dW(s). \end{aligned} \tag{5.11}$$

Furthermore, the same proof as that of Theorem 4.3 shows that u^n and u^n_x are bounded uniformly in $n \geq 1$ (note that the higher derivatives are not necessarily uniformly bounded since we only have (5.9)–(5.10)). Next, similar to (5.3)–(5.4), we have

$$E\|u^n(t)\|^2_{H^2} + E\int_t^T \left\{\|u^n(s)\|^2_{H^2} + \|q^n(s)\|^2_{H^1}\right\}ds \leq C\left\{1 + E\|g\|^2_{H^2}\right\}. \tag{5.12}$$

Now, we set $(u^{n,m}, q^{n,m}) = (u^n - u^m, q^n - q^m)$. Then $(u^{n,m}, q^{n,m})$ is a strong adapted solution to the following (linear) BSPDE:

$$\begin{aligned} u^{n,m}(t,x) = g^n(x) - g^m(x) + \int_t^T &\Big\{\mathcal{L}^n u^{n,m}(s,x) + \mathcal{M} q^{n,m}(s,x) \\ &+(\mathcal{L}^n - \mathcal{L}^m)u^m(s,x) + (f^n(s,x,u^n) - f^m(s,x,u^m)\Big\}ds \\ &- \int_t^T q^{n,m}(s,x)dW(s). \end{aligned} \tag{5.13}$$

Applying Theorem 3.1, we have

$$\begin{aligned} &E\|u^{n,m}(t)\|_{H^0} + E\int_t^T \|u^{n,m}(s)\|^2_{H^0} ds \\ &\quad\le CE\Big\{\int_t^T \Big[\|(\mathcal{L}^n - \mathcal{L}^m)u^m\|^2_{H^0} + \|f^n(s,x,u^n) - f^m(s,x,u^m)\|^2_{H^0}\Big]ds \\ &\qquad + \|g^n - g^m\|^2_{H_0}\Big\} \\ &\quad\le CE\Big\{\int_t^T \Big[\|u^{n,m}(s)\|^2_{H^0} + \|(\mathcal{L}^n - \mathcal{L}^m)u^m\|^2_{H^0} \\ &\qquad + \|f^n(s,x,u^m) - f^m(s,x,u^m)\|^2_{H^0}\Big]ds + \|g^n - g^m\|^2_{H_0}\Big\}. \end{aligned} \tag{5.14}$$

By Gronwall's inequality, we obtain

$$\begin{aligned} &E\|u^{n,m}(t)\|_{H^0} + E\int_t^T \|u^{n,m}(s)\|^2_{H^0} ds \\ &\quad\le CE\Big\{\int_t^T \Big[\|(\mathcal{L}^n - \mathcal{L}^m)u^m\|^2_{H^0} + \|f^n(s,x,u^m) - f^m(s,x,u^m)\|^2_{H^0}\Big]ds \\ &\qquad + \|g^n - g^m\|^2_{H_0}\Big\}. \end{aligned} \tag{5.15}$$

Let us now estimate each term on the right hand side of (5.15). It is clear by (5.10) that the third term goes to 0 as $n, m \to \infty$. Next, using the boundedness and convergence of f^n, and the Dominated Convergence Theorem, we see that the second term on the right side of (5.15) tends to 0 as well. Finally, from (5.9)–(5.10) and (5.12), we have

$$\begin{aligned} &E\int_t^T \|(\mathcal{L}^n - \mathcal{L}^m)u^m\|^2_{H^0} ds \\ &\quad\le CE\int_t^T \Big[\|(a^n - a^m)u^m_{xx}\|^2_{H^0} + \|(b^n - b^m)u^m_x\|^2_{H^0} + \|(c^n - c^m)u^m\|^2_{H^0}\Big]ds \\ &\quad\le C\Big(\|a^n - a^m\|^2_{L^\infty_{\mathcal{F}}(0,T;C^0_b)} + \|b^n - b^m\|^2_{L^\infty_{\mathcal{F}}(0,T;C^0_b)} \\ &\qquad + \|c^n - c^m\|_{L^\infty_{\mathcal{F}}(0,T;C^0_b)}\Big) E\int_0^T \|u^m(s)\|^2_{H^2} ds \to 0. \end{aligned} \tag{5.16}$$

Hence, we obtain that u^n is Cauchy in $L^2_{\mathcal{F}}(0, T; H^0)$, and denote $\overline{u}$ to be the limit of u^n in $L^2_{\mathcal{F}}(0, T; H^0)$.

Next, from (5.12) we see that, along some subsequence $\{n_i\}$, one has

$$\begin{cases} u^{n_i} \to u, & \text{weakly in } L^2_{\mathcal{F}}(0, T; H^2), \\ q^{n_i} \to q, & \text{weakly in } L^2_{\mathcal{F}}(0, T; H^1). \end{cases} \tag{5.17}$$

Clearly, u has to be the same as $\overline{u}$ obtained above. Furthermore, by Mazur's Theorem (cf. e.g., [5]) we can find numbers $\lambda_{ij} \geq 0$, $\sum_{j\geq 0} \lambda_{ij} = 1$, such that

$$\begin{cases} \widetilde{u}^i \stackrel{\Delta}{=} \sum_{j\geq 0} \lambda_{ij} u^{n_{i+j}} \to u, & \text{strongly in } L^2_{\mathcal{F}}(0, T; H^2), \\ \widetilde{q}^i \stackrel{\Delta}{=} \sum_{j\geq 0} \lambda_{ij} q^{n_{i+j}} \to q, & \text{strongly in } L^2_{\mathcal{F}}(0, T; H^1). \end{cases} \tag{5.18}$$

Now, for any $\zeta \in C^\infty([0, T]\times\mathbb{R}; \mathbb{R})$ such that $\zeta(s, \cdot) \in C_0^\infty(\mathbb{R})$ for each $s \in [0, T]$, and $\zeta(t, x) = 0$ for any $x \in \mathbb{R}$, applying Itô's formula to $u^n(s, x)\zeta(s, x)$, and using the BSPDE for u^n, we obtain that

$$\begin{aligned} &-\int_{\mathbb{R}} u^n(T, x)\zeta(T, x)dx \\ &= \int_t^T \int_{\mathbb{R}} \Big\{ -u^n\zeta_s - \frac{1}{2}(a^n)^2 u^n_x \zeta_x + \widetilde{b}^n u^n_x \zeta - \alpha q^n \zeta_x + \widetilde{\gamma} q^n \zeta \\ &\qquad + f^n(s, x, u^n)\zeta \Big\} dxds - \int_t^T \int_{\mathbb{R}} q^n \zeta dxdW(s). \end{aligned} \tag{5.19}$$

Making convex combination of (5.19), we have

$$\begin{aligned} &-\int_{\mathbb{R}} \widetilde{u}^i(T, x)\zeta(T, x)dx \\ &= \int_t^T \int_{\mathbb{R}} \Big\{ -\widetilde{u}^i\zeta_s - \frac{1}{2}a^2 \widetilde{u}^i_x \zeta_x + \widetilde{b}\widetilde{u}^i_x \zeta - \alpha \widetilde{q}^i \zeta_x + \widetilde{\gamma}\widetilde{q}^i \zeta \\ &\quad + \sum_{j\geq 0} \lambda_{ij} f^{n_{i+j}}(s, x, u)\zeta \\ &\quad + \frac{1}{2}\sum_{j\geq 0} \lambda_{ij}[(a^{n_{i+j}})^2 - a^2] u_x^{n_{i+j}} \zeta_x + \sum_{j\geq 0} \lambda_{ij}[\widetilde{b}^{n_{i+j}} - \widetilde{b}] u_x^{n_{i+j}} \zeta \\ &+ \sum_{j\geq 0} \lambda_{ij}[f^{n_{i+j}}(s, x, u^{n_{i+j}}) - f^{n_{i+j}}(s, x, u)]\zeta \Big\} dxds - \int_t^T \int_{\mathbb{R}} \widetilde{q}^i \zeta dxdW(s). \end{aligned} \tag{5.20}$$

Passing to the limit in the above yields

$$\begin{aligned}&\int_t^T\int_{\mathbb{R}} u\zeta_s dxds-\int_{\mathbb{R}} u(T,x)\zeta(T,x)dx\\&\quad=\int_t^T\int_{\mathbb{R}}\Big\{-\frac{1}{2}a^2u_x\zeta_x+\widetilde{b}u_x\zeta-\alpha q\zeta_x+\widetilde{\gamma}q\zeta+f(s,x,u)\zeta\Big\}dxds\\&\qquad-\int_t^T\int_{\mathbb{R}} q\zeta dxdW(s).\end{aligned}\tag{5.21}$$

Finally, for any $\varphi\in C_0^\infty$, $t\in[0,T)$ and $\varepsilon>0$ with $t+\varepsilon\le T$, we take

$$\zeta(s,x)=\begin{cases}\varphi(x), & s\in[t+\varepsilon,T],\\ \frac{s-t}{\varepsilon}\varphi(x), & s\in(t,t+\varepsilon),\\ 0, & s\in[0,t].\end{cases}\tag{5.22}$$

Then (5.21) becomes

$$\begin{aligned}&\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}} u(s,x)\varphi(x)dxds-\int_{\mathbb{R}} u(T,x)\varphi(x)dx\\&=\int_{t+\varepsilon}^T\int_{\mathbb{R}}\Big\{-\frac{1}{2}a^2u_x\varphi_x+\widetilde{b}u_x\varphi-\alpha q\varphi_x+\widetilde{\gamma}q\varphi+f(s,x,u)\varphi\Big\}dxds\\&\quad-\int_{t+\varepsilon}^T\int_{\mathbb{R}} q\varphi dxdW(s)\\&\quad+\int_t^{t+\varepsilon}\int_{\mathbb{R}}\frac{s-t}{\varepsilon}\Big\{-\frac{1}{2}a^2u_x\varphi_x+\widetilde{b}u_x^n\varphi-\alpha q\varphi_x+\widetilde{\gamma}q\varphi+f(s,x,u)\varphi\Big\}dxds\\&\quad-\int_t^{t+\varepsilon}\int_{\mathbb{R}}\frac{s-t}{\varepsilon}q\varphi dxdW(s).\end{aligned}\tag{5.23}$$

Now, sending $\varepsilon\to 0$, we obtain that

$$\begin{aligned}&\int_{\mathbb{R}} u(t,x)\varphi(x)dxds-\int_{\mathbb{R}} u(T,x)\varphi(x)dx\\&\quad=\int_t^T\int_{\mathbb{R}}\Big\{-\frac{1}{2}a^2u_x\varphi_x+\widetilde{b}u_x\varphi-\alpha q\varphi_x+\widetilde{\gamma}q\varphi+f(s,x,u)\varphi\Big\}dxds\\&\qquad-\int_t^T\int_{\mathbb{R}} q\varphi dxdW(s),\quad dP\times dt\text{-a.e.}\end{aligned}\tag{5.24}$$

After a possible modification on a set of $dP\times dt$-measure 0, and noting that $u=\overline{u}$, we see that (2.14) holds and, hence, (u,q) is a weak solution of (1.3).

Moreover, due to the convergence (5.17), we see that $(u,q)\in\mathcal{H}^2$. Therefore, (u,q) is actually a strong solution of (1.3). □

It is not clear to us at this point that if there exists a weak adapted solution under, say, parabolicity condition (1.4), $(\mathbf{A}_c^1)$, $(\mathbf{F}_c^1)$, $(\mathbf{G}_c^{1,2})$ and $(\widehat{\mathbf{A}})$. It seems to us that this should be related to some kind of viscosity solution to BSPDEs. We hope to explore that in our future publications.

Remark 5.2. We note that there are some essential difficulties here mainly due to the stochastic feature of the equation. For example, in deterministic PDE theory, if $\{u^n\}$ are solutions to some perturbed parabolic equations, with $\int_0^T \|u^n(t)\|_{H^1}^2 dt$ being uniformly bounded, then one should be able to obtain some kind of strong convergence of u^n, at least along a subsequence. In other words, the H^1-norm (in the spatial variable x), together with the equation, should give some kind of strong compactness of the sequence $\{u^n\}$, in both variables (t, x). However, this is no longer clear for SPDEs, with only the estimate on $E\int_0^T \|u^n(t)\|_{H^1}^2 dt$. Because no conclusion can be drawn in general on the "strong compactness" on the variable ω! The application of the Mazur theorem in the proof of Theorem 5.1 is to overcome this difficulty.

6. Some related results

In this section we present some direct consequences of the main results derived in the previous sections. The proof of these results are either identical or very similar to those seen in the linear case (cf. [9,10] or [11]), we shall thus give only sketches.

A. Comparison Theorems

We first look at the comparison theorem for adapted strong solutions to semilinear BSPDEs studied in §4 and §5. We begin with the superparabolic case. To make the notations more specific, in what follows we shall denote BSPDE(f, g) to be the BSPDE

$$\begin{aligned} u(t,x) = \ & g(x) + \int_t^T \{\mathcal{L}u(s,x) + \mathcal{M}q(s,x) + f(s,x,u(s,x))\}ds \\ & - \int_t^T q(s,x)dW(s), \qquad (t,x) \in [0,T]\times\mathbb{R}. \end{aligned} \tag{6.1}$$

We have the following *comparison theorem*:

Theorem 6.1. *Suppose that the super-parabolicity condition (1.5) and assumptions* $(\tilde{\mathbf{A}}^1)$ *and* $(\hat{\mathbf{A}})$ *are all in force. Let* (f, g) *and* $(\bar{f}, \bar{g})$ *satisfy* $(\mathbf{F}^1)$ *and* $(\mathbf{G}^{1,2})$*, and let* (u, q) *and* $(\bar{u}, \bar{q})$ *be the adapted strong solution to BSPDE(*f, g*) and BSPDE(*$\bar{f}$, $\bar{g}$*), respectively. Suppose further that* $g(x) \geq \bar{g}(x)$ *and* $f(t,x,u) \geq \bar{f}(t,x,u)$*, for all* $(t,x,u) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}$*, a.s., then* $u(t,x) \geq \bar{u}(t,x)$, $\forall (t,x) \in [0,T]\times\mathbb{R}^n$*, a.s.*

Proof. The proof follows the same idea of [10, Theorem 7.1], with slight modification. Define a function $\varphi : \mathbb{R} \to [0, +\infty)$ as follows:

$$\varphi(r) = \begin{cases} r^2, & r < -1, \\ (6r^3 + 8r^4 + 3r^5)^2, & -1 \leq r \leq 0, \\ 0, & r > 0. \end{cases}$$

One can check directly that φ is C^2 and $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$, $\varphi(-1) = 1$, $\varphi'(-1) = -2$, and $\varphi''(-1) = 2$. Next, for any $\epsilon > 0$, we let $\varphi_\epsilon(r) = \epsilon^2\varphi(\frac{r}{\epsilon})$. One shows that

$$\begin{cases} \lim_{\varepsilon\to 0} \varphi_\varepsilon(r) = [r^-]^2, \ \lim_{\varepsilon\to 0} \varphi'_\varepsilon(r) = -2r^-, \ \text{uniformly;} \\ |\varphi''_\varepsilon(r)| \le C, \forall \varepsilon > 0, r \in \mathbb{R}, \ \lim_{\varepsilon\to 0} \varphi''_\varepsilon(r) = \begin{cases} 2, r < 0, \\ 0, r > 0. \end{cases} \end{cases} \tag{6.2}$$

Set $\hat{u} = u - \bar{u}, \ \hat{g} = g - \bar{g}, \ \hat{q} = q - \bar{q}$. Then

$$\begin{aligned}
&E\int_{\mathbb{R}} \varphi_\varepsilon(\hat{g}(x))dx - E\int_{\mathbb{R}} \varphi_\varepsilon(\hat{u}(t,x))dx \\
&= E\int_{\mathbb{R}_t} \Big\{\frac{1}{2}\varphi''_\varepsilon(\hat{u})[a^2\hat{u}_x^2 + 2\alpha\hat{q}\hat{u}_x + \hat{q}^2] \\
&\quad -\varphi'_\varepsilon(\hat{u})[\tilde{b}\hat{u}_x + c\hat{u} + \gamma\hat{q} + (f(s,x,u) - \bar{f}(s,x,\bar{u}))]\Big\}dxds \\
&= E\int_{\mathbb{R}_t} \Big\{\frac{1}{2}\varphi''_\varepsilon(\hat{u})[(a^2-\alpha^2)\hat{u}_x^2 + (\alpha\hat{u}_x + \hat{q} - \gamma\hat{u})^2] \\
&\quad +\frac{1}{2}\varphi''_\varepsilon(\hat{u})[-\gamma^2\hat{u}^2 + 2\alpha\hat{u}_x\gamma\hat{u} + 2\tilde{\gamma}\hat{u}\hat{q}] \\
&\quad -\tilde{b}D\varphi_\varepsilon(\hat{u}) - \varphi'_\varepsilon(\hat{u})[c\hat{u} + \gamma\hat{q} + f(s,x,u) - \bar{f}(s,x,\bar{u})]\Big\}dxds \\
&\ge E\int_{\mathbb{R}_t} \Big\{-\frac{1}{2}\varphi''_\varepsilon(\hat{u})\gamma^2\hat{u}^2 + \alpha\gamma D(\int_0^{\hat{u}} \varphi''_\varepsilon(r)rdr) \\
&\quad +[\varphi''_\varepsilon(\hat{u})\hat{u} - \varphi'_\varepsilon(\hat{u})](\gamma\hat{q}) + \tilde{b}_x\varphi_\varepsilon(\hat{u}) - \varphi'_\varepsilon(\hat{u})(c\hat{u} + f(s,x,u) - \bar{f}(s,x,\bar{u}))\Big\}dxds.
\end{aligned} \tag{6.3}$$

Note that

$$\begin{cases} \int_0^u \varphi''_\varepsilon(r)rdr = \varphi'_\varepsilon(u)u - \varphi_\varepsilon(u), \\ \lim_{\varepsilon\to 0}[\varphi''_\varepsilon(u)u - \varphi'_\varepsilon(u)] = 2u\mathbf{1}_{\{u\le 0\}} + 2u^- = 0, \end{cases}$$

letting $\varepsilon \to 0$ in (6.3) and recalling (6.2) we obtain:

$$\begin{aligned}
&-E\int_{\mathbb{R}} [\hat{u}^-(t,x)]^2 dx \\
&\quad \ge E\int_{\mathbb{R}_t} \Big\{-\mathbf{1}_{\{\hat{u}\le 0\}}\gamma^2\hat{u}^2 + \tilde{b}_x|\hat{u}^-|^2 + 2\hat{u}^-(c\hat{u} + f(s,x,u) - \bar{f}(s,x,\bar{u}))\Big\}dxds.
\end{aligned}$$

Finally, observe that the comparison between f and $\bar{f}$ tells us that

$$\hat{u}^-[f(s,x,u) - \bar{f}(s,x,\bar{u})] \ge \hat{u}^-[\bar{f}(s,x,u) - \bar{f}(s,x,\bar{u})] \ge -C|\hat{u}^-|^2.$$

Therefore, one has

$$-E\int_{\mathbb{R}} [\hat{u}^-(t,x)]^2 dx \ge -CE\int_{\mathbb{R}_t} |\hat{u}^-(x,s)|^2 dxds.$$

Consequently, the Gronwall inequality leads to that $\hat{u}^- = 0$ as desired. This proves the theorem. □

The comparison theorem in the parabolic case can be treated in the similar way and thus, we only state the result here.

Theorem 6.2. *Suppose that the parabolicity condition (1.4) and assumptions* $(\tilde{\mathbf{A}}_c^2)$, $(\hat{\mathbf{A}})$ *are all in force. Let* (f, g) *and* $(\bar{f}, \bar{g})$ *satisfy* $(\mathbf{F}_c^2)$ *and* $(\mathbf{G}_c^{2,\infty})$*, and let* (u, q) *and* $(\bar{u}, \bar{q})$ *are the strong adapted solutions to BSPDE(*f, g*) and BSPDE(*$\bar{f}, \bar{g}$*), respectively. Suppose that* $g(x) \geq \bar{g}(x)$, $f(t, x, u) \geq \bar{f}(t, x, u), \forall (t, x, u)$*, a.s., then it holds that*

$$u(t, x) \geq \bar{u}(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, a.s.$$

B. A nonlinear, stochastic Feynman-Kac formula

The well-posedness of the semilinear BSPDE leads immediately to the following nonlinear, stochastic Feynman-Kac formula, extending the linear version presented in [9]. We note here that the deterministic version of such a nonlinear Feynman-Kac formula can be found in [15], and its application, among others, in homogenization of nonlinear PDEs can be found in [6].

In light of the nonlinear Feynman-Kac formula established by Pardoux-Peng [15], we consider the following (decoupled) forward-backward SDE (FBSDE, for short):

$$\begin{cases} X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dW_r; \\ Y_s = E\left\{g(X_T) + \int_s^T f(r, X_r, Y_r)dr \middle| \mathcal{F}_s\right\} \end{cases} \quad s \in [t, T]. \tag{6.4}$$

The FBSDE (6.4) has proved to be useful in mathematical finance, either as a recursive utility model or as a term structure of interests model (see [3], [4] or [11]). Note that in (6.4) all the coefficients are allowed to be random, thus if we denote the solution to (6.4) by $(X^{t,x}, Y^{t,x})$, then $v(t, x) \stackrel{\Delta}{=} Y_t^{t,x}$ is an $\mathcal{F}_t$-measurable random variable, for each fixed (t, x). (Unlike the deterministic coefficients case, v is not necessarily a deterministic function!). Following the idea of "Four Step Scheme" (cf. [8] or [11]) we assume that the BSPDE

$$\begin{cases} du(t, x) = \{-\frac{1}{2}\sigma^2(t, x)u_{xx} - b(t, x)u_x - \sigma(t, x)q_x - f(t, x, u)\}dt + qdW_t \\ u(T, x) = g(x) \end{cases} \tag{6.5}$$

has a classical solution (u, q). Then by applying the Itô-Ventzell formula to $u(s, X_s^{t,x})$ from t to T and then comparing to (6.4) one shows that $Y_s^{t,x} = u(s, X_s^{t,x})$, for all $s \in [t, T]$, P-a.s. We thus have the following *non-linear Feynman-Kac formula*, which, to our best knowledge, is new.

Theorem 6.3. *Let the parabolicity condition (1.4) hold, and assume* $(\mathbf{A}_c^3)$, $(\mathbf{F}_c^3)$, $(\mathbf{G}_c^{3,\infty})$ *and* $(\tilde{\mathbf{A}})$. *Then the BSPDE (6.5) admits a unique adapted classical solution. Furthermore, the following relation hold:*

$$\begin{cases} Y_s^{t,x} = u(s, X_s^{t,x}), \\ Z_s^{t,x} = \sigma(s, X_s^{t,x})u_x(s, X_s^{t,x}) + q(s, X_s^{t,x}), \qquad \forall s \in [t, T], \end{cases}$$

where $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *is the adapted solution to the FBSDE (6.4).*

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