A General Conditional McKean-Vlasov Stochastic Differential Equation

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Abstract

In this paper we consider a class of *conditional McKean-Vlasov SDEs* (CMVSDE for short). Such an SDE can be considered as an extended version of McKean-Vlasov SDEs with common noises, as well as the general version of the so-called *conditional mean-field SDEs* (CMFSDE) studied previously by the authors [1, 16], but with some fundamental differences. In particular, due to the lack of compactness of the iterated conditional laws, the existing arguments of Schauder's fixed point theorem do not seem to apply in this situation, and the heavy nonlinearity on the conditional laws caused by change of probability measure adds more technical subtleties. Under some structural assumptions on the coefficients of the observation equation, we prove the well-posedness of the solutions in a weak sense along a more direct approach. Our result is the first that deals with McKean-Vlasov type SDEs involving state-dependent conditional laws.

Keywords. Conditional McKean-Vlasov SDEs, Kantorovich-Rubinstein's duality, weak solution.

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1 Introduction

In this paper we are interested in the well-posedness of the following general form of *conditional McKean-Vlasov* stochastic differential equations (SDEs), defined on a certain filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\})$:

$$\begin{cases} dX_t = b(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu^{X|Y}_{\cdot\wedge t})dt + \sum_{i=1}^{2} \sigma_i(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu^{X|Y}_{\cdot\wedge t})dB^i_t, & X_0 = x; \\ dY_t = h(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu^{X|Y}_{\cdot\wedge t})dt + \hat{\sigma} dB^2_t, & Y_0 = 0, \end{cases} \tag{1.1}$$

where b, h, σ_1, σ_2 are measurable functions defined on appropriate spaces, $\hat{\sigma}$ is a constant, (B^1, B^2) is an $(\mathbb{F}, \mathbb{P})$-Brownian motion, and $\mu^{X|Y}_t(\cdot) := \mathbb{P}\{X_t \in \cdot \,|\mathcal{F}^Y_t\}$ denotes the regular conditional distribution of X_t given $\mathcal{F}^Y_t = \sigma\{Y_s : s \le t\}$.

Special forms of SDE (1.1) have appeared in many applications, especially those involving partial information, and have been studied by the authors in different co-authorships in the past (see, for example, [1], [16]). In many of these applications the conditional law appears in the form of conditional expectations $\mathbb{E}[X_t|\mathcal{F}^Y_t]$, in the spirit of the nonlinear filtering problems, and hence are often referred to as *conditional mean-field* SDEs. Consequently, the coefficients of these SDEs depend either linearly on $\mathbb{E}[X_t|\mathcal{F}^Y_t]$ (see, e.g., [16]), or linearly on the law of $\mathbb{E}[X_t|\mathcal{F}^Y_t]$ (see, e.g., [1]). We should note that SDEs with a general coupling between the solutions and their conditional law in the coefficients such as (1.1) have not been completely explored yet in the literature. In fact, there seem to be some fundamental difficulties when the usual solution methods are employed.

SDE (1.1) can also be viewed from another angle. Assuming for example $h \equiv 0$, then $\mathcal{F}^Y \equiv \mathcal{F}^B$, and the SDE becomes the so-called McKean-Vlasov SDE with common noise. We refer to [3–5, 9, 15] and the references cited therein for various recent studies for SDEs with similar natures and their applications. In this case, two facts are worth noting: 1) the probability measure determining the conditional law is fixed throughout; and 2) the conditioning filtration is given exogenously, and is independent of the state X. The case when the coefficient $h \ne 0$, however, is quite different. Since the "observation" process Y depends on X, the conditioning filtration $\mathcal{F}^Y$ becomes *state-dependent*, whence endogenous. Among other complications caused by such a "coupling" nature, one of the severe consequences is that the conditional laws $\{\mu^{X|Y}\}$ are no longer compact, losing an important technical basis of the well-posedness arguments for McKean-Vlasov SDEs with common noises (see, e.g., [9]).

To illustrate this point, let us ask the following simple question one would encounter naturally in constructing any iteration scheme in seeking the solution for SDE (1.1): Given a pair of random variables taking values in any metric space, does the strong convergence $(X^n, Y^n) \to (X, Y)$ imply the convergence $\mu^{X^n|Y^n} \to \mu^{X|Y}$, in the sense of probability distributions? The answer to this question unfortunately negative. For example, let $X^n \equiv U$, where U is a random variable such that $\mathrm{Var}(U) > 0$ (whence $\mathbb{P}\{U \neq \mathbb{E}^{\mathbb{P}}[U]\} > 0$), and $Y^n = \frac{1}{n}X_n = \frac{1}{n}U$. Then $X^n \to X = U$, and $Y^n \to Y \equiv 0$, as $n \to \infty$. Obviously, for suitable non-constant, bounded measurable function f, we have, for any n, $\mathbb{E}[f(X^n)|\mathcal{F}^{Y^n}] = f(U) \neq \mathbb{E}[f(U)] = \mathbb{E}[f(X)|\mathcal{F}^Y]$. This shows, in particular, that $\mu^{X^n|Y^n}$ does not converge to $\mu^{X|Y}$. We note, however, that in the usual common noise case the conditioning σ-field is fixed (i.e., $Y^n = Y = B^2$), so such a problem does not occur.

In light of the nonlinear filtering theory, a tempting remedy to "fix" the conditioning fitration is to consider the so-called *reference measure* $\mathbb{Q}^0$, a *prior* probability measure that is equivalent to $\mathbb{P}$ but under which (B^1, Y) is a Brownian motion. But doing so would lead to another dilemma: The conditional law in (1.1) is defined under the original probability $\mathbb{P}$ (under which (B^1, B^2) is a Brownian motion), not the reference measure $\mathbb{Q}^0$. The two conditional laws can be connected via the Bayes rule (known as the *Kallianpur-Strieble* formula), but will inevitably cause some serious technical issues, especially when the conditional law $\mu^{X|Y}$ (under $\mathbb{P}$) is now a part of the solution of the CMVSDE (1.1).

Our plan of attack is based on the following basic ideas. We shall design an iteration scheme which would include the conditional law $\mu^{X|Y}$ as a component, considered as *measure-valued* process defined on an appropriate space where the weak convergence can be more conveniently analyzed. More specifically, we shall argue that $\mu^{X|Y}$ is a measure-valued process that has continuous paths in the space of probability laws under the Wasserstein metric, and that it can be identified as part of the fixed point, along with the processes (X, Y). The main difficulty in implementing such an idea is that throughout the process we need to use the reference probability $\mathbb{Q}^0$, via the Kallianpur-Strieble formula. This leads to some new technicalities that are not commonly seen in the existing literature of nonlinear filtering or the McKean-Vlasov SDEs with common noises. In particular, it seems that a certain boundedness of the Girsanov kernel involved in connecting reference measure and the original ones (in both directions) becomes inevitable, and it essentially amounts to asking for a pathwise bound for the solution of a linear SDE (or a martingale), which is next to impossible. As a consequence, we shall impose a structural assumption on the observation drift coefficient h, and we hope to be able to remove such restrictions in our future works.

This paper is organized as follows. In Section 2 we introduce the basic notations, definitions, and assumptions. In particular, we shall define the processes of the conditional laws, and establish some basic facts on its path regularities in terms of the Wasserstein metric. In Section 3 we introduce our solution scheme and give some justifications of our main ideas. In Section 4 we establish our fundamental estimates. In the Sections 5 and 6 we prove the existence and uniqueness (in law) of the weak solution, respectively.

2 Preliminaries

Throughout this paper we denote $\mathbb{C}_T := \mathbb{C}([0,T],\mathbb{R})$, and $\mathbb{P}^0$ to be the Wiener measure on $\mathbb{C}_T$. We shall consider the following *canonical space* $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$:

$$\Omega^0 := \mathbb{C}_T^2 := \mathbb{C}([0,T];\mathbb{R}^2), \quad \mathcal{F}^0 := \mathscr{B}(\mathbb{C}_T^2), \quad \mathbb{Q}^0 := \mathbb{P}^0 \otimes \mathbb{P}^0. \tag{2.1}$$

In the above, $\mathscr{B}(\mathbb{C}_T^2)$ denotes the Borel σ-field on $\mathbb{C}_T^2$. Furthermore, we denote (B^1, Y) to be the canonical process, that is, $(B_t^1, Y_t)(\omega) = (\omega^1(t), \omega^2(t))$, $t \in [0,T]$, where $\omega = (\omega^1, \omega^2) \in \mathbb{C}_T^2$. Then (B^1, Y) is a 2-dimensional-Brownian motion under $\mathbb{Q}^0$. Also, we define $\mathbb{F}^0 = \{\mathcal{F}_t^0\}_{t\in[0,T]} := \{\mathscr{B}_t(\mathbb{C}_T^2)\}_{t\in[0,T]}$, where $\mathscr{B}_t(\mathbb{C}_T^2) := \sigma\{\omega(\cdot \wedge t) : \omega \in \mathbb{C}_T^2\}$, to be the natural filtration generated by (B^1, Y), and we denote $\mathbb{F} := \overline{\mathbb{F}^0}^{\mathbb{Q}^0}$, the augmentation of $\mathbb{F}^0$ under $\mathbb{Q}^0$, so that $\mathbb{F}$ satisfies the *usual hypotheses*.

Now let $(\mathscr{X}, d)$ be any metric space, and $\mathscr{B}(\mathscr{X})$ the topological Borel σ-field on $\mathscr{X}$. For any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}^0$, and $p \geq 1$, we denote $\mathbb{L}_{\mathcal{G}}^p(\mathscr{X})$ to be the space of all random variables $\xi \mapsto \mathscr{X}$, such that ξ is $\mathcal{G}$-measurable and for any/some $e \in \mathscr{X}$, $\mathbb{E}^{\mathbb{Q}^0}[d(e,\xi)^p] < \infty$. Similarly, for a sub-filtration $\mathbb{G} \subseteq \mathbb{F}$, and $p \geq 1$, we let $\mathbb{L}_{\mathbb{G}}^p([0,T];\mathscr{X})$ be the space of all $\mathscr{X}$-valued, $\mathbb{L}^p$-integrable, $\mathbb{G}$-adapted processes on $[0,T]$. Furthermore, we denote $\mathscr{C}_T(\mathscr{X})$ to be all $\mathscr{X}$-valued continuous functions defined on $[0,T]$, and denote $\mathbb{L}_{\mathbb{G}}^0(\mathscr{C}_T(\mathscr{X}))$ to be the space of all $\mathscr{X}$-valued, $\mathbb{G}$-adapted continuous processes. Finally, for any $p \geq 1$, we define

$$\begin{cases} \mathbb{S}_{\mathbb{G}}^p(\mathscr{X}) := \{z \in \mathbb{L}_{\mathbb{G}}^0(\mathscr{C}_T(\mathscr{X})) : \ \exists\, e \in \mathscr{X}, \ \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{t\in[0,T]} d(z_t, e)^p\Big] < +\infty\}; \\ \\ \mathbb{S}_{\mathbb{G}}^{\infty-}(\mathscr{X}) := \bigcap_{p\geq 1} \mathbb{S}_{\mathbb{G}}^p(\mathscr{X}). \end{cases} \tag{2.2}$$

Let us now denote $\mathscr{P}(\mathscr{X})$ to be the space of all probability measures on the metric space $\mathscr{X}$, and $\mathscr{P}_p(\mathscr{X}) = \{\gamma \in \mathscr{P}(\mathscr{X}) : \exists\, e \in \mathscr{X}, \ \int_{\mathscr{X}} d(z,e)^p \gamma(dz) < +\infty\} \subset \mathscr{P}(\mathscr{X})$, $p \geq 1$. For $p \geq 1$, we endow $\mathscr{P}_p(\mathscr{X})$ with the *p-Wasserstein metric*:

$$\begin{aligned} W_p^p(\gamma_1, \gamma_2) \ &:= \ \inf\{\int_{\mathscr{X}^2} d(z_1,z_2)^p \rho(dz_1, dz_2) : \rho \in \mathscr{P}(\mathscr{X}^2), \rho(\cdot \times \mathscr{X}) = \gamma_1, \rho(\mathscr{X} \times \cdot) = \gamma_2\} \\ &= \ \inf\{\mathbb{E}^{\mathbb{Q}^0}[d(\xi_1,\xi_2)^p] \ : \xi_1, \xi_2 \in \mathbb{L}_{\mathcal{F}^0}^1(\mathscr{X}), \text{with } P_{\xi_1} = \gamma_1, \text{ and } P_{\xi_2} = \gamma_2\}. \end{aligned} \tag{2.3}$$

Recall that, if the metric space $(\mathscr{X}, d)$ is complete, then also $(\mathscr{P}_p(\mathscr{X}), W_p)$ is complete, for all $p \geq 1$.

In what follows we shall focus on the case $p = 1$. It is well-known that, for $\mathscr{X} = \mathbb{R}$, $(\mathscr{P}_1(\mathbb{R}), W_1(\cdot,\cdot))$ is a complete and separable metric space. Furthermore, since $\Omega^0 = \mathbb{C}_T^2 = \mathbb{C}_T \otimes \mathbb{C}_T$ is Polish, we know that for each $t \in [0,T]$, the regular conditional probability $\mathbb{Q}_t^{\omega^2}(\,\cdot\,) := \mathbb{Q}^0(\,\cdot\,|\mathcal{F}_t^Y\}(\omega^2)$ exists, that is, for any $A \in \mathcal{F}^0$, $\omega^2 \mapsto \mathbb{Q}_t^{\omega^2}(A)$ is $\mathscr{B}_t(\mathbb{C}_T)/\mathscr{B}(\mathbb{R})$ measurable and, for any $\omega^2 \in \Omega^0$, $\mathbb{Q}_t^{\omega^2}(\cdot)$ is a probability measure. Since $\mathscr{B}_t(\mathbb{C}_T)$ is generated by the paths $Y_{\cdot\wedge t}(\omega^2) = \omega^2(\cdot \wedge t)$, $\omega^2 \in \mathbb{C}_T$, we will denote $\mathbb{Q}_t^{\omega^2} = \mathbb{Q}^{Y_{\cdot\wedge t}} = \mathbb{Q}^{\omega^2_{\cdot\wedge t}}$, when there is no confusion.

Now for any random variable ξ defined on $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$, and $t \in [0,T]$, we consider the regular conditional distribution:

$$\mathbb{P}_\xi^{Y_{\cdot\wedge t}}(\,\cdot\,)(\omega^2) := \mathbb{Q}^0[\xi \in \cdot\,|\mathcal{F}_t^Y](\omega^2) = \mathbb{Q}_t^{\omega^2} \circ \xi^{-1}(\cdot) =: \mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}(\cdot) \in \mathscr{P}_1(:= \mathscr{P}_1(\mathbb{R})). \tag{2.4}$$

We would like to show that the mapping $(t, \omega^2) \mapsto \mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}(\cdot) \in \mathscr{P}_1$ actually defines a measure-valued process as it should. More precisely, we have the following result.

Lemma 2.1. *Let ξ be a random variable defined on $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$. Then for each $t \in [0,T]$, the mapping $\omega^2 \mapsto \mathbb{P}_\xi^{Y_{\cdot\wedge t}}(\cdot)(\omega^2) = \mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}(\cdot)$ is $\mathscr{B}_t(\mathbb{C}_T)/\mathscr{B}(\mathscr{P}_1)$-measurable.*

Proof. To begin with, note that $\mathscr{B}(\mathscr{P}_1) = \sigma\{B_r(\mu) : \mu \in \mathscr{P}_1, r > 0\}$, where $B_r(\mu) = \{\nu \in \mathscr{P}_1 : W_1(\nu, \mu) \le r\}$. Thus, $[\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}]^{-1}(B_r(\mu)) = \{\omega^2 \in \mathbb{C}_T : W_1(\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}, \mu) \le r\}$. Next, we recall the *Kantorovich-Rubinstein formula* (cf. [12] or [13]):

$$W_1(\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}, \mu) = \sup\Big\{\Big|\int_{\mathbb{R}} \varphi d\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}} - \int_{\mathbb{R}} \varphi d\mu\Big| :\ \varphi \in \mathrm{Lip}_1(\mathbb{R})\Big\}, \tag{2.5}$$

where $\mathrm{Lip}_1(\mathbb{R})$ is the space of all Lipschitz functions with Lipschitz constant 1. We claim that there exists a countable subset $\Lambda \subset \mathrm{Lip}_1(\mathbb{R})$ such that (2.5) can be replaced by

$$W_1(\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}, \mu) = \sup\Big\{\Big|\int_{\mathbb{R}} \varphi d\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}} - \int_{\mathbb{R}} \varphi d\mu\Big| :\ \varphi \in \Lambda\Big\}, \tag{2.6}$$

and we can then conclude that

$$\begin{aligned}
& [\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}]^{-1}(B_r(\mu)) = \{\omega^2 \in \mathbb{C}_T : W_1(\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}, \mu) \le r\} \\
= \;& \{\omega^2 \in \mathbb{C}_T : \sup_{\varphi\in\Lambda}\Big|\int_{\mathbb{R}} \varphi d\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}} - \int_{\mathbb{R}} \varphi d\mu\Big| \le r\} \\
= \;& \bigcap_{\varphi\in\Lambda}\{\omega^2 \in \mathbb{C}_T : \Big|\int_{\mathbb{R}} \varphi d\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}} - \int_{\mathbb{R}} \varphi d\mu\Big| \le r\} \\
= \;& \bigcap_{\varphi\in\Lambda}\{\omega^2 \in \mathbb{C}_T : \Big|\mathbb{E}^{\mathbb{Q}^0}[\varphi(\xi)|\mathcal{F}_t^Y](\omega^2) - \bar\varphi\Big| \le r\} = \bigcap_{\varphi\in\Lambda} \mathbb{E}^{\mathbb{Q}^0}[\varphi(\xi)|\mathcal{F}_t^Y]^{-1}(B_r(\bar\varphi)),
\end{aligned}$$

where $\bar\varphi := \int_{\mathbb{R}} \varphi d\mu \in \mathbb{R}$. Since for each $\varphi \in \Lambda$, the mapping $\omega^2 \mapsto \mathbb{E}^{\mathbb{Q}^0}[\varphi(\xi)|\mathcal{F}_t^Y](\omega^2)$ is $\mathscr{B}_t(\mathbb{C}_T)$-measurable, we conclude that $[\mathbb{P}_\xi^{\omega^2_{\cdot\wedge t}}]^{-1}(B_r(\mu)) \in \mathscr{B}_t(\mathbb{C}_T)$.

It remains to find the countable subset $\Lambda \in \mathrm{Lip}_1(\mathbb{R})$ so that (2.6) holds. To this end, we consider the following subset of $C^1(\mathbb{R})$ (the space of differentiable functions which are defined on $\mathbb{R}$):

$$\mathbb{H}_0 := \{f \in C^1(\mathbb{R}) : |f|_{\mathbb{H}}^2 := |f(0)|^2 + \int_{\mathbb{R}} |f'(y)|^2 e^{-|y|} dy < \infty\}, \tag{2.7}$$

and let $\mathbb{H} := \overline{\mathbb{H}}_0$, the closure of $\mathbb{H}_0$ under the norm $|\cdot|_{\mathbb{H}}$. Then $(\mathbb{H}, |\cdot|_{\mathbb{H}})$ is a separable Banach space (in fact, a Hilbert space), and clearly $\mathrm{Lip}_1(\mathbb{R}) \subset \mathbb{H}$.

Now, for each $f \in \mathbb{H}$, define $\varphi_f(x) = f(0) + \int_0^x [(f'(y) \wedge 1) \vee (-1)]dy$, $x \in \mathbb{R}$, then $\varphi_f \in \mathrm{Lip}_1(\mathbb{R})$. Furthermore, since $(\mathbb{H}, |\cdot|_{\mathbb{H}})$ is separable, there exists a countable dense subset $\Lambda_{\mathbb{H}} \subset \mathbb{H}$, and then it is not hard to check that $\Lambda := \{\varphi_f : f \in \Lambda_{\mathbb{H}}\}$ is a countable dense subset of $\mathrm{Lip}_1(\mathbb{R})$ under the norm $|\cdot|_{\mathbb{H}}$. Consequently, for any $\mu \in \mathscr{P}_1(\mathbb{R})$ and $f \in \mathrm{Lip}_1(\mathbb{R})$, we can find $\{f_n\} \subset \Lambda \subset \mathrm{Lip}_1(\mathbb{R})$ such that $|f - f_n|_{\mathbb{H}} \to 0$, as $n \to \infty$. Since f and f_n's are all absolutely continuous, we have

$$\Big| \int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f_n d\mu \Big| \le |f(0) - f_n(0)| + \Big| \int_{\mathbb{R}} \Big| \int_0^x |f'(y) - f_n'(y)| dy \Big| \mu(dx) \Big|. \tag{2.8}$$

Note that for each $x \in \mathbb{R}$, we have $\Big[\int_0^x |f'(y) - f_n'(y)| dy \Big]^2 \le |x| e^{|x|} |f - f_n|_{\mathbb{H}} \to 0$, as $n \to \infty$, and since $\{f, f_n, n \in \mathbb{N}\} \subset \mathrm{Lip}_1(\mathbb{R})$, we have $\big| \int_0^x |f'(y) - f_n'(y)| dy \big| \le 2|x|$. But $\mu \in \mathscr{P}_1(\mathbb{R})$ implies that $\int_{\mathbb{R}} |x| \mu(dx) < \infty$. The Dominated Convergence Theorem and (2.8) thus imply that $\int_{\mathbb{R}} f_n d\mu \to \int_{\mathbb{R}} f d\mu$, as $n \to \infty$. This, together with (2.5), easily leads to (2.6). The proof is now complete. ∎

We remark that Lemma 2.1 does not imply directly that the mapping $(t, \omega^2) \mapsto \mathbb{P}_{\xi}^{Y_{\cdot \wedge t}}$ is $\mathscr{B}([0,T]) \otimes \mathscr{B}(\mathbb{C}_T)/\mathscr{B}(\mathscr{P}_1)$ jointly measurable. But as we shall argue in the next section (see, also [1]), for fixed $\omega^2 \in \mathbb{C}_T$, the mapping $t \mapsto \mathbb{P}_{\xi}^{\omega^2_{\cdot \wedge t}}$ is a $\mathscr{P}_1$-valued continuous function, which then renders the desired joint measurability. Throughout our paper we shall focus on the case $\mathscr{X} = \mathscr{C}_T(\mathscr{P}_1)$, the space of all $\mathscr{P}_1$-valued continuous functions defined on $[0,T]$. The set $\mathbb{S}^p_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$ defined by (2.2) as well as the set $\mathbb{S}^{\infty-}_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$ will therefore be particularly useful in our discussion.

To conclude this section we introduce the following standard assumptions on the coefficients b, σ_1, σ_2, and h of SDE (1.1). For convenience, in what follows we shall assume $\sigma_1 = \sigma$, $\sigma_2 = 0$, and $\hat{\sigma} = 1$.

Assumption 2.2. *The function $\varphi = (b, \sigma, h) : [0,T] \times \Omega \times \mathbb{C}_T^2 \times \mathscr{C}_T(\mathscr{P}_1) \mapsto \mathbb{R}^3$ is bounded, progressively measurable, and for some constant $C > 0$, it holds that*

$$|\varphi(t, x, y, \mu) - \varphi(t, x', y, \mu')| \le C\Big[\sup_{s \in [0,t]} |x_s - x_s'| + \sup_{s \in [0,t]} W_1(\mu_s, \mu_s') \Big], \tag{2.9}$$
$$t \in [0,T],\ x, x', y \in \mathbb{C}_T,\ \mu, \mu' \in \mathscr{C}_T(\mathscr{P}_1).$$

Remark 2.3. We should note that the case when $\sigma_2 \neq 0$ is known as the "correlated noise case" in the nonlinear filtering theory, which is well-understood and without substantial difficulties, although technically slightly more tedious. We prefer not to pursue such complexity in this paper, but focus on the *conditional McKean-Vlasov* nature instead. ■

We also recall the notion of a *weak solution* to SDE (1.1), which will be the main objective of this paper.

Definition 2.4. *A six-tuple* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (B^1, B^2), (X, Y))$ *is called a weak solution of (1.1), if:*

(i) $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *is a filtered probability space satisfying the usual assumptions;*

(ii) (B^1, B^2) *is an* $(\mathbb{F}, \mathbb{P})$*-Brownian motion;*

(iii) $(X, Y) \in \mathbb{L}^2_{\mathbb{F}}([0, T]; \mathbb{R}^2)$ *such that all terms in (1.1) are well-defined and (1.1) holds for all* $t \in [0, T]$*,* $\mathbb{P}$*-a.s.*

Remark 2.5. It is worth noting that Definition 2.4 only defines processes (X, Y), along with a probability set-up including the Brownian motion (B^1, B^2). The conditional law $\mu^{X|Y}$ then comes naturally as the function of X and Y, under probability $\mathbb{P}$. But the example in the introduction shows that, unless some more structural information on the process $\mu^{X|Y}$ is known, the simple minded iteration scheme will likely fail. The main idea of our solution scheme is to add the conditional law $\mu^{X|Y}$ into the iteration process itself to help the convergence analysis. ■

3 The Solution Scheme

In this section we introduce the iteration scheme that will lead to the desired weak solution. A key element in this scheme is the process of conditional laws, $\mu^{X|Y} = \{\mu^{X|Y}_t\}$ which, by Lemma 2.1, is a $\mathscr{P}_1(\mathbb{R})$-valued measurable process, and will be used to "decouple" the SDEs for X and Y in (1.1). In light of the analysis in our previous work [1], we shall argue that it is actually a $\mathscr{P}_1(\mathbb{R})$-valued continuous process. That is, $\mu^{X|Y} \in \mathscr{C}_T(\mathscr{P}_1)$. We therefore shall start our scheme by considering μ as a free variable taking values in $\mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, and then try to find the desired conditional law by a fixed-point argument. All our arguments are essentially independent of the drift coefficient b, under Assumption 2.2. Thus, for notational simplicity, in what follows we shall assume that $b \equiv 0$, as adding it back does not cause substantial difficulties.

To begin with, for $\mu \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, we consider the following simplified system of SDEs on $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$:

$$\begin{cases} dX_t = \sigma(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu_{\cdot\wedge t})dB^1_t, & X_0 = x; \\ dL_t = h(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu_{\cdot\wedge t})L_t dY_t, & L_0 = 1. \end{cases} \tag{3.1}$$

Since μ is $\mathbb{F}^Y$-adapted, we can write $\mu_{\cdot\wedge t} = \Phi_t(Y_{\cdot\wedge t})$, $t \in [0,T]$, $\mathbb{Q}^0$-a.s., for some progressively measurable functional $\Phi : [0,T] \times \mathbb{C}_T \mapsto \mathscr{C}_T(\mathscr{P}_1)$. But Y is part of the canonical process, therefore SDE (3.1) can be thought of as one that has random and functional type coefficients. Thus under Assumption 2.2, it has a unique strong solution on the probability space $(\Omega^0, \mathcal{F}^0, \mathbb{Q}^0)$, and we denote it by (X^μ, L^μ). Since h is bounded, we see that the process L^μ is an $(\mathbb{F}, \mathbb{Q}^0)$-martingale, and can be written as the Doléans-Dade stochastic exponential:

$$L^\mu_t = \exp\Big\{ \int_0^t h(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})dY_s - \frac{1}{2}\int_0^t |h(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})|^2 ds \Big\}, \tag{3.2}$$

$t \in [0,T]$. Moreover, since also σ is bounded, it is not hard to show that $(X^\mu, L^\mu) \in \mathbb{S}^{\infty-}_{\mathbb{F}}(\mathbb{R}^2)$. Furthermore, as a strong solution, there exists a measurable non-anticipating functional $\Psi : \mathbb{C}^2_T \times \mathscr{C}(\mathscr{P}_1) \mapsto \mathbb{C}^2_T$ such that $(X^\mu, L^\mu) = \Psi(B^1, Y, \mu)$, $\mathbb{Q}^0$-a.s.

Next, the $\mathbb{Q}^0$-martingale L^μ defines a new probability measure on $(\Omega^0, \mathcal{F})$: $\mathbb{P}^\mu(d\omega) := L^\mu_T \mathbb{Q}^0(d\omega)$. Then, under the new probability $\mathbb{P}^\mu$, the process $(B^1, B^2 = Y - \int_0^\cdot h(s, X^\mu_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu_{\cdot\wedge t})dt)$ is a Brownian motion. Furthermore, we denote the *regular conditional probability distribution* of the process X^μ, given $\mathcal{F}^Y$, under the probability measure $\mathbb{P}^\mu$ by $\tilde{\mu}_t$, $t \in [0,T]$. Since $\tilde{\mu}$ is obviously uniquely determined for each $\mu \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, we can then define the so-called *solution mapping* by $\mathscr{T}(\mu) := \tilde{\mu}$. That is, $\mathscr{T}$ is a mapping from $\mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$ to $\mathbb{L}^0_{\mathbb{F}^Y}([0,T]; \mathscr{P}_1)$ (the space of all $\mathbb{F}^Y$-adapted $\mathscr{P}_1$-valued processes), and by the *Kallianpur-Striebel formula* (cf., e.g., [10, 11]) we see that, for $A \in \mathscr{B}(\Omega)$ and $t \in [0,T]$, one has

$$[\mathscr{T}(\mu)]_t(A) = \tilde{\mu}_t(A) \stackrel{\triangle}{=} \mathbb{P}^\mu\{X^\mu_t \in A | \mathcal{F}^Y_t\} = \frac{\mathbb{E}^{\mathbb{Q}^0}\big[L^\mu_t \mathbf{1}_{\{X^\mu_t \in A\}} | \mathcal{F}^Y_t\big]}{\mathbb{E}^{\mathbb{Q}^0}\big[L^\mu_t | \mathcal{F}^Y_t\big]}. \tag{3.3}$$

Let us now assume that the mapping $\mathscr{T}$ has a fixed point. That is, there exists $\hat{\mu} \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, such that $\mathscr{T}(\hat{\mu}) = \hat{\mu}$. Then, denoting $(\hat{X}, \hat{L}) := (X^{\hat{\mu}}, L^{\hat{\mu}})$ to be the corresponding solution to (3.1) with $\mu = \hat{\mu}$, and $\hat{\mathbb{P}} = \mathbb{P}^{\hat{\mu}}$, the Kallianpur-Strieble formula (3.3) implies that $\hat{\mu}_t = \mu^{\hat{X}|Y}_t$, $t \in [0,T]$, under $\hat{\mathbb{P}}$. In other words, writing

$$\hat{B}^2_t = Y_t - \int_0^t h(s, \hat{X}_{\cdot\wedge s}, Y_{\cdot\wedge s}, \hat{\mu}_{\cdot\wedge s})ds,$$

we see that $(\Omega^0, \mathcal{F}^0, \mathbb{F}^0, \hat{\mathbb{P}}, (B^1, \hat{B}^2), (\hat{X}, Y))$ is a weak solution of (1.1).

In order to make our scheme to work we shall carry out the following tasks in the next sections.

(i) Identify a subspace $\mathscr{E} \subset \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, and show that the solution mapping $\mathscr{T}$ is from $\mathscr{E}$ to itself.

(ii) Show that we can at least find a sequence of $\mathcal{F}^Y$-stopping times $\{\tau_N\}$, such that $\mathscr{T}^N := \mathscr{T}\big|_{[0,\tau_N]}$ is a contraction, hence has a fixed point $\hat{\mu}^N$ on $[0, \tau_N]$. We then argue that these $\hat{\mu}^N$'s can be "patched" together to become a fixed point $\hat{\mu} \in \mathscr{E}$.

(iii) Show that the law of the solution (X, Y) is unique.

Remark 3.1. We should note that the "localization" procedure is merely technical, in order to deal with the unboundedness caused by the fraction in (3.3). In fact, such a technicality only occurs in CMVSDEs when $h \neq 0$, and it is the fundamental difference between SDE (1.1) and CMVSDEs of the "common noise" type that we often see in the literature. ■

To simplify notation, in what follows we denote $L^{-\mu} = (L^\mu)^{-1}$, for $\mu \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$ and the corresponding solution (X^μ, L^μ) to SDE (3.1). We note that $L^{-\mu}$ is the inverse Girsanov kernel of L^μ, and it is a $\mathbb{P}^\mu$-martingale, but not a $\mathbb{Q}^0$-martingale. Last but not least, for any $\xi \in \mathbb{C}_T$, we shall also use the notation $\xi^*_t = \sup_{s\in[0,t]} |\xi_s|$, and for any $p > 0$, $\xi^{*,p}_t = [\xi^*_t]^p$, $t \in [0, T]$.

4 The Main Estimates

In this section we establish the main estimates that will be crucial for us to implement the solution scheme. Before we start, we emphasize again that the conditional law $\mu^{X|Y}$ in CMVSDE (1.1) is under the probability $\mathbb{P}$ (under which (B^1, B^2) is a Brownian motion), but our scheme is defined under the reference measure $\mathbb{Q}^0$, connected to $\mathbb{P}$ via a Girsanov kernel L, defined by SDE (3.1) or explicitly by (3.2).

Now, for any $\mu, \mu' \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$, denote $\widetilde{\mu} := \mathscr{T}(\mu)$, $\widetilde{\mu}' := \mathscr{T}(\mu')$. Let (X^μ, L^μ), $(X^{\mu'}, L^{\mu'})$ be the solution of SDE (3.1) and define

$$\zeta_t(\mu, \mu') := \mathbb{E}^{\mathbb{Q}^0}[(L^\mu)^{*,4}_t + (L^{-\mu})^{*,4}_t + (L^{\mu'})^{*,4}_t + (L^{-\mu'})^{*,4}_t \big| \mathcal{F}^Y_t], \quad t \in [0, T]. \tag{4.1}$$

Then it is not hard to check that $\zeta(\mu, \mu')$ is a continuous, increasing $\mathbb{F}^Y$-adapted process with $\zeta_0(\mu, \mu') = 4$. We have the following result.

Proposition 4.1. *Let Assumption 2.2 be in force. Then, for all $0 \le s \le t \le T$, it holds $\mathbb{Q}^0$-almost surely that*

$$W_1(\widetilde{\mu}_s, \widetilde{\mu}'_t) \le C\zeta_t(\mu,\mu')\{(\mathbb{E}^{\mathbb{Q}^0}[|X^\mu_s - X^{\mu'}_t|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}} + (\mathbb{E}^{\mathbb{Q}^0}[|L^\mu_s - L^{\mu'}_t|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}}\}, \tag{4.2}$$

where $\zeta(\mu,\mu')$ is defined by (4.1). Furthermore, for each $p \ge 1$, there exists a $C_p > 0$, such that $\mathbb{E}^{\mathbb{Q}^0}[\zeta^p_T(\mu,\mu')] \le C_p$, for all $\mu, \mu' \in \mathbb{L}^0_{\mathbb{F}^Y}([0,T];\mathscr{P}_1)$.

Proof. First recall the *Kantorovich-Rubinstein formula* (see (2.5)): For $0 \le s \le t \le T$,

$$W_1(\widetilde{\mu}_s, \widetilde{\mu}'_t) = \sup\Big\{\Big|\int_{\mathbb{R}} \varphi d\widetilde{\mu}_s - \int_{\mathbb{R}} \varphi d\widetilde{\mu}'_t\Big|,\ \varphi \in \mathrm{Lip}_1(\mathbb{R})\Big\}.$$

Since both $\widetilde{\mu}_s, \widetilde{\mu}'_t$ are probability measures, it suffices to consider only those test functions $\varphi \in \mathrm{Lip}_1(\mathbb{R})$ with $\varphi(0) = 0$ so that $|\varphi(z)| \le |z|,\ z \in \mathbb{R}$. In other words, we can write:

$$W_1(\widetilde{\mu}_s, \widetilde{\mu}'_t) = \sup\Big\{\Big|\int_{\mathbb{R}} \varphi d\widetilde{\mu}_s - \int_{\mathbb{R}} \varphi d\widetilde{\mu}'_t\Big|,\ \varphi \in \mathrm{Lip}_1(\mathbb{R}),\ \varphi(0) = 0\Big\}. \tag{4.3}$$

Note that for any $\varphi \in Lip_1(\mathbb{R})$ with $\varphi(0) = 0$, by definition of $\tilde{\mu}$, $\tilde{\mu}'$ (see (3.3)) we have

$$\Big|\int_{\mathbb{R}} \varphi d\widetilde{\mu}_s - \int_{\mathbb{R}} \varphi d\widetilde{\mu}'_t\Big| = \big|\mathbb{E}^{\mathbb{P}}[\varphi(X^\mu_s)\big|\mathcal{F}^Y_s] - \mathbb{E}^{\mathbb{P}'}[\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_t]\big|,$$

where $\mathbb{P} = \mathbb{P}^\mu$ and $\mathbb{P}' = \mathbb{P}^{\mu'}$. By the Kalliapur-Strieble formula (3.3) we have

$$\mathbb{E}^{\mathbb{P}}[\varphi(X^\mu_s)\big|\mathcal{F}^Y_s] = \frac{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\varphi(X^\mu_s)\big|\mathcal{F}^Y_s]}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\big|\mathcal{F}^Y_s]} = \frac{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\varphi(X^\mu_s)\big|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\big|\mathcal{F}^Y_T]},\ \mathbb{Q}^0\text{-a.s.},$$

where the second equality follows from the fact that $\mathcal{F}^Y_T = \mathcal{F}^Y_s \vee \mathcal{F}^Y_{s,T}$, and $\mathcal{F}_s = \mathcal{F}^{B^1,Y}_s$ and $\mathcal{F}^Y_{s,T}$ are independent under $\mathbb{Q}^0$. Similarly, we have

$$\mathbb{E}^{\mathbb{P}'}[\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_t] = \frac{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_t]}{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\big|\mathcal{F}^Y_t]} = \frac{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\big|\mathcal{F}^Y_T]},\ \mathbb{Q}^0\text{-a.s.}$$

Hence, we deduce that, $\mathbb{Q}^0$-almost surely,

$$\begin{aligned}
\Big|\int_{\mathbb{R}} \varphi d\widetilde{\mu}_s - \int_{\mathbb{R}} \varphi d\widetilde{\mu}'_t\Big| &= \Big|\frac{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\varphi(X^\mu_s)\big|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\big|\mathcal{F}^Y_T]} - \frac{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\big|\mathcal{F}^Y_T]}\Big| \\
\le \quad & \frac{1}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\big|\mathcal{F}^Y_T]}\mathbb{E}^{\mathbb{Q}^0}[|L^\mu_s\varphi(X^\mu_s) - L^{\mu'}_t\varphi(X^{\mu'}_t)|\big|\mathcal{F}^Y_T] \\
&+\frac{|\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\varphi(X^{\mu'}_t)\big|\mathcal{F}^Y_T]|}{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\big|\mathcal{F}^Y_T]\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s\big|\mathcal{F}^Y_T]}\mathbb{E}^{\mathbb{Q}^0}[|L^\mu_s - L^{\mu'}_t|\big|\mathcal{F}^Y_T] =: I^1_{s,t} + I^2_{s,t},
\end{aligned} \tag{4.4}$$

where $I^i_{s,t}$, $i=1,2$, are defined in the obvious way. Now by Jensen's inequality we have (recall the definition of $L^{-\mu}$),

$$(\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s|\mathcal{F}^Y_T])^{-1} \le \mathbb{E}^{\mathbb{Q}^0}[L^{-\mu}_s|\mathcal{F}^Y_T] \quad \text{and} \quad (\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t|\mathcal{F}^Y_T])^{-1} \le \mathbb{E}^{\mathbb{Q}^0}[L^{-\mu'}_t|\mathcal{F}^Y_T], \tag{4.5}$$

and recalling the notation ξ^* for $\xi \in \mathbb{C}_T$, and that $\varphi \in Lip_1(\mathbb{R})$ with $\varphi(0)=0$, we have

$$\begin{aligned} &\Big|\frac{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t\varphi(X^{\mu'}_t)|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^{\mu'}_t|\mathcal{F}^Y_T]\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s|\mathcal{F}^Y_T]}\Big| \qquad (4.6)\\ \le\ & \big(\mathbb{E}^{\mathbb{Q}^0}[(L^{\mu'})^{*,2}_t|\mathcal{F}^Y_T]\big)^{\frac12}\big(\mathbb{E}^{\mathbb{Q}^0}[(X^{\mu'})^{*,2}_t|\mathcal{F}^Y_T]\big)^{\frac12}\mathbb{E}^{\mathbb{Q}^0}[(L^{-\mu'})^*_t|\mathcal{F}^Y_T]\mathbb{E}^{\mathbb{Q}^0}[(L^{-\mu})^*_t|\mathcal{F}^Y_T] =: \zeta^1_t. \end{aligned}$$

To analyze ζ^1 we first recall that, under $\mathbb{Q}^0$, the coordinate process (B^1,Y) on $\Omega^0=\mathbb{C}^2_T=\mathbb{C}_T\otimes\mathbb{C}_T$ is a 2-dimensional Brownian motion. Therefore, if we denote the conditional probability $\mathbb{Q}^0[A|\mathcal{F}^Y_T](\omega^2)=\mathbb{Q}^{\omega^2}[A]$, $A\in\mathscr{B}(\mathbb{C}^2_T)$, then we can consider the SDE for X^μ in (3.1) as on the probability space $(\mathbb{C}_T,\mathscr{B}(\mathbb{C}_T),\mathbb{Q}^{\omega^2})$ for $\mathbb{P}^0$-a.e. $\omega^2\in\mathbb{C}_T$. Note that for fixed ω^2, the process

$$X^\mu_t(\cdot,\omega^2) = x + \int_0^t \sigma(s, X^\mu_s(\cdot,\omega^2), \omega^2_{\cdot\wedge s}, \mu_{\cdot\wedge s}(\omega^2))dB^1_s, \quad t\in[0,T],$$

is a $\mathbb{Q}^{\omega^2}$-martingale, and as σ is bounded, by the Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}^{\omega^2}[(X^\mu)^{*,2}_T] \le C\mathbb{E}^{\omega^2}[\langle X^\mu\rangle_T] = C\mathbb{E}^{\omega^2}\Big[\int_0^T \sigma^2(\cdots)ds\Big] \le C, \quad \mathbb{P}^0\text{-a.e. } \omega^2\in\mathbb{C}_T,$$

where $\mathbb{E}^{\omega^2}[\,\cdot\,]=\mathbb{E}^{\mathbb{Q}^{\omega^2}}[\,\cdot\,]=\mathbb{E}^{\mathbb{Q}^0}[\,\cdot\,|\mathcal{F}^Y_T](\omega^2)$, and $C>0$ is a generic constant depending only on $T>0$ and the bounds of σ and h, which is allowed to vary between expressions. Thus we have $\mathbb{E}^{\mathbb{Q}^0}[(X^\mu)^{*,2}_T|\mathcal{F}^Y_T]\le C$, $\mathbb{Q}^0$-a.s.

Now repeatedly applying Hölder's inequality and the fact $abc\le a^3+b^3+c^3$, for $a,b,c\ge 0$, we obtain from the definition of ζ^1_t in (4.6)

$$\begin{aligned} \zeta^1_t &\le C\mathbb{E}^{\mathbb{Q}^0}[(L^{\mu'})^{*,3}_t + (L^{-\mu'})^{*,3}_t + (L^{-\mu})^{*,3}_t|\mathcal{F}^Y_t] \qquad (4.7)\\ &= C\mathbb{E}^{\mathbb{Q}^0}[(L^{\mu'})^{*,3}_t + (L^{-\mu'})^{*,3}_t + (L^{-\mu})^{*,3}_t|\mathcal{F}^Y_T] =: \zeta^2_t, \quad t\in[0,T]. \end{aligned}$$

Now, for notational simplicity we denote $\Delta L^{\mu,\mu'}_{s,t} := L^\mu_s - L^{\mu'}_t$, and $\Delta X^{\mu,\mu'}_{s,t} := X^\mu_s - X^{\mu'}_t$. Then, combining (4.5)–(4.7), and recalling the definition of $I^2_{s,t}$ (see (4.4)), we have

$$I^2_{s,t} \le \zeta^2_t\mathbb{E}^{\mathbb{Q}^0}[|\Delta L^{\mu,\mu'}_{s,t}||\mathcal{F}^Y_T] \le \zeta^2_t\big(\mathbb{E}^{\mathbb{Q}^0}[|\Delta L^{\mu,\mu'}_{s,t}|^2|\mathcal{F}^Y_T]\big)^{\frac12}, \ \ \mathbb{Q}^0\text{-}a.s.,\ 0\le s\le t\le T. \tag{4.8}$$

Similarly, we have the estimate for $I^1_{s,t}$ (noting that $(L^{\mu'})^*_t \ge L^{\mu'}_0 = 1$), for $0 \le s \le t \le T$,

$$
\begin{aligned}
I^1_{s,t} \;&\le\; \mathbb{E}^{\mathbb{Q}^0}[(L^{-\mu})^*_t \big| \mathcal{F}^Y_T]\big\{\big(\mathbb{E}^{\mathbb{Q}^0}[(X^{\mu})^{*,2}_t \big| \mathcal{F}^Y_T]\big)^{\frac{1}{2}}\big(\mathbb{E}^{\mathbb{Q}^0}[|\Delta L^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T]\big)^{\frac{1}{2}} \\
&\quad + \big(\mathbb{E}^{\mathbb{Q}^0}[(L^{\mu'})^{*,2}_t \big| \mathcal{F}^Y_T]\big)^{\frac{1}{2}}\big(\mathbb{E}^{\mathbb{Q}^0}[|\Delta X^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T]\big)^{\frac{1}{2}}\big\} \qquad (4.9)\\
&\le\; C\mathbb{E}^{\mathbb{Q}^0}[(L^{-\mu})^{*,2}_t + (L^{\mu'})^{*,2}_t \big| \mathcal{F}^Y_T]\big\{(\mathbb{E}^{\mathbb{Q}^0}[|\Delta L^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}} + (\mathbb{E}^{\mathbb{Q}^0}[|\Delta X^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}}\big\}.
\end{aligned}
$$

Using (4.8) and (4.9), we deduce easily from (4.4) that, for all $0 \le s \le t \le T$, $\mathbb{Q}^0$-a.s.,

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}} \varphi d\widetilde{\mu}_s - \int_{\mathbb{R}} \varphi d\widetilde{\mu}'_t \Big| \qquad (4.10)\\
&\le\; C\zeta_t(\mu,\mu')\Big((\mathbb{E}^{\mathbb{Q}^0}[|\Delta X^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}} + (\mathbb{E}^{\mathbb{Q}^0}[|\Delta L^{\mu,\mu'}_{s,t}|^2 \big| \mathcal{F}^Y_T])^{\frac{1}{2}}\Big),
\end{aligned}
$$

where $C > 0$ is a constant depending only on T and the bounds of σ, h, and

$$
\begin{aligned}
\zeta_t(\mu,\mu') \;&:=\; \mathbb{E}^{\mathbb{Q}^0}[(L^{\mu})^{*,4}_t + (L^{-\mu})^{*,4}_t + (L^{\mu'})^{*,4}_t + (L^{-\mu'})^{*,4}_t \big| \mathcal{F}^Y_t] \qquad (4.11)\\
&=\; \mathbb{E}^{\mathbb{Q}^0}[(L^{\mu})^{*,4}_t + (L^{-\mu})^{*,4}_t + (L^{\mu'})^{*,4}_t + (L^{-\mu'})^{*,4}_t \big| \mathcal{F}^Y_T],\ t \in [0,T].
\end{aligned}
$$

From its definition we can easily see that $\zeta_t(\mu,\mu')$, $t \in [0,T]$, is an $\mathbb{F}^Y$-adapted, increasing process with $\zeta_0(\mu,\mu') = 4$. Moreover, by the last expression of (4.11) we see that it is $L^2(\mathbb{Q}^0)$-continuous. Thus, the continuity of $t \to \zeta_t(\mu,\mu')$ follows. Finally, for each $p \ge 1$, there exists some constant $C_p > 0$, depending only on p and the bounds of coefficients, such that

$$
\mathbb{E}^{\mathbb{Q}^0}[\zeta^p_T(\mu,\mu')] \le C_p, \quad \text{for all } \mu,\ \mu' \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1)),\ p \ge 1.
$$

This proves the proposition.

■

We now consider the following subspace of $\mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$:

$$
\mathscr{E} := \mathbb{S}^{\infty-}_{\mathbb{F}^Y}(\mathscr{P}_1). \qquad (4.12)
$$

We shall argue that the conclusion of Proposition 4.1 is strong enough to imply the following important property of the solution mapping $\mathscr{T}$.

Corollary 4.2. *Let Assumption 2.2 hold. Then $\mathscr{T}(\mathscr{E}) \subseteq \mathscr{E}$.*

Proof. For any $\mu \in \mathscr{E}$ we put $\widetilde{\mu} = \mathscr{T}(\mu)$. Setting $\mu' = \mu$ in Proposition 4.1, we deduce from (4.2) that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}[W_1(\widetilde{\mu}_s,\widetilde{\mu}_t)^4] \;&\le\; C\big(\mathbb{E}^{\mathbb{Q}^0}[\zeta_T(\mu,\mu)^8]\big)^{\frac{1}{2}}\Big\{\big(\mathbb{E}^{\mathbb{Q}^0}[|X^{\mu}_s - X^{\mu}_t|^8]\big)^{\frac{1}{2}} + \big(\mathbb{E}^{\mathbb{Q}^0}[|L^{\mu}_s - L^{\mu}_t|^8]\big)^{\frac{1}{2}}\Big\}\\
&\le\; C\Big\{\big(\mathbb{E}^{\mathbb{Q}^0}[|X^{\mu}_s - X^{\mu}_t|^8]\big)^{\frac{1}{2}} + \big(\mathbb{E}^{\mathbb{Q}^0}[|L^{\mu}_s - L^{\mu}_t|^8]\big)^{\frac{1}{2}}\Big\}. \qquad (4.13)
\end{aligned}
$$

Here and in what follows we shall denote $C > 0$ to be a generic constant depending only on T and the bounds of h, which varies from line to line. Since σ and h are bounded, it is clear that $\mathbb{E}^{\mathbb{Q}^0}[(L^\mu_T)^p] \le C_p$, for all $\mu \in \mathscr{C}_T(\mathscr{P}_1)$, and it follows by standard estimates that

$$\mathbb{E}^{\mathbb{Q}^0}[|X^\mu_s - X^\mu_t|^8 + |L^\mu_s - L^\mu_t|^8] \le C|s-t|^4, \quad 0 \le s \le t \le T.$$

Hence, by (4.13) we have $\mathbb{E}^{\mathbb{Q}^0}[W_1(\widetilde{\mu}_s, \widetilde{\mu}_t)^4] \le C|s-t|^2$, $s, t \in [0,T]$. Thus, by Kolmogorov's continuity criterion, it follows that $\widetilde{\mu} = (\widetilde{\mu})_{t\in[0,T]}$ admits a continuous modification, which we shall use from now on. In other words, we have proved that $\mathscr{T}(\mu) = \widetilde{\mu}$ is $\mathscr{C}_T(\mathscr{P}_1)$-valued.

It remains to check that $\mathscr{T}(\mu) \in \mathscr{E}$. To see this we fix $p \ge 1$, and note that for any $\mu' \in \mathscr{E}$ we always have $\mathscr{T}(\mu')_0 = \mathbb{P}' \circ (X^{\mu'}_0)^{-1} = \delta_{\{x\}}$. Applying (4.2) again we see that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{t\in[0,T]} W_1(\widetilde{\mu}_t, \delta_{x_0})^p\Big] = \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{t\in[0,T]} W_1(\widetilde{\mu}_t, \widetilde{\mu}'_0)^p\Big] \\ \le\ & C\mathbb{E}^{\mathbb{Q}^0}\Big[\zeta_T(\mu,\mu')^p \cdot \sup_{t\in[0,T]} \Big(\big(\mathbb{E}^{\mathbb{Q}^0}[|X^\mu_t - x|^2|\mathcal{F}^Y_T]\big)^{\frac{p}{2}} + \big(\mathbb{E}^{\mathbb{Q}^0}[|L^\mu_t - 1|^2|\mathcal{F}^Y_T]\big)^{\frac{p}{2}}\Big)\Big] \\ \le\ & C\Big(\mathbb{E}^{\mathbb{Q}^0}\big[\zeta_T(\mu,\mu')^{2p}\big]\Big)^{\frac12}\Big(1 + \mathbb{E}^{\mathbb{Q}^0}[(X^\mu)^{*,2p}_T] + \mathbb{E}^{\mathbb{Q}^0}[(L^\mu)^{*,2p}_T]\Big)^{\frac12} < +\infty. \end{aligned}$$

Since $\widetilde{\mu}$ is obviously $\mathbb{F}^Y$-adapted, by definition (2.2) we then have $\widetilde{\mu} \in \mathbb{S}^p_{\mathbb{F}^Y}(\mathscr{P}_1)$. Note now that the above argument holds for all $p \ge 1$, we conclude that $\mathscr{T}(\mu) = \widetilde{\mu} \in \mathscr{E}$. The proof is now complete. ■

Remark 4.3. As we pointed out before, Proposition 4.1 actually shows that $\mathscr{T}(\mu) \in \mathbb{S}^{\infty-}_{\mathbb{F}^Y}(\mathscr{P}_1)$, for any $\mu \in \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T(\mathscr{P}_1))$. This is due largely to the fact that we have assumed that all coefficients σ and h are bounded. In general, we should have, for any $p \ge 1$, the solution mapping $\mathscr{T} : \mathbb{S}^p_{\mathbb{F}^Y}(\mathscr{P}_1) \mapsto \mathscr{E} = \mathbb{S}^{\infty-}_{\mathbb{F}^Y}(\mathscr{P}_1) \subseteq \mathbb{S}^p_{\mathbb{F}^Y}(\mathscr{P}_1)$. The case when $p = 2$ is frequently used. ■

5 Existence of a Weak Solution

We are now ready to prove the existence of the weak solution to SDE (1.1). To begin with, we note that Proposition 4.1 only shows that (assuming, for example, $s = t$), the (Wasserstein) distance between $\tilde{\mu}_t = \mathscr{T}(\mu)_t$ and $\tilde{\mu}'_t = \mathscr{T}(\mu')_t$ can be controlled by the distances of the corresponding solutions (X^μ, L^μ) and $(X^{\mu'}, L^{\mu'})$ at each fixed $t \in [0,T]$. But in order to look for a fixed point in the space $\mathscr{E}$, we need to strengthen the estimate in terms of the distance in $\mathbb{L}^p_{\mathbb{F}^Y}(\mathscr{C}(\mathscr{P}_1))$. In light of Remark 4.3, we shall only consider the case $p = 2$.

We begin by a brief analysis. Let $\mu, \mu' \in \mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$. For notational simplicity we denote the corresponding triplets $(X, L, \mathbb{P}) := (X^\mu, L^\mu, \mathbb{P}^\mu)$ and $(X', L', \mathbb{P}') := (X^{\mu'}, L^{\mu'}, \mathbb{P}^{\mu'})$, respectively, and put $\widetilde{\mu} := \mathscr{T}(\mu)$ and $\widetilde{\mu}' := \mathscr{T}(\mu')$ as before. We also set $\Delta X := X - X'$, $\Delta L := L - L'$, and

$$\delta\varphi(t, x, x', y, \mu, \mu') := \varphi(t, x, y, \mu) - \varphi(t, x', y, \mu'), \quad \varphi = \sigma,\ h.$$

Our goal is to use estimate (4.2) in Proposition 4.1 to obtain the desired contraction estimate: For some constant $C < 1$,

$$\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le t\le T} W_1(\tilde{\mu}, \tilde{\mu}')^2\Big] \le C\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\le t\le T} W_1(\mu, \mu')^2\Big]. \tag{5.1}$$

To begin with, we note that (4.2) only gives us

$$\sup_{s\le t} W_1(\widetilde{\mu}_s, \widetilde{\mu}'_s)^2 \le C\zeta_t(\mu, \mu')^2 \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t}|\Delta X_s|^2 + \sup_{s\le t}|\Delta L_s|^2 \Big| \mathcal{F}^Y_T\Big], \quad t \in [0, T]. \tag{5.2}$$

But on the other hand, since X and X' satisfy (3.1), following the standard arguments using the Burkholder-Davis-Gundy inequality and Assumption 2.2, one can easily check that, for $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}[\sup_{s\in[0,t]}|\Delta X_s|^4 \big|\mathcal{F}^Y_T] \le\ & C\mathbb{E}^{\mathbb{Q}^0}\Big[\Big(\int_0^t \big|\delta\sigma(s, X_{\cdot\wedge s}, X'_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s}, \mu'_{\cdot\wedge s})\big|^2 ds\Big)^2 \Big|\mathcal{F}^Y_T\Big] \\
\le\ & C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t \sup_{r\le s}|\Delta X_r|^4 ds \Big|\mathcal{F}^Y_T\Big] + C\mathbb{E}^{\mathbb{Q}^0}\Big[\Big(\int_0^t \sup_{r\le s} W_1(\mu_r, \mu'_r)^2 ds\Big)^2 \Big|\mathcal{F}^Y_T\Big] \\
=\ & C\int_0^t \mathbb{E}^{\mathbb{Q}^0}[\sup_{r\le s}|\Delta X_r|^4 |\mathcal{F}^Y_T] ds + C\Big(\int_0^t \sup_{r\le s} W_1(\mu_r, \mu'_r)^2 ds\Big)^2.
\end{aligned}$$

Observe that in the last equality above we used the fact that μ and μ' are $\mathbb{F}^Y$-adapted. Now applying Gronwall's inequality we obtain that

$$\Big(\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\in[0,t]}|\Delta X_s|^4 \big|\mathcal{F}^Y_T\Big]\Big)^{\frac{1}{2}} \le C\int_0^t \sup_{r\le s} W_1(\mu_r, \mu'_r)^2 ds,\ t \in [0, T], \quad \mathbb{Q}\text{-}a.s. \tag{5.3}$$

Similarly, since h is bounded, we also obtain from (3.1) that, for $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t}|\Delta L_s|^2\Big] \ \le\ & C\Big\{\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t |\Delta L_s|^2 ds\Big] \\
& +\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t L_s^2 \big|\delta h(s, X_{\cdot\wedge s}, X'_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s}, \mu'_{\cdot\wedge s})\big|^2 ds\Big],
\end{aligned}$$

and again applying Gronwall's inequality we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\in[0,t]}|\Delta L_s|^2\Big] \le\ & C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t L_s^2\Big(\sup_{r\le s}|\Delta X_r|^2 + \sup_{r\in[0,s]} W_1(\mu_r, \mu'_r)^2\Big) ds\Big] \\
\le\ & C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t \Big(\mathbb{E}^{\mathbb{Q}^0}\big[L_s^4 \big|\mathcal{F}^Y_T\big]\Big)^{\frac{1}{2}}\Big[\Big(\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le s}|\Delta X_r|^4 \big|\mathcal{F}^Y_T\Big]\Big)^{\frac{1}{2}} + \sup_{r\le s} W_1(\mu_r, \mu'_r)^2\Big] ds\Big].
\end{aligned}$$

Now by (5.3) we conclude from the above that, for $t \in [0,T]$, $\mathbb{Q}^0$-a.s.,

$$\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\in[0,t]}|\Delta L_s|^2\Big] \le\; C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t \big(\mathbb{E}^{\mathbb{Q}^0}\big[L_s^4\big|\mathcal{F}_T^Y\big]\big)^{\frac{1}{2}}\cdot\sup_{r\le s}W_1(\mu_r,\mu_r')^2ds\Big]. \tag{5.4}$$

Moreover, noting that $\mathbb{E}^{\mathbb{Q}^0}\big[L_s^4\big|\mathcal{F}_T^Y\big] \le \zeta_t(\mu,\mu')$, we see from (5.2), (5.3), and (5.4) that we would easily have the desired estimate (5.1) so the Contraction Mapping Theorem can be applied (at least in the case when the time duration is small) if we could find a bound for $\zeta_t(\mu,\mu')$ that is independent of μ, μ'. But this is in general difficult, since each L^μ is the solution to a linear SDE driven by the $\mathbb{Q}^0$-Brownian motion Y, thus under the conditional expectation $\mathbb{E}^{\mathbb{Q}^0}[\cdot|\mathcal{F}_t^Y]$, this essentially amounts to asking a pathwise uniform bound for a family of martingales, which is generally impossible. We shall therefore impose the following extra structural assumption on the coefficient h in SDE (1.1).

Assumption 5.1. *The function h in (1.1) is of the form:*

$$h(t,x,y_{\cdot\wedge t}) = \sum_{i=1}^N f_i(t,x)g_i(t,y_{\cdot\wedge t}), \tag{5.5}$$

where $f_i \in \mathbb{C}_b^{1,2}([0,T]\times\mathbb{R})$, $1\le i\le N$, and g_i's are bounded and measurable. ■

We remark that Assumption 5.1 trivially contains all the traditional nonlinear filtering problems, in which $h = h(t,x)$. In what follows, without loss of generality we shall assume $N=1$, and $f := f_1$, $g := g_1$. We have the following crucial result regarding the process $\zeta(\mu,\mu')$ defined by (4.1), for any $\mu,\ \mu' \in \mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$.

Proposition 5.2. *Let Assumptions 2.2 and 5.1 be in force. Then there exists a continuous, increasing, $\mathbb{F}^Y$-adapted process $A = \{A_t\}_{t\in[0,T]}$, with $A_0 > 0$, such that for any $\mu,\mu' \in S^2_{\mathbb{F}^Y}(\mathscr{P}_1)$, it holds that $\zeta_t(\mu,\mu') \le A_t$, $t\in[0,T]$, $\mathbb{Q}^0$-a.s.*

Proof. For any $\mu \in \mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$, let $X = X^\mu$ be the solution to (3.1). Since $f \in \mathbb{C}^{1,2}([0,T]\times\mathbb{R})$, thanks to Assumption 5.1, applying Itô's formula we get:

$$df(t,X_t) = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial^2_{xx}f(t,X_t)d\langle X\rangle_t, \quad t\in[0,T]. \tag{5.6}$$

Now let us consider the following two processes: For $t\in[0,T]$,

$$\begin{cases} Z_t = \displaystyle\int_0^t g(s,Y_{\cdot\wedge s})dY_s; \\ M_t = M_t^\mu := \displaystyle\int_0^t Z_s\partial_x f(s,X_s)\sigma(s,X_{\cdot\wedge s},Y_{\cdot\wedge s},\mu_{\cdot\wedge s})dB_s^1. \end{cases} \tag{5.7}$$

Recalling that B^1 and Y are independent under $\mathbb{Q}^0$, we have $d\langle M, Z\rangle_t = d\langle X, Z\rangle_t \equiv 0$. Thus (5.6) and integrating by parts yield, for $t \in [0,T]$,

$$\begin{aligned}
&\int_0^t h(s, X_s, Y_{\cdot\wedge s})dY_s = \int_0^t f(s, X_s)g(s, Y_{\cdot\wedge s})dY_s = \int_0^t f(s, X_s)dZ_s \\
&= f(t, X_t)Z_t - \int_0^t Z_s\big[\partial_t f(s, X_s)ds + \partial_x f(t, X_s)dX_s + \frac{1}{2}\partial^2_{xx} f(s, X_s)d\langle X\rangle_s\big] \qquad (5.8)\\
&= f(t, X_t)Z_t - M_t - \int_0^t Z_s\big[\partial_t f(s, X_s) + \frac{1}{2}\partial^2_{xx} f(s, X_s)|\sigma(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})|^2\big]ds.
\end{aligned}$$

Since σ is bounded and $f \in \mathbb{C}^{1,2}_b$ we easily deduce that

$$\begin{cases}
\Big|f(t, X_t)Z_t - \displaystyle\int_0^t Z_s\big[\partial_t f(s, X_s) + \frac{1}{2}\partial^2_{xx} f(s, X_s)|\sigma(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})|^2\big]ds\Big| \le CZ^*_t, \\
\langle M\rangle_t = \displaystyle\int_0^t |Z_s\partial_x f(s, X_s)\sigma(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})|^2 ds \le CZ^{*,2}_t, \quad t \in [0,T].
\end{cases} \qquad (5.9)$$

Here $C > 0$ is a generic constant depending only on the bounds of f and σ. Since Z^* is $\mathbb{F}^Y$-adapted, a direct computation using (5.8) and (5.9) shows that, for all $p > 0$, $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t}\Big(\exp\Big\{p\int_0^s h(r, X_r, Y_{\cdot\wedge r})dY_r\Big\}\Big)\Big|\mathcal{F}^Y_T\Big] \\
&= \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t}\Big(\exp\Big\{pf(s, X_s)Z_s - pM_s - p\int_0^s Z_r[\partial_t f(r, X_r) \\
&\qquad + \frac{1}{2}\partial^2_{xx} f(r, X_r)|\sigma(r, X_{\cdot\wedge r}, Y_{\cdot\wedge r}, \mu_{\cdot\wedge r})|^2]dr\Big\}\Big)\Big|\mathcal{F}^Y_T\Big] \\
&\le C_p\Big(\mathbb{E}^{\mathbb{Q}^0}\big[\sup_{s\le t}\big(\exp\{-pM_s - p^2\langle M\rangle_s\}\big)^2|\mathcal{F}^Y_T\big]\Big)^{\frac{1}{2}} e^{C_p Z^{*,2}_t} \qquad (5.10)\\
&\le C_p\Big(\mathbb{E}^{\mathbb{Q}^0}[\exp\{-2pM_t - 2p^2\langle M\rangle_t\}|\mathcal{F}^Y_T]\Big)^{\frac{1}{2}} e^{C_p Z^{*,2}_t} = C_p\big(\mathbb{E}^{\mathbb{Q}^0}[\mathcal{E}_t|\mathcal{F}^Y_T]\big)^{\frac{1}{2}} e^{C_p Z^{*,2}_t},
\end{aligned}$$

where $C_p > 0$ is some generic constant that may depend on p, and is allowed to vary from line to line, and $\mathcal{E}_t := \exp\{-2pM_t - \frac{1}{2}\langle 2pM\rangle_t\}$ is the Doléans-Dade stochastic exponential of the process $2pM$. That is, $\mathcal{E}$ solves the linear SDE:

$$\begin{aligned}
\mathcal{E}_t &= 1 - \int_0^t \mathcal{E}_s d(2pM_s) \\
&= 1 - 2p\int_0^t \mathcal{E}_s Z_s \partial_x f(s, X_s)\sigma(s, X_{\cdot\wedge s}, Y_{\cdot\wedge s}, \mu_{\cdot\wedge s})dB^1_s, \quad t \in [0,T]. \qquad (5.11)
\end{aligned}$$

Now consider the regular conditional probability $\mathbb{P}^{\omega^2}_T(\cdot) := \mathbb{Q}^0[\,\cdot\,|\mathcal{F}^Y_T](\omega^2)$, for $\mathbb{P}_0$-a.e. $\omega^2 \in \mathbb{C}_T$. For an $\mathbb{F}$-adapted process ξ we denote $\xi^{\omega^2}(\omega^1) = \xi(\omega^1, \omega^2)$, $(\omega^1, \omega^2) \in \mathbb{C}^2_T$. Then, since μ is $\mathbb{F}^Y$-adapted, (5.11) means that for $\mathcal{E}^{\omega^2}$, for $\mathbb{P}_0$-a.e. $\omega^2 \in \mathbb{C}_T$, it holds $\mathbb{P}^{\omega^2}_T$ almost surely:

$$\mathcal{E}^{\omega^2}_t = 1 - 2p\int_0^t \mathcal{E}^{\omega^2}_s Z^{\omega^2}_s \partial_x f(s, X^{\omega^2}_s)\sigma(s, X^{\omega^2}_{\cdot\wedge s}, \omega^2_{\cdot\wedge s}, \mu^{\omega^2}_{\cdot\wedge s})dB^1_s, \quad t \in [0,T].$$

That is, $\mathcal{E}^{\omega^2}$ is an exponential martingale under $\mathbb{P}_T^{\omega^2}$, and thus $\mathbb{E}^{\mathbb{Q}^0}[\mathcal{E}_t|\mathcal{F}_T^Y](\omega^2) = 1$, for $\mathbb{P}_0$-a.e. $\omega^2 \in \mathbb{C}_T$. Consequently, since h is bounded, for $\mu \in \mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$ and $p \geq 1$ we see from the definition of L^μ (3.2) and (5.10) that, for some generic constant $C_p > 0$,

$$\mathbb{E}^{\mathbb{Q}^0}[(L^\mu)_t^{*,p}|\mathcal{F}_T^Y] \leq C_p\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{0\leq s\leq t}\Big(\exp\Big\{p\int_0^s h(r, X_r, Y_{\cdot\wedge r})dY_r\Big\}\Big)\Big|\mathcal{F}_T^Y\Big] \leq C_p e^{C_p Z_t^{*,2}}.$$

But this particularly implies that, for $p = 4$ (Recall the definition (4.1) of $\zeta_t(\mu, \mu'))$ there exists a constant $C(= C_p) > 0$, such that for any $\mu, \mu' \in \mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$, it holds that

$$\zeta_t(\mu, \mu') \leq C \exp\{CZ_t^{*,2}\} =: A_t, \quad t \in [0, T], \tag{5.12}$$

where Z is defined by (5.7). Clearly, the process A is continuous, $\mathbb{F}^Y$-adapted, increasing, and is independent of the choice of μ, proving the proposition. ∎

We now give the main result of this section.

Theorem 5.3. *Let Assumptions 2.2 and 5.1 be in force. Then, the solution mapping $\mathscr{T}(\cdot)$ defined by (3.3) has a unique fixed point in $\mathbb{S}^2_{\mathbb{F}^Y}(\mathscr{P}_1)$.*

Proof. First consider the process A in Proposition 5.2. For $N \geq 1$, define the $\mathbb{F}^Y$-stopping time $\tau_N := \inf\{t \geq 0 : A_t > N\} \wedge T$. Then, $\mathbb{Q}^0\{\tau_N \nearrow T\} = 1$. Moreover, let us now define, for $N \in \mathbb{N}$ and $p \geq 1$, $\mathbb{S}^{p,N}_{\mathbb{F}^Y}(\mathscr{P}_1) := \{\mu_{\cdot\wedge\tau_N} : \mu \in \mathbb{S}^p_{\mathbb{F}^Y}(\mathscr{P}_1)\}$, and

$$\mathscr{T}_N(\mu)_t := \mathscr{T}(\mu)_{t\wedge\tau_N}, \quad t \in [0, T], \quad \mu \in \mathbb{S}^{p,N}_{\mathbb{F}^Y}(\mathscr{P}_1). \tag{5.13}$$

Then, applying Proposition 4.1 and Corollary 4.2 we conclude that $\mathscr{T}_N$ is a mapping from $\mathbb{S}^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)$ to itself.

We first show that each $\mathscr{T}_N$, $N \in \mathbb{N}$, has a fixed point. To this end, let $\mu \in \mathbb{S}^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)$ and let (X^μ, L^μ) be the corresponding solution of (3.1). Consider the function $(t, x, \omega) \mapsto h(t, x, Y_{\cdot\wedge t}(\omega))\mathbf{1}_{[0,\tau_N]}(t, \omega)$. Since τ_N is an $\mathbb{F}^Y$-stopping time, and Y is a canonical Brownian motion $(Y_t(\omega) = \omega_t^2)$ under $\mathbb{Q}^0$, there is some bounded and measurable functional h^N : $[0, T] \times \mathbb{R} \times \mathbb{C}_T \to \mathbb{R}$, such that

(i) for each $x \in \mathbb{R}$, and $(t, \omega) \in [0, T] \times \Omega^0$, $h^N(t, x, \omega) = h^N(t, x, \omega^2_{\cdot\wedge t})$. In other words, the mapping $(t, \omega) \mapsto h^N(t, x, \omega)$ is $\mathbb{F}^Y$-progressively measurable; and

(ii) it holds that

$$h^N(t, X_t^\mu, Y_{\cdot\wedge t}) = h(t, X_t^\mu, Y_{\cdot\wedge t})\mathbf{1}_{[0,\tau_N]}(t, \omega), \quad t \in [0, T]. \tag{5.14}$$

Using the function h^N we can solve the SDE:

$$L_t^{\mu,N} = 1 + \int_0^t h^N(s, X_s^\mu, Y_{\cdot\wedge s}) L_s^{\mu,N} dY_s, \quad t\in[0,T], \tag{5.15}$$

Then, by uniqueness, it is easy to check that $L^{\mu,N} \equiv L^\mu_{\cdot\wedge\tau_N}$, where L^μ solves (3.1).

Now for $\mu,\mu' \in \mathbb{S}^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)$, let (X^μ, L^μ), $(X^{\mu'}, L^{\mu'})$ be the corresponding solutions to (3.1), respectively. We shall denote $X := X^\mu$, $X' := X^{\mu'}$, and $L^N := L^{\mu,N}$, $L'^N := L^{\mu',N}$ for simplicity. Since h^N is uniformly Lipschitz continuous in x with the same Lipschitz constant as h given in Assumption 2.2, we deduce from (5.3)-(5.4) that, for $t\in[0,T]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\in[0,t]} |X_s - X'_s|^4 \Big| \mathcal{F}_T^Y\Big] &\le C\Big[\int_0^t \sup_{r\le s} W_1(\mu_r,\mu'_r)^2 ds\Big]^2, && (5.16)\\
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\in[0,t]} |L_s^N - L'^N_s|^2\Big] &\le C\mathbb{E}^{\mathbb{Q}^0}\Big[\int_0^t (\mathbb{E}^{\mathbb{Q}^0}[L^4_{s\wedge\tau_N}|\mathcal{F}_T^Y])^{\frac12}\cdot \sup_{r\le s\wedge\tau_N} W_1(\mu_r,\mu'_r)^2 ds\Big]\\
&\le C\sqrt{N}\int_0^t \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le s\wedge\tau_N} W_1(\mu_r,\mu'_r)^2\Big] ds. && (5.17)
\end{aligned}$$

Here in the above we used the facts that $\mathbb{E}^{\mathbb{Q}^0}[L^4_{s\wedge\tau_N}|\mathcal{F}^Y_T] \le A_{t\wedge\tau_N} \le N$ by definition of τ_N. Let us denote $\widetilde{\mu}^N_t(\cdot) := \mathbb{P}^N\{X_t\in\cdot|\mathcal{F}^Y_t\}$ and $\widetilde{\mu}'^N_t(\cdot) := \mathbb{P}'^N\{X'_t\in\cdot|\mathcal{F}^Y_t\}$, $t\in[0,T]$, where $d\mathbb{P}^N := L^N_T d\mathbb{Q}^0$, and $d\mathbb{P}'^N := L'^N_T d\mathbb{Q}^0$, respectively.

Recall again that τ_N is an $\mathbb{F}^Y$-stopping time, and observe for $B\in\mathcal{B}(\mathbb{R})$ we have

$$\begin{aligned}
\widetilde{\mu}^N_t(B) &= \mathbb{P}^N\{X_t\in B|\mathcal{F}^Y_t\} = \frac{\mathbb{E}^{\mathbb{Q}^0}[L^N_t \mathbf{1}_{\{X_t\in B\}}|\mathcal{F}^Y_t]}{\mathbb{E}^{\mathbb{Q}^0}[L^N_t|\mathcal{F}^Y_t]} = \frac{\mathbb{E}^{\mathbb{Q}^0}[L_{t\wedge\tau_N}\mathbf{1}_{\{X_t\in B\}}|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L_{t\wedge\tau_N}|\mathcal{F}^Y_T]}\\
&= \frac{\mathbb{E}^{\mathbb{Q}^0}[L_t\mathbf{1}_{\{X_t\in B\}}|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L_t|\mathcal{F}^Y_T]} = \widetilde{\mu}_t(B), \quad t\le\tau_N, \quad \mathbb{Q}^0\text{-}a.s.
\end{aligned}$$

In other words, we have $\tilde{\mu}^N_{\cdot\wedge\tau_N} = \tilde{\mu}_{\cdot\wedge\tau_N} = \mathscr{T}_N(\mu)$, by definition (5.13). Similarly, we have $\widetilde{\mu}'^N_{\cdot\wedge\tau_N} = \widetilde{\mu}'_{\cdot\wedge\tau_N} = \mathscr{T}_N(\mu')$. Consequently, from (5.2) and Proposition 5.2, as τ_N is an $\mathbb{F}^Y$-stopping time and, hence, $\mathcal{F}^Y_T$-measurable, for $t\in[0,T]$,

$$\begin{aligned}
&\sup_{s\le t\wedge\tau_N} W_1(\widetilde{\mu}^N_s, \widetilde{\mu}'^N_s)^2 = \sup_{s\le t\wedge\tau_N} W_1(\widetilde{\mu}_s,\widetilde{\mu}'_s)^2 && (5.18)\\
&\le N^2\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t\wedge\tau_N}|X_s - X'_s|^2 + \sup_{s\le t\wedge\tau_N}|L_s - L'_s|^2\Big|\mathcal{F}^Y_T\Big].
\end{aligned}$$

Combining (5.18) with (5.16) and (5.17) we derive, for all $\mu,\mu'\in S^{2,N}_{\mathbb{F}^Y}(\mathcal{P}_1)$, that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t} W_1(\mathscr{T}_N(\mu)_s, \mathscr{T}_N(\mu')_s)^2\Big] &\le C_N\int_0^t \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le s\wedge\tau_N} W_1(\mu_r,\mu'_r)^2\Big] ds && (5.19)\\
&\le C_N\int_0^t \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le s} W_1(\mu_r,\mu'_r)^2\Big] ds, \quad t\in[0,T].
\end{aligned}$$

Consequently, iterating (5.19) k times we have, for $\mu,\ \mu' \in S^{2,N}_{\mathbb{F}^Y}(\mathcal{P}_1)$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{s\le t} W_1\big(\mathscr{T}_N^k(\mu)_s, \mathscr{T}_N^k(\mu')_s\big)^2\Big] &\le C_N^k \int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}} \mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le t_k} W_1(\mu_r,\mu'_r)^2\Big] dt_k\cdots dt_1 \\ &\le \frac{(C_N t)^k}{k!}\mathbb{E}^{\mathbb{Q}^0}\Big[\sup_{r\le t} W_1(\mu_r,\mu'_r)^2\Big], \quad t\in[0,T].\end{aligned}$$

Choosing k large enough, we see that $\mathscr{T}_N^k:\ S^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)\to S^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)$ is a contraction, thus there is a unique $\bar\mu^N \in S^{2,N}_{\mathbb{F}^Y}(\mathscr{P}_1)$ such that $\mathscr{T}_N^k(\bar\mu^N)=\bar\mu^N$, which implies that $\mathscr{T}_N(\bar\mu^N) = \mathscr{T}_N(\mathscr{T}_N^k(\bar\mu^N)) = \mathscr{T}_N^k(\mathscr{T}_N(\bar\mu^N))$. Then by uniqueness we obtain $\mathscr{T}_N(\bar\mu^N)=\bar\mu^N$.

To construct the desired fixed point for $\mathscr{T}$ on $[0,T]$ we shall argue that there is a standard extension, $\bar\mu$, of the family $\{\bar\mu^N\}_{N\ge1}$: $\bar\mu_t=\bar\mu^N_t$, whenever $t\in[0,\tau_N]$, as $\mathbb{Q}^0\{\tau_N\nearrow T\}=1$. For this, we first claim that, given $\mu\in S^2_{\mathbb{F}^Y}(\mathscr{P}_1)$ and any $\mathbb{F}^Y$-stopping time $\tau\le T$, one has

$$\mathscr{T}(\mu)_{t\wedge\tau} = \mathscr{T}(\mu_{\cdot\wedge\tau})_t, \qquad t\in[0,T]. \tag{5.20}$$

Indeed, for any bounded measurable function φ and $t\in[0,T]$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\varphi(x)\mathscr{T}(\mu)_{t\wedge\tau}(dx) = \mathbb{E}^{\mathbb{P}^\mu}[\varphi(X^\mu_s)|\mathcal{F}^Y_s]\big|_{s=t\wedge\tau} = \frac{\mathbb{E}^{\mathbb{Q}^0}[\varphi(X^\mu_s)L^\mu_s|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_s|\mathcal{F}^Y_T]}\Big|_{s=t\wedge\tau} \\ =\ & \frac{\mathbb{E}^{\mathbb{Q}^0}[\varphi(X^\mu_{t\wedge\tau})L^\mu_{t\wedge\tau}|\mathcal{F}^Y_T]}{\mathbb{E}^{\mathbb{Q}^0}[L^\mu_{t\wedge\tau}|\mathcal{F}^Y_T]} = \mathbb{E}^{\mathbb{P}^\mu}[\varphi(X^\mu_{t\wedge\tau})|\mathcal{F}^Y_{t\wedge\tau}] = \int_{\mathbb{R}}\varphi(x)\mathscr{T}(\mu_{\cdot\wedge\tau})_t(dx).\end{aligned}$$

This proves (5.20). Now, using (5.20) and the definition of $\mathscr{T}_N$ we can further deduce that

$$\mathscr{T}_N(\bar\mu^{N+1}_{\cdot\wedge\tau_N}) = \mathscr{T}(\bar\mu^{N+1}_{\cdot\wedge\tau_N})_{\cdot\wedge\tau_N} = (\mathscr{T}(\bar\mu^{N+1})_{\cdot\wedge\tau_{N+1}})_{\cdot\wedge\tau_N} = \mathscr{T}_{N+1}(\bar\mu^{N+1})_{\cdot\wedge\tau_N} = \bar\mu^{N+1}_{\cdot\wedge\tau_N}.$$

Thus $\bar\mu^{N+1}_{\cdot\wedge\tau_N}$ is also a fixed point of $\mathscr{T}_N$. By the uniqueness of the fixed point for $\mathscr{T}_N$ we must have $\bar\mu^{N+1}_{\cdot\wedge\tau_N}=\bar\mu^N$ on $[0,\tau_N]$. That is, $\bar\mu^{N+1}$ is an "extension" of $\bar\mu^N$.

The rest of the proof is now standard. We can "patch" all the $\bar\mu$'s together by defining a measure-valued process

$$\bar\mu_t := \bar\mu^N_t, \quad t\in[0,\tau_N], \quad N\ge1.$$

Then $\bar\mu$ is well-defined on $[0,T]$ and one can easily check $\bar\mu\in S^2_{\mathbb{F}^Y}(\mathscr{P}_1)$. Furthermore, using (5.20) again we have, for any $N\ge1$,

$$\mathscr{T}(\bar\mu)_{\cdot\wedge\tau_N} = \mathscr{T}(\bar\mu_{\cdot\wedge\tau_N})_{\cdot\wedge\tau_N} = \mathscr{T}(\bar\mu^N)_{\cdot\wedge\tau_N} = \mathscr{T}_N(\bar\mu^N) = \bar\mu^N = \bar\mu_{\cdot\wedge\tau_N}.$$

Thus, $\bar\mu$ is a fixed point of $\mathscr{T}$ on $[0,T]=\cup_{N=1}^\infty[0,\tau_N]$. Finally, note that if ν is another fixed point of $\mathscr{T}$, then by definition, for each $N\ge1$, $\nu_{\cdot\wedge\tau_N}$ must be a fixed point of $\mathscr{T}_N$.

The uniqueness of the fixed point then implies that $\bar{\mu}_{\cdot\wedge\tau_N} = \nu_{\cdot\wedge\tau_N}$, which in turn implies the uniqueness of the fixed point of $\mathscr{T}$. The proof is now complete. ■

Now let $\bar{\mu} \in S^2_{\mathbb{F}^Y}(\mathscr{P}_1)$ be the fixed point of $\mathscr{T}$, and denote $(X, L) := (X^{\bar{\mu}}, L^{\bar{\mu}})$. Recalling the construction of $\mathscr{T}(\bar{\mu})$ we see that, under $\mathbb{Q}^0$, the couple of processes (X, L) satisfies the following SDE:

$$\begin{cases} dX_t = \sigma(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \bar{\mu}_{\cdot\wedge t})dB^1_1, & X_0 = x_0, \\ dL_t = h(t, X_t, Y_{\cdot\wedge t})L_t dY_t, & L_0 = 1, \qquad t \in [0, T]. \end{cases} \tag{5.21}$$

Furthermore, by construction (3.3) we see that $\bar{\mu}_t = \mathbb{P}\{X_t \in \cdot | \mathcal{F}^Y_t\}$, where $d\mathbb{P} := L_T d\mathbb{Q}^0$. Now, denoting $\bar{\mu} = \mu^{X|Y}$, we have the following theorem.

Theorem 5.4. *Let Assumptions 2.2 and 5.1 be in force. Then the SDE (1.1) possesses a weak solution.*

Proof. First note that, given the fixed point $\bar{\mu}$ of the mapping $\mathscr{T}$, and the corresponding solution (X, L) to SDE (5.21) on the probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, if we define $d\mathbb{P} := L_T d\mathbb{Q}^0$, and $B^2_t := Y_t - \int_0^t h(s, X_s, Y_{\cdot\wedge s})ds$, $t \in [0, T]$, then the Girsanov theorem tells us that the process (B^1, B^2) is an $(\mathbb{F}, \mathbb{P})$-Brownian motion.

Now, recalling that $\mu^{X|Y}_t(\cdot) = \bar{\mu}_t(\cdot) = \mathbb{P}\{X_t \in \cdot | \mathcal{F}^Y_t\}$, $t \in [0, T]$, we have, for $t \in [0, T]$,

$$\begin{cases} dX_t = \sigma(t, X_{\cdot\wedge t}, Y_{\cdot\wedge t}, \mu^{X|Y}_{\cdot\wedge t})dB^1_t, & X_0 = x_0; \\ dY_t = h(t, X_t, Y_{\cdot\wedge t})dt + dB^2_t, & Y_0 = 0. \end{cases}$$

In other words, the six-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (B^1, B^2), (X, Y))$ is a weak solution of SDE (1.1), proving the theorem. ■

6 Uniqueness in Law

In this section we shall address the issue of uniqueness of the weak solution to SDE (1.1). Namely, we shall prove that the weak solution of (1.1) is unique in law. Our main idea extends the one in our previous work [1] in a non-trivial way. That is, we note the fact that if $(X, Y, \mathbb{P})$ is a weak solution to (1.1), and $\mu^{X|Y}$ is the conditional law of X given $\mathbb{F}^Y$, under $\mathbb{P}$, then as we argued before we must have $\mu^{X|Y} \in S^2_{\mathbb{F}^Y}(\mathscr{P}_1) \subset \mathbb{L}^0_{\mathbb{F}^Y}(\mathscr{C}_T)$. Therefore, there exists a progressively measurable Borel functional $\Phi : \mathbb{C}_T \to \mathscr{C}_T(\mathscr{P}_1)$, such that

$$\mu^{X|Y}_t = \Phi(Y)_t = \Phi(Y_{\cdot\wedge t})_t, \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \tag{6.1}$$

We shall use this function Φ as the bridge to connect two weak solutions, and then argue that they must be unique in law. More precisely, we have the following theorem.

Theorem 6.1. *Let Assumptions 2.2 and 5.1 hold. Let* $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, (B^{1,i}, B^{2,i}), (X^i, Y^i))$, $i = 1, 2$, *be two weak solutions of (1.1). Then, it holds that*

$$\mathbb{P}^1 \circ (B^{1,1}, B^{2,1}, X^1, Y^1)^{-1} = \mathbb{P}^2 \circ (B^{1,2}, B^{2,2}, X^2, Y^2)^{-1}.$$

Proof. Consider the following SDEs on $(\Omega^i, \mathbb{F}^i, \mathbb{P}^i)$, $i = 1, 2$, respectively:

$$d\hat{L}^i_t = -h(t, X^i_t, Y^i_{\cdot\wedge t})\hat{L}^i_t dB^{2,i}_t, \quad \bar{L}^i_0 = 1, \quad t \in [0, T]. \tag{6.2}$$

(Note the difference between this SDE and the one in (3.1)!) Since h is bounded, we know that $\mathbb{E}^{\mathbb{P}^i}[\hat{L}^i_T] = 1$, and $d\mathbb{Q}^i := \hat{L}^i_T d\mathbb{P}^i$ defines a probability measure such that $(B^{1,i}, Y^i)$ is an $(\mathbb{F}^i, \mathbb{Q}^i)$-Brownian motion, $i = 1, 2$. Denote $L^i = [\hat{L}^i]^{-1}$.

Now let $\Phi^i : \mathbb{C}_T \to \mathscr{C}_T(\mathscr{P}_1)$, $i = 1, 2$, be the progressively measurable Borel functionals, such that (6.1) holds for $\mu^{X^i|Y^i}$, $i = 1, 2$, respectively. Then, the process $(X^1, L^1 = [\hat{L}^1]^{-1})$ must satisfy the following SDE on $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{Q}^1)$:

$$\begin{cases} dX^1_t = \sigma(t, X^1_{\cdot\wedge t}, Y^1_{\cdot\wedge t}, \Phi^1_{\cdot\wedge t}(Y^1_{\cdot\wedge t}))dB^{1,1}_t, & X^1_0 = x_0, \\ dL^1_t = L^1_t h(t, X^1_t, Y^1_{\cdot\wedge t})dY^1_t, & L^1_0 = 1, \quad t \in [0, T]. \end{cases} \tag{6.3}$$

Note that under $\mathbb{Q}^1$, $(B^{1,1}, Y^1)$ is a Brownian motion, thus (6.3) is just an SDE with random coefficients, and under the Assumptions 2.2 and 5.1, it has a pathwisely unique strong solution. That is, there exists a progressively measurable Borel functional $\psi : \mathbb{C}^2_T \mapsto \mathbb{C}^2_T$, such that $(X^1, L^1) = \psi(B^{1,1}, Y^1)$.

We now consider the following auxiliary SDE on the filtered space $(\Omega^2, \mathcal{F}^2, \mathbb{F}^2, \mathbb{Q}^2)$:

$$\begin{cases} d\bar{X}^2_t = \sigma(t, \bar{X}^2_{\cdot\wedge t}, Y^2_{\cdot\wedge t}, \Phi^1_{\cdot\wedge t}(Y^2_{\cdot\wedge t}))dB^{1,2}_t, & \bar{X}^2_0 = x_0, \\ d\bar{L}^2_t = \bar{L}^2_t h(t, \bar{X}^2_t, Y^2_{\cdot\wedge t})dY^2_t, & \bar{L}^2_0 = 1, \qquad t \in [0, T]. \end{cases} \tag{6.4}$$

Note that SDE (6.4) actually has the same coefficients as (6.3), hence by pathwise uniqueness we deduce that $(\bar{X}^2, \bar{L}^2) = \psi(B^{1,2}, Y^2)$ as well. But since $\mathbb{Q}^1 \circ (B^{1,1}, Y^1)^{-1} = \mathbb{Q}^2 \circ (B^{1,2}, Y^2)^{-1}$ is the Wiener measure on $(\Omega^0, \mathcal{F}^0) = (\mathbb{C}^2_T, \mathscr{B}(\mathbb{C}^2_T))$, we conclude that

$$\mathbb{Q}^1 \circ ((B^{1,1}, Y^1), (X^1, L^1))^{-1} = \mathbb{Q}^2 \circ ((B^{1,2}, Y^2), (\bar{X}^2, \bar{L}^2))^{-1}. \tag{6.5}$$

Our next step is to use $(\bar{X}^2, \bar{L}^2)$ to build a bridge that links the laws of (X^1, L^1) and (X^2, L^2). To this end, let us now define a new probability $\bar{\mathbb{P}}^2$ by $d\bar{\mathbb{P}}^2 = \bar{L}^2_T d\mathbb{Q}^2$, and consider the conditional law $\bar{\mu}^2 = \bar{\mu}^{\bar{X}^2|Y^2}$, under $\bar{\mathbb{P}}^2$. That is, for $A \in \mathscr{B}(\mathbb{R})$, it holds that

$$\bar{\mu}^2_t(A) = \bar{\mu}^{\bar{X}^2|Y^2}_t(A) := \bar{\mathbb{P}}^2\big\{\bar{X}^2_t \in A\big|\mathcal{F}^{Y^2}_t\big\} = \frac{\mathbb{E}^{\mathbb{Q}^2}\big[\bar{L}^2_t \cdot \mathbf{1}_{\{\bar{X}^2_t \in A\}}\big|\mathcal{F}^{Y^2}_T\big]}{\mathbb{E}^{\mathbb{Q}^2}\big[\bar{L}^2_t\big|\mathcal{F}^{Y^2}_T\big]}. \tag{6.6}$$

We shall assume without loss of generality that $\bar{\mu}^2$ is a regular conditional probability. As before, we can show that $\bar{\mu}^2 \in S^2_{\mathbb{F}^Y}(\mathscr{P}_1)$ (under $\mathbb{Q}^2$). Furthermore, it might be chosen that

$$\bar{\mu}^2_t(\cdot) = \Phi^1_t(Y^2_{\cdot\wedge t})(\cdot), \qquad t \in [0,T]. \tag{6.7}$$

Indeed, recall that $\Phi^1 : \mathbb{C}_T \to \mathscr{C}_T(\mathscr{P}_1)$ and observe that, for any bounded Borel functionals $\varphi : \mathbb{R} \mapsto \mathbb{R}$ and $f : \mathbb{C}_T \to \mathbb{R}$, (6.5) implies that

$$\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}^2}\Big[\varphi\big(Y^2_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\Phi^1_t(Y^2_{\cdot\wedge t})(dx)\Big] = \mathbb{E}^{\mathbb{Q}^2}\Big[\bar{L}^2_t\varphi\big(Y^2_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\Phi^1_t(Y^2_{\cdot\wedge t})(dx)\Big]\\
=\;& \mathbb{E}^{\mathbb{Q}^1}\Big[L^1_t\varphi\big(Y^1_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\Phi^1_t(Y^1_{\cdot\wedge t})(dx)\Big].
\end{aligned} \tag{6.8}$$

Recalling that $\Phi^1_t(Y^1_{\cdot\wedge t}) = \mu^{X^1|Y^1}_t(\cdot) = \mathbb{P}^1\big\{X^1_t \in \cdot\,\big|\mathcal{F}^{Y^1}_t\big\}$, $t \in [0,T]$, we have

$$\int_{\mathbb{R}} f(x)\Phi^1_t(Y^1_{\cdot\wedge t})(dx) = \mathbb{E}^{\mathbb{P}^1}\big[f(X^1_t)\big|\mathcal{F}^{Y^1}_t\big].$$

Thus (6.8) now reads

$$\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}^2}\Big[\varphi\big(Y^2_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\Phi^1_t(Y^2_{\cdot\wedge t})(dx)\Big] = \mathbb{E}^{\mathbb{Q}^1}\Big[L^1_t\varphi\big(Y^1_{\cdot\wedge t}\big)\mathbb{E}^{\mathbb{P}^1}\big[f(X^1_t)\big|\mathcal{F}^{Y^1}_t\big]\Big] \qquad (6.9)\\
=\;& \mathbb{E}^{\mathbb{P}^1}\big[\varphi\big(Y^1_{\cdot\wedge t}\big)\mathbb{E}^{\mathbb{P}^1}\big[f(X^1_t)\big|\mathcal{F}^{Y^1}_t\big]\big] = \mathbb{E}^{\mathbb{P}^1}\big[\varphi\big(Y^1_{\cdot\wedge t}\big)f(X^1_t)\big] = \mathbb{E}^{\mathbb{Q}^1}\Big[L^1_t\varphi\big(Y^1_{\cdot\wedge t}\big)f(X^1_t)\Big].
\end{aligned}$$

On the other hand, (6.9), together with (6.5), also shows that

$$\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}^2}\Big[\varphi\big(Y^2_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\Phi^1_t(Y^2_{\cdot\wedge t})(dx)\Big] = \mathbb{E}^{\mathbb{Q}^1}\big[L^1_t\varphi\big(Y^1_{\cdot\wedge t}\big)f(X^1_t)\big]\\
=\;& \mathbb{E}^{\mathbb{Q}^2}\big[\bar{L}^2_t\varphi\big(Y^2_{\cdot\wedge t}\big)f(\bar{X}^2_t)\big] = \mathbb{E}^{\bar{\mathbb{P}}^2}\big[\varphi\big(Y^2_{\cdot\wedge t}\big)f(\bar{X}^2_t)\big]\\
=\;& \mathbb{E}^{\bar{\mathbb{P}}^2}\Big[\varphi\big(Y^2_{\cdot\wedge t}\big)\mathbb{E}^{\bar{\mathbb{P}}^2}\big[f(\bar{X}^2_t)\big|\mathcal{F}^{Y^2}_t\big]\Big] = \mathbb{E}^{\bar{\mathbb{P}}^2}\Big[\varphi\big(Y^2_{\cdot\wedge t}\big)\int_{\mathbb{R}} f(x)\bar{\mu}^2_t(dx)\Big].
\end{aligned}$$

Since both φ and f are arbitrary, we have proved the claim (6.7). We note that using (6.7) SDE (6.4) can be rewritten as

$$\begin{cases} d\bar{X}^2_t = \sigma(t, \bar{X}^2_{\cdot\wedge t}, Y^2_{\cdot\wedge t}, \bar{\mu}^2_{\cdot\wedge t})dB^{1,2}_t, & \bar{X}^2_0 = x_0,\\ d\bar{L}^2_t = \bar{L}^2_t h(t, \bar{X}^2_t, Y^2_{\cdot\wedge t})dY^2_t, & \bar{L}^2_0 = 1, \qquad t \in [0,T], \end{cases} \tag{6.10}$$

with $\bar{\mu}^2_{\cdot\wedge t}(\cdot) = \bar{\mathbb{P}}^2\big\{\bar{X}^2_t \in \cdot\big|\mathcal{F}^{Y^2}_t\big\}$, $t \in [0,T]$, which satisfies (6.6).

Our final observation is that, by the construction of the solution mapping $\mathscr{T}$ (3.3) and the definition of μ^2, we see that both μ^2 and $\bar{\mu}^2$ are in $S^2_{\mathbb{F}^{Y^2}}(\mathscr{P}_1)$ under the probability $\mathbb{Q}^2$, and they satisfy

$$\mathscr{T}(\bar{\mu}^2) = \bar{\mu}^2, \qquad \mathscr{T}(\mu^2) = \mu^2.$$

Namely, both μ^2 and $\bar\mu^2$ are the fixed points of the solution mapping $\mathscr{T}$. Thus, the uniqueness of the fixed point implies that $\bar\mu^2 = \mu^2$.

Finally, recall that the process $(X^2, L^2 = [\hat L^2]^{-1})$ satisfies the SDE on $(\Omega^2, \mathcal{F}^2, \mathbb{Q}^2)$:

$$\begin{cases} dX^2_t = \sigma(t, X^2_{\cdot\wedge t}, Y^2_{\cdot\wedge t}, \mu^2_{\cdot\wedge t})dB^{1,2}_t, & X^2_0 = x_0, \\ dL^2_t = L^2_t h(t, X^2_t, Y^2_{\cdot\wedge t})dY^2_t, & L^2_0 = 1, \qquad t \in [0,T], \end{cases} \tag{6.11}$$

where $\mu^2_{\cdot\wedge t} = \mathbb{P}^2\{X^2_t \in \cdot | \mathcal{F}^{Y^2}_t\}$, $t\in[0,T]$, $d\mathbb{P}^2 = L^2_T d\mathbb{Q}^2$. Consequently, both SDEs (6.10) and (6.11) are defined on $(\Omega^2, \mathcal{F}^2, \mathbb{Q}^2)$, have the same coefficients (given μ^2), and are driven by the same $(\mathbb{F}^2, \mathbb{Q}^2)$-Brownian motion $(B^{1,2}, Y^2)$. Thus the pathwise uniqueness of SDE (given μ^2) leads to that $(\bar X^2, \bar L^2) \equiv (X^2, L^2)$, $\mathbb{Q}^2$-a.s.

Consequently, we now have $d\bar{\mathbb{P}}^2 = \bar L^2_T d\mathbb{Q}^2 = L^2_T d\mathbb{Q}^2 = d\mathbb{P}^2$. Combining this with (6.5) we get

$$\mathbb{Q}^1 \circ (B^{1,1}, Y^1, X^1, L^1)^{-1} = \mathbb{Q}^2 \circ (B^{1,2}, Y^2, \bar X^2, \bar L^2)^{-1} = \mathbb{Q}^2 \circ (B^{1,2}, Y^2, X^2, L^2)^{-1}. \tag{6.12}$$

Since $B^{2,1}_t = Y^1_t - \int_0^t h(s, X^1_s, Y^1_{\cdot\wedge s})ds$, $B^{2,2}_t = Y^2_t - \int_0^t h(s, X^2_s, Y^2_{\cdot\wedge s})ds$, $t\in[0,T]$, we obtain from (6.12) that

$$\mathbb{P}^1 \circ (B^{1,1}, B^{2,1}, X^1, Y^1)^{-1} = \mathbb{P}^2 \circ (B^{1,2}, B^{2,2}, X^2, Y^2)^{-1}.$$

The proof is now complete. ∎

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