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PATHWISE STOCHASTIC CONTROL PROBLEMS AND STOCHASTIC HJB EQUATIONS*

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Abstract. In this paper we study a class of *pathwise stochastic control problems* in which the optimality is allowed to depend on the paths of exogenous noise (or information). Such a phenomenon can be illustrated by considering a particular investor who wants to take advantage of certain extra information but in a completely legal manner. We show that such a control problem may not even have a "minimizing sequence," but nevertheless the (Bellman) dynamical programming principle still holds. We then show that the corresponding Hamilton–Jacobi–Bellman equation is a stochastic partial differential equation, as was predicted by Lion and Souganidis [*C. R. Acad. Sci. Paris Sér. I Math.*, 327 (1998), pp. 735–741]. Our main device is a Doss–Sussmann-type transformation introduced in our previous work [*Stochastic Process. Appl.*, 93 (2001), pp. 181–204] and [*Stochastic Process. Appl.*, 93 (2001), pp. 205–228]. With the help of such a transformation we reduce the pathwise control problem to a more standard relaxed control problem, from which we are able to verify that the value function of the pathwise stochastic control problem is the unique stochastic viscosity solution to this stochastic partial differential equation, in the sense of [*Stochastic Process. Appl.*, 93 (2001), pp. 181–204] and [*Stochastic Process. Appl.*, 93 (2001), pp. 205–228].

Key words. pathwise stochastic control, dynamical programming, Bellman principle, Doss–Sussmann transformation, stochastic viscosity solutions

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1. Introduction.

In this paper we are interested in the so-called *pathwise stochastic control problem*, originally proposed by P.-L. Lions and Souganidis [13]. A version of such a problem can be described by the following example of optimazation problem in finance, in which the underlying risky asset follows a "hidden Markovian" stochastic volatility model:

$$\begin{cases} dS_t = S_t[\mu(t, S_t)dt + \widehat{\sigma}(Y_t)dW_t], \\ dY_t = h(t, Y_t)dt + \sigma_1(t, Y_t)dW_t + \sigma_2(t, Y_t) \circ dB_t, \\ S_0 = \xi, Y_0 = y. \end{cases} \tag{1.1}$$

where W and B are two independent Brownian motions, S_t is the asset value at t, and $\widehat{\sigma}(Y)$ is the volatility process. Here the Stratonovic differential $\circ dB$ in (1.1) is used to simplify our future discussion; it can be replaced by an Itô-type integral if needed.

We note that the extra noise B in (1.1) can be thought of as some extra information that cannot be detected in the market in general, but is available to the particular investor. The problem then is to show how this investor can take advantage of such extra information to optimize his/her utility, but by taking actions that are

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completely "legal," in the sense that the investor has to choose the optimal strategy in the usual class of the admissible portfolios.

Mathematically, we can formulate such an optimization problem for this investor as follows. First recall that if the short rate of the riskless asset in this market is $\{r_t,\ t \geq 0\}$, and we assume that the (self-financing) portfolios $\{\pi_t,\ t \geq 0\}$ are all the $\{\mathcal{F}_t^W\}$-progressively measurable, square-integrable processes, then the dynamics of the wealth process, $\{\mathcal{V}_t, t \geq 0\}$, of the investor satisfies the SDE

$$\begin{cases} d\mathcal{V}_t = [r_t\mathcal{V}_t + \pi_t(\mu(t,S_t) - r_t)]dt + \pi_t\widehat{\sigma}(Y_t)dW_t, \\ \mathcal{V}_0 = v. \end{cases} \tag{1.2}$$

Next, denoting $\{\mathcal{F}_t^B\}_{t\geq 0}$ to be the filtration generated by B, we define the following "cost functional" (given the extra information):

$$J(\pi) = E\left\{H(\mathcal{V}_T) + \int_0^T \ell(t,\mathcal{V}_t,\pi_t)dt\Big|\mathcal{F}_T^B\right\}. \tag{1.3}$$

Clearly, the purpose of conditioning on $\mathcal{F}_T^B$ means that we are seeking optimization given all the possible extra information (some of them might be anticipating!), while the restriction that all strategies are $\{\mathcal{F}_t^W\}$-adapted indicates that they are completely "legal."

To make (1.2) and (1.3) fit more into a stochastic control framework, we combine (1.1) and (1.2) as follows. Define $X \stackrel{\triangle}{=} (Y,S,\mathcal{V})^T$ and

$$\begin{aligned} b(t,X_t,\pi_t) &\stackrel{\triangle}{=} [h(t,Y_t), \mu(t,S_t)S_t, r_t\mathcal{V}_t + \pi_t(\mu(t,S_t) - r_t)]^T, \\ \sigma(t,X_t,\pi_t) &\stackrel{\triangle}{=} [\sigma_1(t,Y_t), \hat{\sigma}(Y_t)S_t, \hat{\sigma}(Y_t)\pi_t]^T, \\ \theta(t,X_t) &\stackrel{\triangle}{=} [\sigma_2(t,Y_t),0,0]^T. \end{aligned}$$

Then the new "state" process X satisfies the following SDE:

$$\begin{cases} dX_t = b(t,X_t,\pi_t)dt + \sigma(t,X_t,\pi_t)dW_t + \theta(t,X_t)\circ dB_t, \\ X_s = (y,\xi,v)^T, \end{cases} \qquad 0 \leq s \leq t \leq T, \tag{1.4}$$

and cost functional (1.3) can be rewritten as

$$J(s,(y,\xi,v);\pi) = E_{s,(y,\xi,v)}\left\{H(X_T) + \int_s^T \ell(t,X_t,\pi_t)dt\Big|\mathcal{F}_T^B\right\}. \tag{1.5}$$

Our *pathwise stochastic control problem* is then to minimize the cost functional (1.5) over the set of "admissible controls" $\mathcal{A}$, which is by definition all the $\{\mathcal{F}_t^W\}$-adapted strategies. The value function of this stochastic control problem is defined by

$$V(s,(y,\xi,v)) = \operatorname*{essinf}_{\pi\in\mathcal{A}} J(s,(y,\xi,v)). \tag{1.6}$$

Here the essential infimum should be understood as one with respect to the indexed family of random variables (see, e.g., [5, 7] or [10, Appendix A]; detailed definition will be given in section 2).

At this point we would like to point out that the pathwise stochastic control problem of this kind was one of the motivations for the study of the "stochastic viscosity solution" for fully nonlinear stochastic PDEs (see Lions and Souganidis [13]). However, while it has long been predicted that the Hamilton–Jacobi–Bellman (HJB) equation for such stochastic control problem is a fully nonlinear stochastic PDE, to date the mathematical content of this problem has not been fully explored. One of the main purposes of this paper is to try to establish a rigorous framework for the pathwise stochastic control problem and provide some necessary machinery for future study. It turns out that many of them are interesting in their own right. Among other things, we shall prove the Bellman dynamic programming principle in this particular situation, from which we then prove that the value function is a *stochastic viscosity* solution of a stochastic HJB equation, in the sense of our previous works (see Buckdahn and Ma [2, 3]). It should be noted that the special measurability issue involved in the "legality" of our admissible controls has not been studied before.

We should note that the pathwise control problem defined above is quite different from a standard stochastic control problem, or even those with partial observations (see, e.g., Bensoussan [1]). The most essential difference is that the cost functional is now a random field instead of a deterministic function, and therefore so is the value function. Consequently, the usual infimum (or supremum) involved in the optimization problem should naturally be replaced by the "essential infimum" (or "essential supremum"). Such a seemingly "routine" change, together with the "legality" requirements for the admissible controls, turns out to be the source of many substantial difficulties, both from a mathematical point of view and from the control theoretical point of view. In fact, in the appendix we shall provide an example which shows that in general there does not exist a minimizing sequence for our pathwise stochastic control problem. The lack of such a sequence seems to be fatal for the dynamic programming method. To overcome this difficulty we introduce an intermediate stochastic control problem (called the "wider-sense" control problem in what follows), in which the stochastic integral against the Brownian motion B is eliminated. We show that this wider-sense control problem is in some sense equivalent to a traditional stochastic control problem, and we can use it as a bridge to reach our goal.

Another immediate problem is the stochastic HJB equation itself, mainly in various forms of measurability issues including the definition of the stochastic integration (one should appreciate the fact that an HJB equation is always "backward"!). One of the main reasons that we insist on using the Stratonovic stochastic integral with regard to the Brownian motion B is that it is "insensitive" to the direction of integration, as we can show. Finally, we would like to remark that the notion of stochastic viscosity solutions has been studied by Lions and Souganidis [12, 13, 14, 15] and Buckdahn and Ma [2, 3, 4]. The definition of the stochastic viscosity solution in this paper is consistent with our previous works, with slight modifications to suit the present situation. We note that such a modification will not alter the uniqueness result from [3].

The rest of the paper is organized as follows. In section 2 we formulate the problem more formally and provide some necessary preliminaries. In section 3 we introduce some wider-sense control problems and establish some properties and relationship among them. Section 4 is devoted to a proof of the Bellman principle. Finally, in section 5 we prove that the value function of the corresponding wider-sense control problem is the viscosity solution to a randomized HJB equations, and in section 6 we extend the result to the original problem.

2. Problem formulation and preliminaries. In this section we give a detailed formulation of our pathwise control problem. Let $W = (W^1, \dots, W^d)$ and $B = (B^1, \dots, B^m)$ be two independent, standard Brownian motions defined on some complete probability space $(\Omega, \mathcal{F}, P)$, and let $T > 0$ be a given finite time horizon. We denote $\mathbf{F}^W = \{\mathcal{F}_t^W\}_{t\in[0,T]}$ and $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{t\in[0,T]}$ to be the two filtrations generated by W and B, respectively, and augmented by the P-null sets in $\mathcal{F}$ so that they satisfy the *usual hypotheses* (see, e.g., [16]). The following two filtrations will be frequently used in the future:

$$
(2.1) \qquad \begin{cases} \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]} \stackrel{\Delta}{=} \{\mathcal{F}_t^W \vee \mathcal{F}_t^B\}_{t\in[0,T]} = \mathbf{F}^W \vee \mathbf{F}^B, \\ \mathbf{G} = \{\mathcal{G}_t\}_{t\in[0,T]} \stackrel{\Delta}{=} \{\mathcal{F}_t^W \vee \mathcal{F}_T^B\}_{t\in[0,T]} = \mathbf{F}^W \vee \mathcal{F}_T^B. \end{cases}
$$

Here, for two σ-fields $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \vee \mathcal{G}$ denotes $\sigma(\mathcal{F} \cup \mathcal{G})$ as usual. The meaning for those involving filtrations is obvious.

Throughout this paper we let $\mathbb{E}$ be a generic Euclidean space, with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We denote $C^k(\mathbb{E}; \mathbb{E}_1)$ to be the usual space of $\mathbb{E}_1$-valued, k-times continuously differentiable functions defined on $\mathbb{E}$. Furthermore, we denote $C_b^k(\mathbb{E}; \mathbb{E}_1) \subset C^k(\mathbb{E}; \mathbb{E}_1)$ to be all functions that have uniformly bounded partial derivatives and $C_p^k(\mathbb{E}; \mathbb{E}_1) \subset C^k(\mathbb{E}; \mathbb{E}_1)$ to be all functions whose partial derivatives are of at most polynomial growth. The spaces $C^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$, $C_b^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$ are defined similarly. Finally, if $\mathbb{E}_1 = \mathbb{R}$, we shall omit $\mathbb{E}_1$ in the notation above.

Now let $\mathbb{B}$ be a generic Banach space and $\mathbf{H} = \{\mathcal{H}_t\}_{t\in[0,T]}$ a generic Filtration on $(\Omega, \mathcal{F}, P)$. We shall denote the following:

- For any $1 \le p < \infty$, $L^p(\mathbf{H}, [0,T]; \mathbb{B})$ denotes all $\mathbb{B}$-valued, $\mathbf{H}$-progressively measurable processes ψ, such that $E\int_0^T \|\psi_t\|_{\mathbb{B}}^p dt < \infty$. In particular, we denote $L^0(\mathbf{H}, [0,T]; \mathbb{B})$ to be all $\mathbb{B}$-valued, $\mathbf{H}$-progressively measurable processes and $L^\infty(\mathbf{H}, [0,T]; \mathbb{B})$ to be a subset of $L^0(\mathbf{H}, [0,T]; \mathbb{B})$ in which all processes are uniformly bounded.
- For any $1 \le p < \infty$, $L_{loc}^p(\mathbf{H}, [0,T] \times \mathbb{E}; \mathbb{E}_1)$ denotes the space of all $\mathbb{E}_1$-valued random fields η defined on $[0,T] \times \Omega \times \mathbb{E}$, such that for fixed $x \in \mathbb{E}$, the mapping $(t,\omega) \mapsto \eta(t,\omega,x)$ is $\mathbf{H}$-progressively measurable, and that $E\int_0^T |\eta(t,\cdot,x)|^p dt < \infty$. In particular, we denote $L_{loc}^p(\mathbf{H}, [0,T]; C^k(\mathbb{E}, \mathbb{E}_1))$ to be all $C^k(\mathbb{E}, \mathbb{E}_1)$-valued, $\mathbf{H}$-progressively measurable random fields η such that η and all the partial derivatives $D_{x_i}\eta$ are elements of $L_{loc}^p(\mathbf{H}, [0,T] \times \mathbb{E}; \mathbb{E}_1)$.
- $C^{k,\ell}(\mathbf{H}, [0,T] \times \mathbb{E}; \mathbb{E}_1)$ (resp., $C_b^{k,\ell}(\mathbf{H}, [0,T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathbf{H}, [0,T] \times \mathbb{E}; \mathbb{E}_1)$) denotes the space of all $\mathbb{E}_1$-valued random fields φ, defined on $[0,T] \times \Omega \times \mathbb{E}$ such that for P-a.e. $\omega \in \Omega$, $\varphi(\cdot, \omega, \cdot) \in C^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$ (resp., $C_b^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}([0,T] \times \mathbb{E}; \mathbb{E}_1)$), and that for fixed $x \in \mathbb{E}$, the process $\varphi(\cdot, \cdot, x)$ is $\mathbf{H}$-progressively measurable.
- $\mathcal{M}_{0,T}(\mathbf{H})$ denotes all the $\mathbf{H}$-stopping times τ such that $0 \le \tau \le T$, P-a.s., and $\mathcal{M}_{0,\infty}(\mathbf{H})$ denotes all $\mathbf{H}$-stopping times that are almost surely finite.

To formulate our control problem let us first specify the *admissible control* sets. Let U be a compact metric space. We denote $\mathcal{A}$ to be the set of all $\mathbf{F}^W$-progressively measurable processes $\alpha : [0,T] \times \Omega \to U$, and denote $\widetilde{\mathcal{A}}$ to be the set of all $\mathbf{G}$-progressively measurable processes $\beta : [0,T] \times \Omega \to U$. We shall refer to the elements in $\mathcal{A}$ as the admissible controls and those in $\widetilde{\mathcal{A}}$ as the *admissible controls in a wider sense*.

For $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\alpha \in \mathcal{A}$, let us consider the following controlled stochastic system: for $0 \le s \le t \le T$,

$$
(2.2) \quad X_t = x + \int_s^t b(r, X_r, \alpha_r)\, dr + \int_s^t \sigma(r, X_r, \alpha_r)\, dW_r + \int_s^t \theta(r, X_r) \circ dB_r.
$$

The solution of SDE (2.2) will be denoted by $X^{\alpha,s,x}$. The superscripts $^{(\alpha,s,x)}$ will often be dropped for notational simplicity if the context is clear.

The *cost functional* of the pathwise control problems is defined by

$$J(\alpha;s,x)\stackrel{\Delta}{=}E\left\{H(X_T^{\alpha,s,x})+\int_s^T\ell(t,X_t^{\alpha,s,x},\alpha_t)dt\Big|\mathcal{F}_T^B\right\},\tag{2.3}$$

where $(s,x)\in[0,T]\times\mathbb{R}^n$, and $\alpha\in\mathcal{A}$, and the *value function* is defined by

$$V(s,x)\stackrel{\Delta}{=}\operatorname*{essinf}_{\alpha\in\mathcal{A}}J(\alpha,s,x).\tag{2.4}$$

We note that the definition of essential infimum for a family of nonnegative random variables can be found in [7] and [10, Appendix A]; we recast it here for ready reference.

Definition 2.1. *Let $\mathcal{X}$ be a nonempty family of nonnegative random variables defined on a probability space $(\Omega,\mathcal{F},P)$. The essential infimum of $\mathcal{X}$, denoted by* essinf $\mathcal{X}$*, is a random variable X^* satisfying the folowing:*

(i) *for all $X\in\mathcal{X}$, $X^*\le X$, P-a.s.; and*

(ii) *if Y is a random variable such that $Y\le X$ for all $X\in\mathcal{X}$, then $Y\le X^*$, P-a.s.*

Throughout this paper we shall make use of the following standing assumptions:

(H1) The functions $b:\mathbb{R}^n\times U\mapsto\mathbb{R}^n$, $\sigma:\mathbb{R}^n\times U\mapsto\mathbb{R}^{n\times d}$ are bounded, uniformly continuous, and uniformly Lipschitz with respect to $x\in\mathbb{R}^n$, uniformly in $u\in U$.

(H2) The function θ belongs to $C^4_{\ell,b}(\mathbb{R}^n;\mathbb{R}^{n\times m})$.

(H3) The function $H:\mathbb{R}^n\mapsto\mathbb{R}$ is uniformly bounded and continuous.

The Lipschitz constants in (H1)–(H3) will be denoted by a generic one $K>0$.

We would like to remark here that in (2.4) the value function $V(\cdot,\cdot)$ is obtained by taking an "essinf" instead of an "inf" as in the usual stochastic control problem. Such a change on the one hand is necessary due to the randomness of the cost functionals, as it is often seen in optimization problems involving random objectives (see, e.g., [10]); it does, on the other hand, generate a great deal of subtleties to the otherwise standard control problem. In fact, in the Appendix we shall provide a counterexample which shows that with such an "essinf" one does not even have a "minimizing sequence" to the control problem. This gives rise to some substantial difficulties in proving the dynamic programming principle, as well as in deriving the stochastic HJB equations. We will show how to get around of this difficulty by studying the related "wider-sense control problems" in section 4.

We should also note that since we do not require any "nondegeneracy" condition on the coefficient σ (or $\sigma\sigma^T$) in this framework, we can apply a standard treatment to reduce the system to a time-homogeneous one and to eliminate the running cost ℓ in (2.3) by adding the extra states

$$X_t^0=t\qquad\text{and}\qquad X_t^{n+1}=\int_s^t\ell(r,X_r,\alpha_r)dr.$$

For example, in this case the cost functional can be written as

$$J(\alpha;s,(s,x,0))=E\{\widetilde{H}(X_T^{\alpha,s,(s,x,0)})\big|\mathcal{F}_T^B\},$$

where $\widetilde{H}(x^0,x,x^{n+1})=H(x)+x^{n+1}$. Therefore, in the rest of the paper we shall consider the following simplified version of (2.2), (2.3), and (2.4):

$$X_t = x + \int_s^t b(X_r, \alpha_r)\, dr + \int_s^t \sigma(X_r, \alpha_r)\, dW_r + \int_s^t \theta(X_r) \circ dB_r, \tag{2.5}$$

$$J(\alpha; s, x) \stackrel{\triangle}{=} E\left\{ H(X_T^{\alpha,s,x}) \Big| \mathcal{F}_T^B \right\}, \tag{2.6}$$

and

$$V(s,x) \stackrel{\triangle}{=} \operatorname*{essinf}_{\alpha\in\mathcal{A}} J(\alpha, s, x). \tag{2.7}$$

To conclude this section we remark that under (H1) and (H2), for any admissible control $\alpha \in \mathcal{A}$ the SDE (2.2) has a unique ($\mathbf{F}$-adapted) solution. But if $\alpha \in \widetilde{\mathcal{A}}$, the situation would be much more complicated, although it could be made sensible if we allow the integral $\int \theta(X_r) \circ dB_r$ to be the anticipating Stratonovic integral. But we shall avoid such complexity by introducing the wider-sense problems. Finally, we note that the value function defined by (2.7) is an $\mathcal{F}_T^B$-measurable random field. We shall prove that it is indeed $\mathcal{F}_{s,T}^B$-measurable for any $s \in [0,T]$, and hence the stochastic HJB equation, which is "backwardly" defined (given the terminal condition at time T), would simply be a time-reversed stochastic PDE in the usual sense.

3. A Doss–Sussmann-type transformation. In this section we introduce the first step towards our wider-sense control problem. In light of the idea of "stochastic characteristics" (cf. Lions and Souganidis [12, 13, 14, 15]) and/or "Doss–Sussmann" transformation (cf. Buckdahn and Ma [2, 3, 4]), we would like to remove the stochastic integral "$\int \theta(X) \circ dB$" so it becomes less "problematic." We note that such a step is essential in the study of stochastic viscosity solutions as well.

We proceed as follows. First consider the following SDE with parameters: for any $0 \le s \le T$, and $z \in \mathbb{R}^n$,

$$\eta_t^s(z) = z + \int_s^t \theta(\eta_r^s(z)) \circ dB_r \qquad s \le t \le T. \tag{3.1}$$

We note that this SDE can be converted easily into the following "Itô form":

$$\eta_t^s(z) = z + \int_s^t \theta(\eta_r^s(z)) dB_r + \frac{1}{2}\int_s^t [D_x\theta \otimes \theta](\eta_r^s(z)) dr, \qquad s \le t \le T. \tag{3.2}$$

Here $[D_x\theta \otimes \theta]$ denotes the product of the tensor $D_x\theta$ and the matrix θ, defined by

$$[D_x\theta \otimes \theta]_i \stackrel{\triangle}{=} \sum_{j=1}^m \sum_{k=1}^n \frac{\partial \theta^{ij}}{\partial x^k}\theta^{kj} = \operatorname{tr}\{[D_x\theta^i]\theta^T\}, \qquad i = 1, \dots, n, \tag{3.3}$$

where θ^i is the ith row vector of the matrix θ. We note that in this paper the dimensions of the Brownian motion are not essential. Thus, to simplify notation, in what follows we shall assume $m = 1$. Thus the tensor product is simplified to the usual matrix product: $D\theta \otimes \theta = [D\theta]\theta$.

Next, we note that the stochastic flow $z \mapsto \eta_t^s(z)$ is a diffeomorphism for all $s \le t \le T$, P-a.s., and

$$D_z\eta_t^s(z) = I + \int_s^t D_z\eta_r^s(z) D\theta(\eta_r^s(z)) \circ dB_r, \quad s \le t \le T. \tag{3.4}$$

Consequently, $[D_z\eta_t^s]^{-1}(\cdot)$ exists; and it can be shown (see, e.g., [2]) that the inverse flow of $\eta_t^s(\cdot)$, denoted by $\zeta_t^s(\cdot)$, exists and satisfies the first order stochastic PDE:

$$\zeta_t^s(z) = z - \int_s^t D_z\zeta_r^s(z)\theta(z)\circ dB_r, \qquad 0\le s\le t\le T,\ z\in\mathbb{R}^n, \tag{3.5}$$

and it holds that $\eta_t^s(\zeta_t^s(z)) = \zeta_t^s(\eta_t^s(z)) = z$ for $0\le s\le t\le T$, $z\in\mathbb{R}^n$, P-a.s. Let us now define the following random fields: for $(t,z,u)\in[0,T]\times\mathbb{R}^n\times U$,

$$\widetilde{\sigma}(t,z,u) \stackrel{\triangle}{=} [D_z\eta_t^0(z)]^{-1}\sigma(\eta_t^0(z),u); \tag{3.6}$$

$$\widetilde{b}(t,z,u) \stackrel{\triangle}{=} [D_z\eta_t^0(z)]^{-1}\Big\{b(\eta_t^0(z),u) \tag{3.7}$$
$$-\frac{1}{2}\mathrm{tr}\,\{[D_z\eta_t^0(z)]^{-1}\sigma\sigma^T(\eta_t^0(z),u)[(D_z\eta_t^0(z))^T]^{-1}[D_{zz}^2\eta_t^0(z)]\}\Big\}.$$

Clearly, $\widetilde{b}$ and $\widetilde{\sigma}$ are $\mathbf{F}$-progressively measurable. The following lemma is essential.

LEMMA 3.1. *Assume* (H1) *and* (H2). *Then for any $0<\gamma<\frac{1}{12}$, there exist a sequence of $\mathbf{F}^B$-stopping times $0\le\tau_1\le\tau_2\le\cdots\le\tau_\ell\le\cdots$ satisfying $P\{\tau_\ell=T\}\uparrow 1$, as $\ell\to\infty$, and some constant $C>0$, depending only on the coefficients b, σ, and θ, such that for all $\ell\ge 1$ and $t\in[0,T]$, $z,z'\in\mathbb{R}^n$, and $u\in U$, it holds P-a.s. on $\{\tau_\ell=T\}$ that*

$$\begin{aligned}
|\widetilde{\sigma}(t,z,u)| &\le C\ell(1+|z|^2)^\gamma \le C\ell(1+|z|);\\
|\widetilde{b}(t,z,u)| &\le C\ell^4(1+|z|^2)^{4\gamma} \le C\ell^4(1+|z|);\\
|\widetilde{\sigma}(t,z,u)-\widetilde{\sigma}(t,z',u)| &\le C\ell^3(1+|z|^2+|z'|^2)^{3\gamma}|z-z'|\\
&\le C\ell^3(1+|z|+|z'|)|z-z'|;\\
|\widetilde{b}(t,z,u)-\widetilde{b}(t,z',u)| &\le C\ell^6(1+|z|^2+|z'|^2)^{6\gamma}|z-z'|\\
&\le C\ell^6(1+|z|+|z'|)|z-z'|.
\end{aligned}$$

Proof. First, let $\mathbb{E}$ be any Euclidean space, and let $f\in L^2_{loc}(\mathbf{G};[0,T],C^1(\mathbb{R}^n;\mathbb{E}))$ (that is, $f(t,\omega,\cdot)\in C^1(\mathbb{R}^n;\mathbb{E})$, and all components of f and D_zf belong to the space $L^2(\mathbf{G};[0,T]\times\mathbb{R}^n;\mathbb{E})$). For any constant $\gamma>0$ and $p\ge 2\vee n$, we can apply the Sobolev imbedding theorem (cf. section 7.10 of [9]), the Burkholder–Gundy–Davis inequality, and the Hölder inequality to conclude that there exists a constant $C_{p,\gamma}>0$ (which we allow to vary from line to line) such that, for all $s\in[0,T]$,

$$\begin{aligned}
&E\left\{\sup_{\substack{t\in[s,T]\\ z\in\mathbb{R}^n}}\left[(1+|z|^2)^{-\gamma}\Big|\int_s^t f(r,z)dB_r\Big|\right]^2\right\}\\
&\le C_{p,\gamma}E\left\{\sup_{t\in[s,T]}\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}\left\{\Big|\int_s^t f(r,z)dB_r\Big|^p+\Big|\int_s^t D_zf(r,z)dB_r\Big|^p\right\}dz\right\}^{\frac{2}{p}}\\
&\le C_{p,\gamma}\left\{\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}\left\{E\left[\sup_{t\in[s,T]}\Big|\int_s^t f(r,z)dB_r\Big|^p\right]\right.\right.\\
&\qquad\left.\left.+E\left[\sup_{t\in[s,T]}\Big|\int_s^t D_zf(r,z)dB_r\Big|^p\right]\right\}dz\right\}^{\frac{2}{p}}\\
&\le C_{p,\gamma}\left\{\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}\left\{E\left[\int_s^T|f(r,z)|^2dr\right]^{\frac{p}{2}}+E\left[\int_s^T|D_zf(r,z)|^2dr\right]^{\frac{p}{2}}\right\}dz\right\}^{\frac{2}{p}}.
\end{aligned} \tag{3.8}$$

Here we note that, as we pointed out before, if $\mathbb{E} = \mathbb{R}^{n\times m}$, then $D_z f$ should be understood as a "tensor." But such a notational complexity does not cause any substantial difficulty; therefore to simply presentation in the rest of the proof we consider only the case $n = m = 1$.

Now, differentiating (3.1) twice we have, for $s \le t \le T$,

$$D_z\eta_t^s(z) = 1 + \int_s^t \theta'(\eta_r^s(z))D_z\eta_r^s(z)\circ dB_r,$$
$$D_z^2\eta_t^s(z) = \int_s^t \{D_z^2\eta_r^s(z)\theta'(\eta_r^s(z)) + \theta''(\eta_r^s(z))[D_z\eta_t^s(z)]^2\}\circ dB_r.$$

Here and in what follows D_z^k denotes the kth derivative of the flow $z \mapsto \eta_t^s(z)$. Thus, noting assumption (H2) it is readily seen that for any $q \ge 2$,

$$E\left\{\sup_{t\in[s,T]}\left(|D_z\eta_t^s(z)|^q + |D_z^2\eta_t^s(z)|^q\right)\right\} \le \widetilde{C}_q \qquad \forall z \in \mathbb{R}^n. \tag{3.9}$$

Setting $f(s,\cdot,z) = \theta'(\eta_t^s(z))D_z\eta_t^s(z)$ in (3.8) and using (3.9) we obtain that

$$\begin{aligned}
&E\left\{\sup_{\substack{t\in[s,T]\\ z\in\mathbb{R}^n}}\left[\left(1+|z|^2\right)^{-\gamma}\left|\int_s^t \theta'(\eta_t^s(z))D_z\eta_t^s(z)dB_r\right|\right]^2\right\}\\
&\le C_{p,\gamma}\left\{\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}\left[E\left(\int_s^T\left|\theta'(\eta_r^s(z))\right|^2dr\right)^p\right]^{1/2}dz\right\}^{2/p} \qquad (3.10)\\
&\quad + C_{p,\gamma}\left\{\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}\left[E\left(\int_s^T\left|\theta''(\eta_r^s(z))\right|^2dr\right)^p\right]^{1/2}dz\right\}^{2/p}\\
&\le C_{p,\gamma}\left\{\int_{\mathbb{R}^n}(1+|z|^2)^{-p\gamma}dz\right\}^{2/p} \le C_{p,\gamma},
\end{aligned}$$

provided $2p\gamma > 1$. Moreover, from (H2) we also have

$$E\left\{\sup_{\substack{t\in[s,T]\\ z\in\mathbb{R}^n}}\left[\left(1+|z|^2\right)^{-\gamma}\left|\int_s^t[\theta'\theta]'(\eta_r^s(z))D_z\eta_r^s(z)dr\right|\right]^2\right\} \le C_{p,\gamma}. \tag{3.11}$$

This, together with (3.10), gives that

$$E\left\{\sup_{\substack{t\in[s,T]\\ z\in\mathbb{R}^n}}\left[(1+|z|^2)^{-\gamma}\left|\int_s^t\theta'(\eta_r^s(z))D_z\eta_r^s(z)\circ dB_r\right|\right]^2\right\} \le C_{p,\gamma}, \tag{3.12}$$

and consequently

$$E\left\{\sup_{\substack{t\in[s,T]\\ z\in\mathbb{R}^n}}\left[\left(1+|z|^2\right)^{-\gamma}|D_z\eta_t^s(z)|\right]^2\right\} \le C_{p,\gamma}. \tag{3.13}$$

Repeating the similar argument one also shows that, for all $\gamma > 0$ and $2p\gamma > 1$, there is some constant $C_{p,\gamma} > 0$ such that

$$(3.14)\quad E\left\{\sup_{\substack{s\in[t,T]\\ z\in R^n}}\left[(1+|z|^2)^{-\gamma}\left\{|[D_z\eta_t^s]^{-1}(z)| + |D_z^2\eta_t^s(z)| + |D_z^3\eta_t^s(z)|\right\}\right]^2\right\} \le C_{p,\gamma}.$$

We now fix $0 < \gamma < 1/12$, and define a sequence of **F**-stopping times: for $\ell \ge 1$,

$$(3.15)\quad \tau_\ell \stackrel{\triangle}{=} \inf\left\{s \ge 0 : \sup_{z\in\mathbb{R}^n}(1+|z|^2)^{-\gamma}\left[|[D_z\eta_s^0]^{-1}(z)| + \sum_{i=1}^3 |D_z^i\eta_s^0(z)|\right] > \ell\right\} \wedge T.$$

Clearly, by virtue of (3.13) and (3.14), this sequence of stopping times $\{\tau_\ell\}$ satisfies the following properties:

(i) $0 \le \tau_1 \le \tau_2 \le \cdots$;

(ii) $\{\tau_\ell = T\} \uparrow \Omega$, P-a.s., as $\ell \to +\infty$;

(iii) for each $s \in [t, \tau_\ell)$, $z \in \mathbb{R}^n$, and $\ell \ge 1$, it holds P-a.s. that

$$(3.16)\qquad |[D_z\eta_s^0]^{-1}(z)| + |D_z\eta_s^0(z)| + |D_z^2\eta_s^0(z)| + |D_z^3\eta_s^0(z)| \le \ell(1+|z|^2)^\gamma.$$

We can now easily derive the desired properties of $\widetilde{\sigma}$ and $\widetilde{b}$ by combining the definitions (3.6), (3.7) and the estimate (3.16), proving the lemma. $\square$

A direct consequence of Lemma 3.1 is that we can now consider the following "transformed" control system of (2.2). Note that W and B are independent, and W is a **G**-Brownian motion as well. Also, since both $\widetilde{b}$ and $\widetilde{\sigma}$ are **G**-progressively measurable random fields, for any β in the wider-sense admissible control set $\widetilde{\mathcal{A}}$ the following SDE is well-defined:

$$(3.17)\quad F_t^{s,x,\beta} = x + \int_s^t \widetilde{b}(r, F_r^{s,x,\beta}, \beta_r)dr + \int_s^t \widetilde{\sigma}(r, F_r^{s,x,\beta}, \beta_r)dW_r, \qquad t \in [s,T].$$

The following result validates the name "Doss–Sussmann transformation."

LEMMA 3.2. *Assume* (H1) *and* (H2). *Let* $\{\eta_t^s(z) : (t,z) \in [s,T]\times\mathbb{R}^n\}$ *be the stochastic flow given by* (3.1) *or* (3.2), $\{X_t^{s,x,\alpha} : t \in [s,T]\}$ *the solution of the SDE* (2.5), *and* $\{F_t^{s,x,\beta} : t\in[s,T]\}$ *the solution of* (3.17). *Then the following hold:*

(i) *For each* $\alpha \in \mathcal{A}$, *the solution* $F^{s,x,\alpha}$ *is* **F**-*progressively measurable. Moreover, it holds that*

$$\eta_t^s(F_t^{s,x,\alpha}) = X_t^{s,\eta_s^0(x),\alpha}, \qquad t \in [s,T],\ P\text{-a.s.},$$

or, equivalently, $\eta_t^s(F_t^{s,\zeta_s^0(x),\alpha}) = X_t^{s,x,\alpha}$, $t \in [s,T]$, P-*a.s.*

(ii) *For each* $\beta \in \widetilde{\mathcal{A}}$, *we define* $X_t^{s,x,\beta} \stackrel{\triangle}{=} \eta_t^s(F_t^{s,\zeta_s^0(x),\beta})$, $t \in [s,T]$. *Then there exists a sequence of* **F**-*stopping times* $\{\tau_\ell\}_{\ell\ge1}$ *with* $P\{\tau_\ell = T\}\uparrow 1$, *such that*

$$\sup_{\beta\in\widetilde{A}} E\left\{\sup_{t\in[s,\tau_\ell]}\left(\left|X_t^{s,x,\beta}\right|^p + \left|X_t^{s,\eta_s^0(x),\beta}\right|^p\right)\right\} < +\infty \quad \forall\ p \ge 1.$$

Proof. Again, we shall assume $n = 1$ to simplify the presentation. The higher-dimensional case can be treated in exactly the same way without substantial difficulties.

(i) Let $\alpha \in \mathcal{A}$, and denote $F_t^\alpha = F_t^{s,x,\alpha}$ and $\eta_t = \eta_t^0$, $t \geq s$, for simplicity. Define $X_t^\alpha = \eta_t(F_t^\alpha)$, $t \geq s$. Applying the Itô–Ventzell formula (cf., e.g., [11]) in a differential form we get, for $t \in [s,T]$,

$$\begin{aligned}
dX_t^\alpha &= d\eta_t(F_t^\alpha) + D_z\eta_t(F_t^\alpha)dF_t^\alpha + \frac{1}{2}D_z^2\eta_t(F_t^\alpha)d[F^\alpha]_t \\
&= \theta(\eta_t(F_t^\alpha)) \circ dB_t + D_z\eta_t(F_t^\alpha)\widetilde{\sigma}(t,F_t^\alpha,\alpha_t)dW_t + D_z\eta_t(F_t^\alpha)\widetilde{b}(t,F_t^\alpha,\alpha_t)dt \\
&\quad + \frac{1}{2}D_z^2\eta_t(F_t^\alpha)|\widetilde{\sigma}(t,F^\alpha,\alpha_t)|^2 dt \\
&= \theta(\eta_t(F_t^\alpha)) \circ dB_t + \sigma(\eta_t(F_t^\alpha),\alpha_t)dW_t \\
&\quad + \left\{ b(\eta_t(F_t^\alpha),\alpha_t) - \frac{1}{2}\left[\left|[D_z\eta_t(F_t^\alpha)]^{-1}\right|^2 |\sigma(\eta_t(F_t^\alpha),\alpha_t)|^2 D_z^2\eta_t(F_t^\alpha)\right]\right\} dt \\
&\quad + \frac{1}{2}\left|[D_z\eta_t(F_t^\alpha)]^{-1}\right|^2 |\sigma(\eta_t(F_t^\alpha),\alpha_t)|^2 D_z^2\eta_t(F_s^\alpha)dt \\
&= \theta(X_t^\alpha) \circ dB_t + \sigma(X_t^\alpha,\alpha_t)dW_t + b(X_t^\alpha,\alpha_t)dt.
\end{aligned}$$

Furthermore, at $t = s$, one has $X_s^\alpha = \eta_s^0(F_s^{s,x,\alpha}) = \eta_s^0(x)$. Thus by uniqueness of the SDE one must have $X^\alpha \equiv X^{s,\eta_s^0(z),\alpha}$.

(ii) We now let $\beta \in \widetilde{\mathcal{A}}$, and define $X_s^\beta = \eta_t(F_t^\beta)$, $t \in [s,T]$. Note that for all $(t,z) \in [s,T] \times \mathbb{R}^n$ we can write

$$\eta_t(z) = \eta_t(0) + \int_0^1 D_z\eta_t(\theta z)d\theta.$$

One can easily show, thanks to Lemma 3.1, that for all $\gamma > 0$, there exists an increasing sequence of $\mathbb{F}$-stopping times $(\tau_\ell)_{\ell \geq 1}$ with $P\{\tau_\ell = T\} \uparrow 1$, as $\ell \to \infty$, such that all the estimates of Lemma 3.1 are satisfied, and it holds furthermore that

$$|\eta_s(z)|^2 \leq \ell\left(1 + |z|^{2+\gamma}\right), \qquad s \in [t,\tau_\ell],\ z \in \mathbb{R}^n,\ P\text{-a.s.}$$

Consequently, we see that the unique $\mathbb{G}$-adapted solution F^β of (3.17) must satisfy

$$E\left\{ \sup_{s\in[t,\tau_\ell]} |F_s^\beta|^p \right\} < +\infty \qquad \forall\ p > 1.$$

Finally, it is readily seen that $X_t^\beta = \eta_t(F_t^\beta)$, $t \in [s,T]$, is a $\mathbb{G}$-adapted continuous process and it satisfies that $E\{\sup_{s\in[t,\tau_\ell]} |X_s^\beta|^p\} < +\infty$ for all $p > 1$. The other estimate is similar. The proof is complete. □

4. Wider-sense control problems. In this section we introduce two types of wider-sense stochastic control problems which will help us attack the original pathwise control problem. We begin by considering the state equation after the Doss–Sussmann transformation:

$$F_t^\beta = x + \int_s^t \widetilde{\sigma}(r,F_r^\beta,\beta_r)dW_r + \int_s^t \widetilde{b}(r,F_r^\beta,\beta_r)dr, \quad s \in [t,T], \tag{4.1}$$

where $\beta \in \widetilde{\mathcal{A}}$. Lemma 3.1 guarantees the well-posedness of this SDE, and Lemma 3.2 enables us to rewrite the cost functional (2.6) as

$$J(s,x;\alpha) = E\left\{H\ (X_T^{s,x,\alpha}) |\ \mathcal{F}_T^B\right\} = E\left\{H\ (\eta_T^0(F_T^{s,\zeta_0^s(x),\alpha}))\Big|\ \mathcal{F}_T^B\right\}, \qquad P\text{-a.s.},$$

for all $\alpha \in \mathcal{A}$. Let us now define the following two new "cost functionals":

$$\widetilde{J}(s,x,\beta) = E\{H(\eta_T^0(F_T^{s,x,\beta}))|\mathcal{F}_T^B\}, \qquad \beta \in \widetilde{\mathcal{A}}, \tag{4.2}$$

and

$$\widehat{J}(s,x,\beta) = E\{\widetilde{J}(s,x,\beta)\} = E\{H(\eta_T^0(F_T^{s,x,\beta}))\}, \quad \beta \in \widetilde{\mathcal{A}}. \tag{4.3}$$

Then Lemma 3.2(i) tells us that

$$\widetilde{J}(s,x,\alpha) = J(s,\eta_s^0(x),\alpha) \qquad \forall \alpha \in \mathcal{A}. \tag{4.4}$$

In other words $\widetilde{J}$ is in some sense an "extension" of J to the wider-sense admissible control set $\widetilde{\mathcal{A}}$. The following two "wider-sense" stochastic control problems are the building blocks of our method.

Wider-sense control problem I (WSCP-I).
- State: $F^{s,x,\beta}$, $\beta \in \widetilde{\mathcal{A}}$.
- Cost functional: $\widetilde{J}(s,x;\beta)$, $\beta \in \widetilde{\mathcal{A}}$.
- Value function:

$$\widetilde{V}(t,x) = \operatorname*{essinf}_{\beta \in \widetilde{\mathcal{A}}} \widetilde{J}(t,x,\beta)\,. \tag{4.5}$$

Remark 4.1. (i) The main purpose of introducing WSCP-I is to remove the "problematic" term involving the Brownian motion B from the state equation. However, a closer look at the cost functional and the value function should lead to the understanding that it is still a far cry from a standard stochastic control problem. For example, the cost functional not only still contains a conditional expectation, the terminal cost function is actually random (via the flow $\eta_\cdot^0$). As a consequence the value function $\widetilde{V}(s,x)$ is still a random field(!), and thus the "pathwise" nature of the problem remains.

(ii) Although the value function in WSCP-I still involves an "essinf," this time it is much more benign than the original one. The main difference is that in this case there do exist minimizing sequences to this problem.

We make a further modification to completely eliminate the "pathwise" nature of the control problem.

Wider-sense control problem II (WSCP-II).
- State: $F^{s,x,\beta}$, $\beta \in \widetilde{\mathcal{A}}$.
- Cost functional: $\widehat{J}(s,x;\beta)$, $\beta \in \widetilde{\mathcal{A}}$.
- Value function:

$$\widehat{V}(s,x) = \inf_{\beta \in \widetilde{\mathcal{A}}} \widehat{J}(s,x,\beta)\,. \tag{4.6}$$

It is readily seen that WSCP-II looks almost like a standard stochastic control problem, except for the form of the terminal cost function (it is still random via the flow η^0). But it is much easier to handle than the previous two control problems. In the rest of this section we analyze the relationship among the two wider-sense stochastic control problems and the original pathwise control problem.

We begin by observing some more or less obvious facts. First, it is clear that for any $\beta \in \widetilde{\mathcal{A}}$, it holds that

$$E\{\widetilde{V}(s,x)\} = E\bigg\{\operatorname*{essinf}_{\beta \in \widetilde{\mathcal{A}}} \widetilde{J}(s,x;\beta)\bigg\} \le E\{\widetilde{J}(s,x;\beta)\}.$$

Thus we must have

$$E\{\widetilde{V}(s,x)\} \le \hat{V}(s,x), \qquad (s,x) \in [0,T]\times\mathbb{R}^n. \tag{4.7}$$

Next, from (4.4) we see that for all $\alpha \in \mathcal{A}$ and $(s,x) \in [0,T]\times\mathbb{R}^n$, it holds that

$$V(s,\eta^0_s(x)) = \operatorname*{essinf}_{\alpha\in\mathcal{A}} J(s,\eta^0_s(x);\alpha) = \operatorname*{essinf}_{\alpha\in\mathcal{A}} \widetilde{J}(s,x;\alpha) \ge \widetilde{V}(s,x). \tag{4.8}$$

We now give the main result of this section. Among other things, we show that the equalities in (4.7) and (4.8) both hold, and we construct a minimizing sequence for WSCP-I, which is essential for our future discussion.

THEOREM 4.2. *Assume* (H1)–(H3). *Then the following statements hold:*

(i) $\widetilde{V}(s,x) = V(s,\eta^0_s(x))$, *for all* $(s,x)\in[0,T]\times\mathbb{R}^n$, P*-a.s.*

(ii) *There exists some sequence* $\{\beta^k\}_{k\ge1} \subset \widetilde{\mathcal{A}}$ *such that*

$$\widetilde{V}(s,x) = \lim_{k\to\infty}\downarrow \widetilde{J}(s,x;\beta^k), \qquad P\text{-a.s.}$$

Here and in what follows " $\lim\downarrow$*" stands for the monotone decreasing limit.*

(iii) $E\{\widetilde{V}(s,x)\} = \widehat{V}(s,x)$ *for all* $(s,x)\in[0,T]\times\mathbb{R}^n$.

Proof. (i) From (4.8) we know that $\widetilde{V}(s,x) \le V(s,\eta^0_s(x))$. We need only show the reverse inequality. To this end, consider a subset $\widetilde{\mathcal{A}}_0 \subset \widetilde{\mathcal{A}}$ that consists of all elements of the form

$$\beta_t(\omega) = \sum_{1\le i,j\le N} u_{i,j}\mathbf{1}_{[t_{i-1},t_i)\times(A_{i,j}\cap B_{i,j})}(t,\omega), \qquad 0\le s\le t,$$

where $N\ge1$, $s=t_0<t_1<\cdots<t_N=T$, $u_{i,j}\in U$, $A_{i,j}\in\mathcal{F}^W_{t_{i-1}}$, $B_{i,j}\in\mathcal{F}^B_T$ with $B_{i,j}\cap B_{i,k}=\emptyset$ for $j\ne k$, and $\cup_{j=1}^N B_{i,j}=\Omega$.

It is not hard to check that $\widetilde{\mathcal{A}}_0$ is a dense subset of $\widetilde{\mathcal{A}}$ in the space $L^2([s,T]\times\Omega;U)$. That is, for any $\beta\in\widetilde{\mathcal{A}}$ one can find a sequence $(\beta^\ell)_{\ell\ge1}\subset\widetilde{\mathcal{A}}_0$ such that

$$E\left\{\int_s^T |\beta_r-\beta^\ell_r|^2dr\right\} \to 0 \qquad \text{as } \ell\to\infty.$$

It follows that $\{F^{s,x,\beta^\ell}\}_{\ell\ge1}$ converges to $F^{s,x,\beta}$ in $L^0(\Omega;C([0,T];\mathbb{R}^n))$ as $\ell\to\infty$, thanks to Lemma 3.1, and that $\widetilde{J}(s,x;\beta^\ell)\to\widetilde{J}(s,x,\beta)$, in probability, as $\ell\to\infty$. Consequently, one has $\operatorname{essinf}_\ell \widetilde{J}(s,x;\beta^\ell)\le\widetilde{J}(s,x;\beta)$, P-a.s., and thus it suffices to show that

$$V(s,\eta^0_s(x)) \le \widetilde{J}(s,x;\beta) \qquad \forall\ \beta\in\widetilde{\mathcal{A}}_0. \tag{4.9}$$

To this end, we fix arbitrarily a $\beta\in\widetilde{\mathcal{A}}_0$. Denote by Λ the set of all finite sequences $\lambda=\{\lambda_i\}_{1\le i\le N}$, where λ_i's take values in the finite set $\{1,2,\dots,N\}$. For each $\lambda\in\Lambda$, we denote $B_\lambda=\cap_{i=1}^N B_{i,\lambda_i}$, and

$$\alpha^\lambda_t(\omega) = \sum_{1\le i\le N}\mathbf{1}_{[t_{i-1},t_i)}(t)u_{i,\lambda_i}\mathbf{1}_{A_{i,\lambda_i}}(\omega), \qquad (t,\omega)\in[s,T]\times\Omega.$$

Then it is readily seen that all B_λ's are $\mathcal{F}^B_T$-measurable sets, and all α^λ's are elements of the original admissible control set $\mathcal{A}$(!).

Now let us rewrite β as follows:

$$\beta_t(\omega) = \sum_{\lambda\in\Lambda} \alpha_t^\lambda(\omega)\mathbf{1}_{B_\lambda}(\omega), \qquad (t,\omega)\in[s,T]\times\Omega.$$

Observe that $\sum_{\lambda\in\Lambda} B_\lambda = \Omega$, and the SDE (4.1) does not contain a stochastic integral with respect to the Brownian motion B. Using the total probability formula and the uniqueness of solution to the SDE, one can show that the following decomposition holds:

$$F_t^{s,x,\beta} = \sum_{\lambda\in\Lambda} F_t^{s,x,\alpha^\lambda}\mathbf{1}_{B_\lambda}, \qquad t\in[s,T],\ P\text{-a.s.}$$

Similarly, applying the same arguments and noting Lemma 3.2(i) we also have

$$\eta_t^0(F_t^{s,x,\beta}) = \eta_t^0\left(\sum_{\lambda\in\Lambda} F_t^{s,x,\alpha^\lambda}\mathbf{1}_{B_\lambda}\right) = \sum_{\lambda\in\Lambda}\eta_t^0(F_t^{s,x,\alpha^\lambda})\mathbf{1}_{B_\lambda} = \sum_{\lambda\in\Lambda} X_t^{s,\eta^0(x),\alpha^\lambda}\mathbf{1}_{B_\lambda}$$

for all $t\in[s,T]$. Therefore, we have

$$\begin{aligned}\widetilde{J}(s,x,\beta) &= E\left\{H\Big(\eta_T^0(F_T^{s,x,\beta})\Big)\Big|\mathcal{F}_T^B\right\} = \sum_{\lambda\in\Lambda} E\left\{H(X_T^{s,\eta_s^0(x),\alpha^\lambda})\Big|\mathcal{F}_T^B\right\}\mathbf{1}_{B_\lambda}\\ &= \sum_{\lambda\in\Lambda} J(s,\eta^0(x),\alpha^\lambda)\mathbf{1}_{B_\lambda} \ge V(s,\eta^0(x)),\end{aligned}$$

proving (4.9), whence (i).

(ii) Let $(s,x)\in[0,T]\times\mathbb{R}^n$ be fixed. We first choose a sequence $\{\widetilde{\beta}^k\}_{k\ge1}\subset\widetilde{\mathcal{A}}$ such that

$$\widetilde{V}(s,x) = \operatorname*{essinf}_{k\ge1}\widetilde{J}(s,x;\widetilde{\beta}^k).$$

To do this, we borrow the idea of [8]. For any $\varepsilon>0$, define

$$\delta = \inf_{\{\beta^k\}\subset\widetilde{\mathcal{A}}} P\left\{\widetilde{V}(s,x)\le \operatorname*{essinf}_{k\ge1}\widetilde{J}(s,x;\beta^k)-\varepsilon\right\}. \tag{4.10}$$

(Note that the infimum above is taken over all the sequences $\{\beta^k\}$ in $\widetilde{\mathcal{A}}$!) We claim the following two facts:

(a) the infimum in (4.10) is always attained, and

(b) $\delta=0$.

To prove (a), we first use the definition of "inf" to find, for $n=1,2,\dots$, sequences $\{\beta^{n,k}\}_{k\ge1}$, $n=1,2,\dots$, such that

$$\delta\le P\left\{\widetilde{V}(s,x)\le\operatorname*{essinf}_{k\ge1}\widetilde{J}(s,x;\beta^{n,k})-\varepsilon\right\}<\delta+\frac{1}{n}.$$

Let us consider the sequence $\{\beta^{n,k}\}_{n\ge1,k\ge1}$, and denote

$$\begin{aligned}A_\varepsilon^n &\triangleq \left\{\omega:\widetilde{V}(s,x)\le\operatorname*{essinf}_{k\ge1}\widetilde{J}(s,x;\beta^{n,k})-\varepsilon\right\};\\ \bar{A}_\varepsilon &\triangleq \left\{\omega:\widetilde{V}(s,x)\le\operatorname*{essinf}_{n\ge1,k\ge1}\widetilde{J}(s,x;\beta^{n,k})-\varepsilon\right\}.\end{aligned}$$

Then it is easily seen that $\bar{A}_\varepsilon \subseteq A^n_\varepsilon$ for all n. This, together with the definition of δ, leads to

$$\delta \le P\{\bar{A}_\varepsilon\} \le P\{A^n_\varepsilon\} = P\left\{\widetilde{V}(s,x) \le \operatorname*{essinf}_{k\ge 1} \widetilde{J}(s,x;\beta^{n,k}) - \varepsilon\right\} < \delta + \frac{1}{n}$$

for all $n \ge 1$. Letting $n \to \infty$ we obtain that

$$\delta = P\left\{\widetilde{V}(s,x) \le \operatorname*{essinf}_{n\ge 1, k\ge 1} \widetilde{J}(s,x;\beta^{n,k}) - \varepsilon\right\}.$$

This proves (a).

(b) By a rearrangement of indices let us denote the minimizer in part (a) by $\{\widetilde{\beta}^k\}_{k\ge1}$. For any $\varepsilon > 0$ and $\beta \in \widetilde{\mathcal{A}}$ we denote

$$\bar{A}_\varepsilon \stackrel{\triangle}{=} \left\{\widetilde{V}(s,x) \le \operatorname*{essinf}_{k\ge 1} \widetilde{J}(s,x;\widetilde{\beta}^k) - \varepsilon\right\}, \qquad A_\varepsilon(\beta) \stackrel{\triangle}{=} \left\{\widetilde{V}(s,x) \le \widetilde{J}(s,x;\beta) - \varepsilon\right\}.$$

Suppose that $P(\bar{A}_\varepsilon) = \delta > 0$. Then we claim that for each $\varepsilon > 0$ there exists a $\bar{\beta} \in \widetilde{\mathcal{A}}$ such that

$$P\{\bar{A}_\varepsilon \setminus A_{\frac{\varepsilon}{2}}(\bar{\beta})\} > 0. \tag{4.11}$$

Indeed, if for all $\beta \in \widetilde{\mathcal{A}}$ one has $P\{\bar{A}_\varepsilon \setminus A_{\frac{\varepsilon}{2}}(\beta)\} = 0$, then for all $\beta \in \widetilde{\mathcal{A}}$ one must have

$$\widetilde{V}(s,x) + \frac{\varepsilon}{2}\mathbf{1}_{\bar{A}_\varepsilon} \le \widetilde{J}(s,x;\beta), \qquad P\text{-a.s.}$$

But then it follows that

$$\widetilde{V}(s,x) + \frac{\varepsilon}{2}\mathbf{1}_{\bar{A}_\varepsilon} \le \operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} \widetilde{J}(s,x;\beta) = \widetilde{V}(s,x), \qquad P\text{-a.s.},$$

contradicting $P(\bar{A}_\varepsilon) = \delta > 0$. Hence (4.11) must hold, and consequently

$$\begin{aligned} P\left\{\widetilde{V}(s,x) \le \widetilde{J}(s,x;\bar{\beta}) \wedge \operatorname*{essinf}_{k\ge 1} \widetilde{J}(s,x;\widetilde{\beta}^k) - \varepsilon\right\} &\le P\{\bar{A}_e \cap A_\varepsilon(\bar{\beta})\} \\ &\le P\{\bar{A}_e \cap A_{\frac{\varepsilon}{2}}(\bar{\beta})\} = P\{\bar{A}_\varepsilon\} - P\{\bar{A}_\varepsilon \setminus A_{\frac{\varepsilon}{2}}(\bar{\beta})\} < \delta. \end{aligned} \tag{4.12}$$

Let us now modify the sequence $\{\widetilde{\beta}^k\}$ slightly: define

$$\bar{\beta}^\kappa \stackrel{\triangle}{=} \begin{cases} \widetilde{\beta}^k & \text{on } \{\widetilde{J}(s,x;\widetilde{\beta}^k) \le \widetilde{J}(s,x;\bar{\beta})\}, \\ \bar{\beta} & \text{on } \{\widetilde{J}(s,x;\widetilde{\beta}^k) > \widetilde{J}(s,x;\bar{\beta})\}, \end{cases} \qquad k \ge 1. \tag{4.13}$$

Since $J(s,x,\bar{\beta})$ and $J(s,x,\widetilde{\beta}^k)$'s are all $\mathcal{F}^B_T$-measurable, $\{\bar{\beta}^k\}_{k\ge1} \subset \widetilde{\mathcal{A}}$. Furthermore, by definition it is readily seen that

$$\widetilde{J}(s,x;\bar{\beta}^k) \le \widetilde{J}(s,x;\bar{\beta}) \wedge \widetilde{J}(s,x;\widetilde{\beta}^k) \qquad \forall k \ge 1.$$

Thus, using (4.12) we obtain that

$$P\left\{\widetilde{V}(s,x) \le \operatorname*{essinf}_{k\ge 1} \widetilde{J}(s,x;\bar{\beta}^k) - \varepsilon\right\} < \delta.$$

This contradicts the definition of δ, proving (b).

To conclude the proof we use a similar technique to construct the desired minimizing sequence inductively as follows. Let $\beta^1 = \widetilde\beta^1$, and for $k \ge 2$, we define

$$(4.14)\qquad \beta^\kappa \triangleq \begin{cases} \widetilde\beta^k & \text{on } \{\widetilde J(s,x;\widetilde\beta^k) \le \widetilde J(s,x;\beta^{k-1})\}, \\ \beta^{k-1} & \text{on } \{\widetilde J(s,x;\widetilde\beta^k) > \widetilde J(s,x;\beta^{k-1})\}, \end{cases} \qquad k \ge 1.$$

Again, we have $\{\beta^k\}_{k\ge1} \subset \widetilde{\mathcal A}$ and $\widetilde J(s,x;\beta^k) \le \widetilde J(s,x;\beta^{k-1}) \wedge \widetilde J(s,x;\widetilde\beta^k)$ for all $k \ge 1$. Consequently, we have

$$\lim_{k\to\infty} \downarrow \widetilde J(s,x;\beta^k) \le \operatorname*{essinf}_{k\ge1} \widetilde J(s,x;\widetilde\beta^k) = \widetilde V(s,x).$$

Finally, that $\widetilde V(s,x) \le \lim_{k\to\infty} \downarrow \widetilde J(s,x;\beta^k)$ is obvious. We proved (ii).

(iii) is a direct consequence of (ii). Thus proof is complete. □

We remark that part (ii) in Theorem 4.2 provides us the first version of a "minimizing sequence"(!). As we can see, the construction of such a sequence depends heavily on $\widetilde J$, hence $\mathcal F_T^B$-measurable. Thus the wider-sense admissible class $\widetilde{\mathcal A}$ is essential. The counterexample in the appendix shows that this cannot be relaxed.

To end this section let us take a brief look at the existence of optimal control. Note that WSCP-II is now a rather standard optimal control problem; therefore with a possible change of probability space, one should always be able to find an optimal control, at least in a "relaxed" form (cf., e.g., El Karoui, Nguyen, and Jeanblanc-Picqué [8]). We do not pursue this issue here due to the length of the paper. However, we give the following corollary of Theorem 4.2 that more or less explains the benefit of introducing the wider-sense controls.

COROLLARY 4.3. *Assume* (H1)–(H3). *Then any optimal control β^* for the WSCP*-II *is also an optimal control for WSCP*-I. *That is, if for $(s,x) \in [0,T] \times \mathbb R^n$, $\beta^* \in \widetilde{\mathcal A}$ is such that $\widehat V(s,x) = \widehat J(s,x;\beta^*)$, then it must hold that*

$$\widetilde V(s,x) = \widetilde J(s,x;\beta^*), \qquad P\text{-a.s.}$$

Proof. Let $\beta^* \in \widetilde{\mathcal A}$ be an optimal control for WSCP-II, that is, $\widehat V(s,x) = \widehat J(s,x,\beta^*)$. We show that it actually holds that

$$(4.15)\qquad \widetilde J(s,x,\beta^*) \le \widetilde J(s,x,\beta) \qquad \forall\ \beta \in \widetilde{\mathcal A}.$$

To see this let $\beta \in \widetilde{\mathcal A}$ be arbitrary. We define, as before, a new control $\widetilde\beta$ be such that

$$\widetilde\beta = \beta \mathbf 1_{\{\widetilde J(s,x,\beta) < \widetilde J(s,x,\beta*)\}} + \beta^* \mathbf 1_{\{\widetilde J(s,x,\beta) \ge \widetilde J(s,x,\beta^*)\}}.$$

Again, we have $\widetilde\beta \in \widetilde{\mathcal A}$, and the optimality of β^* leads to

$$\begin{aligned} E[\widetilde J(s,x,\beta^*)] &\le E[\widetilde J(s,x,\widetilde\beta)] \\ &= E\left[\widetilde J(s,x,\beta)\mathbf 1_{\{\widetilde J(s,x,\beta)<\widetilde J(s,x,\beta^*)\}} + \widetilde J(s,x,\beta^*)\mathbf 1_{\{\widetilde J(s,x,\beta)\ge\widetilde J(s,x,\beta^*)\}}\right], \end{aligned}$$

or, equivalently, $E[(\widetilde J(s,x,\beta) - \widetilde J(s,x,\beta^*))\mathbf 1_{\{\widetilde J(s,x,\beta)<\widetilde J(s,x,\beta^*)\}}] \ge 0$. Thus we obtain that $P\{\widetilde J(s,x,\beta) < \widetilde J(s,x,\beta^*)\} = 0$, proving (4.15), whence the corollary. □

5. Properties of the value function $\widetilde{V}$. In this section we take a closer look at the value function of WSCP-I, $\widetilde{V}(s,x)$. To be more precise we would like to derive some finer results on its measurability as well as regularity, as a random field. These properties will be important for us to derive the Bellman principle, and ultimately the stochastic HJB equation.

First note that by definition we know immediately that $\widetilde{V}(s,x)$ is $\mathcal{F}^B_T$-measurable for all $(s,x)\in[0,T]\times\mathbb{R}^n$, and therefore it is $\mathbf{G}$-progressively measurable. On the other hand, note that

$$\begin{aligned}\widetilde{J}(s,x,\beta) &= E\left\{H(\eta^0_T(F^{s,x,\beta}_T)\Big|\,\mathcal{F}^B_T\right\} = E\left\{E\left\{H\left(\eta^0_T(F^{s,x,\beta}_T)\right)\Big|\,\mathcal{G}_s\right\}\Big|\,\mathcal{F}^B_T\right\}\\ &\geq E\left\{\operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{H\Big(\eta^0_T(F^{s,x,\beta}_T)\Big)\Big|\,\mathcal{G}_s\right\}\Bigg|\,\mathcal{F}^B_T\right\}.\end{aligned}$$

Therefore we can only have

$$\widetilde{V}(s,x)\geq E\left\{\operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{H\Big(\eta^0_T(F^{s,x,\beta}_T)\Big)\Big|\,\mathcal{G}_s\right\}\Bigg|\,\mathcal{F}^B_T\right\}.$$

Let us first establish a stronger result than the relation above and construct another version of minimizing sequence, which is essential for us to derive the Bellman principle.

THEOREM 5.1. *Assume* (H1)–(H3). *For all $(s,x)\in[0,T]\times\mathbb{R}^n$, there exists a sequence $\{\beta^k\}\subset\widetilde{\mathcal{A}}$, such that*

$$\widetilde{V}(s,x)=\operatorname*{essinf}_{\beta\in\widetilde{A}} E\left\{H\left(\eta^0_T(F^{s,x,\beta}_T)\right)\Big|\,\mathcal{G}_s\right\}=\lim_{k\to\infty}\downarrow E\left\{H\left(\eta^0_T(F^{s,x,\beta^k}_T)\right)\Big|\,\mathcal{G}_s\right\},\ \ P\text{-a.s.}$$

Proof. We first show that, for any $(s,x)\in[0,T]\times\mathbb{R}^n$,

$$\widetilde{V}(s,x)\leq\operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{H(\eta^0_T(F^{s,x,\beta}_T))\Big|\mathcal{G}_s\right\},\qquad P\text{-a.s.}\tag{5.1}$$

To this end, we fix $(s,x)\in[0,T]\times\mathbb{R}^n$ and $\beta\in\widetilde{\mathcal{A}}$, and construct a special sequence $\{\beta^k\}_{k\geq1}$ that approximates β in $L^2([0,T]\times\Omega)$. We proceed as follows. First, let us denote for each $k\geq1$ and $0\leq a<b\leq T$

$$D^{a,b}_k\stackrel{\triangle}{=}\left\{a+\frac{i}{2^k}(b-a): i=0,1,2,\ldots,2^k\right\},$$

and denote the generic elements of $D^{a,b}_k$ to be $t^{k,a,b}_i$ $(=a+\frac{i}{2^k}(b-a))$, $i=0,1,2,\ldots,2^k$. To simplify notation we shall denote $D^a_k=D^{0,a}_k$, and $t^{k,a}_i=t^{k,0,a}_i$ for $a\in[0,T]$. Finally we define $D_k=D^s_k\cup D^{s,T}_k$.

Now consider the probability space $([0,T]\times\Omega,\mathcal{P},\mu)$, where $\mathcal{P}$ is the $\mathbf{G}$-predictable σ-field on $[0,T]\times\Omega$ and $\mu(dtd\omega)=\frac{1}{T}dtP(d\omega)$. Let $\mathcal{G}^s_t\stackrel{\triangle}{=}\sigma\{W_r-W_s,\ r\in[s,t]\}\vee\mathcal{F}^B_T$, $t\geq s$, and introduce the following σ-fields:

$$\begin{aligned}\mathcal{G}^{k,s}_\ell &\stackrel{\triangle}{=}\sigma\{\Delta W_{t^{k,s}_i},\ \Delta B_{t^{k,T}_j}:\ 0\leq i\leq\ell-1;\ 0\leq j\leq 2^k-1\},\\ \widetilde{\mathcal{G}}^{k,s,T}_\ell &\stackrel{\triangle}{=}\sigma\{\Delta W_{t^{k,s}_i},\ \Delta W_{t^{k,s,T}_{i'}},\ \Delta B_{t^{k,T}_j}:\ 0\leq i'\leq\ell-1;\ 0\leq i,j\leq 2^k-1\}.\end{aligned}$$

where $\Delta\xi_{t_i^{k,a,b}} \stackrel{\triangle}{=} \xi_{t_{i+1}^{k,a,b}} - \xi_{t_i^{k,a,b}}$, $\xi = W, B$. Now let

$$\mathcal{P}_k \stackrel{\triangle}{=} \sigma\left\{(t_\ell^{k,s}, t_{\ell+1}^{k,s}] \times A_\ell;\ (t_\ell^{k,s,T}, t_{\ell+1}^{k,s,T}] \times \widetilde{A}_\ell :\ A_\ell \in \mathcal{G}_\ell^{k,s},\ \widetilde{A}_\ell \in \widetilde{\mathcal{G}}_\ell^{k,s,T}\right\}.$$

It is then clear that $\{\mathcal{P}_k\}_{k\ge 1}$ is an increasing family of σ-fields, and $\mathcal{P}_k \uparrow \mathcal{P}$ as $k \to \infty$ (cf., e.g., Dellacherie and Meyer [6]). Furthermore, if we define $\beta^k \stackrel{\triangle}{=} E^\mu\{\beta|\mathcal{P}_k\}$, then we must have $\beta^k \in \widetilde{\mathcal{A}}$, and $\beta^k \to \beta$ in $L^2([0,T]\times\Omega)$ as $k \to \infty$. Consequently, possibly along a subsequence (we may again denote it by $\{\beta^k\}$), one has

$$E\left\{H(\eta_T^0(F_T^{s,x,\beta^k}))\Big|\mathcal{G}_s\right\} \to E\left\{H(\eta_T^0(F_T^{s,x,\beta}))\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.} \quad \text{as } k\to\infty.$$

Now let us use the sequence $\{\beta^k\}$ to prove (5.1). First note that by a monotone-class argument and the structure of the σ-fields $\mathcal{P}_k$'s, it can be shown that the processes β^k's have the following representations:

$$\beta_t^k(\omega) = \gamma^k(t, W_{t_1^{k,s}}(\omega), W_{t_2^{k,s}}(\omega), \dots, W_{t_{2^k-1}^{k,s}}(\omega), \omega), \qquad (t,\omega) \in [0,T]\times\Omega,$$

where, for $y = (y_1, \dots, y_{2^k}) \in \mathbb{R}^{2^k d}$,

$$\begin{aligned}\gamma^k(t,y,\omega) = &\sum_{\ell=0}^{2^k-1} f_{k,\ell}(y1,\dots,y_\ell, B_{t_1^{k,T}}(\omega),\dots,B_{t_{2^k-1}^{k,T}}(\omega))\mathbf{1}_{(t_\ell^{k,s},t_{\ell+1}^{k,s}]}(t)\\ &+\sum_{\ell=0}^{2^k-1} g_{k,\ell}(y, W_{t_1^{t,s,T}},\dots,W_{t_{2^k-1}^{k,s,T}}(\omega), B_{t_1^{k,T}}(\omega), B_{t_{2^k-1}^{k,T}}(\omega))\mathbf{1}_{(t_\ell^{k,s,T},t_{\ell+1}^{k,s,T}]}(t),\end{aligned}$$

where $f_{k,\ell} : \mathbb{R}^{\ell d}\times\mathbb{R}^{2^k m} \mapsto U$ and $g_{k,\ell} : \mathbb{R}^{2^k d}\times\mathbb{R}^{\ell d}\times\mathbb{R}^{2^k m} \mapsto U$ are Borel measurable functions. Note that $\gamma^k : [0,T]\times\mathbb{R}^{2^k d}\times\Omega \mapsto \mathbb{R}^n$ is a $\mathbf{G}^s$-progressive random field (that is, for each $y \in \mathbb{R}^{2^k n}$, the mapping $(t,\omega)\mapsto\gamma^k(t,y,\omega)$ is $\mathcal{G}_t^s$-progressively measurable). We see that for each $y\in\mathbb{R}^{2^k d}$, the random variable $H(\eta_T^0(F_T^{s,x,\gamma^k(y,\cdot)}))$ is independent of $\mathcal{F}_s^W$. Therefore,

$$\begin{aligned}E\left\{H\left(\eta_T^0(F_T^{s,x,\beta})\right)\Big|\mathcal{G}_s\right\} &\ge \operatorname*{essinf}_{k\ge1} E\left\{H\left(\eta_T^0(F_T^{s,x,\beta^k})\right)\Big|\mathcal{G}_s\right\}\\ &\ge \operatorname*{essinf}_{k\ge1}\inf_{y\in\mathbb{R}^{2^k d}} E\left\{H\left(\eta_T^0(F_T^{s,x,\gamma^k(y,\cdot)})\right)\Big|\mathcal{G}_s\right\}\\ &= \operatorname*{essinf}_{k\ge1}\inf_{y\in\mathbb{R}^{2^k d}} E\left\{H\left(\eta_T^0(F_T^{s,x,\gamma^k(y,\cdot)})\right)\Big|\mathcal{F}_T^B\right\}\\ &\ge \operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{H\left(\eta_T^0(F_T^{s,x,\beta})\right)\Big|\mathcal{F}_T^B\right\} = \widetilde{V}(s,x), \qquad P\text{-a.s.}\end{aligned}$$

We now prove the reversed inequality of (5.1) and construct another "minimizing sequence." In fact we will show that there exists a sequence $\{\beta^k\}\subset\widetilde{\mathcal{A}}$ such that

$$\widetilde{V}(s,x) = \lim_{k\to\infty}\downarrow E\left\{H\left(\eta_T^0(F_T^{s,x,\beta^k})\right)\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.} \tag{5.2}$$

Since the right side above is obviously no less than $\operatorname{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\{H(\eta_T^0(F_T^{s,x,\beta}))|\mathcal{G}_s\}$, the reversed inequality of (5.1), whence the theorem, will follow.

To construct the desired minimizing sequence, let us first choose a sequence $\{\hat\beta^k\}_{k\ge1}\subset\widetilde{\mathcal{A}}$ such that $\widetilde V(s,x)=\lim_{k\to\infty}\downarrow\widetilde J(s,x,\hat\beta^k)$, P-a.s., thanks to Theorem 4.2(ii), and then modify it as follows. Let $\beta^1\triangleq\hat\beta^1$, and for $k\ge2$, we denote recursively that

$$A^k\triangleq\Big\{E\Big\{H\Big(\eta^0_T(F_T^{s,x,\hat\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\le E\Big\{H\Big(\eta^0_T(F_T^{s,x,\beta^{k-1}})\Big)\Big|\mathcal{G}_s\Big\}\Big\},$$

and define, for $(t,\omega)\in[s,T]\times\Omega$,

$$\beta^k_t(\omega)=\hat\beta^k_t(\omega)\mathbf{1}_{([s,T]\times A^k)\cup([0,s]\times\Omega)}(t,\omega)+\beta^{k-1}_t(\omega)\mathbf{1}_{[s,T]\times[A^k]^c}(t,\omega).$$

Then obviously $\{\beta^k\}_{k\ge1}\subset\widetilde A$, since A^k's are all $\mathbf{G}$-progressively measurable. Further, it holds that

$$H\Big(\eta^0_T(F_T^{s,x,\beta^k})\Big)=H\Big(\eta^0_T(F_T^{s,x,\hat\beta^k})\Big)\mathbf{1}_{A^k}+H\Big(\eta^0_T(F_T^{s,x,\beta^{k-1}})\Big)\mathbf{1}_{(A^k)^c}.$$

Therefore, we have

$$E\Big\{H\Big(\eta^0_T(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\le E\Big\{H\Big(\eta^0_T(F_T^{s,x,\hat\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\wedge E\Big\{H\Big(\eta^0_T(F_T^{s,x,\beta^{k-1}})\Big)\Big|\mathcal{G}_s\Big\}. \tag{5.3}$$

Consequently, the sequence $\{E\{H(\eta^0_T(F^{s,x,\beta^k}))|\mathcal{G}_s\}:k\ge1\}$ is monotone decreasing, and by (5.1),

$$\widetilde V(s,x)\le\lim_{k\to\infty}\downarrow E\Big\{H\Big(\eta^0_T(F_T^{t,x,\beta^k})\Big)\Big|\mathcal{G}_s\Big\},\qquad P\text{-a.s.} \tag{5.4}$$

But on the other hand, noting the definition of $\{\hat\beta^k\}_{k\ge1}$, the fact (5.3), and applying the monotone convergence theorem, we also have

$$\begin{aligned}&E[\widetilde V(s,x)]=E\Big\{\lim_{k\to\infty}\downarrow\widetilde J(s,x,\hat\beta^k)\Big\}=\lim_{k\to\infty}\downarrow E\{\widetilde J(s,x,\hat\beta^k)\}\\&=\lim_{k\to\infty}\downarrow E\Big\{E\Big\{H\Big(\eta^0_T(F_T^{s,x,\hat\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\Big\}\ge\lim_{k\to\infty}\downarrow E\Big\{E\Big\{H\Big(\eta^0_T(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\Big\}\\&=E\Big\{\lim_{k\to\infty}\downarrow E\Big\{H\Big(\eta^0_T(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\Big\}\Big\}.\end{aligned}$$

This, together with (5.4), leads to (5.2), completing the proof. ∎

We now turn to the regularity of the value function $\widetilde V(s,x)$. It is well understood that in a standard stochastic control problem the value function is usually (locally) Lipschitz in the spatial variable, and Hölder-1/2 in the temporal variable. We will show in the next theorem that this is in principle still true, but with a slight modification.

We first give a lemma that concerns the solution $F^{s,x,\beta}$ of the SDE (3.17). Let us begin by recalling Lemma 3.1 and the sequence of $\mathcal{F}^B_T$-measurable random times (whence $\mathbf{G}$-stopping times!) $\{\tau_\ell\}_{\ell\ge1}$ there (see (3.15)):

$$\tau_\ell=\inf\Big\{s\ge0:\sup_{z\in\mathbb{R}^n}(1+|z|^2)^{-\gamma}\Big(\Big|[D_z\eta^0_s(z)]^{-1}\Big|+\sum_{i=1}^3\Big|D^i_z\eta^0_s(z)\Big|\Big)>\ell\Big\}\wedge T, \tag{5.5}$$

for $\ell=1,2,\dots$. We can localize $F^{s,x,\beta}$ even further: for any $M>0$, and fixed $(s,x)\in[0,T]\times\mathbb{R}^n$ and $\beta\in\widetilde{\mathcal{A}}$, we define

$$\tau^{s,x,\beta}_{\ell,M}=\inf\Big\{t\ge s:|F^{s,x,\beta}_t|>M,\ \tau_\ell=T\Big\}\wedge T. \tag{5.6}$$

Then it is clear that $\tau^{s,x,\beta}_{\ell,M}$ is again a $\mathbf{G}$-stopping time. We have the following lemma.

LEMMA 5.2. *Assume* (H1)—H(3). *Then, for any $p \geq 1$ and $\ell \geq 1$, there exists a constant $C_{\ell,p} > 0$ such that for any $(s,x), (s',x') \in [0,T] \times \mathbb{R}^n$, $\beta \in \widetilde{\mathcal{A}}$, and $M > 0$, it holds P-a.s. on the set $\{\tau_\ell = T\}$ that*

$$(5.7)\quad E\Big\{\sup_{t\in[s,T]} |F_t^{s,x,\beta}|^p \Big| \mathcal{G}_s\Big\} \leq C_{\ell,p}(1+|x|^p);$$

$$(5.8)\quad P\Big\{\tau_{\ell,M}^{s,x,\beta} < M \Big| \mathcal{G}_s\Big\} \leq \frac{1}{M^p} E\Big\{\sup_{t\in[s,T]} |F_t^{s,x,\beta}|^p \Big| \mathcal{G}_s\Big\} \leq C_{\ell,p}\frac{(1+|x|^p)}{M^p};$$

$$(5.9)\quad E\Big\{\sup_{s\leq t\leq \tau_{\ell,M}^{s,x,\beta}\wedge\tau_{\ell,M}^{s,x',\beta}} |F_t^{s,x,\beta} - F_t^{s,x',\beta}|^p \Big| \mathcal{G}_s\Big\} \leq C_{\ell,p}e^{M^p}|x-x'|^p;$$

$$(5.10)\quad E\Big\{\sup_{s\leq t\leq \tau_{\ell,M}^{s,x,\beta}\wedge\tau_{\ell,M}^{s',x,\beta}} |F_t^{s,x,\beta} - F_t^{s',x,\beta}|^p \Big| \mathcal{G}_s\Big\} \leq C_{\ell,p}e^{M^p}(1+|x|^p)|s-s'|^{\frac{p}{2}}.$$

Proof. Note that all the following discussions are restricted to the set $\{\tau_\ell = T\}$. By virtue of the (localized) Lipschitz conditions and the linear growth properties of the coefficients $\widetilde{b}$ and $\widetilde{\sigma}$, thanks to Lemma 3.1, the estimates (5.7), (5.9), and (5.10) follow easily from some standard arguments for SDEs. Further, the estimate (5.8) is a direct consequence of definition (5.6), Chebyshev's inequality, and (5.7). We leave the details to the reader. ☐

The second main result of this section is the following. One should note that the parameter $\varepsilon > 0$ makes our result different from similar ones in standard stochastic control theory.

THEOREM 5.3. *Assume* (H1)–(H3). *Then, for any $\ell \geq 1$, $R > 0$, and $\varepsilon > 0$, there exists a $C_{\ell,R,\varepsilon} > 0$, such that for all $(s,x),(s',x') \in [0,T] \times \mathbb{B}_R^n$, it holds P-a.s. on $\{\tau_\ell = T\}$ that*

$$(5.11)\qquad |\widetilde{V}(s,x) - \widetilde{V}(s',x')| \leq \varepsilon + C_{\ell,R,\varepsilon}\|H\|_\infty(|s-s'|^{\frac{1}{2}} + |x-x'|).$$

Here $\mathbb{B}_R^n \stackrel{\triangle}{=} \{z \in \mathbb{R}^n : |z| \leq R\}$, and $\|\cdot\|_\infty$ is the sup-norm of $L^\infty(\mathbb{R}^n)$.

Proof. We first fix $R > 0$ and let $(s,x),(s',x') \in [0,T] \times \mathbb{B}_R^n$ be arbitrarily given. Then by Theorem 5.1 we can choose a sequence $\{\beta^k\}_{k\geq1}$ such that

$$(5.12)\qquad \widetilde{V}(s,x) = \lim_{k\to\infty} \downarrow E\Big\{H\Big(\eta_T^0(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\Big\}, \qquad P\text{-a.s.}$$

Since H is uniformly continuous by (H3), for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$(5.13)\qquad |H(z) - H(z')| < \frac{\varepsilon}{4}, \qquad \text{whenever } |z-z'| < \delta.$$

Now fix ℓ such that $P\{\tau_\ell = T\} > 0$. For each $M > 0$ and $k > 0$, we recall the stopping times defined by (5.6) and denote, for simplicity, that

$$\tau_{k,M} \stackrel{\triangle}{=} \tau_{\ell,M}^{s,x,\beta^k}, \qquad \tau'_{k,M} \stackrel{\triangle}{=} \tau_{\ell,M}^{s,x',\beta^k}, \qquad \tau''_{k,M} \stackrel{\triangle}{=} \tau_{\ell,M}^{s',x',\beta^k},$$

and define $\bar{\tau}_{k,M} \stackrel{\triangle}{=} \tau_{k,M} \wedge \tau'_{k,M} \wedge \tau''_{k,M}$. Then using (5.8) (with $p=2$) we have

$$(5.14)\qquad P\{\bar{\tau}_{k,M} < T|\mathcal{G}_{s\wedge s'}\} \leq \frac{1}{M^2}C_{\ell,2}(1+|x|^2+|x'|^2) \leq C_{\ell,2}\frac{(1+2R^2)}{M^2}.$$

Bearing in mind that $\widetilde{V}(s,x)$ is $\mathcal{F}^B_T$-measurable, and $\mathcal{F}^B_T \subset \mathcal{G}_t$ for all $t \geq 0$, we can easily check that the minimizing sequence in Theorem 5.1 also satisfies

$$\widetilde{V}(s,x) = \lim_{k\to\infty} \downarrow E\left\{H\Big(\eta^0_T(F^{s,x,\beta^k}_T)\Big)\Big|\mathcal{G}_{s\wedge s'}\right\}, \qquad P\text{-a.s.}$$

On the other hand, we observe that

$$\widetilde{V}(s',x') = \operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{H\Big(\eta^0_T(F^{s',x',\beta}_T)\Big)\Big|\mathcal{G}_{s\wedge s'}\right\}, \qquad P\text{-a.s.}$$

Keeping these in mind we now define

$$A^k_{\varepsilon,\ell} = \left\{\tau_\ell = T; \widetilde{V}(s,x) \geq E\left\{H\Big(\eta^0_T(F^{s,x,\beta^k}_T)\Big)\Big|\mathcal{G}_{s\wedge s'}\right\} - \frac{\varepsilon}{4}\right\}.$$

Then $\lim_{k\to\infty} P\{A^k_{\varepsilon,\ell}\} = P\{\tau_\ell = T\} > 0$, and hence $P\{A^k_{\varepsilon,\ell}\} > 0$ for k large enough. Also, on the set $A^k_{\varepsilon,\ell}$ we have

$$\begin{aligned}(5.15)\qquad &\widetilde{V}(s',x') - \widetilde{V}(s,x)\\ &\leq \frac{\varepsilon}{4} + \left|E\left\{H\Big(\eta^0_T(F^{s',x',\beta^k}_T)\Big)\Big|\mathcal{G}_{s\wedge s'}\right\} - E\left\{H\Big(\eta^0_T(F^{s,x,\beta^k}_T)\Big)\Big|\mathcal{G}_{s\wedge s'}\right\}\right|\end{aligned}$$

We now analyze the right-hand side above. Again let us simplify the notation a little bit. Denote

$$\begin{aligned}&\Delta F^k_T \triangleq F^{s',x',\beta^k}_T - F^{s,x,\beta^k}_T, \quad \Delta\eta(F^k_T) \triangleq \eta^0_T(F^{s',x',\beta^k}_T) - \eta^0_T(F^{s,x,\beta^k}_T),\\ &\Delta H(\eta(F^k_T)) \triangleq H\left(\eta^0_T(F^{s',x',\beta^k}_T)\right) - H\left(\eta^0_T(F^{s,x,\beta^k}_T)\right).\end{aligned}$$

Recalling the definition of $\bar\tau_{k,M}$ and δ we see that the second term on the right-hand side of (5.15) becomes

$$\begin{aligned}&\left|E\left\{\Delta H(\eta(F^k_T))\Big|\mathcal{G}_{s\wedge s'}\right\}\right|\\ &= \left|E\left\{\Delta H(\eta(F^k_T))\mathbf{1}_{\{\bar\tau_{k,M}<T\}}\Big|\mathcal{G}_{s\wedge s'}\right\} + E\left\{\Delta H(\eta(F^k_T))\mathbf{1}_{\{\bar\tau_{k,M}=T\}}\Big|\mathcal{G}_{s\wedge s'}\right\}\right|\\ &\leq E\left\{\left|\Delta H(\eta(F^k_T))\right|\mathbf{1}_{\{\bar\tau_{k,M}<T\}}\Big|\mathcal{G}_{s\wedge s'}\right\} + E\left\{\left|\Delta H(\eta(F^k_T))\right|\mathbf{1}_{\{|\Delta\eta(F^k_T)|<\delta,\bar\tau_{k,M}=T\}}\Big|\mathcal{G}_{s\wedge s'}\right\}\\ &\quad + E\left\{\left|\Delta H(\eta(F^k_T))\right|\mathbf{1}_{\{|\Delta\eta(F^k_T)|\geq\delta,\bar\tau_{k,M}=T\}}\Big|\mathcal{G}_{s\wedge s'}\right\} = I_1 + I_2 + I_3,\end{aligned}\tag{5.16}$$

where I_1, I_2, and I_3 are defined in an obvious way. Clearly, by (5.14) we see that

$$\begin{aligned}&I_1 = E\left\{\left|\Delta H(\eta(F^k_T))\right|\mathbf{1}_{\{\bar\tau_{k,M}<T\}}\Big|\mathcal{G}_{s\wedge s'}\right\} \leq 2\|H\|_\infty P\left\{\bar\tau_{k,M} < T\Big|\mathcal{G}_{s\wedge s'}\right\}\\ (5.17)\qquad &\leq C_{\ell,2}\frac{1+2R^2}{M^2}2\|H\|_\infty.\end{aligned}$$

Also, in light of (5.13) we see that $|\Delta H(\eta(F^k_T))| < \frac{\varepsilon}{4}$ on the set $\{|\Delta\eta(F^k_T)| < \delta\}$, which implies that

$$I_2 = E\left\{\left|\Delta H(\eta(F^k_T))\right|\mathbf{1}_{\{|\Delta\eta(F^k_T)|<\delta,\bar\tau_{k,M}=T\}}\Big|\mathcal{G}_{s\wedge s'}\right\} < \frac{\varepsilon}{4}.\tag{5.18}$$

Now, first applying the Chebyshev inequality and then applying Lemma 5.2 with $p=1$, we have, P-a.s. on $\{\tau_\ell = T\}$,

$$
\begin{aligned}
I_3 &= E\left\{ \left|\Delta H(\eta(F_T^k))\right| \mathbf{1}_{\{|\Delta\eta(F_T^k)|\ge\delta,\bar\tau_{k,M}=T\}} \Big| \mathcal{G}_{s\wedge s'}\right\}\\
&\le 2\|H\|_\infty P\left\{|\Delta\eta(F_T^k)|\ge\delta, \bar\tau_{k,M}=T \Big| \mathcal{G}_{s\wedge s'}\right\}\\
&\le \frac{2}{\delta}\|H\|_\infty E\left\{\ell\left(1+|F_T^{s',x',\beta^k}|+|F_T^{s,x,\beta^k}|\right)|\Delta F_T^k|\mathbf{1}_{\{\bar\tau_{k,M}=T\}}\Big|\mathcal{G}_{s\wedge s'}\right\}\\
&\le \frac{2(1+2M)\ell}{\delta}\|H\|_\infty E\left\{|F_{\bar\tau^k}^{s,x,\beta^k}-F_{\bar\tau^k}^{s,x',\beta^k}|+|F_{\bar\tau^k}^{s',x',\beta^k}-F_{\bar\tau^k}^{s,x',\beta^k}|\Big|\mathcal{G}_{s\wedge s'}\right\}\\
&\le \frac{2(1+2M)C_{\ell,1}\ell e^M(1+R)}{\delta}\|H\|_\infty(|s-s'|^{\frac12}+|x-x'|).
\end{aligned}\tag{5.19}
$$

Plugging (5.16)–(5.19) into (5.15) we see that on the set $A^k_{\varepsilon,\ell}$ one has

$$
\begin{aligned}
\widetilde V(s',x')-\widetilde V(s,x) &< \frac{\varepsilon}{2}+C_{\ell,2}\frac{1+2R^2}{M^2}2\|H\|_\infty\\
&\quad+\frac{2(1+2M)C_{\ell,1}\ell e^M(1+R)}{\delta}\|H\|_\infty(|s-s'|^{\frac12}+|x-x'|).
\end{aligned}\tag{5.20}
$$

We note that since $\{\tau_\ell=T\}=\cup_{k\ge1}A^k_{\varepsilon,\ell}$, (5.20) actually holds on the set $\{\tau_\ell=T\}$.

To conclude, for fixed $\varepsilon>0$, $\ell>0$, and $R>0$, we first choose $\delta=\delta(\varepsilon)>0$ such that (5.13) holds, and then choose $M=M_{\ell,R,\varepsilon}>2\sqrt{C_{\ell,2}(1+2R^2)\|H\|_\infty/\varepsilon}$. Denoting

$$
C_{\ell,R,\varepsilon}\triangleq\frac{2(1+2M)C_{\ell,1}\ell e^M(1+R)}{\delta(\varepsilon)},
$$

it is then easily seen that (5.20) becomes

$$
\widetilde V(s',x')-\widetilde V(s,x)<\varepsilon+C_{\ell,R,\varepsilon}\|H\|_\infty(|s-s'|^{\frac12}+|x-x'|),\quad P\text{-a.s. on } \{\tau_\ell=T\}.
$$

Reversing the role of (s,x) and (s',x'), we have proved the theorem. ∎

A closer look at the proof of Theorem 5.3 would lead to the following dependence result for the wider-sense state process $X_t^{s,x,\beta}=\eta_t^0(F_t^{s,x,\beta})$, $t\in[s,T]$, $\beta\in\widetilde{\mathcal{A}}$.

COROLLARY 5.4. *Assume* (H1)–(H3). *Then for any* $\ell\ge1$, $R>0$, *and* $\varepsilon>0$ *there exists some constant* $C_{\ell,R,\varepsilon}>0$, *depending only on* ℓ, R, *and* ε, *such that for all* $\beta\in\widetilde{\mathcal{A}}$ *and all* $(s,x),(s',x')\in[0,T]\times\mathbb{R}^n$ *with* $x,x'\in\mathbb{B}_R$, *it holds* P*-a.s. on* $\{\tau_\ell=T\}$ *that*

$$
\left|E\left\{X_T^{s',x',\beta}-X_T^{s,x,\beta}\Big|\mathcal{G}_{s\wedge s'}\right\}\right|\le\varepsilon+C_{\ell,R,\varepsilon}((|s-s'|^{\frac12}+|x-x'|).\tag{5.21}
$$

Proof. Noting that $X_T^{s,x,\beta}=\eta_T^0(F_T^{s,x,\beta})$, the proof follows from similar arguments of Theorem 5.3. We leave it to the interested reader. ∎

To conclude this section we note that Theorem 5.3 does not provide us the usual regularity of the value function (that is, Lipschitz in x, and Hölder-1/2 in s). In fact, as a random field, even the following property of $\widetilde V(\cdot,\cdot)$ is not completely trivial.

THEOREM 5.5. *Assume* (H1)–(H3). *The random field* $\widetilde V(s,x)$ *possesses a continuous version on* $[0,T]\times\mathbb{R}^n$.

Proof. We would like to construct a P-null exceptional set beyond which the function $(s,x)\mapsto\widetilde V(s,x,\omega)$ is continuous for all fixed ω.

To this end, let us denote $\mathbf{Q}$ to be all the rationals in $\mathbb{R}$. Denote also $\mathbf{Q}_+^* = \mathbf{Q}\cap(0,\infty)$ and $\mathbf{Q}_T = \mathbf{Q}\cap[0,T]$. The notation for the n-dimensional version $\mathbf{Q}^n \subset \mathbb{R}^n$ is defined in an obvious way.

For $\ell \ge 1$ we define the following subset of Ω:

$$\Omega_\ell \stackrel{\triangle}{=} \bigcap_{\substack{R\in\mathbf{Q}_+^*\\ \varepsilon\in\mathbf{Q}_+^*}} \{\tau_\ell = T \text{ and (5.11) holds for all } (s,x),(s',x')\in \mathbf{Q}_T\times\mathbb{B}_R^n\cap\mathbf{Q}^n\},$$

$$\widetilde{\Omega} = \cup_{\ell=1}^\infty \Omega_\ell.$$

Applying Theorem 5.3 we see that for fixed $\ell > 0$, $R > 0$, and $\varepsilon > 0$, (5.11) holds P-a.s. on the set $\{\tau_\ell = T\}$. In other words, one must have $\Omega_\ell \subseteq \{\tau_\ell = T\}$ and $P\{\{\tau_\ell = T\}\setminus\Omega_\ell\} = 0$ for all ℓ. Since $\cup_{\ell=1}^\infty\{\tau_\ell = T\} = \Omega$, modulo a P-null set, one has

$$\Omega\setminus\widetilde{\Omega} \subseteq \bigcup_{\ell=1}^\infty \{\{\tau_\ell = T\}\setminus\Omega_\ell\}, \qquad \text{almost surely.}$$

It then follows that $P\{\widetilde{\Omega}\} = 1$.

Now let us fix $\omega\in\widetilde{\Omega}$. By (5.11) we see that the mapping $(s,x)\mapsto \widetilde{V}(s,x,\omega)$ is continuous on $\mathbf{Q}_T\times\mathbf{Q}^n$. For general $(s,x)\in[0,T]\times\mathbb{R}^n$, we choose a sequence $\{(s_k,x_k)\}_{k\ge1}\subset\mathbf{Q}_T\times\mathbf{Q}^n$, such that $(s_k,x_k)\to(s,x)$, as $k\to\infty$. Applying (5.11) again we see that $\{\widetilde{V}(s_k,x_k)\}_{k\ge1}$ is Cauchy, and we can define the limit by $\bar{V}(s,x,\omega)$ and show that it is independent of the choice of the sequence $\{(s_k,x_k)\}$. Now define a random field

$$\bar{V}(s,x,\omega) = \begin{cases} \lim\limits_{k\to\infty}\widetilde{V}(s_k,x_k,\omega), & \mathbf{Q}_T\times\mathbf{Q}^n \ni (s_k,x_k)\to(s,x),\ \omega\in\widetilde{\Omega},\\ 0, & \omega\in\Omega\setminus\widetilde{\Omega}.\end{cases}$$

Then, using a standard "3ε-argument," one shows that $\bar{V}$ is continuous on $[0,T]\times\mathbb{R}^n$ for all $\omega\in\widetilde{\Omega}$. It remains to verify that $\bar{V}$ is a version of $\widetilde{V}$. But by continuity of $\bar{V}$ we need only check that $\bar{V}$ is a modification of $\widetilde{V}$. To wit, for any $(s,x)\in[0,T]\times\mathbb{R}^n$, it holds that $\bar{V}(s,x) = \widetilde{V}(s,x)$, P-a.s. But by virtue of Theorem 5.3 and the definition of $\bar{V}$ one can check that for fixed (s,x) and any sequence $\{(s_k,x_k)\}_{k\ge1}\subset\mathbf{Q}_T\times\mathbf{Q}^n$ such that $(s_k,x_k)\to(s,x)$, it must hold that

$$\widetilde{V}(s,x) = \lim_{k\to\infty}\widetilde{V}(s_k,x_k) = \bar{V}(s,x), \qquad P\text{-a.s.}$$

We leave the details to the readers. The proof is now complete. $\square$

6. The Bellman principle. We are now ready to establish the first fundamental result of this paper— the "Bellman principle" of dynamic programming for the wider-sense control problem. We should note that our reduction of the original problem to problems WSCP-I and WSCP-II enables us to find different versions of minimizing sequences, which will be essential in our discussions in this section.

Our main result of this section is the following.

THEOREM 6.1 (Bellman principle). *Assume* (H1)–(H3). *For any* $(s,x)\in[0,T)\times\mathbb{R}^n$ *and* $h>0$ *such that* $0\le s\le s+h\le T$, *it holds that*

$$\widetilde{V}(s,x) = \operatorname*{essinf}_{\beta\in\widetilde{A}} E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta})\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.}$$

The proof of Theorem 6.1 is rather lengthy, so we will split it into two lemmas, each taking care of one direction of the inequality, and each using a special technique.

LEMMA 6.2. *Assume* (H1)–(H3). *For all* $(s,x) \in [0,T) \times \mathbb{R}^n$, $h > 0$ *such that* $s + h \le T$, *it holds that*

$$\widetilde{V}(s,x) \le \operatorname*{essinf}_{\beta \in \widetilde{\mathcal{A}}} E\left\{\widetilde{V}(s+h, F^{s,x,\beta}_{s+h})\Big|\mathcal{G}_s\right\}. \tag{6.1}$$

Proof. Clearly, we need only show that for all $(s,x) \in [0,T) \times \mathbb{R}^n$, $h > 0$ such that $s+h \le T$, and $\beta \in \widetilde{A}$, it holds that

$$\widetilde{V}(s,x) \le E\left\{\widetilde{V}(s+h, F^{s,x,\beta}_{s+h})\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.} \tag{6.2}$$

The proof depends heavily on the estimates established in Theorem 5.3. First let us fix $\ell > 0$ and $\varepsilon > 0$, and let $0 \le s < s+h \le T$ and $x \in \mathbb{R}^n$ be given and fixed as well. Applying Lemma 5.2 and the Chebyshev inequality, we have, for any $R > 0$, P-a.s. on $\{\tau_\ell = T\}$,

$$P\left\{|F^{s,x,\beta}_{s+h}| > R\Big|\mathcal{G}_t\right\} \le \frac{1}{R^2} E\left\{\sup_{t\in[s,T]} |F^{s,x,\beta}_t|^2\Big|\mathcal{G}_t\right\} \le \frac{C_{\ell,2}}{R^2}(1+|x|^2).$$

Let us now fix $R = R_{\ell,x,\varepsilon} > \sqrt{C_{\ell,2}(1+|x|^2)\|H\|_\infty/\varepsilon}$, so that

$$P\left\{|F^{s,x,\beta}_{s+h}| > R\Big|\mathcal{G}_s\right\} < \Big(\frac{\varepsilon}{\|H\|_\infty}\Big) \wedge \varepsilon, \quad P\text{-a.s.} \ \text{ on } \{\tau_\ell = T\}. \tag{6.3}$$

Next, following the main ideas of the proof of Theorem 5.3, we can find a constant $C_{\ell,R,\varepsilon}$ such that for all $y, y' \in \mathbb{B}^n_R$, it holds almost surely on $\{\tau_\ell = T\}$ that

$$\begin{aligned}&\left|E\left\{H\Big(\eta^0_T(F^{s+h,y,\beta}_T)\Big)\Big|\mathcal{G}_{s+h}\right\} - E\left\{H\Big(\eta^0_T(F^{s+h,y',\beta}_T)\Big)\Big|\mathcal{G}_{s+h}\right\}\right| \\ &\qquad \le \frac{\varepsilon}{2} + C_{\ell,R,\varepsilon}\|H\|_\infty|y-y'|\end{aligned} \tag{6.4}$$

and

$$\left|\widetilde{V}(s+h,y) - \widetilde{V}(s+h,y')\right| \le \frac{\varepsilon}{2} + C_{\ell,R,\varepsilon}\|H\|_\infty|y-y'|. \tag{6.5}$$

Now let us fix $\delta = \delta_{\ell,x,\varepsilon} < \frac{\varepsilon}{2C_{\ell,R,\varepsilon}\|H\|_\infty}$, and choose a finite set of open balls $\{\mathcal{O}_\delta(y^k)\}_{k=1}^N$, centered at y^k's and with radius δ such that it covers the (compact) ball $\mathbb{B}^n_R$. That is, $\mathbb{B}^n_R \subset \bigcup_{k=1}^N \mathcal{O}_\delta(y^k)$. Clearly, on the set $\{\tau_\ell = T\}$, whenever $y \in \mathcal{O}_\delta(y^k)$ we have

$$\left|E\left\{H\Big(\eta^0_T(F^{s+h,y,\beta}_T)\Big)\Big|\mathcal{G}_{s+h}\right\} - E\left\{H\Big(\eta^0_T(F^{s+h,y^k,\beta}_T)\Big)\Big|\mathcal{G}_{s+h}\right\}\right| < \varepsilon \tag{6.6}$$

and

$$\left|\widetilde{V}(s+h,y) - \widetilde{V}((s+h,y^k))\right| < \varepsilon. \tag{6.7}$$

Furthermore, we define a partition of $\mathbb{B}^n_R$ as follows:

$$\Gamma_{R,1} \triangleq \mathcal{O}_\delta(y^1) \cap \mathbb{B}^n_R; \qquad \Gamma_{R,k} \triangleq \left(\mathcal{O}_\delta(y^k) \setminus \bigcup_{l=1}^{k-1} \mathcal{O}_\delta(y^l)\right) \bigcap \mathbb{B}^n_R, \qquad k > 1.$$

Thus it follows from (6.7) that, for any $\beta \in \widetilde{\mathcal{A}}$, P-a.s. on $\{|F^{s,x,\beta}_{s+h}| \le R, \tau_\ell = T\}$,

$$\begin{aligned}\widetilde{V}(s+h, F^{s,x,\beta}_{s+h}) &= \sum_{k=1}^N \widetilde{V}(s+h, F^{s,x,\beta}_{s+h})\mathbf{1}_{\Gamma_{R,k}}(F^{s,x,\beta}_{s+h})\\ &\ge \sum_{k=1}^N \widetilde{V}(s+h, y^k)\mathbf{1}_{\Gamma_{R,k}}(F^{s,x,\beta}_{s+h}) - \varepsilon.\end{aligned}\tag{6.8}$$

Now for each k we apply Theorem 5.1 (or, in particular, (5.2)) to find a $\beta^k \in \widetilde{\mathcal{A}}$ such that the set

$$A_{\varepsilon,k} \stackrel{\triangle}{=} \left\{\omega : \widetilde{V}(s+h, y^k, \omega) < E\left\{H\Big(\eta^0_T(F^{s+h,y^k,\beta^k}_T)\Big)\Big|\mathcal{G}_{s+h}\right\}(\omega) - \varepsilon\right\}$$

satisfies $P(A_{\varepsilon,k}) \le 2^{-k}\varepsilon$. Next, define

$$\widetilde{A}_{\ell,\varepsilon,k} \stackrel{\triangle}{=} \{|F^{s,x,\beta}_{s+h}| \le R, \tau_\ell = T\} \cap A^c_{\varepsilon,k} \quad \text{and} \quad \widetilde{\Omega}_{\ell,\varepsilon} \stackrel{\triangle}{=} \bigcap_{k=1}^N \widetilde{A}_{\ell,\varepsilon,k}.\tag{6.9}$$

Then

$$\begin{aligned}P\{\widetilde{\Omega}^c_{\ell,\varepsilon}\} &\le P\left\{\{\tau_\ell < T\} \cup \{|F^{s,x,\beta}_{s+h}| > R, \tau_\ell = T\} \cup \left(\bigcup_{k=1}^N A_{\varepsilon,k}\right)\right\}\\ &\le P\{\tau_\ell < T\} + P\{|F^{s,x,\beta}_{s+h}| > R, \tau_\ell = T\} + \sum_{k=1}^N P(A_{\varepsilon,k})\\ &\le P\{\tau_\ell < T\} + 2\varepsilon \to 0 \qquad \text{as} \quad \ell \uparrow \infty \text{ and } \varepsilon \downarrow 0.\end{aligned}\tag{6.10}$$

Let us now restrict ourselves to the set $\widetilde{\Omega}_{\ell,\varepsilon}$. Clearly, almost surely on $\widetilde{\Omega}_{\ell,\varepsilon}$ we have

$$\widetilde{V}(s+h, y^k) \ge E\left\{H\Big(\eta^0_T(F^{s+h,y^k,\beta^k}_T)\Big)\Big|\mathcal{G}_{s+h}\right\} - \varepsilon.$$

Thus on the set $\widetilde{\Omega}_{\ell,\varepsilon} \in \mathcal{G}_{s+h}$ the estimate (6.8) can further be written as

$$\widetilde{V}(s+h, F^{s,x,\beta}_{s+h}) \ge \sum_{k=1}^N E\left\{H\Big(\eta^0_T(F^{s+h,y^k,\beta^k}_T)\Big)\Big|\mathcal{G}_{s+h}\right\}\mathbf{1}_{\{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\}} - 2\varepsilon.\tag{6.11}$$

Now let $\beta \in \widetilde{\mathcal{A}}$ be any control. We modify β as follows: for $(t,\omega) \in [s,T]\times\Omega$,

$$\widetilde{\beta}_t(\omega) = \begin{cases} \beta^k_t(\omega) & \text{if } t \ge s+h, \omega \in \{\tau_\ell = T\} \cap \{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\},\ 1 \le k \le N,\\ \beta_t(\omega) & \text{otherwise.}\end{cases}$$

Then clearly $\widetilde{\beta} \in \widetilde{\mathcal{A}}$ as well, and, moreover, the pathwise uniqueness of SDE (3.17), together with the estimate (6.6), shows that for each k, P-a.s. on $\{\tau_\ell = T\}$,

$$\begin{aligned}&E\left\{H\Big(\eta^0_T(F^{s,x,\widetilde{\beta}}_T)\Big)\Big|\mathcal{G}_{s+h}\right\}\mathbf{1}_{\{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\}}\\ &= E\left\{H\Big(\eta^0_T(F^{s+h,F^{s,x,\beta}_{s+h},\beta^k}_T)\Big)\mathbf{1}_{\{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\}}\Big|\mathcal{G}_{s+h}\right\}\\ &\le E\left\{H\Big(\eta^0_T(F^{s+h,y^k,\beta^k}_T)\Big)\mathbf{1}_{\{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\}}\Big|\mathcal{G}_{s+h}\right\} + \varepsilon\mathbf{1}_{\{F^{s,x,\beta}_{s+h} \in \Gamma_{R,k}\}}.\end{aligned}\tag{6.12}$$

Thus, noting that $\cup_{k=1}^N \Gamma_{R,k} = \mathbb{B}_R^n$, we can continue from (6.11) to get

$$
\begin{aligned}
&\widetilde{V}(s+h, F_{s+h}^{s,x,\beta})\\
&\geq \sum_{k=1}^N E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\mathbf{1}_{\{F_{s+h}^{s,x,\beta}\in\Gamma_{R,k}\}}\Big|\mathcal{G}_{s+h}\right\} - 2\varepsilon - \varepsilon\sum_{k=1}^N \mathbf{1}_{\{F_{s+h}^{s,x,\beta}\in\Gamma_{R,k}\}}\\
&= E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\Big|\mathcal{G}_{s+h}\right\}\sum_{k=1}^N \mathbf{1}_{\{F_{s+h}^{s,x,\beta}\in\Gamma_{R,k}\}} - 2\varepsilon - \varepsilon\sum_{k=1}^N \mathbf{1}_{\{F_{s+h}^{s,x,\beta}\in\Gamma_{R,k}\}}\\
&= \left[E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\Big|\mathcal{G}_{s+h}\right\} - \varepsilon\right] - 2\varepsilon = E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\Big|\mathcal{G}_{s+h}\right\} - 3\varepsilon.
\end{aligned}
$$

Note that the above inequality holds only on the $\widetilde{\Omega}_{\ell,\varepsilon}$. Taking conditional expectation $E\{\cdot|\mathcal{G}_s\}$ over $\widetilde{\Omega}_{\ell,\varepsilon} \in \mathcal{G}_{s+h}$ on both sides, we have

$$
\begin{aligned}
&E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta})\Big|\mathcal{G}_s\right\} \geq E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta})\mathbf{1}_{\widetilde{\Omega}_{\ell,\varepsilon}}\Big|\mathcal{G}_s\right\}\\
&\geq E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\mathbf{1}_{\widetilde{\Omega}_{\ell,\varepsilon}}\Big|\mathcal{G}_s\right\} - 3\varepsilon = E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)[1-\mathbf{1}_{\widetilde{\Omega}^c_{\ell,\varepsilon}}]\Big|\mathcal{G}_s\right\} - 3\varepsilon\\
&\geq E\left\{ H\Big(\eta_T^0(F_T^{s,x,\widetilde{\beta}})\Big)\Big|\mathcal{G}_s\right\} - \|H\|_\infty P\{\widetilde{\Omega}^c_{\ell,\varepsilon}|\mathcal{G}_s\} - 3\varepsilon\\
&\geq \widetilde{V}(s,x) - \|H\|_\infty P\{\widetilde{\Omega}^c_{\ell,\varepsilon}|\mathcal{G}_s\} - 3\varepsilon.
\end{aligned}
$$

Letting $\ell\to\infty$ and then $\varepsilon\downarrow 0$, we obtain (6.2), whence (6.1). ∎

The next lemma will show the reversed inequality.

LEMMA 6.3. *Assume* (H1)–(H3). *Then, for all* $(s,x)\in[0,T)\times\mathbb{R}^n$, *and* $s+h\leq T$,

$$
\widetilde{V}(s,x) \geq \operatorname*{essinf}_{\beta\in\widetilde{\mathcal{A}}} E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta})\Big|\mathcal{G}_s\right\}. \tag{6.13}
$$

Proof. This time we shall prove that for all $(s,x)\in[0,T)\times\mathbb{R}^n$, and $s+h\leq T$, there exists a sequence $\{\beta^k\}\subset\widetilde{A}$, such that

$$
\widetilde{V}(s,x) = \lim_{k\to\infty}\downarrow E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta^k})\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.} \tag{6.14}
$$

We note that this "minimizing sequence" is different from all the previous ones, and that (6.13) follows from (6.14) trivially.

To prove (6.14), we fix $(s,x)\in[0,T)\times\mathbb{R}^n$ and $h>0$ such that $s+h\leq T$. First applying Theorem 5.1 we know that there exists a sequence $\{\beta^k\}\subset\widetilde{\mathcal{A}}$ such that

$$
\widetilde{V}(s,x) = \lim_{k\to\infty}\downarrow E\left\{ H\Big(\eta_T^0(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\right\}.
$$

Next, applying Lemma 6.2 we have

$$
\begin{aligned}
\widetilde{V}(s,x) &\leq E\left\{\widetilde{V}(s+h, F_{s+h}^{s,x,\beta^k})\Big|\mathcal{G}_s\right\}\\
&\leq E\left\{E\Big\{H\Big(\eta_T^0\Big(F_T^{s+h,F_{s+h}^{s,x,\beta^k},\beta^k}\Big)\Big)\Big|\mathcal{G}_{s+h}\Big\}\Big|\mathcal{G}_s\right\}\\
&= E\left\{ H\Big(\eta_T^0(F_T^{s,x,\beta^k})\Big)\Big|\mathcal{G}_s\right\}, \qquad P-\text{a.s.}, \forall k\geq 1.
\end{aligned} \tag{6.15}
$$

We will use the by now standard technique to modify the sequence $\{\beta^k\}$ to derive the desired minimizing sequence. Let $\widetilde{\beta}^1 = \beta^1$. For $k > 1$, we define, for $t \in [s, T]$,

$$\widetilde{\beta}^k_t = \begin{cases} \beta^k_t & \text{on } \left\{ E\Big\{\widetilde{V}(s+h, F^{s,x,\beta^k}_{s+h})\Big|\mathcal{G}_s\Big\} \le E\Big\{\widetilde{V}(s+h, F^{s,x,\widetilde{\beta}^{k-1}}_{s+h})\Big|\mathcal{G}_s\Big\}\right\}, \\ \widetilde{\beta}^{k-1}_t & \text{otherwise,} \end{cases}$$

and set $\widetilde{\beta}^k_t \equiv \widetilde{\beta}^1_t$ for all k for $t \in [0, s)$. Then, clearly $\{\widetilde{\beta}^k\} \subset \widetilde{\mathcal{A}}$, and for all $k \ge 1$, one has

$$\begin{aligned} &E\left\{\widetilde{V}(s+h, F^{s,x,\widetilde{\beta}^k}_{s+h})\Big|\mathcal{G}_s\right\} \\ &\le E\left\{\widetilde{V}(s+h, F^{s,x,\widetilde{\beta}^{k-1}}_{s+h})\Big|\mathcal{G}_s\right\} \wedge E\left\{\widetilde{V}(s+h, F^{s,x,\beta^k}_{s+h})\Big|\mathcal{G}_s\right\} \\ &\le E\left\{\widetilde{V}(s+h, F^{s,x,\widetilde{\beta}^{k-1}}_{s+h})\Big|\mathcal{G}_s\right\} \wedge E\left\{H\Big(\eta^0_T(F^{s,x,\beta^k}_T)\Big)\Big|\mathcal{G}_s\right\}, \qquad P\text{-a.s.} \end{aligned}$$

Consequently we have

$$\begin{aligned} \widetilde{V}(s,x) &\le \lim_{k\to\infty} \downarrow E\left\{\widetilde{V}(s+h, F^{s,x,\widetilde{\beta}^k}_{s+h})\Big|\mathcal{G}_s\right\} \\ &\le \lim_{k\to\infty} \downarrow E\left\{H\Big(\eta^0_T(F^{s,x,\beta^k}_T)\Big)\Big|\mathcal{G}_s\right\} = \widetilde{V}(s,x), \end{aligned}$$

thanks to Theorem 5.1 again. This proves the lemma. ∎

Remark 6.4. By modifying the proof slightly one can easily show that the Bellman principle (Theorem 6.1) also holds if the initial states (s, x) and the increment h are replaced by $\mathcal{F}^B_T$-measurable random variables, since $\mathcal{F}^B_T \subset \mathcal{G}_t$ for all $t \ge 0$. In particular, if (s, x) is replaced by $\mathcal{F}^B_T$-measurable random variables (τ, ξ) and $h > 0$ remains a deterministic constant, then we make the convention that $\tau + h = (\tau + h) \wedge T$.

7. Stochastic HJB equation. In this section we shall derive two versions of the stochastic HJB equations: one for the value function of WSCP-I and the other for the original pathwise control problem.

7.1. HJB equation for WSCP-I. We first consider the HJB equation for the value function $\widetilde{V}$. Let us denote

$$\widetilde{\mathcal{L}}_{s,x,u} \triangleq \frac{1}{2}\mathrm{tr}\,\{\widetilde{\sigma}\widetilde{\sigma}^T(s,x,u)D^2_{xx}\} + \widetilde{b}(t,x,u)D_x.$$

Note that in the above the coefficients $\widetilde{b}$ and $\widetilde{\sigma}$ are $\mathcal{F}^B_T$-measurable random fields, and thus $\widetilde{\mathcal{L}}$ is a "random" differential operator. We then consider the following (random) PDE:

$$(7.1) \qquad \begin{cases} -\dfrac{\partial v}{\partial s}(s,x) - \operatorname*{essinf}_{u\in U}[\widetilde{\mathcal{L}}_{s,x,u} v](s,x) = 0, & (s,x) \in [0,T] \times \mathbb{R}^n; \\ v(T,x) = H\left(\eta^0_T(x)\right), & x \in \mathbb{R}^n. \end{cases}$$

We should note that since this randomized PDE does not involve any stochastic integrals, it could be studied in an ω-wise manner. However, the "adaptedness" nature of the random field is by no means obvious from such a treatment. As an alternative, in what follows we shall consider the random PDE as a special (degenerate) stochastic PDE with a time reversal, and introduce a definition of "stochastic viscosity solution" of this equation that is in the spirit of Buckdahn and Ma [2, 3, 4].

DEFINITION 7.1. *A continuous, $\mathcal{B}([0,T]\times\mathbb{R}^n)\otimes\mathcal{F}^B_T$-measurable random field v is a stochastic viscosity subsolution (resp., supersolution) if*

(i) $v(T,x)\le$ *(resp.,* $\ge$*)* $H\left(\eta^0_T(x)\right)$, $x\in\mathbb{R}^n$*;*

(ii) *for all* $\tau\in L^0_{\mathcal{F}^B_T}(\Omega;[0,T])$, $\xi\in L^2_{\mathcal{F}^B_T}(\Omega;\mathbb{R}^n)$, *and* $\varphi\in C^{1,2}_{\mathcal{F}^B_T}([0,T]\times\mathbb{R}^n)$, *it holds that*

$$-\frac{\partial\varphi}{\partial s}(\tau,\xi)-\operatorname*{essinf}_{u\in U}[\widetilde{\mathcal{L}}_{\tau,\xi,u}\varphi](\tau,\xi)\le 0\quad (resp., \ge 0), \tag{7.2}$$

on the (ω-)set $\{v-\varphi$ achieves a local maximum (resp., minimum) at $(\tau,\xi)\}\in\mathcal{F}^B_T$.

A random field v is called a stochastic viscosity solution if it is both a stochastic subsolution and a stochastic supersolution.

We remark here that the main difference between Definition 7.1 and those of [2, 3, 4], besides the time reversal, is that in [2, 3, 4] we require τ to be a stopping time and ξ to be an $\mathcal{F}^B_\tau$-measurable random variable. Due to the special structure here we assume only that τ is an $\mathcal{F}^B_T$-measurable random time. But one should appreciate again that any $\mathcal{F}^B_T$-measurable random time τ is a $\mathbf{G}$-stopping time!

Our first main result of this section is the following theorem.

THEOREM 7.2. *Assume* (H1)—(H3). *The value function $\widetilde{V}$ is a stochastic viscosity solution of* (7.1) *in the sense of Definition* 7.1. *Furthermore, the solution is unique in the class $C_{\mathcal{F}^B_T}([0,T]\times\mathbb{R}^n)$.*

Proof. We first show that $\widetilde{V}$ is a subsolution. Let $\tau\in L^0_{\mathcal{F}^B_T}(\Omega;[0,T])$, $\xi\in L^2_{\mathcal{F}^B_T}(\Omega;\mathbb{R}^n)$, and $\varphi\in C^{1,2}_{\mathcal{F}^B_T}([0,T]\times\mathbb{R}^n)$ be given. We assume that the set

$$\Gamma=\Gamma^{\tau,\xi}\stackrel{\triangle}{=}\{\widetilde{V}-\varphi \text{ achieves a local maximum at } (\tau,\xi)\}$$

satisfies $P(\Gamma)>0$ (for otherwise there is nothing to prove).

We begin by the following "localization" procedure. First, let $\{\tau_\ell\}$ be the sequence of $\mathbf{G}$-stopping times defined by (3.15). We can then choose $\ell>0$ and $\varepsilon>0$ such that the set

$$\Gamma_{\ell,\varepsilon}\stackrel{\triangle}{=}\{\widetilde{V}-\varphi \text{ achieves in } \mathbb{B}^{n+1}_\varepsilon(\tau,\xi) \text{ its maximum at } (\tau,\xi),\quad \tau_\ell=T\} \tag{7.3}$$

also satisfies $P(\Gamma_{\ell,\varepsilon})>0$, where $\mathbb{B}^{n+1}_\varepsilon(s,x)$ denotes the (closed) ball in $\mathbb{R}^{n+1}$ centered at (s,x) with radius ε.

Next, noting that the random fields $\widetilde{V}$ and φ are both $\mathcal{B}([0,T]\times\mathbb{R}^n)\otimes\mathcal{F}^B_T$-measurable, and $\widetilde{V}$ is uniformly bounded (by $\|H\|_\infty$!), we can modify the value of φ outside of $\mathbb{B}^{n+1}_{\varepsilon/2}(\tau,\xi)$ so that φ is uniformly bounded,

$$P\Big\{\sup_{(s,x)\in[0,T]\times\mathbb{R}^n}\Big[|D_t\varphi(s,x)|+|D_x\varphi(s,x)|+|D^2_{xx}\varphi(s,x)|\Big]<\infty\Big\}=1,$$

and $\Gamma_{\ell,\varepsilon}\subset\{(\tau,\xi)\in\operatorname{argmax}\{\widetilde{V}-\varphi\}\}\cap\{\tau_\ell=T\}$. Consequently, we can find $\rho>0$ small enough, and a set $\Gamma^\rho_{\ell,\varepsilon}\in\mathcal{F}^B_T$, such that

(i) $P(\Gamma^\rho_{\ell,\varepsilon})>P(\Gamma_{\ell,\varepsilon})-\rho>0$;

(ii) $\Gamma^{\rho}_{\ell,\varepsilon} \subseteq \{(\tau,\xi) \in \mathrm{argmax}\{\widetilde{V}-\varphi\}\} \cap \{\tau_\ell = T\}$;

(iii) $D_t\varphi(\tau,\xi)$, $D_x\varphi(\tau,\xi)$, $D^2_{xx}\varphi(\tau,\xi)$ are all bounded on $\Gamma^{\rho}_{\ell,\varepsilon}$.

Now let us recall from Remark 6.4 that the Bellman principle (Theorem 6.1) can be extended to the case where the initial state (τ,ξ) is a pair of $\mathcal{F}^B_T$-measurable random variables (with the convention that $\tau+h=(\tau+h)\wedge T$). Thus, for any $h>0$ and $\beta_s \equiv u \in U$ we can apply such a version of Theorem 6.1 and Itô's formula to get the following: on the set $\Gamma^{\rho}_{\ell,\varepsilon}$,

$$
\begin{aligned}
0 &\le \frac{1}{h}E\left\{\widetilde{V}(\tau+h,F^{\tau,\xi,\beta}_{\tau+h})-\widetilde{V}(\tau,\xi)\Big|\,\mathcal{G}_\tau\right\} \le \frac{1}{h}E\left\{\varphi(\tau+h,F^{\tau,\xi,\beta}_{\tau+h})-\varphi(\tau,\xi)\Big|\mathcal{G}_\tau\right\}\\
&= \frac{1}{h}E\left\{\int_\tau^{\tau+h}\left[\partial_t+\widetilde{\mathcal{L}}_{s,F^{\tau,\xi,\beta}_s,u}\right]\varphi(s,F^{\tau,\xi,\beta}_s)ds\bigg|\,\mathcal{G}_\tau\right\} \qquad (7.4)\\
&\le \Big[\partial_t+\widetilde{\mathcal{L}}_{\tau,\xi,u}\Big]\varphi(\tau,\xi)+r_{\tau,\xi}(h),
\end{aligned}
$$

where

$$
r_{\tau,\xi}(h) \stackrel{\triangle}{=} E\left\{\sup_{s\in[\tau,\tau+h]}\Big|\left[\partial_t+\widetilde{\mathcal{L}}_{s,F^{\tau,\xi,\beta}_s,u}\right]\varphi(s,F^{\tau,\xi,\beta}_s)-\left[\partial_t+\widetilde{\mathcal{L}}_{\tau,\xi,u}\right]\varphi(\tau,\xi)\Big|\,\bigg|\,\mathcal{G}_\tau\right\}.
$$

Note that $\Gamma^{\rho}_{\ell,\varepsilon} \subset \{\tau_\ell = T\}$, and applying Lemma 3.1 and 3.2(ii) we have, on the set $\Gamma^{\rho}_{\ell,\varepsilon}$,

$$
\begin{aligned}
&E\left\{\sup_{s\in[\tau,T]}|(\partial_t+\widetilde{\mathcal{L}}_{s,F^{\tau,\xi,\beta}_s,u})\varphi(s,F^{\tau,\xi,\beta}_s)\right\}\\
&\le C\Big(1+E\Big\{\sup_{s\in[\tau,T]}(|\widetilde{\sigma}(s,F^{\tau,\xi,\beta}_s,u)|^2+|\widetilde{b}(s,F^{\tau,\xi,\beta}_s,u)|)\Big\}\\
&\le C\ell^2\Big(1+E\Big\{\sup_{s\in[\tau,T]}|F^{\tau,\xi,\beta}_s|^2\Big\}\Big)<\infty.
\end{aligned}
$$

Thus, applying the dominated convergence theorem we see that $\lim_{h\downarrow 0} r_{\tau,\xi}(h)=0$, in probability, on the set $\Gamma^{\rho}_{\ell,\varepsilon}$, and (7.4) leads to

$$
\operatorname*{essinf}_{u\in U}\left[\partial_t+\widetilde{\mathcal{L}}_{\tau,\xi,u}\right]\varphi(\tau,\xi)\ge 0, \qquad P\text{-a.s. on the set } \Gamma^{\rho}_{\ell,\varepsilon}.
$$

Therefore, first letting $\rho\to 0$, then $\varepsilon\to 0$, and then $\ell\to\infty$, we obtain that

$$
-\operatorname*{essinf}_{u\in U}\left[\partial_t+\widetilde{\mathcal{L}}_{\tau,\xi,u}\right]\varphi(\tau,\xi)\le 0, \quad P\text{-a.s. on } \Gamma.
$$

In other words, $\widetilde{V}$ is a stochastic subsolution.

The proof that $\widetilde{V}$ is also a supersolution is a little more involved. We again fix $\tau\in L^0_{\mathcal{F}^B_T}(\Omega;[0,T])$, $\xi\in L^2_{\mathcal{F}^B_T}(\Omega;\mathbb{R}^n)$ and $\varphi\in C^{1,2}_{\mathcal{F}^B_T}([0,T]\times\mathbb{R}^n)$. Also, for any $\varepsilon>0$, $\ell>0$, $\rho>0$, we again find $\Gamma^{\rho}_{\ell,\varepsilon}$ that satisfies (i)–(iii) before, with "max" being replaced by "min."

To derive the desired inequality, we argue slightly differently. First we modify Lemma 6.3 to obtain a sequence $\{\beta^k\}\subset\widetilde{\mathcal{A}}$ such that (6.14) holds for the $\mathcal{F}^B_T$-measurable pair (τ,ξ) and $0<h\ll 1$, that is,

$$
\widetilde{V}(\tau,\xi)=\lim_{k\to\infty}\downarrow E\left\{\widetilde{V}(\tau+h,F^{\tau,\xi,\beta^k}_{\tau+h})\Big|\,\mathcal{G}_\tau\right\}, \quad P\text{-a.s.}
$$

Here we note that the convention $\tau + h = (\tau + h) \wedge T$ is used again and that the sequence $\{\beta^k\}$ may depend on h. Let us denote, for fixed $h > 0$ and each $k \geq 1$,

$$A_h^k \triangleq \left\{\omega : E\left\{\widetilde{V}(\tau+h, F_{\tau+h}^{\tau,\xi,\beta^k})\Big|\,\mathcal{G}_\tau\right\}(\omega) - \widetilde{V}(\tau,\xi,\omega) \leq h^2\right\}.$$

Then by definition of $\{\beta^k\}$ we have $\lim_{k\to\infty} P\{\Omega \setminus A_h^k\} = 0$. Since $P(\Gamma_{\ell,e}^\rho) > 0$, we must also have $\lim_{k\to\infty} P\{A_h^k \cap \Gamma_{\ell,\varepsilon}^\rho\} = P(\Gamma_{\ell,e}^\rho) > 0$.

Now, using the definition of φ and applying Itô's formula, one has, P-a.s. on the set $A_h^k \cap \Gamma_{\ell,\varepsilon}^\rho$,

$$\begin{aligned}
h &\geq \frac{1}{h} E\left\{\widetilde{V}(\tau+h, F_{\tau+h}^{\tau,\xi,\beta^k}) - \widetilde{V}(\tau,\xi)\Big|\,\mathcal{G}_\tau\right\}\\
&\geq \frac{1}{h} E\left\{\varphi(\tau+h, F_{\tau+h}^{\tau,\xi,\beta^k}) - \varphi(\tau,\xi)\Big|\,\mathcal{G}_\tau\right\}\\
&= \frac{1}{h} E\left\{\int_\tau^{\tau+h}\Big(\partial_t + \widetilde{\mathcal{L}}_{s,F_s^{\tau,\xi,\beta^k},\beta^k}\Big)\varphi(s, F_s^{\tau,\xi,\beta^k})ds\Big|\,\mathcal{G}_\tau\right\}\\
&\geq \operatorname*{essinf}_{u\in U}\Big[\partial_t + \widetilde{\mathcal{L}}_{\tau,\xi,u}\Big]\varphi(\tau,\xi) - \widetilde{r}_{\tau,\xi}(h),
\end{aligned} \tag{7.5}$$

where

$$\widetilde{r}_{\tau,\xi}(h) \triangleq \sup_{k\geq 1} E\left\{\sup_{s\in[\tau,\tau+h]}\left|\Big[\partial_t + \widetilde{\mathcal{L}}_{s,F_s^{\tau,\xi,\beta^k},\beta_s^k}\Big]\varphi(s, F_s^{\tau,\xi,\beta^k}) - \Big[\partial_t + \widetilde{\mathcal{L}}_{\tau,\xi,\beta_s^k}\Big]\varphi(\tau,\xi)\right|\,\Big|\,\mathcal{G}_\tau\right\}.$$

Let us now denote $\Psi = (\partial_t\varphi, D_x\varphi, D_{xx}^2\varphi)$ and for fixed (τ,ξ),

$$G_{\tau,\xi}^k(s) \triangleq |\widetilde{\sigma}\widetilde{\sigma}^T(s,\xi,\beta_s^k) - \widetilde{\sigma}\widetilde{\sigma}^T(\tau,\xi,\beta_s^k)| + |\widetilde{b}(s,\xi,\beta_s^k) - \widetilde{b}(\tau,\xi,\beta_s^k)|, \qquad s \in [0,T].$$

Clearly, Ψ is bounded and continuous on $\Gamma_{\ell,\varepsilon}^\rho$. Furthermore, a tedious but straightforward argument using Lemmas 3.1 and 5.2 ((5.7) in particular) and the Chebyshev inequality, one can show that for any $h > 0, k \geq 1$, and $\delta > 0$, there exists a constant $C_{\ell,\varepsilon,\rho} > 0$ such that, P-a.s., on $\Gamma_{\ell,\varepsilon}^\rho$, it holds that

$$\begin{aligned}
&E\left\{\sup_{s\in[\tau,\tau+h]}\Big[\partial_t + \widetilde{\mathcal{L}}_{s,F_s^{\tau,\xi,\beta^k},\beta_s^k}\Big]\varphi(s, F_s^{\tau,\xi,\beta^k}) - \Big[\partial_t + \widetilde{\mathcal{L}}_{\tau,\xi,\beta_s^k}\Big]\varphi(\tau,\xi)\Big|\mathcal{G}_\tau\right\}\\
&\leq C_{\ell,\varepsilon,\rho}\left(1+\frac{1}{\delta^2}\right)\left(1+|\xi|^4\right)\left\{\sup_{\substack{s\in[\tau,\tau+h],\\ |x|\leq\delta}}|\Psi(s,\xi+x) - \Psi(\tau,\xi)| + \sup_{\tau\leq s\leq\tau+h} E\{G_{\tau,\xi}^k(s)|\mathcal{G}_\tau\}\right\}.
\end{aligned}$$

Consequently, for all $h > 0, \delta > 0$, it holds, P-a.s. on $\Gamma_{\ell,\varepsilon}^\rho$ that

$$\begin{aligned}
\widetilde{r}_{\tau,\xi}(h) &\leq C_{\ell,\varepsilon\rho}\left(1+\frac{1}{\delta^2}\right)(1+|\xi|^4)\\
&\quad\times\left\{\sup_{\substack{s\in[\tau,\tau+h],\\ |x|\leq\delta}}|\Psi(s,\xi+x) - \Psi(\tau,\xi)|\Big\} + \sup_{\substack{s\in[\tau,\tau+h]\\ k\geq 1}} E\{G_{\tau,\xi}^k(s)|\mathcal{G}_\tau\}\right\}.
\end{aligned}$$

Now from the definition of $\widetilde{\sigma}$ and $\widetilde{b}$ ((3.6) and (3.7)) and the assumption (H1) we see that $G_{\tau,\xi}^k(s)$ is continuous in s, uniformly in k. Thus applying the dominated convergence theorem we have

$$\lim_{h\downarrow 0}\sup_{s\in[\tau,\tau+h],k\geq 1} E\{G_{\tau,\xi}^k(s)|\mathcal{G}_\tau\} = 0, \qquad P\text{-a.s.}$$

It then follows that $\widetilde{r}_{\tau,\xi}(h) \to 0$, P-a.s. on $\Gamma^{\rho}_{\ell,\varepsilon}$ as $h \downarrow 0$. In other words, we obtain

$$\operatorname*{essinf}_{u\in U} \left[\partial_t + \widetilde{\mathcal{L}}_{\tau,\xi,u}\right] \varphi(\tau,\xi) \le 0, \qquad P\text{-a.s. on } \Gamma^{\rho}_{\ell,\varepsilon}.$$

Again, first sending $\rho \to 0$, then $\varepsilon \to 0$, and then $\ell \to \infty$, we obtain that

$$-\operatorname*{essinf}_{u\in U} \left[\partial_t + \widetilde{\mathcal{L}}_{\tau,\xi,u}\right] \varphi(\tau,\xi) \ge 0, \quad P\text{-a.s.}$$

That is, $\widetilde{V}$ is a stochastic viscosity supersolution of (7.1).

To conclude, note that the uniqueness of the viscosity solution can be proved by following the almost identical idea of Buckdahn and Ma [3], with even easier arguments since in the present case there is no "martingale" term in the degenerated stochastic PDF (7.1). The proof is complete. □

7.2. HJB equation for the original control problem. We now turn to our last objective of this paper: to derive the stochastic HJB equation for the original value function V. The idea is straightforward, that is, we shall apply the "inverse" Doss–Sussmann transformation and see what will happen to the stochastic HJB equation (7.1). We first observe the following simple fact: if $\varphi \in C^{1,2}_{\mathcal{F}^B_T}([0,T]\times R^n)$, then by the Itô–Ventzell formula (cf., e.g., Kunita [11]), one has (recall the process $\{\zeta^s_t\}$ from (3.5))

$$\begin{aligned} d[\varphi(t,\zeta^0_t(x))] &= \frac{\partial\varphi}{\partial t}(t,\zeta^0_t(x))dt + (D_x\varphi)(t,\zeta^0_t(x)) \circ d\zeta^0_t(x) \\ &= \frac{\partial\varphi}{\partial t}(t,\zeta^0_t(x))dt - (D_x\varphi)(t,\zeta^0_t(x)) D_z\zeta^0_t(x)\theta(x) \circ dB_t \\ &= \frac{\partial\varphi}{\partial t}(t,\zeta^0_t(x))dt - D_x[\varphi(t,\zeta^0_t(x))]\theta(x) \circ dB_t. \end{aligned} \tag{7.6}$$

Now recall from Theorem 4.2(i) that $\widetilde{V}(s,x) = V(s,\eta^0_s(x))$, and hence $V(s,x) = \widetilde{V}(s,\zeta^0_s(x))$. Thus if $\widetilde{V} \in C^{1,2}_{\mathcal{F}^B_T}$, then in light of (7.6) we should have

$$dV(s,x) = d[\widetilde{V}(s,\zeta^0_s(x))] = \frac{\partial\widetilde{V}}{\partial t}(t,\zeta^0_s(x))ds - D_x[\widetilde{V}(t,\zeta^0_s(x))]\theta(x) \circ dB_s. \tag{7.7}$$

Combining (7.1) for $\widetilde{V}$ and relation (7.7) we can then formally write down the stochastic HJB equation for the original value function V:

$$\begin{cases} dV(s,x) + \operatorname*{essinf}_{u\in U}[\mathcal{L}_{x,u}V](s,x)ds + D_xV(s,x)\theta(x) \circ dB_s = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (s,x) \in [0,T)\times\mathbb{R}^n, \\ V(T,x) = H(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{7.8}$$

Again, we remark that this stochastic PDE is a "terminal value" problem; therefore it would become a complicated issue if we were to seek an $\{\mathcal{F}^B_t\}$-adapted solution. However, the following modification of the definition of stochastic viscosity solution for the stochastic PDE (7.8) given in [2, 3, 4] proves sufficient for our purpose.

DEFINITION 7.3. *A random field $v \in C_{\mathcal{F}^B_T}([0,T]\times\mathbb{R}^n)$ is said to be a (stochastic) viscosity subsolution (resp., supersolution) of equation* (7.8) *if*

(i) *$v(T,x) \le$ (resp., $\ge$) $H(x)$ for all $x \in \mathbb{R}^n$;*

(ii) *for any* $(\tau,\xi)\in L^2(\mathcal{F}_T^B;[0,T]\times\mathbb{R}^n)$, *and* $\varphi\in C^{1,2}_{\mathcal{F}_T^B}([0,T]\times\mathbb{R}^n)$, *it holds that*

$$-\frac{\partial\varphi}{\partial t}\left(\tau,\zeta_s^0(\xi)\right)\le \textit{(resp., }\ge\textit{)}\operatorname*{essinf}_{u\in U}\mathcal{L}_{\tau,\xi,u}\left[\varphi(\tau,\zeta_\tau^0(\cdot))\right](\xi),$$

P-a.s. on the set

$$\Gamma^\varphi_{\tau,\xi}\stackrel{\triangle}{=}\{\omega:(\tau(\omega),\xi(\omega))\in\operatorname{argmax}_{loc}[v(s,x,\omega)-\varphi(s,\zeta_0^s(x),\omega)]\ \textit{(resp., }\operatorname{argmin}_{loc}\textit{)}\}$$

We have the following theorem.

THEOREM 7.4. *Assume* (H1)–(H3). *Then the value function V is the unique stochastic viscosity solution of* (7.8).

Proof. Having proved Theorem 7.2, we need only prove the following equivalence relation: a random field $v\in C_{\mathcal{F}_T^B}([0,T]\times\mathbb{R}^n)$ is a stochastic viscosity subsolution (resp., supersolution) of the stochastic HJB equation (7.8) if and only if

$$\widetilde{v}(t.x)=v(t,\eta_t^0(x)),\qquad (t,x)\in[0,T]\times\mathbb{R}^n,$$

is a stochastic viscosity subsolution (supersolution) of the (random) HJB equation (7.1).

To this end, we observe that for any given $(\tau,\xi)\in L^2(\mathcal{F}_T^B;[0,T]\times\mathbb{R}^n)$ and $\varphi\in C^{1,2}_{\mathcal{F}_T^B}([0,T]\times\mathbb{R}^n\}$,

$$\begin{aligned}\Gamma^\varphi_{\tau,\xi}&=\{\omega:(\tau(\omega),\xi(\omega))\in\operatorname{argmax}_{loc}[v(s,x,\omega)-\varphi(s,\zeta_s^0(x),\omega)]\}\\&=\{\omega:(\tau(\omega),\zeta_\tau^0(\xi)(\omega))\in\operatorname{argmax}_{loc}[v(s,\eta_s^0(x),\omega)-\varphi(s,x,\omega)]\}\\&=\{\omega:(\tau(\omega),\zeta_\tau^0(\xi)(\omega))\in\operatorname{argmax}_{loc}[\widetilde{v}(s,x,\omega)-\varphi(s,x,\omega)]\}.\end{aligned}$$

On the other hand, it is easy to check that

$$\mathcal{L}_{x,u}[\varphi(t,\zeta_t^0(\cdot))](x)=(\widetilde{\mathcal{L}}_{t,\zeta_t^0(x),u}\varphi)(t,\zeta_t^0(x))\qquad\forall(t,x,u)\in[0,T]\times\mathbb{R}^n\times U;$$

we see that $\widetilde{v}$ is a viscosity subsolution of equation (7.1) if and only if for all $(\tau,\xi)\in L^2(\mathcal{F}_T^B;[0,T]\times\mathbb{R}^n)$ and $\varphi\in C^{1,2}_{\mathcal{F}_T^B}([0,T]\times\mathbb{R}^n\}$, it holds, P-a.s. on the set $\Gamma^\varphi_{\tau,\xi}$, that

$$-\left[\left(\frac{\partial}{\partial t}+\operatorname*{essinf}_{u\in U}\widetilde{\mathcal{L}}_{\tau,\zeta_\tau^0(\xi),u}\right)\varphi\right](\tau,\zeta_\tau^0(\xi))\le 0.$$

But this amounts to saying that for any such given (τ,ξ) and φ, it holds, P-a.s. on $\Gamma^\varphi_{\tau,\xi}$, that

$$-\frac{\partial\varphi}{\partial t}\left(\tau,\zeta_0^s\left(\xi\right)\right)\le\operatorname*{essinf}_{u\in U}\mathcal{L}_{\tau,\xi,u}\left[\varphi\left(\tau,\zeta_0^\tau(\cdot)\right)\right](\xi)\,.$$

But by Definition 7.3 it shows exactly that v is a stochastic viscosity subsolution of (7.8). The proof for the supersolutions is analogous. Thus the theorem follows from Theorem 7.2. ◻

8. Appendix (a counterexample). We give an example showing that the original pathwise stochastic control problem does not possess any minimizing sequence in $\mathcal{A}$. Let us assume that $n=1$, $U=[0,1]$, and $(s,x)=(0,0)$. Assume also that

$b(t,x,u) \equiv u$, $\sigma \equiv 0$, and $\theta(x) \equiv x$. Finally, let $H(x) = |x-1|$. That is, the system dynamics is

$$\begin{cases} dX_t^\alpha = \alpha_t dt + X_t^\alpha \circ dB_t, & t \in [0,T], \\ X_0^\alpha = 0. \end{cases}$$

Furthermore, the Doss–Sussmann transformation is $\eta_t^0(z) = z + \int_0^t \eta_r^0(z) \circ dB_r$, which can be solved explicitly as $\eta_t^0(z) = ze^{B_t}$, $t \in [0,T]$. Therefore, the "transformed" coefficients are $\widetilde{\sigma} \equiv 0$ and $\widetilde{b}(t,z,u) = \left[D_z \eta_t^0(z)\right]^{-1} b(\eta_t^0(z), u) = e^{-B_t}u$; and the "transformed" system equation becomes

$$dF_t^\beta = \widetilde{b}(t, F_t^\beta, \beta_t)dt = e^{-B_t}\beta_t dt, \qquad \beta \in \widetilde{A},$$

which can also be solved explicitly as $F_t^\beta = \int_0^t e^{-B_r}\beta_r dr$. Consequently we have $\eta_T^0(F_T^\beta) = \int_0^T e^{B_T - B_r}\beta_r dr$.

We can now easily check that, at $(s,x) = (0,0)$,

$$V(0,0) = \widetilde{V}(0,0) = \operatorname*{essinf}_{\beta \in \widetilde{A}} E\left\{ |\eta_T^0(F_T^\beta) - 1| \,\Big|\, \mathcal{F}_T^B \right\} = \left[1 - \int_0^T e^{B_T - B_r} dr\right]^+,$$

with a wider sense optimal control being

$$\beta_t^* = \begin{cases} \left(\int_0^T e^{B_T - B_r} dr\right)^{-1} & \text{on } \left\{ \int_0^T e^{B_T - B_r} dr > 1 \right\}, \\ 1 & \text{on } \left\{ \int_0^T e^{B_T - B_r} dr \le 1 \right\}. \end{cases}$$

We should note that such an optimal control is unique on the set $\{\int_0^T e^{B_T - B_r} dr \le 1\}$.

On the other hand, we observe that if $\alpha \in \mathcal{A}$, then so is $E[\alpha]$, and by Jensen's inequality we have

$$\begin{aligned} J(0,0;\alpha) &= E\left\{ \left| \int_0^T e^{B_T - B_r}\alpha_r dr - 1 \right| \Bigg| \mathcal{F}_T^B \right\} \ge \left| \int_0^T e^{B_T - B_r} E[\alpha_r] dr - 1 \right| \\ &= J(0,0,E[\alpha]) \ge V(0,0). \end{aligned}$$

Now let us assume that there exists a minimizing sequence $\{\alpha^k\}_{k \ge 1} \subset \mathcal{A}$. Then it holds, P-a.s., that $\lim_{k\to\infty} J(0,0;\alpha^k) = V(0,0)$. But the above argument shows that

$$J(0,0,E[\alpha^k]) = \left| \int_0^T e^{B_T - B_r} E[\alpha_r^k] dr - 1 \right| \longrightarrow V(0,0) = \left[1 - \int_0^T e^{B_T - B_r} dr\right]^+ \tag{8.1}$$

as well. Since $0 \le E[\alpha_t^k] \le 1$ for all t, we see that on the set $\{\int_0^t e^{B_T - B_t} dr \le 1\}$ it must hold that

$$\int_0^T e^{B_T - B_r} E[\alpha_r^k] dr \longrightarrow \int_0^T e^{B_T - B_r} dr \qquad \text{as } k \to \infty,$$

or, equivalently, $\int_0^T e^{B_T - B_r}[1 - E[\alpha_r^k]]dr \to 0$, on $\{\int_0^t e^{B_T - B_t} dr \le 1\}$. But again notice that $0 \le E[\alpha_t^k] \le 1$, which implies that $E[\alpha_\cdot^k] \longrightarrow 1$, as $k \to \infty$ in (Lebesgue) measure, and consequently one must have

$$\left|\int_0^T e^{B_T-B_r}E[\alpha_r^k]dr-1\right| \longrightarrow \left|\int_0^T e^{B_T-B_r}dr-1\right|, \quad P\text{-a.s.}$$

Since

$$\left|\int_0^T e^{B_T-B_r}dr-1\right| > V(0,0) \qquad \text{on} \quad \left\{\int_0^T e^{B_T-B_r}dr>1\right\},$$

and $P\{\int_0^T e^{B_T-B_r}dr>1\}>0$, we obtain that

$$\lim_{k\to\infty} J(0,0,E[\alpha^k]) = \left|\int_0^T e^{B_T-B_r}dr-1\right| > V(0,0) \qquad \text{on } \left\{\int_0^T e^{B_T-B_r}dr>1\right\}.$$

This contradicts (8.1). Therefore such a minimizing sequence cannot exist.

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