Stochastic Processes and their Applications 93 (2001) 205–228

stochastic processes and their applications

www.elsevier.com/locate/spa

Stochastic viscosity solutions for nonlinear stochastic partial differential equations. Part II

Rainer Buckdahn[a], Jin Ma[b,*]

[a] *Département de Mathématiques, Université de Bretagne-Occidentale, F-29285 Brest Cedex, France*

[b] *Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395, USA*

Received 7 December 1999; received in revised form 9 October 2000; accepted 26 October 2000

Abstract

This paper is a continuation of our previous work (Part I, Stochastic Process. Appl. 93 (2001) 181–204), with the main purpose of establishing the uniqueness of the stochastic viscosity solution introduced there. We shall prove a comparison theorem between a stochastic viscosity solution and an ω-wise stochastic viscosity solution, which will lead to the uniqueness results. As the byproducts we extend the measurable section theorem of Dellacherie and Meyer (1978), and a fundamental lemma of Crandall et al. (1992). © 2001 Elsevier Science B.V. All rights reserved.

MSC: 60H15,30; 60G46,60; 35R60

Keywords: Stochastic PDEs; Stochastic viscosity solutions; Uniqueness; (Optional) section theorem

1. Introduction

In this paper we continue to study the following nonlinear stochastic PDE (SPDE):

$$\mathrm{d}u(t,x) = \{\mathscr{A}u(t,x) + f(t,x,u(t,x),\sigma^*(x)Du(t,x))\}\,\mathrm{d}t$$

$$+\sum_{i=1}^{d} g_i(t,x,u(t,x))\circ \mathrm{d}B_t^i \quad (t,x)\in(0,T)\times\mathbb{R}^n,$$

$$u(0,x) = u_0(x), \quad x\in\mathbb{R}^n, \tag{1.1}$$

where $B=(B^1,\dots,B^d)$ is a standard d-dimensional Brownian motion defined on some complete filtered probability space $(\Omega,\mathscr{F},P;\boldsymbol{F})$, with $\boldsymbol{F}=\{\mathscr{F}_t\}_{t\geqslant 0}$ being a filtration satisfying the *usual hypotheses* (see, e.g. Protter, 1990); and the stochastic integral is in the sense of Stratonovich. Further, the second-order differential operator $\mathscr{A}$ in (1.1)

* Corresponding author.

E-mail addresses: rainer.buckdahn@univ-brest.fr (R. Buckdahn), majin@math.purdue.edu (J. Ma).

0304-4149/01/$ - see front matter

PII: S0304-4149(00)00092-2

is defined by

$$\mathscr{A} = \tfrac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)^* D^2) + \langle \beta(x), D\rangle, \tag{1.2}$$

where $D = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})^{\mathrm{T}}$, $D^2 = (\partial^2_{x_i x_j})^n_{i,j=1}$; the functions σ, β are assumed to be measurable; and $\sigma^*(\cdot)$ denotes the transpose of $\sigma(\cdot)$. To simplify notations we also denote, as in Buckdahn and Ma (2001),

$$F(\omega, t, x, y, p, A) \triangleq \tfrac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)^* A) + \langle \beta(x), p\rangle + f(\omega, t, x, y, \sigma^*(x)p)$$

$$(\omega, t, x, y, p, A) \in \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}^n, \tag{1.3}$$

where $\mathscr{S}^n$ is the space of all symmetric $n \times n$-matrices. In the sequel we will refer to (1.1) as SPDE(f, g).

In Buckdahn and Ma (2001) we introduced a notion of *stochastic viscosity solution*, inspired by the earlier results of Lions and Souganidis (1998a,b). By using a Doss–Sussmann-type transformation and the so-called *backward doubly stochastic differential equations* introduced by Pardoux and Peng (1994) we established the existence of stochastic viscosity solution to SPDE(f, g) (1.1). In this paper we shall prove that the stochastic viscosity solution to SPDE(f, g) is unique among the class of *stochastically bounded* random fields by establishing a main comparison theorem for stochastic viscosity solutions, as is done in the deterministic case.

Our line of attack can be roughly described as follows. Following the idea of Buckdahn and Ma (2001) we shall first convert the SPDE(f, g) (via the Doss–Sussmann-type transformation) to SPDE$(\tilde{f}, 0)$, an SPDE without martingale term. Then we prove a comparison theorem between the stochastic viscosity solution and the ω-wise viscosity solution of SPDE$(\tilde{f}, 0)$, from which the uniqueness will follow. We should note that the price for such a convertion is that the new non-homogeneous term $\tilde{f}$ becomes rather ill-behaved, which results in some technicalities with different nature. But we nevertheless think that this is a good "trade-off" because SPDE$(\tilde{f}, 0)$ is essentially a PDE with random coefficients, which seems to be much easier to handle.

As in Buckdahn and Ma (2001) when we introduce the notion of stochastic viscosity solutions, the main difficulty here is again how to "translate" the pivotal results from the deterministic theory to the stochastic case. One of the subtle issues is conceivably the measurability with regard to the variable ω, or in particular, the adaptedness of all the devices involved, so that the stochastic calculus can be applied. In fact, it is this simple requirement that causes most of the tedious work, as we will see in the paper. Two main byproducts of this paper are a generalized version of the optional section theorem of Dellacherie and Meyer (1978) to the combined "space–time" random vectors; and consequently a generalization of a fundamental theorem of Crandall et al. (1992, Theorem 3.2) to the stochastic case.

This paper is organized as follows. In Section 2 we recall the notations, definitions, and relevant results from Buckdahn and Ma (2001). In Section 3 we state the main theorem, and introduce some useful auxiliary functions and discuss their properties. In Section 4 we prove the measurable selection theorem and the fundamental lemma, and in Section 5 we prove the main theorem.

2. Preliminaries

In this paper we inherit all the notations from Buckdahn and Ma (2001). Let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which is defined a d-dimensional Brownian motion $B=(B_t)_{t\geqslant 0}$, let $\boldsymbol{F}^B \triangleq \{\mathscr{F}_t^B\}_{t\geqslant 0}$ be the natural filtration generated by B, augmented by the P-null sets of $\mathscr{F}$; and let $\mathscr{F}^B=\mathscr{F}^B_\infty$. Let $\mathbb{E}$ and $\mathbb{E}_1$ be two Euclidean spaces, whose inner products and norms will be denoted as the same $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. Further, we denote

- $L^2(\mathscr{F}^B;\mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathscr{F}^B$-measurable, square-integrable random variables;
- for any sub-σ-field $\mathscr{G}\subseteq\mathscr{F}_T^B$, let $L^0(\mathscr{G};\mathbb{E})$ be all $\mathbb{E}$-valued, $\mathscr{G}$-measurable random variables (when there is no danger of confusion, we often write $\xi\in\mathscr{G}$, instead of $\xi\in L^0(\mathscr{G};\mathbb{E})$, for simplicity);
- $L^q(\boldsymbol{F}^B,[0,T];\mathbb{E})$ to be all $\mathbb{E}$-valued, $\boldsymbol{F}^B$-progressively measurable processes ψ, such that $E\int_0^{\mathrm{T}}|\psi_t|^q\,\mathrm{d}t<\infty$. In particular, let $L^0(\boldsymbol{F}^B,[0,T];\mathbb{E})$ denote all $\mathbb{E}$-valued, $\boldsymbol{F}^B$-progressively measurable processes; and let $L^\infty(\boldsymbol{F}^B,[0,T];\mathbb{E})$ denote those processes in $L^0(\boldsymbol{F}^B,[0,T];\mathbb{E})$ that are uniformly bounded;
- $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ to be the space of all $\mathbb{E}_1$-valued functions defined on $[0,T]\times\mathbb{E}$ which are k-times continuously differentiable in $t\in[0,T]$ and ℓ-times continuously differentiable in $x\in\mathbb{E}$; let $C_b^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ be the subspace of $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ in which all functions have uniformly bounded partial derivatives; and let $C_p^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ be the subspace of $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ in which all the partial derivatives are of at most polynomial growth;
- for any sub-σ-field $\mathscr{G}\subseteq\mathscr{F}_T^B$, let $C^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$) be the space of all $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$)-valued random variables that are $\mathscr{G}\otimes\mathscr{B}([0,T]\times\mathbb{E})$-measurable;
- $C^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$) to be the space of all random fields $\varphi\in C^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$), such that for fixed $x\in\mathbb{E}$, the mapping $(t,\omega)\mapsto\varphi(t,x,\omega)$ is $\boldsymbol{F}^B$-progressively measurable.

If $\mathbb{E}_1=\mathbb{R}$, we shall simply denote $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{R})$ as $C^{k,\ell}([0,T]\times\mathbb{E})$, and so on; and we denote $C^{0,0}([0,T]\times\mathbb{E};\mathbb{E}_1)=C([0,T]\times\mathbb{E};\mathbb{E}_1)$, and $C^{0,0}(\boldsymbol{F}^B,[0,T]\times\mathbb{E})=C(\boldsymbol{F}^B,[0,T]\times\mathbb{E})$, etc.

In light of the results of Buckdahn and Ma (2001), throughout this paper we shall make use of the following *standing assumptions*:

(A1) The functions $\sigma:\mathbb{R}^n\to\mathbb{R}^{n\times k}$ and $\beta:\mathbb{R}^n\to\mathbb{R}$ are uniformly Lipschitz continuous, with a common Lipschitz constant $K>0$.

(A2) The function $f:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\mapsto\mathbb{R}$ is a continuous random field such that for fixed (x,y,p), $f(\cdot,\cdot,x,y,\sigma^*(x)p)$ is $\boldsymbol{F}^B$-progressively measurable; and there exists some constant $K>0$ such that for P-a.e. $\omega\in\Omega$,

$$\begin{aligned}
&|f(\omega,t,x,0,0)|\leqslant K \quad \forall(t,x)\in[0,T]\times\mathbb{R},\\
&|f(\omega,t,x,y,z)-f(\omega,t',x',y',z')|\leqslant K(|t-t'|+|x-x'|+|y-y'|+|z-z'|);\\
&\qquad\qquad \forall(t,x,y,z),\,(t',x',y',z')\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k. \qquad (2.1)
\end{aligned}$$

(A3) The function $u_0: \mathbb{R}^n \mapsto \mathbb{R}$ is continuous and, such that for some constants $K, p > 0$,

$$|u_0(x)| \leqslant K(1 + |x|^p), \qquad x \in \mathbb{R}^n. \tag{2.2}$$

(A4) The function $g \in C_b^{0,2,3}([0,T] \times \mathbb{R}^n \times \mathbb{R};\ \mathbb{R}^d)$.

Recall that in order to obtain the so-called *uniform stochastic boundedness* of the stochastic viscosity solution (see Definition 2.4 below), we need to strengthen Assumption (A4) to the following:

(A4′) The function g satisfies (A4); and for any $\varepsilon > 0$, there exists a function $G^\varepsilon \in C^{1,2,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R})$, such that

$$\frac{\partial G^\varepsilon}{\partial t}(t,w,x,y) = \varepsilon; \quad \frac{\partial G^\varepsilon}{\partial w^i} = g_i(t,x,G^\varepsilon(t,w,x,y)), \quad i = 1,\dots,d;$$
$$G^\varepsilon(0,0,x,y) = y. \tag{2.3}$$

We remark that when g is independent of t and $d=1$, then (A4′) is trivially satisfied, since one can always first solve the ODE (with parameter x):

$$\frac{\mathrm{d}G}{\mathrm{d}w} = g(x,G), \quad G(0) = y,$$

and then set $G^\varepsilon(t,w,x,y) = G(w,x,y) + \varepsilon t$.

To define a stochastic viscosity solution, we first consider the following SDE in the Stratonovich sense: for each $(x,y) \in \mathbb{R}^n \times \mathbb{R}$,

$$\eta(t,x,y) = y + \sum_{i=1}^d \int_0^t g_i(s,x,\eta(s,x,y)) \circ \mathrm{d}B_s^i$$
$$\triangleq y + \int_0^t \langle g(s,x,\eta(s,x,y)), \circ \mathrm{d}B_s \rangle, \quad t \geqslant 0, \tag{2.4}$$

or equivalently, an Itô SDE (with parameter)

$$\eta(t,x,y) = y + \frac{1}{2}\int_0^t \langle g, D_y g\rangle(s,x,\eta(s,x,y))\mathrm{d}s + \int_0^t \langle g(s,x,\eta(s,x,y)), \mathrm{d}B_s\rangle. \tag{2.5}$$

Denote the (unique) solution of (2.4) or (2.5) by $\eta(t,x,y)$, $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}$. From the theory of SDEs we know that, as a stochastic flow, $\eta \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n \times \mathbb{R})$. Since under (A4) the mapping $y \mapsto \eta(t,x,y,\omega)$ defines a diffeomorphism for all (t,x), P-a.s. (see, e.g., Protter, 1990, Chapter V), we can denote the y-inverse of $\eta(t,x,y)$ by $\mathscr{E}(t,x,y)$, and show that $\mathscr{E}(t,x,y)$ is the solution to the following SPDE (see Buckdahn and Ma, 2001):

$$\mathscr{E}(t,x,y) = y - \int_0^t \langle D_y \mathscr{E}(s,x,y) g(s,x,y), \circ \mathrm{d}B_s\rangle \quad \forall (t,x,y),\ P\text{-a.s.} \tag{2.6}$$

The following lemma of Buckdahn and Ma (2001) gives important estimates for the random fields η and $\mathscr{E}$:

Lemma 2.1. *Assume* (A4′). *Let* $\alpha = (\alpha_0,\dots,\alpha_n)$ *be a multiindex, with* $|\alpha| \triangleq \sum_{k=0}^n |\alpha_i|$; *and let* $D^\alpha \triangleq D_y^{\alpha_0} D_{x_1}^{\alpha_1} \dots, D_{x_n}^{\alpha_n}$. *Then there exists a constant* $C > 0$, *depending only on* *the bounds of g and its partial derivatives, such that for $\zeta = \eta, \mathscr{E}$, it holds P-almost surely that*

$$|\zeta(t,x,y)| \leqslant |y| + C|B_t|; \quad |D^\alpha\zeta| \leqslant C\exp\{C|B_t|\},$$
$$(t,x,y) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}, \quad |\alpha|\leqslant 3.$$

The *stochastic viscosity solution* for SPDE(f,g) (1.1) is then defined as follows. Let $\mathscr{M}^B_{0,T}$ denote all the $\boldsymbol{F}^B$-stopping times τ such that $0\leqslant\tau\leqslant T$, P-a.s.; and $\mathscr{M}^B_{0,\infty}$ be all the $\boldsymbol{F}^B$-stopping times that are almost surely finite.

Definition 2.2. A random field $u \in C(\boldsymbol{F}^B, [0,T]\times\mathbb{R}^n)$ is called a stochastic viscosity subsolution (resp. supersolution) of SPDE(f,g), if $u(0,x)\leqslant$ (resp. $\geqslant$) $u_0(x)$ $\forall x\in\mathbb{R}^n$; and if for any $\tau\in\mathscr{M}^B_{0,T}$, $\xi\in L^0(\mathscr{F}^B_\tau;\ \mathbb{R}^n)$, and any random field $\varphi\in C^{1,2}(\mathscr{F}^B_\tau,[0,T]\times R^n)$ satisfying

$$u(t,x)-\eta(t,x,\varphi(t,x))\leqslant \text{ (resp.}\geqslant\text{) } 0 = u(\tau,\xi)-\eta(\tau,\xi,\varphi(\tau,\xi)),$$

for all (t,x) in a neighborhood of (τ,ξ), P-a.e. on the set $\{0<\tau<T\}$, it holds that

$$\mathscr{A}\psi(\tau,\xi)+f(\tau,\xi,\psi(\tau,\xi),\sigma^*(\xi)D\psi(\tau,\xi)) \geqslant \text{(resp.} \leqslant\text{)} D_y\eta(\tau,\xi,\varphi(\tau,\xi))D_t\varphi(\tau,\xi), \tag{2.7}$$

P-a.e. on $\{0<\tau<T\}$, where $\psi(t,x)\triangleq\eta(t,x,\varphi(t,x))$.

A random field $u\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$ is called a stochastic viscosity solution of SPDE(f,g), if it is both a stochastic viscosity subsolution and a supersolution.

In the special case when $g\equiv 0$, one can view SPDE$(f,0)$ as a PDE with random coefficients. Therefore, for each $\omega\in\Omega$ one can define the viscosity solution to SPDE$(f,0)$ in the deterministic sense. Taking the ω-measurability into account we have the following definition which is important for the study of uniqueness.

Definition 2.3. A random field $u\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$ is called an ω-wise viscosity (sub-, super-) solution if for P-a.e. $\omega\in\Omega$, $u(\omega,\cdot,\cdot)$ is a (deterministic) viscosity (sub-, super-) solution of the SPDE$(f,0)$.

The following notion of "boundedness" for a random field defined in Buckdahn and Ma (2001) gives the main characterization of the class of stochastic viscosity solutions on which the uniqueness holds.

Definition 2.4. A random field $u\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$ is called stochastically uniformly bounded if there exists a positive, increasing process $\Theta\in L^0(\boldsymbol{F}^B,[0,T])$, such that P-almost surely, it holds that $|u(t,x)|\leqslant\Theta_t$ $\forall(t,x)\in[0,T]\times\mathbb{R}^n$.

One of the main devices used in Buckdahn and Ma (2001) is a transformation of random fields in the spirit of the so-called Doss–Sussmann transformation seen in the standard SDE theory, which we now describe.

Note that the random fields $\eta, \mathscr{E} \in C^{0,2,2}(\boldsymbol{F}^B, [0,T]\times\mathbb{R}^n\times\mathbb{R})$; for any random field $\psi: [0,T]\times\mathbb{R}^n\times\Omega \mapsto \mathbb{R}$, we consider the transformation introduced in the Definition 2.2:

$$\varphi(t,x) = \mathscr{E}(t,x,\psi(t,x)), \quad (t,x)\in[0,T]\times\mathbb{R}^n, \tag{2.8}$$

or equivalently, $\psi(t,x) = \eta(t,x,\varphi(t,x))$ $\forall(t,x)$, P-a.s. One can easily check that $\varphi \in C^{0,p}(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$, for $p=0,1,2$. Moreover, if $\varphi\in C^{0,2}(\mathscr{F}^B,[0,T]\times\mathbb{R}^n)$, then one shows that

$$\begin{aligned} &D_y\mathscr{E}(t,x,\psi(t,x))\{\mathscr{A}\psi(t,x) + f(t,x,\psi(t,x),D\psi(t,x)\sigma(x))\}\\ &\quad = \mathscr{A}\varphi(t,x) + \tilde{f}(t,x,\varphi(t,x),D\varphi(t,x)\sigma(x)), \end{aligned} \tag{2.9}$$

for all $(t,x)\in(0,T)\times\mathbb{R}^n$, P-a.e., where

$$\begin{aligned} \tilde{f}(t,x,y,z) \triangleq \frac{1}{D_y\eta(t,x,y)}\Big\{ & f(t,x,\eta(t,x,y),\sigma^*(x)D_x\eta(t,x,y)+D_y\eta(t,x,y)z)\\ & +\mathscr{A}_x\eta(t,x,y)+\langle\sigma^*(x)D_{xy}\eta(t,x,y),z\rangle+\frac{1}{2}D_{yy}\eta(t,x,y)|z|^2\Big\}, \end{aligned} \tag{2.10}$$

for all $(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k$, P-a.s. Here $\mathscr{A}_x$ is the same as the operator $\mathscr{A}$ in (1.2), with the emphasis that all the partial derivatives are with respect to x. We often omit the subscript x in the sequel when there is no danger of confusion. Further, we should note that in the above all the partial derivatives of the random field $\mathscr{E}(\cdot,\cdot,\cdot)$ are evaluated at $(t,x,\eta(t,x,y))$; and all those of $\eta(\cdot,\cdot,\cdot)$ are evaluated at (t,x,y).

In what follows we refer to (2.8) as the Doss–Sussmann-type transformation. The following results of Buckdahn and Ma (2001) are the basis of our discussion in this paper.

Theorem 2.5. *Assume* (A1)–(A4). *A random field u is a stochastic viscosity sub- (resp. super-)solution to* SPDE(f,g) (1.1) *if and only if $v(\cdot,\cdot)=\mathscr{E}(\cdot,\cdot,u(\cdot,\cdot))$ is a stochastic viscosity sub- (resp. super-)solution to* SPDE$(\tilde{f},0)$.

Consequently, u is a stochastic viscosity solution of SPDE(f,g) (1.1) *if and only if $v(\cdot,\cdot)=\mathscr{E}(\cdot,\cdot,u(\cdot,\cdot))$ is a stochastic viscosity solution to* SPDE$(\tilde{f},0)$.

Theorem 2.6. *Assume* (A1)–(A4). *Then the* SPDE(f,g) *admits a stochastic viscosity solution $u\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$; and* SPDE$(\tilde{f},0)$ *admits a stochastic viscosity solution $v\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$. Furthermore, a pair of these solutions u, v can be related as*

$$u(t,x)=\eta(t,x,v(t,x)); \quad v(t,x)=\mathscr{E}(t,x,u(t,x)),$$

where η and $\mathscr{E}$ are the solutions to (2.5) *and* (2.6), *respectively.*

Finally, if in addition (A4$'$) *holds and u_0 is uniformly bounded, then the random fields u and v are both stochastically uniformly bounded.*

3. Statement of main results

In this section we state our main results and give some preparation of the proof. While most of the technical details of the proofs will be carried out in the rest of the paper, some important auxiliary functions will be introduced and studied in this and next sections to facilitate our future discussion.

To begin with, we note that, in light of Theorem 2.5, we need only prove the uniqueness of the stochastic viscosity solution to SPDE($\tilde{f}, 0$):

$$D_t v(t,x) = \mathscr{A} v(t,x) + \tilde{f}(t,x,v(t,x), Dv(t,x)\sigma(x)), \quad (t,x) \in [0,T] \times R^n. \tag{3.1}$$

We should point out that although SPDE (3.1) does not have a martingale term, the definitions of a stochastic viscosity solution and an ω-wise stochastic viscosity solution to (3.1) are still different! However, since the uniqueness for the ω-wise stochastic viscosity solution is essentially parallel to the deterministic case, we shall establish the uniqueness by identifying the two definitions in the case of (3.1).

To do this, we shall make use of a technical assumption on the random field $\tilde{f}$:

(A5) There exists an increasing, $\boldsymbol{F}$-adapted process $\Theta = \{\Theta_t;\ t \geqslant 0\}$, such that

$$|\tilde{f}(t,x,y+h,z) - \tilde{f}(t,x,y,z)| \leqslant \Theta_t h \quad \forall h > 0,\ \forall (t,x,y,z),\ P\text{-a.e.}$$

We remark here that Assumption (A5) is merely technical. A trivial special case is that the function g is such that the corresponding solution to (2.4) satisfies that $D_{xy}\eta \equiv 0$ and $D_{yy}\eta \equiv 0$ (e.g., $g(t,x,y) = \lambda_1(t)y + \lambda_2(t,x)$). Removal of this condition seems to be possible by restricting slightly the class of random fields on which the uniqueness is considered, which requires some extra properties of the stochastic viscosity solutions. We would prefer to address this in a separate publication, in order not to over-complicate this already technical paper. We should point out, however, that the main difficulties in this paper, such as the measurable selection issue, are by no means eased by such an assumption.

The main result of this paper is the following *comparison theorem*:

Theorem 3.1. *Assume* (A1)–(A5). *Suppose that* $v_1 \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ *is a stochastic viscosity sub- (resp. super-)solution of* (3.1), *and* $v_2 \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ *is an* ω*-wise viscosity super- (sub-)solution of* (3.1), *such that they are both stochastically uniformly bounded. Then it holds that*

$$v_1(t,x) \leqslant (\text{resp. } \geqslant) v_2(t,x) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n,\ P\text{-a.e} \tag{3.2}$$

The uniqueness results are contained in the following corollary. Recall that in the case of $g = 0$, an ω-wise viscosity solution is necessary stochastic viscosity solution.

Corollary 3.2. *Assume* (A1)–(A5). *Then*

(i) *If* $v_1 \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ *is a stochastic viscosity solution and* $v_2 \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ *is an* ω*-wise viscosity solution of* (3.1), *and both are uniformly stochastically bounded, then* $v_1(t,x) \equiv v_2(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$, *P-a.s.*

(ii) *The uniformly stochastically bounded* ω*-wise viscosity solution to* (3.1) *is unique. In particular, if* $\tilde{f}$ *is deterministic, then the uniformly bounded, deterministic viscosity solution of* (3.1) *is unique.*

(iii) *If in addition* (A4$'$) *also holds, then the stochastic viscosity solution to* SPDE(f, g) *is unique among uniformly stochastically bounded random fields in* $C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$.

Proof. (i) Obvious.

(ii) Since an ω-wise viscosity supersolution is also a stochastic viscosity super solution by definition, Theorem 3.1 gives the comparison between the ω-wise super and subsolutions, and the uniqueness follows. We note that if $\tilde{f}$ is deterministic, then the result is well-known (cf. Pardoux and Peng, 1992).

(iii) Let $u_1, u_2 \in C(\boldsymbol{F}^B, [0,T]\times\mathbb{R}^n)$ be two stochastic viscosity solutions of SPDE(f,g), such that they both are uniformly stochastically bounded. Define $v_i(\cdot,\cdot) = \mathscr{E}(\cdot,\cdot,u_i(\cdot,\cdot))$, $i=1,2$. Then by Theorem 2.5, v_1 and v_2 are both stochastic viscosity solution to the SPDE$(\tilde{f},0)$, where $\tilde{f}$ is defined by (2.10); and are both uniformly stochastically bounded, thanks to Lemma 2.1. The uniqueness then follows from part (i) and Theorem 4.3 of Pardoux and Peng (1992).

The proof of Theorem 3.1 is quite lengthy, we break it into a sequence of lemmas, which will be proved throughout the rest of the paper. To begin with, let us assume without loss of generality that v_1 is a stochastic viscosity subsolution and v_2 is a ω-wise viscosity supersolution of (3.1), and define

$$\zeta \triangleq \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} (v_1(t,x) - v_2(t,x)).$$

Clearly, $P\{\zeta > 0\} = 0$ means that the conclusion of Theorem 3.1 holds, so we shall assume $P\{\zeta>0\}>0$, and then try to find a set $\Gamma^* \in \mathscr{F}_T^B$ with $P(\Gamma^*)>0$, on which a contradiction can be drawn.

Let us first make some reductions. Recall that v_1 and v_2 are both stochastically uniformly bounded, that is, there exists a positive, increasing process $\Theta \in L^0(\boldsymbol{F}^B,[0,T])$ such that $|\zeta| \leqslant 2\Theta_T < \infty$, P-a.e., so $P\{\zeta>0\}>0$ implies that for some $\ell > 0$, $P\{\zeta>0, \Theta_T \leqslant \ell\} > 0$. Consequently,

$$\zeta^\ell \triangleq \operatorname*{esssup}_{\omega\in\{\Theta_T\leqslant\ell\}} [\zeta(\omega)] > 0. \tag{3.3}$$

Next, for each $N\in\mathbb{N}$, define $K_N \triangleq [0, T-1/N] \times \{x\in\mathbb{R}^n\colon |x|\leqslant N\}$ and

$$\zeta_N(\omega) = \sup_{(t,x)\in K_N} \{v_1(\omega,t,x) - v_2(\omega,t,x)\}. \tag{3.4}$$

Since $\lim_{N\to\infty}\zeta_N = \zeta$, P-a.s., for each $\varepsilon>0$ we can define the following integer-valued random variable:

$$N(\varepsilon,\omega) \triangleq \inf\{N\in\mathbb{N}\colon \zeta_N(\omega) \geqslant \zeta(\omega) - \varepsilon/3\}. \tag{3.5}$$

Further, since $v_2(\omega,\cdot,\cdot)$ is uniform continuous on each K_N, for a.e. $\omega\in\Omega$ and $N > N(\varepsilon,\omega)$ we can then find some $\delta(N,\omega)>0$ such that

$$\alpha(\omega,\delta,N) \triangleq \sup_{(t,x)\in K_N,\,\delta'\in(0,\delta)} |v_2(\omega,t,x) - v_2(\omega,t-\delta',x)| \leqslant \frac{\varepsilon}{3} \quad \forall \delta\in(0,\delta(N,\omega)). \tag{3.6}$$

Keeping the random variables ζ_N, $N(\varepsilon,\cdot)$, $\alpha(\cdot,\delta,N)$ in mind, for any $\varepsilon\in(0,\zeta^\ell)$, $N\in\mathbb{N}$, and $\delta>0$, we now define

$$\Gamma_{\varepsilon,N,\delta} \triangleq \{\omega\colon \zeta(\omega) \geqslant \zeta^\ell - \varepsilon, \Theta_T(\omega)\leqslant\ell, N\geqslant N(\varepsilon,\omega), \alpha(\omega,\delta,N)\leqslant\varepsilon/3\}. \tag{3.7}$$

We shall prove that the desired set Γ^* can be chosen from the family $\{\Gamma_{\varepsilon,N,\delta}\colon \varepsilon>0,\ N\in\mathbb{N},\ \delta>0\}$.

To end this section, let us define some "auxiliary functions" similar to those that are often used in the uniqueness proof for the deterministic case. We point out that the main difficulty in the stochastic case is that the time variable and the spatial variable *cannot* be treated the same way. In fact, even two time variables will have to be treated differently in order to obtain the appropriate measurability (such as "adaptedness"), as we will see in a moment.

Let us first introduce, for $\delta>0$, a (generalized) convex function

$$\psi_\delta(r)=\begin{cases} -\ln\left[1-\left(\dfrac{\delta-r}{\delta}\right)^2\right], & r\in(0,2\delta);\\ +\infty, & r\notin(0,2\delta).\end{cases} \tag{3.8}$$

Now defining $v_2(t,x)\equiv v_2(0,x)$ $\forall t\leqslant 0$, for any $\delta_1,\delta_2>0$, we define a random field $\Psi_{\delta_1,\delta_2}\colon \Omega\times[0,T)\times\mathbb{R}^n\times(-\infty,T)\times\mathbb{R}^n\mapsto\mathbb{R}$:

$$\begin{aligned}\Psi_{\delta_1,\delta_2}(\omega,t,x,t',x') &= v_1(\omega,t,x)-v_2(\omega,t',x')\\ &\quad-\left\{\frac{1}{2\delta_1}|x-x'|^2+\frac{1}{2}\psi_{\delta_1}(t-t')+\frac{\delta_2}{2}(|x|^2+|x'|^2)+\frac{\delta_2}{2}\,\frac{1}{T-t}\right\},\end{aligned} \tag{3.9}$$

and a process

$$\Phi_{\delta_1,\delta_2}(t)\triangleq\sup_{\substack{x,x'\in\mathbb{R}^n,\\ t'\in(-\infty,T)}}\Psi_{\delta_1,\delta_2}(t,x,t',x'),\quad t\in[0,T]. \tag{3.10}$$

Clearly, by the definition of ψ_δ, the "sup" in (3.10) is actually taken over $t'\in(t-2\delta_1,t)$, for any $t\in[0,T]$.

The following lemma relates the set $\Gamma_{\varepsilon,N,\delta}$ and the auxiliary function Ψ_{δ_1,δ_2}:

Lemma 3.3. (i) *For each* $\varepsilon\in(0,\zeta^\ell)$, $N\in\mathbb{N}$ *and* $\delta>0$,

$$\Gamma_{\varepsilon,N,\delta}\subseteq\bigcap_{\substack{\delta_1\in(0,\delta)\\ \delta_2\in(0,\varepsilon/6N^2)}}\left\{\sup_{\substack{(t,x)\in[0,T)\times\mathbb{R}^n\\ (t',x')\in(-\infty,T)\times\mathbb{R}^n}}\Psi_{\delta_1,\delta_2}(t,x,t',x')\geqslant\zeta^\ell-2\varepsilon\right\}. \tag{3.11}$$

(ii) *For any* $\varepsilon\in(0,\zeta^\ell)$, *there exist* $N^*(\varepsilon)\geqslant 1$ *and* $\delta^*(\varepsilon)=\delta(N^*(\varepsilon))>0$ *such that* $P(\Gamma_{\varepsilon,N^*(\varepsilon),\delta^*(\varepsilon)})>0$.

Proof. (i) Assume $\omega\in\Gamma_{\varepsilon,N,\delta}$. For any $\delta_1\in(0,\delta)$ and $\delta_2\leqslant\varepsilon/6N^2$, we derive from (3.5), (3.6), and (3.7) that

$$\begin{aligned}&\sup_{\substack{(t,x)\in(0,T)\times\mathbb{R}^n\\ (t',x')\in(-\infty,T)\times\mathbb{R}^n}}\Psi_{\delta_1,\delta_2}(\omega,t,x,t',x')\\ &\quad\geqslant\sup_{(t,x)\in K_N}\left\{v_1(\omega,t,x)-v_2(\omega,t-\delta_1,x)-\delta_2|x|^2-\delta_2\,\frac{1}{T-t}\right\}\end{aligned}$$

$$\geqslant \sup_{(t,x)\in K_N}\{v_1(\omega,t,x)-v_2(\omega,t,x)\}-\sup_{(t,x)\in K_N}\{v_2(\omega,t,x)-v_2(\omega,t-\delta_1,x)\}-2\delta_2N^2$$

$$\geqslant \zeta_N(\omega)-\alpha(\omega,\delta,N)-2\delta_2N^2\geqslant\left(\zeta(\omega)-\frac{\varepsilon}{3}\right)-\frac{\varepsilon}{3}-\frac{\varepsilon}{3}\geqslant\zeta(\omega)-\varepsilon\geqslant\zeta^\ell-2\varepsilon,$$

proving (i).
(ii) Since $\bigcup_{N\geqslant 1}\bigcup_{\delta>0}\Gamma_{\varepsilon,N,\delta}=\{\zeta\geqslant\zeta^\ell-\varepsilon,\Theta_T\leqslant\ell\}$; and since the definition of ζ^ℓ implies that $P\{\zeta>\zeta^\ell-\varepsilon,\Theta_T\leqslant\ell\}>0$ for any $\varepsilon\in(0,\zeta^\ell)$. The conclusion follows easily. □

Note that if we denote for each $\varepsilon\in(0,\zeta^\ell)$, $\Gamma^*_\varepsilon\triangleq\Gamma_{\varepsilon,N^*(\varepsilon),\delta^*(\varepsilon)}$, then Lemma 3.3(ii) shows that we need only find $\varepsilon_0\in(0,\zeta^\ell)$ so that the contradiction occurs on $\Gamma^*=\Gamma^*_{\varepsilon_0}$.

4. Properties of functions Φ_{δ_1,δ_2} and Ψ_{δ_1,δ_2}

In this section we study the auxiliary functions Φ_{δ_1,δ_2} and Ψ_{δ_1,δ_2} in detail. Recall the quantities $N^*(\varepsilon)$, $\delta^*(\varepsilon)$, $\varepsilon\in(0,\zeta^\ell)$ defined in Lemma 3.3(ii). We have the following two lemmas.

Lemma 4.1. *Let $\varepsilon>0$ be fixed, and let $N^*=N^*(\varepsilon)$, $\delta^*=\delta^*(\varepsilon)$. Then for all $\delta_1\in(0,\delta^*)$ and $\delta_2\in(0,\varepsilon/6(N^*)^2)$, the process Φ_{δ_1,δ_2} is $\mathbf{F}^B$-adapted and continuous on $[0,T)$. Furthermore, if we define*

$$\hat t\triangleq\inf\{t\in[0,T)\colon \Phi_{\delta_1,\delta_2}\ \text{attains a local maximum at}\ t;\ \Phi_{\delta_1,\delta_2}(t)>\zeta^\ell-2\varepsilon\},\qquad(4.1)$$

then $\hat t$ is an $\mathbf{F}^B$-stopping time; and for any $\delta_2>0$ there exists $\delta^{**}>0$ such that $\Gamma^*_\varepsilon\subseteq\{0<\hat t<T\}$, whenever $\delta_1\leqslant\delta^{**}$.

Proof. The adaptedness of Φ_{δ_1,δ_2} is a consequence of the choice of the auxiliary function $\psi_{\delta_1}(r)$. To see the continuity, note that as the supremum of a family of continuous processes, Φ_{δ_1,δ_2} is lower semicontinuous. Thus it suffices to show that it is upper semicontinuous at any $t\in[0,T)$.

For any $\ell>0$, consider the sets $\Omega_\ell\triangleq\{\omega\colon|\Theta_T(\omega)|\leqslant\ell\}$. Since $\Omega_\ell\uparrow\Omega$, P-a.e., it suffices to show that $\Phi_{\delta_1,\delta_2}(\cdot,\omega)$ is upper semicontinuous for $\omega\in\Omega_\ell$. To this end let $\ell>0$, $\omega\in\Omega_\ell$, and $\eta>0$ be fixed, and define

$$\eta_1=\sqrt{1-\tfrac{1}{2}\exp(-8\ell-2\eta)};\qquad \eta_2=\sqrt{1-\exp(-8\ell-2\eta)}.\qquad(4.2)$$

Let $t\in[0,T)$. For any $\bar t\in[0,T)$ such that $|\bar t-t|\leqslant\delta_1^*\triangleq\delta_1(\eta_1-\eta_2)$, we let $(\bar x,\bar t',\bar x')$ be an $\mathbb{R}^n\times[0,T)\times\mathbb{R}^n$-valued random vector such that

$$\Phi_{\delta_1,\delta_2}(\bar t)\leqslant\Psi_{\delta_1,\delta_2}(\bar t,\bar x,\bar t',\bar x')+\eta.\qquad(4.3)$$

Therefore, the definition of Φ_{δ_1,δ_2} and Ψ_{δ_1,δ_2} (see (3.9) and (3.10)) yields that, for P-a.e. $\omega\in\Omega_\ell$,

$$\begin{aligned}\Phi_{\delta_1,\delta_2}(\bar t)-\Phi_{\delta_1,\delta_2}(t)&\leqslant\Psi_{\delta_1,\delta_2}(\bar t,\bar x,\bar t',\bar x')-\Psi_{\delta_1,\delta_2}(t,\bar x,\bar t',\bar x')+\eta\\&\leqslant[v_1(\bar t,\bar x)-v_1(t,\bar x)]-\tfrac{1}{2}(\psi_{\delta_1}(\bar t-\bar t')-\psi_{\delta_1}(t-\bar t'))-\frac{\delta_2}{2}\left(\frac{1}{T-\bar t}-\frac{1}{T-t}\right).\end{aligned}\qquad(4.4)$$

Since v_1 and v_2 both have continuous paths, it is readily seen from (4.4) that

$$\overline{\lim_{\bar{t}\to t}}\, \Phi_{\delta_1,\delta_2}(\bar{t},\omega) \leqslant \Phi_{\delta_1,\delta_2}(t,\omega) - \underline{\lim}_{\bar{t}\to t}\, \tfrac{1}{2}(\psi_{\delta_1}(\bar{t}-\bar{t}') - \psi_{\delta_1}(t-\bar{t}')).$$

Thus it suffices to show that

$$\underline{\lim}_{\bar{t}\to t}\, \tfrac{1}{2}(\psi_{\delta_1}(\bar{t}-\bar{t}') - \psi_{\delta_1}(t-\bar{t}')) \geqslant 0. \tag{4.5}$$

To see this we observe that

$$\Phi_{\delta_1,\delta_2}(\bar{t}) \geqslant v_1(\bar{t},0) - v_2(\bar{t}-\delta_1,0) - \frac{\delta_2}{2}\frac{1}{T-\bar{t}} \geqslant -2\ell - \frac{\delta_2}{2}\frac{1}{T-\bar{t}}.$$

Therefore

$$\begin{aligned}
&\frac{\delta_2}{2}(|\bar{x}|^2 + |\bar{x}'|^2) + \frac{1}{2}\psi_{\delta_1}(\bar{t}-\bar{t}') \\
&\quad = \left\{ v_1(\bar{t},\bar{x}) - v_2(\bar{t}',\bar{x}') - \frac{1}{2\delta_1}|\bar{x}-\bar{x}'|^2 - \frac{\delta_2}{2}\frac{1}{T-\bar{t}} \right\} - \Psi_{\delta_1,\delta_2}(\bar{t},\bar{x},\bar{t}',\bar{x}') \\
&\quad \leqslant \left(2\ell - \frac{\delta_2}{2}\frac{1}{T-\bar{t}} \right) - \Phi_{\delta_1,\delta_2}(\bar{t}) + \eta \leqslant 4\ell + \eta.
\end{aligned} \tag{4.6}$$

Consequently we have $\psi_{\delta_1}(\bar{t}-\bar{t}') \leqslant 8\ell + 2\eta$. Now recall the definitions of ψ_{δ_1} (3.8), and η_1, η_2 (4.2), as well as δ_1^*, we see that

$$|\delta_1 - (\bar{t}-\bar{t}')| \leqslant \delta_1\eta_2 \quad \text{whenever } \bar{t}\in[0,T), |\bar{t}-t| \leqslant \delta_1^*,$$

and hence

$$|\delta_1 - (t-\bar{t}')| \leqslant |\delta_1 - (\bar{t}-\bar{t}')| + |t-\bar{t}| \leqslant \delta_1\eta_2 + \delta_1^* = \delta_1\eta_1.$$

Since $\eta_2 < \eta_1 < 1$, one has $(\bar{t}-\bar{t}'), (t-\bar{t}') \in [\delta_1(1-\eta_1), \delta_1(1+\eta_1)] \subset (0, 2\delta_1)$. Consequently, by definition of ψ_{δ_1} one checks easily that there exists a constant $C_1 > 0$, depending on δ_1, ℓ and η, such that

$$|\psi_{\delta_1}(\bar{t}-\bar{t}') - \psi_{\delta_1}(t-\bar{t}')| \leqslant C_1|\bar{t}-t|, \quad \forall \bar{t}\in[0,T),\ |\bar{t}-t| \leqslant \delta_1^*.$$

Thus (4.5) holds, proving the upper semicontinuity of $\Phi_{\delta_1,\delta_2}(\cdot,\omega)$ at t.

To see that the random time $\hat{t}$ defined by (4.1) is an $\boldsymbol{F}^B$-stopping time, we need only observe that the adaptedness and continuity of Ψ_{δ_1,δ_2} imply the following fact: for any $t\in[0,T]$,

$$\{\hat{t} < t\} = \bigcup_{\substack{0\leqslant r<s\leqslant t \\ r,s\in\boldsymbol{Q}}} \{\zeta^\ell - 2\varepsilon < \Phi_{\delta_1,\delta_2}(r); \Phi_{\delta_1,\delta_2}(r) \geqslant \Phi_{\delta_1,\delta_2}(s)\} \in \mathscr{F}_t^B,$$

where $\boldsymbol{Q}$ is all the rationals in $\mathbb{R}$.

It remains to prove the last assertion of Lemma 4.1, that is, $\Gamma_\varepsilon^* \subseteq \{0 < \hat{t} < T\}$. In light of (3.11) we need only show that

$$\left\{ \sup_{\substack{(t,x,x')\in(0,T)\times(\mathbb{R}^n)^2 \\ t'\in(-\infty,T)}} \Psi_{\delta_1,\delta_2}(t,x,t',x') \geqslant \zeta^\ell - 2\varepsilon \right\} \subseteq \{0 < \hat{t} < T\}, \tag{4.7}$$

whenever $\delta_1 \leqslant \delta^{**}$. However, since Φ_{δ_1,δ_2} is continuous on $[0,T)$ and $\Phi_{\delta_1,\delta_2}(t) \to -\infty$, as $t\to T$, it suffices to show that $\Phi_{\delta_1,\delta_2}(0) \leqslant 0$.

To see this, define the "modulus of continuity" of the (deterministic) function $v_2(0,\cdot)$: $\mathbb{R}^n \to \mathbb{R}$:

$$\omega(r,s) \triangleq \sup\{|v_2(0,x) - v_2(0,x')| : |x|^2 + |x'|^2 \leqslant s, |x-x'|^2 \leqslant r\}, r,s > 0. \tag{4.8}$$

Since $\omega(0,s) = 0\ \forall s > 0$, for any $\delta_2 > 0$ there exists $\delta^{**} > 0$ such that

$$\omega(4\ell\delta_1, 4\ell/\delta_2) < \delta_2/2T \quad \text{whenever } \delta_1 \leqslant \delta^{**}. \tag{4.9}$$

Now recall that $v_2(t',x) = v_2(0,x) \geqslant v_1(0,x)$, and $v_2(0,x) - v_2(0,x') \leqslant 2\ell$. For $t' \leqslant 0$, $x,x' \in \mathbb{R}^n$, and $\delta_1 \in (0,\delta^{**})$ we have

$$\begin{aligned}
\Psi_{\delta_1,\delta_2}(0,x,t',x') &= (v_1(0,x) - v_2(0,x')) - \left(\frac{1}{2\delta_1}|x-x'|^2 + \tfrac{1}{2}\psi_{\delta_1}(-t')\right) \\
&\quad - \frac{\delta_2}{2}(|x|^2 + |x'|^2) - \frac{\delta_2}{2T} \\
&\leqslant (v_2(0,x) - v_2(0,x')) - \frac{1}{2\delta_1}|x-x'|^2 - \frac{\delta_2}{2}(|x|^2 + |x'|^2) - \frac{\delta_2}{2T} \\
&\leqslant \begin{cases} -\dfrac{\delta_2}{2T} < 0, & |x|^2 + |x'|^2 \geqslant \dfrac{4\ell}{\delta_2} \text{ or } |x-x'|^2 \geqslant 4\ell\delta_1; \\ \omega\left(4\ell\delta_1, \dfrac{4\ell}{\delta_2}\right) - \dfrac{\delta_2}{2T} < 0, & |x|^2 + |x'|^2 < \dfrac{4\ell}{\delta_2}, |x-x'|^2 < 4\ell\delta_1, \end{cases}
\end{aligned}$$

thanks to (4.8) and (4.9). Now, taking supremum over $t' \in (-\infty, T)$ and $x,x' \in \mathbb{R}^n$ we have $\Phi_{\delta_1,\delta_2}(0) \leqslant 0$, proving the lemma. □

Lemma 4.2. *For fixed $\delta_1 \in (0, \delta^* \wedge \delta^{**}))$ and $\delta_2 \in (0, \varepsilon/6(N^*)^2)$, where $\varepsilon \in (0, \zeta^\ell/3)$, there exist $\mathscr{F}^B_{\hat{t}}$-measurable random variables $\hat{t}' \in (-\infty, T)$, and $\hat{x}, \hat{x}' \in \mathbb{R}^n$, such that $\hat{t}(\omega) - 2\delta_1 < \hat{t}'(\omega) < \hat{t}(\omega)$, for P-a.e. $\omega \in \Omega$; and $\Phi_{\delta_1,\delta_2}(\hat{t}) = \Psi_{\delta_1,\delta_2}(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$. Furthermore, P-a.e. on $\{0 < \hat{t} < T\}$, it holds that*
(i) *Ψ_{δ_1,δ_2} attains at a local maximum at $(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$.*
(ii) *$\Psi_{\delta_1,\delta_2}(t,x,t',x') \leqslant \Psi_{\delta_1,\delta_2}(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$, $\forall (t,x,t',x') \in [0,\hat{t}] \times \mathbb{R}^n \times [0,T) \times \mathbb{R}^n$*

Proof. Since $\Phi_{\delta_1,\delta_2}(\hat{t}) \geqslant \zeta^\ell - 2\varepsilon$, except for a P-null set, for any $\omega \in \Omega$ we should have

$$\Phi_{\delta_1,\delta_2}(\hat{t}(\omega)) = \sup_{(t',x,x') \in \mathscr{O}(\omega)} \Psi_{\delta_1,\delta_2}(\omega, \hat{t}(\omega), x, t', x'),$$

where $\mathscr{O}(\omega) \triangleq \{(t',x,x') \in (-\infty,T) \times \mathbb{R}^n \times \mathbb{R}^n | \Psi_{\delta_1,\delta_2}(\omega, \hat{t}(\omega), x, t', x') > \zeta^\ell - 3\varepsilon\}$. Further, note that for $\omega \in \Gamma^*_\varepsilon$ and $(t',x,x') \in \mathscr{O}(\omega)$ one has

$$\begin{aligned}
\zeta^\ell - 3\varepsilon < \Psi_{\delta_1,\delta_2}(\omega, \hat{t}(\omega), x, t', x') &= v_1(\omega, \hat{t}(\omega), x) - v_2(\omega, t', x') \\
&\quad - \left\{\frac{1}{2\delta_1}|x-x'|^2 + \frac{1}{2}\psi_{\delta_1}(\hat{t}(\omega) - t') + \frac{\delta_2}{2}(|x|^2 + |x'|^2) + \frac{\delta_2}{2}\frac{1}{T - \hat{t}(\omega)}\right\} \\
&\leqslant 2\ell - \frac{1}{2}\psi_{\delta_1}(\hat{t}(\omega) - t') - \frac{\delta_2}{2}\frac{1}{T - \hat{t}(\omega)} - \frac{\delta_2}{2}(|x|^2 + |x'|^2).
\end{aligned} \tag{4.10}$$

Therefore $2\ell - (\zeta^\ell - 3\varepsilon) > 0$; and on $\mathcal{O}(\omega)$ one has $|x|^2 + |x'|^2 \leqslant (2/\delta_2)[2\ell - (\zeta^\ell - 3\varepsilon)]$. Now by the definition of ψ_δ (3.8), one can check that

$$|t' - (\hat{t}(\omega) - \delta_1)| \leqslant \delta_1 \sqrt{1 - \exp\{-2[2\ell - (\zeta^\ell - 3)]\}} < \delta_1. \tag{4.11}$$

Consequently, $\mathcal{O}(\omega)$ is a bounded set. Thus for each fixed ω we can find $(\hat{t}', \hat{x}, \hat{x}')$ in the closure of $\mathcal{O}(\omega)$ such that (i) and (ii) hold. Moreover, (4.11) implies that $\hat{t}(\omega) - 2\delta_1 < \hat{t}'(\omega) < \hat{t}(\omega)$ must hold. The lemma then follows from the standard measurable selection theorem. □

Remark 4.3. In what follows we call the quadruple $(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$ in Lemma 4.2 an $\boldsymbol{F}^B$-*maximizer* of Ψ_{δ_1,δ_2}.

5. Measurable selections

In this section we try to generalize the Theorem 3.2 of Crandall et al. (1992) to a stochastic setting. Since the result depends heavily on the measurable selection theorem as well as the optional section theorem (of Dellacherie–Meyer), we thus simply name this section *measurable selections*.

Let us first recall some notions of Crandall et al. (1992). For given $u \in C([0,T] \times \mathbb{R}^n)$ and $(\bar{t}, \bar{x}) \in [0,T] \times \mathbb{R}^n$, a triplet $(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ is called a *parabolic superjet* of u at $(\bar{t}, \bar{x})$ if for any (t, x) in a neighborhood of $(\bar{t}, \bar{x})$, it holds that

$$\begin{aligned} u(t,x) &\leqslant u(\bar{t},\bar{x}) + a(t-\bar{t}) + \langle p, x-\bar{x}\rangle + \tfrac{1}{2}\langle X(x-\bar{x}), x-\bar{x}\rangle \\ &\quad + o(|t-\bar{t}|) + o(|x-\bar{x}|^2). \end{aligned} \tag{5.1}$$

We denote the set of all parabolic superjets of u at $(\bar{t}, \bar{x})$ by $\mathcal{P}^{1,2,+}u(\bar{t}, \bar{x})$. The set of parabolic *subjets*, defined in a similar way by reversing the direction of the inequality in (5.1), is denoted by $\mathcal{P}^{1,2,-}u(\bar{t}, \bar{x})$. Further, if u is independent of t, then the set of the second-order superjets of u at $\bar{x}$, denoted by $\mathcal{P}^{2,+}u(\bar{x})$, are those pairs (p, X) such that (5.1) holds with the obvious modifications. The set $\mathcal{P}^{2,-}u(\bar{x})$ is defined likewise. Finally, we denote the closure of $\mathcal{P}^{1,2,+}u(\bar{t}, \bar{x})$ by $\bar{\mathcal{P}}^{1,2,+}u(\bar{t}, \bar{x})$. The closure of the set of other "jets" are defined similarly.

We now introduce another auxiliary function for notational convenience:

$$\varphi(t,x,t',x') \triangleq \frac{1}{2\delta_1}|x-x'|^2 + \frac{1}{2}\psi_{\delta_1}(t-t') + \frac{\delta_2}{2}\left(|x|^2 + |x'|^2 + \frac{1}{T-t}\right), \tag{5.2}$$

for $t, t' \in (0, T)$ with $0 < t - t' < 2\delta$, and $x, x' \in \mathbb{R}^n$. We have

Theorem 5.1. *Assume* (A1)–(A4). *Let $(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$ be an $\boldsymbol{F}^B$-maximizer of Ψ_{δ_1,δ_2} defined in Lemma* 4.2. *Then, for any $\varepsilon, \gamma > 0$, there exist two $\mathcal{F}^B_{\hat{t}}$-measurable $\mathcal{S}^n$-valued random variables $\mathcal{X}, \mathcal{Y}$ such that for P-a.e. $\omega \in \Gamma^*_\varepsilon$, it holds that*

(i) $((D_t, D_x)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}')(\omega), \mathcal{X}(\omega)) \in \bar{\mathcal{P}}^{1,2,+} v_1(\omega, \hat{t}(\omega), \hat{x}(\omega))$;
(ii) $(-(D_s, D_y)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}')(\omega), \mathcal{Y}(\omega)) \in \bar{\mathcal{P}}^{1,2,-} v_2(\omega, \hat{t}'(\omega), \hat{x}'(\omega))$;

(iii) *with*

$$B=\begin{pmatrix} D_{xx}\varphi & D_{xy}\varphi \\ D_{xy}\varphi & D_{yy}\varphi \end{pmatrix}(\hat{t},\hat{x},\hat{t}',\hat{x}')(\omega)=\frac{1}{\delta_1}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}+\delta_2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

it holds that

$$\begin{pmatrix} \mathscr{X}(\omega) & 0 \\ 0 & -\mathscr{Y}(\omega) \end{pmatrix} \leqslant B+\gamma\left\{B^2+\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}. \tag{5.3}$$

Furthermore, there exists a sequence of $\boldsymbol{F}^B$*-stopping times* $\{\hat{t}_m\}$, *a sequence of sets* Γ^m_ε *with* $P\{\Gamma^m_\varepsilon\}>0$ *and* $\underline{\lim}_{m\to\infty}\Gamma^m_\varepsilon=\Gamma^*_\varepsilon$; *and a sequence of* $\mathscr{F}^B_{\hat{t}_m}$*-measurable,* $[0,T]\times(\mathbb{R}^n)^2\times(\mathbb{R}\times\mathbb{R}^n\times\mathscr{S}^{n+1})^2$*-valued random variables* $\{(\hat{t}'_m,\hat{x}_m,\hat{x}'_m,(\hat{a}^m_1,\hat{p}^m_1,\hat{S}^m_1),(\hat{a}^m_2,\hat{p}^m_2,\hat{S}^m_2))\}$, *which enjoys, P-a.e on* Γ^m_ε,

(i′) $(\hat{a}^m_1,\hat{p}^m_1,\hat{S}^m_1)\in\mathscr{P}^{2,+}v_1(\hat{t}_m,\hat{x}_m)$ *and* $(\hat{a}^m_2,\hat{p}^m_2,\hat{S}^m_2)\in\mathscr{P}^{2,-}v_2(\hat{t}'_m,\hat{x}'_m)$;
(ii′) P-a.e. on Γ^*_ε, *it holds that*

$$\lim_{m\to\infty}(\hat{t}_m,\hat{t}'_m,\hat{x}_m,\hat{x}'_m)=(\hat{t},\hat{t}',\hat{x},\hat{x}');$$

$$\lim_{m\to\infty}((\hat{a}^m_1,\hat{p}^m_1),(\hat{a}^m_2,\hat{p}^m_2))=((D_t,D_x)\varphi(\hat{t},\hat{x},\hat{t}',\hat{x}'),-(D_s,D_y)\varphi(\hat{t},\hat{x},\hat{t}',\hat{x}'));$$

(iii′) *There exists a pair of* $\mathscr{S}^{n+1}$*-valued,* $\mathscr{F}_{\hat{t}}$*-measurable random variables* $(\hat{\mathscr{X}},\hat{\mathscr{Y}})$ *such that, for P-a.e.* $\omega\in\Gamma^*_\varepsilon$, $(\hat{\mathscr{X}}(\omega),\hat{\mathscr{Y}}(\omega))$ *is a limit point of* $\{(\hat{S}^m_1(\omega),\hat{S}^m_2(\omega))\}_{m>0}$, *and that*

$$\hat{\mathscr{X}}=\begin{pmatrix} x_{11} & * \\ * & \mathscr{X} \end{pmatrix},\quad \hat{\mathscr{Y}}=\begin{pmatrix} y_{11} & * \\ * & \mathscr{Y} \end{pmatrix}.$$

To prove Theorem 5.1, we first need to generalize the *optional section theorem* of Dellacherie and Meyer (1978) to the space–time random vectors. Let us first recall some notions from Dellacherie and Meyer (1978). First, for any *paved* set $(G,\mathscr{G})$ let us denote all the $\mathscr{G}$-analytic sets by $\mathscr{A}(\mathscr{G})$. Next, denote $\mathscr{O}$ to be the optional σ-field on $\Omega\times[0,T]$, that is, the σ-field generated by all the stochastic intervals $[\![S_1,S_2[\![\triangleq\{(\omega,t)\colon S_1(\omega)\leqslant t<S_2(\omega)\}$, where S_1 and S_2 are $\boldsymbol{F}^B$-stopping times. Now let $(\boldsymbol{X},\mathscr{B}(\boldsymbol{X}))$ be any Borel space. The following generalization of the Dellacherie–Meyer optional selection theorem seems to be new.

Lemma 5.2. *Let* $\Sigma\subseteq\Omega\times[0,T)\times\boldsymbol{X}$ *be an* $\mathscr{O}\otimes\mathscr{B}(\boldsymbol{X})$*-measurable set. Then for any* $\varepsilon>0$, *there exists a mapping* $(\hat{t},\hat{\boldsymbol{x}})\colon\Omega\mapsto[0,T]\times\boldsymbol{X}$, *such that*

(i) $\hat{t}$ *is an* $\boldsymbol{F}^B$*-stopping time and* $\hat{\boldsymbol{x}}$ *is an* $\mathscr{F}^B_{\hat{t}}$*-measurable random variable*;
(ii) *for P-a.e.* $\omega\in\Omega$ *such that* $\hat{t}(\omega)<T$, $(\omega,\hat{t}(\omega),\hat{\boldsymbol{x}}(\omega))\in\Sigma$;
(iii) $P\{\hat{t}<T\}\geqslant P(\pi(\Sigma))-\varepsilon$, *where* $\pi(\Sigma)\triangleq\mathrm{Proj}_\Omega(\Sigma)$.

Proof. Let $\mathscr{I}_0$ be the paving of all the stochastic intervals $[\![S_1,S_2[\![\triangleq\{(\omega,t)\colon S_1(\omega)\leqslant t<S_2(\omega)\}$, where S_1 and S_2 are $\boldsymbol{F}^B$-stopping times such that $0\leqslant S_1\leqslant S_2\leqslant T$, a.s. (i.e., $S_1,S_2\in\mathscr{M}^B_{0,T}$); and $\mathscr{I}$ be the Boolean algebra generated by $\mathscr{I}_0$. Since $\mathscr{I}$ generates the σ-field $\mathscr{O}$, $\Sigma\in\mathscr{A}(\mathscr{I}\times\mathscr{K}(\boldsymbol{X}))$, where $\mathscr{K}(\boldsymbol{X})$ is any compact paving of $\boldsymbol{X}$ that generates the Borel field $\mathscr{B}(\boldsymbol{X})$. Now by the Jankov–von Neumann Theorem (cf., e.g.,

Bertsekas and Shreve, 1978) or Theorem III.9 of Dellacherie and Meyer (1978), the set $\Pi(\Sigma) \triangleq \mathrm{Proj}_{\Omega\times[0,T]}(\Sigma)$ is $\mathscr{I}$-analytic; and there exists an $\mathscr{I}$-analytically measurable mapping $\Theta : \Pi(\Sigma) \mapsto \boldsymbol{X}$, such that $(\omega, t, \Theta(\omega, t)) \in \Sigma$, for all $(\omega, t) \in \Pi(\Sigma)$.

Next, recall that for any subset $A \subseteq \Omega \times [0,T]$, the *debut* of A is defined by

$$D_A(\omega) = \inf\{t \in [0,T): (\omega,t) \in A\} \quad \forall \omega \in \Omega, \tag{5.4}$$

with the convention that $D_A(\omega) = T$, if $\{\cdots\} = \emptyset$. Since $\mathscr{A}(\mathscr{I}) \subseteq \mathscr{A}(\mathscr{F}_T^B \otimes \mathscr{B}([0,T))$, $\Pi(\Sigma)$ is $\mathscr{F}_T^B \times \mathscr{B}([0,T))$-analytic. Thus by Theorem III.44 of Dellacherie and Meyer (1978) (replacing $\mathbb{R}_+$ there by $[0,T)$), we know that the debut of $\Pi(\Sigma)$, $D_{\Pi(\Sigma)}$, is $\mathscr{F}_T^B$-measurable; and that there exists an $\mathscr{F}_T^B$-measurable random variable θ: $\Omega \mapsto [0,T]$ such that $(\omega, \theta(\omega)) \in \Pi(\Sigma)$, whenever $\theta(\omega) < T$; and $P(\theta < T) = P(D_{\Pi(\Sigma)} < T)$. Since $\Pi(\Sigma) \subseteq \Omega \times [0,T)$ implies that $\{D_{\Pi(\Sigma)} < T\} = \pi(\Sigma)$, we further have

$$P(\theta < T) = P(\pi(\Sigma)). \tag{5.5}$$

We now follow the arguments of Dellacherie and Meyer (1978, Theorem IV.76 or Theorem IV.84) to find the desired stopping time $\hat{t}$. First, we define a measure μ on $\Omega \times [0,T]$ by

$$\mu(A) = \int \mathbf{1}_A(\omega, \theta(\omega))\mathbf{1}_{\{\theta<T\}}(\omega)P(\mathrm{d}\omega), \quad A \in \mathscr{F}_T^B \otimes \mathscr{B}([0,T]). \tag{5.6}$$

Clearly, if $[\![\theta]\!] \triangleq \{(\omega,t): t \in [0,T], t = \theta(\omega)\}$ denotes the *graph* of θ, then the previous argument shows that (noting (5.5))

$$\mu([\![\theta]\!]) = \mu(\Pi(\Sigma)) = P(\theta < T) = P(\pi(\Sigma)). \tag{5.7}$$

Recall that $\Pi(\Sigma)$ is in fact $\mathscr{I}$-analytic, so by viewing μ as a capacity on $\Omega \times [0,T]$ and applying Choquet's theorem (cf. Dellacherie and Meyer, 1978, Theorem III.28), we know that $\Pi(\Sigma)$ is μ-capacitable and, for any $\varepsilon > 0$, there exists a set $B_\varepsilon \in \mathscr{I}_\delta$, such that $B_\varepsilon \subseteq \Pi(\Sigma)$ and that $\mu(B_\varepsilon) \geqslant \mu(\Pi(\Sigma)) - \varepsilon = P(\pi(\Sigma)) - \varepsilon$, thanks to (5.7).

Next, let $\hat{t} = D_{B_\varepsilon}$, the debut of the set B_ε (see (5.4)). Since $B_\varepsilon \in \mathscr{I}_\delta \subseteq \sigma(\mathscr{I}) = \mathscr{O}$, and $B_\varepsilon \subseteq \Pi(\Sigma) \subseteq \Omega \times [0,T)$, Theorem IV.50 of Dellacherie and Meyer (1978) then tells us that $\hat{t}$ is an $\boldsymbol{F}^B$-stopping time such that

$$P(\hat{t} < T) = P(\pi(B_\varepsilon)), \tag{5.8}$$

where $\pi(B_\varepsilon) = \mathrm{Proj}_\Omega(B_\varepsilon)$. Furthermore, since $\boldsymbol{F}^B$ is a Brownian filtration, any stopping time is predictable. Thus following the arguments of Dellacherie and Meyer (1978, Theorem IV.84) we see that, for $B_\varepsilon \in \mathscr{I}_\delta$, one must have $(\omega, \hat{t}(\omega)) \in B_\varepsilon$, whenever $\hat{t}(\omega) < T$. We claim that $\hat{t}$ satisfies part (iii) of the lemma, i.e., $P(\hat{t} < T) \geqslant P(\pi(\Sigma)) - \varepsilon$.

Indeed, since $B_\varepsilon \subseteq \pi(B_\varepsilon) \times [0,T]$ and $\mathbf{1}_{\pi(B_\varepsilon)\times[0,T]}(\omega, \theta(\omega)) = \mathbf{1}_{\pi(B_\varepsilon)}(\omega)$ $\forall \omega \in \Omega$, we deduce from (5.6) that

$$\mu(B_\varepsilon) \leqslant \mu(\pi(B_\varepsilon) \times [0,T]) = P(\pi(B_\varepsilon) \cap \{\theta < T\}) \leqslant P(\pi(B_\varepsilon)). \tag{5.9}$$

Combining (5.7)–(5.9) and recalling the definition of B_ε we obtain that

$$P(\hat{t} < T) \geqslant \mu(B_\varepsilon) \geqslant \mu(\Pi(\Sigma)) - \varepsilon = P(\theta < T) - \varepsilon = P(\pi(\Sigma)) - \varepsilon,$$

proving (iii).

It remains to construct the random vector $\hat{\boldsymbol{x}}$ that verifies (i) and (ii) of the lemma. The difficulty here is that the intermediate mapping $\Theta : \Pi(\Sigma) \mapsto \boldsymbol{X}$ is only analytically measurable; thus the composition $\omega \mapsto \Theta(\omega, \hat{t}(\omega))$ is analytically measurable at best, much less an $\mathscr{F}^B_{\hat{t}}$-measurable random variable as desired. Therefore more careful consideration is needed here, and we argue as follows.

First let us define another measure on $\Omega \times [0, T]$, similar to μ defined by (5.6):

$$\mu_1(A) = \int \mathbf{1}_A(\omega, \hat{t}(\omega)) \mathbf{1}_{\{\hat{t}<T\}}(\omega) P(\mathrm{d}\omega), \quad A \in \mathscr{F}^B_T \otimes \mathscr{B}([0,T]).$$

Then μ_1 is carried by the set B_ε. Since Θ is $\mathscr{I}$-analytically measurable, it is $\mathscr{O}$-analytically measurable, whence $\mathscr{O}$-universally measurable. Thus for the measure μ_1 there exists an $\mathscr{O}$-measurable function, denoted by Θ_1, such that $\mu_1(\{\Theta_1 \neq \Theta\}) = 0$. Now by definition of μ_1 we see that this means for P-a.e. ω such that $\hat{t}(\omega) < T$, one must have $\Theta_1(\omega, \hat{t}(\omega)) = \Theta(\omega, \hat{t}(\omega))$. Consequently, if we define $\hat{\boldsymbol{x}}(\omega) = \Theta_1(\omega, \hat{t}(\omega))\ \forall \omega \in \Omega$ then $\hat{\boldsymbol{x}}$ is $\mathscr{F}^B_{\hat{t}}$-measurable; and P-a.e. on $\{\hat{t} < T\}$,

$$(\omega, \hat{t}(\omega), \hat{\boldsymbol{x}}(\omega)) = (\omega, \hat{t}(\omega), \Theta_1(\omega, \hat{t}(\omega))) = (\omega, \hat{t}(\omega), \Theta(\omega, \hat{t}(\omega))) \in \Sigma,$$

proving (i) and (ii), whence the lemma. □

Proof of Theorem 5.1. Let $(\hat{t}, \hat{x}, \hat{t}', \hat{x}')$ be an $\boldsymbol{F}^B$-maximizer of the random field Ψ_{δ_1,δ_2}, that is, $\hat{t}$ is an $\boldsymbol{F}^B$-stopping time, and $(\hat{x}, \hat{t}', \hat{x}')$ is $\mathscr{F}^B_{\hat{t}}$-measurable. For each fixed $\omega \in \Gamma^*_\varepsilon (\subseteq \{0 < \hat{t} < T\})$ we can apply Theorem 3.2 of Crandall et al. (1992) to find matrices $\bar{\mathscr{X}}^\omega = (x_{ij}), \bar{\mathscr{Y}}^\omega = (y_{ij}) \in \mathscr{S}^{n+1}$ such that,

$$(\hat{a}_1(\omega), \hat{p}_1(\omega), \bar{\mathscr{X}}^\omega) \in \bar{\mathscr{P}}^{2,+} v_1(\omega, \hat{t}(\omega), \hat{x}(\omega));$$

$$(\hat{a}_2(\omega), \hat{p}_2(\omega), \bar{\mathscr{Y}}^\omega) \in \bar{\mathscr{P}}^{2,-} v_2(\omega, \hat{t}'(\omega), \hat{x}'(\omega)), \tag{5.10}$$

and that

$$-\left(\frac{1}{\gamma} + |\bar{B}|\right) I_{2(n+1)} \leqslant \begin{pmatrix} \bar{\mathscr{X}}^\omega & 0 \\ 0 & -\bar{\mathscr{Y}}^\omega \end{pmatrix} \leqslant \bar{B} + \gamma \bar{B}^2, \tag{5.11}$$

where (and in the sequel) I_k denotes the $k \times k$ identity matrix and

$$(\hat{a}_1, \hat{p}_1) = (D_t, D_x)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}'); \quad (\hat{a}_2, \hat{p}_2) = -(D_s, D_y)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}');$$

$$\bar{B} = (D_t, D_x, D_s, D_y) \otimes (D_t, D_x, D_s, D_y)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}')(\omega). \tag{5.12}$$

Using definition (5.2) one shows easily that, on the set Γ^*_ε, $|\bar{B}|$ is bounded by some constant $C_{\delta_1,\delta_2} > 0$. In order to translate our argument to the matrix B $(=(D_x, D_y) \otimes (D_x, D_y))$, let us introduce the following matrices:

$$\Lambda \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}; \qquad \tilde{B}_{\delta_1,\delta_2} \triangleq \begin{pmatrix} C_{\delta_1,\delta_2} I_2 & 0 \\ 0 & B \end{pmatrix}. \tag{5.13}$$

Since $\Lambda(D_t, D_x, D_s, D_y)^{\mathrm{T}} = (D_t, D_s, D_x, D_y)^{\mathrm{T}}$, and $\Lambda\Lambda^{\mathrm{T}} = I_{2(n+1)}$, it can be checked that

$$\bar{B} + \gamma \bar{B}^2 \leqslant \Lambda(\tilde{B}_{\delta_1,\delta_2} + \gamma \tilde{B}^2_{\delta_1,\delta_2})\Lambda^{\mathrm{T}}.$$

Consequently, (5.11) yields

$$-\left(\frac{1}{\gamma}+|\bar{B}|\right)I_{2(n+1)}\leqslant\begin{pmatrix}\bar{\mathscr{X}}^{\omega} & 0\\ 0 & -\bar{\mathscr{Y}}^{\omega}\end{pmatrix}\leqslant\Lambda(\tilde{B}_{\delta_1,\delta_2}+\gamma\tilde{B}^2_{\delta_1,\delta_2})\Lambda^{\mathrm{T}}. \tag{5.14}$$

Further, by definitions of $\bar{\mathscr{P}}^{2,+}$ and $\bar{\mathscr{P}}^{2,-}$ we see that, for each ω, there exists a sequence $\{(t^\kappa, x^\kappa, a_1^\kappa, p_1^\kappa, S_1^\kappa),(t'^\kappa, x'^\kappa, a_2^\kappa, p_2^\kappa, S_2^\kappa)\}_{\kappa\geqslant 1}$ such that

$$\begin{aligned}&(a_1^\kappa, p_1^\kappa, S_1^\kappa)\in\mathscr{P}^{2,+}v_1(\omega,t^\kappa,x^\kappa);(a_2^\kappa, p_2^\kappa, S_2^\kappa)\in\mathscr{P}^{2,-}v_1(\omega,t'^\kappa,x'^\kappa);\\&(t^\kappa, x^\kappa, a_1^\kappa, p_1^\kappa, S_1^\kappa)\to(\hat{t}(\omega),\hat{x}(\omega),\hat{a}_1(\omega),\hat{p}_1(\omega),\bar{\mathscr{X}}^{\omega}),\\&(t'^\kappa, x'^\kappa, a_2^\kappa, p_2^\kappa, S_2^\kappa)\to(\hat{t}'(\omega),\hat{x}'(\omega),\hat{a}_2(\omega),\hat{p}_2(\omega),\bar{\mathscr{Y}}^{\omega}),\quad\text{as }\kappa\to\infty.\end{aligned} \tag{5.15}$$

In light of (4.11) and (5.14) we may assume without loss of generality that $t'^\kappa<t^\kappa$, $\forall\kappa\geqslant 1$; and there is some integer $K(\omega,\gamma)\geqslant 1$ such that whenever $\kappa\geqslant K(\omega,\gamma)$, one has

$$-\left(\frac{2}{\gamma}+C_{\delta_1,\delta_2}\right)I_{2(n+1)}\leqslant\begin{pmatrix}S_1^\kappa & 0\\ 0 & -S_2^\kappa\end{pmatrix}\leqslant\Lambda\left(\tilde{B}_{\delta_1,\delta_2}+\gamma(\tilde{B}^2_{\delta_1,\delta_2}+\frac{1}{2}I_{2(n+1)})\right)\Lambda^{\mathrm{T}}. \tag{5.16}$$

To facilitate our measurable selection procedure we now show that the matrices S_1^κ and S_2^κ can actually be chosen as countable valued. Indeed, for each $M\in\mathbb{N}$ and $\gamma\in(0,\infty)$ let us introduce two subsets of $\mathscr{S}^{n+1}$:

$$\mathscr{S}^{n+1}(\boldsymbol{Z}/2^M)\triangleq\{S=(s_{ij})\in\mathscr{S}^{n+1}:\ \text{all } s_{ij}\text{'s are of the form } k/2^M,\ k\in\boldsymbol{Z}\};$$

$$\mathscr{H}_\gamma^{n+1}\triangleq\left\{H=(h_{ij})\in\mathscr{S}^{n+1}:0\leqslant h_{ii}-\sum_{j\neq i}|h_{ij}|\leqslant\sum_{j=1}^{d}|h_{ij}|\leqslant\frac{1}{2}\gamma\ \ \forall i\right\}.$$

Then, it is not hard to prove by elementary algebra that for any given $\gamma>0$, any $S\in\mathscr{S}^{n+1}$ can be decomposed into the form: $S=\hat{S}-H$, where $\hat{S}\in\mathscr{S}^{n+1}(\boldsymbol{Z}/2^M)$ and $H\in\mathscr{H}_\gamma^{n+1}$, for M large enough. We will denote such a decomposition by $S\sim(\hat{S},H)$ in the sequel.

Now for any $\gamma>0$, let $M\in\mathbb{N}$, $\hat{S}_1^\kappa,\hat{S}_2^\kappa\in\mathscr{S}^{n+1}(\boldsymbol{Z}/2^M)$, and $H_1^\kappa,H_1^\kappa\in\mathscr{H}_\gamma^{n+1}$ be such that $S_i^\kappa\sim(\hat{S}_i^\kappa,H_i^\kappa)$, $i=1,2$. It is readily seen that $(a_1^\kappa,p_1^\kappa,\hat{S}_1^k)\in\mathscr{P}^{2,+}v_1(\omega,t^\kappa,x^\kappa)$ and $(a_2^\kappa,p_2^\kappa,\hat{S}_2^k)\in\mathscr{P}^{2,-}v_2(\omega,t'^\kappa,x'^\kappa)$ still hold, whereas (5.16) becomes, with $I=I_{2(n+1)}$,

$$-\left(\frac{1}{2}\gamma+\frac{2}{\gamma}+C_{\delta_1,\delta_2}\right)I\leqslant\begin{pmatrix}\hat{S}_1^\kappa & 0\\ 0 & -\hat{S}_2^\kappa\end{pmatrix}\leqslant\Lambda(\tilde{B}_{\delta_1,\delta_2}+\gamma(\tilde{B}^2_{\delta_1,\delta_2}+I))\Lambda^{\mathrm{T}},\quad\kappa\geqslant\kappa_\gamma^\omega. \tag{5.17}$$

In particular, for some constant $C_{\delta_1,\delta_2}(\gamma)$,

$$|\hat{S}_1^\kappa|,|\hat{S}_2^\kappa|\leqslant C_{\delta_1,\delta_2}(\gamma),\quad\kappa\geqslant\kappa_\gamma^\omega. \tag{5.18}$$

Our task is clear now: from the sequence $(t^\kappa, t'^\kappa, x^\kappa, x'^\kappa,(a_1^\kappa,p_1^\kappa,\hat{S}_1^\kappa),(a_2^\kappa,p_2^\kappa,\hat{S}_2^\kappa))$, $\kappa\geqslant 0$, which depend on ω, we would like to "select" a sequence of random variables that satisfies (5.15)–(5.17), and t^κ's are replaced by $\boldsymbol{F}^B$-stopping times.

To this end we first note that since the filtration $\boldsymbol{F}^B$ is Brownian, all $\boldsymbol{F}^B$-stopping times are predictable. To wit, there exists an increasing sequence of $\boldsymbol{F}^B$-stopping times $\{\tau_\kappa\}$ that announces $\hat{t}$ (i.e., $\tau_\kappa \uparrow \hat{t}$ as $\kappa \to \infty$ and $\tau_\kappa < \hat{t}$ on $\{\hat{t} > 0\}$, $\kappa \geqslant 1$). Since

$$E\left\{ \sup_{\tau_\kappa < t < \hat{t}+1/\kappa} |E\{(\hat{t}',\hat{x},\hat{x}',(\hat{a}_1,\hat{p}_1),(\hat{a}_2,\hat{p}_2))|\mathscr{F}_t^B\} - (\hat{t}',\hat{x},\hat{x}',(\hat{a}_1,\hat{p}_1),(\hat{a}_2,\hat{p}_2))|^2 \right\} \to 0,$$

as $k \to \infty$, one can select a sequence $\{\rho_m\}$ with $0 < \rho_m \leqslant 1/m$, $\forall m \geqslant 1$, such that the sets

$$\Gamma_\varepsilon^m \triangleq \Gamma_\varepsilon^* \cap \left\{ \sup_{\tau_m < t \leqslant \hat{t}+\rho_m} |E\{(\hat{t}',\hat{x},\hat{x}',(\hat{a}_1,\hat{p}_1),(\hat{a}_2,\hat{p}_2))|\mathscr{F}_t^B\} - (\hat{t}',\hat{x},\hat{x}',(\hat{a}_1,\hat{p}_1),(\hat{a}_2,\hat{p}_2))| \leqslant \frac{1}{2m} \right\} \tag{5.19}$$

satisfy that $P(\Gamma_\varepsilon^* \setminus \Gamma_\varepsilon^m) \leqslant 4^{-m} P(\Gamma_\varepsilon^*)$, $m \geqslant 1$.

Next, for each $m \geqslant 1$, let Σ_m be the set of all $(\omega,t,t',x,x',(a_1,p_1,S_1),(a_2,p_2,S_2)) \in \Omega \times [0,T]^2 \times (\mathbb{R}^n)^2 \times (\mathbb{R} \times \mathbb{R}^n \times \mathscr{S}^{n+1}(\boldsymbol{Z}/2^M))^2$ such that the following holds:

$$\tau_m(\omega) < t \leqslant (\hat{t}(\omega) + \rho_m) \wedge T; \quad 0 \leqslant t' < t;$$

$$|(t',x,x',(a_1,p_1),(a_2,p_2)) - E\{(\hat{t}',\hat{x},\hat{x}',(\hat{a}_1,\hat{p}_1),(\hat{a}_2,\hat{p}_2))|\mathscr{F}_t^B\}| \leqslant \frac{1}{m};$$

$$(a_1,p_1,S_1) \in \mathscr{P}^{2,+}v_1(\omega,t,x); \quad (a_2,p_2,S_2) \in \mathscr{P}^{2,-}v_2(\omega,t',x');$$

$$|S_1|,|S_2| \leqslant C_{\delta_1,\delta_2}(\gamma); \quad \begin{pmatrix} S_1 & 0 \\ 0 & -S_2 \end{pmatrix} \leqslant \Lambda(\tilde{B}_{\delta_1,\delta_2} + \gamma(\tilde{B}^2_{\delta_1,\delta_2} + I_{2(n+1)}))\Lambda^{\mathrm{T}}. \tag{5.20}$$

Now let $\mathscr{O}$ be the *optional* σ-field on $\Omega \times [0,T]$. Denote $\boldsymbol{X} \triangleq [0,T] \times (\mathbb{R}^n)^2 \times (\mathbb{R} \times \mathbb{R}^n \times \mathscr{S}^{n+1}(\boldsymbol{Z}/2^M))^2$ with generic element of $\boldsymbol{X}$ being $\boldsymbol{x} = (t',x,x',(a_1,p_1,S_1),(a_2,p_2,S_2))$.

Since $(a_1,p_1,S_1) \in \mathscr{P}^{2,+}v_1(\omega,t,x)$, $(a_2,p_2,S_2) \in \mathscr{P}^{2,-}v_2(\omega,t',x')$ if and only if

$$\overline{\lim_{\substack{s\to 0\\ y\to 0}}} \frac{1}{|s|+|y|^2} \left\{ v_1(\omega,t+s,x+y) - v_1(\omega,t,x) - \langle (a_1,p_1),(s,y)\rangle - \frac{1}{2}\langle S_1(s,y),(s,y)\rangle \right\} \leqslant 0;$$

$$\overline{\lim_{\substack{S\to 0\\ y\to 0}}} \frac{1}{|s|+|y|^2} \left\{ v_2(\omega,t'+s,x'+y) - v_2(\omega,t',x') - \langle (a_2,p_2),(s,y)\rangle - \frac{1}{2}\langle S_2(s,y),(s,y)\rangle \right\} \geqslant 0,$$

from (5.20) we see that Σ_m is $\mathscr{O} \otimes \mathscr{B}(\boldsymbol{X})$-measurable.

Now for each $m \geqslant 1$ we apply Lemma 5.2 to get that, for each $m \geqslant 1$ there exists a mapping $(\hat{t}_m, \hat{\boldsymbol{x}}_m) : \Omega \mapsto [0,T] \times \boldsymbol{X}$, such that

(i) $\hat{t}_m$ is a $\boldsymbol{F}^B$-stopping time and $\hat{\boldsymbol{x}}_m$ is an $\mathscr{F}^B_{\hat{t}_m}$-measurable random variable;
(ii) for P-a.e. $\omega \in \{\hat{t}_m < T\}$, $(\omega, \hat{t}_m(\omega), \hat{\boldsymbol{x}}_m(\omega)) \in \Sigma_m$;
(iii) $P\{\hat{t}_m < T\} \geqslant P(\Gamma_\varepsilon^m) - 1/m$.

Recall now $(\hat{t}'_m, \hat{\boldsymbol{x}}_m) \triangleq (\hat{t}'_m, \hat{x}_m, \hat{x}'_m, (\hat{a}_1^m, \hat{p}_1^m, \hat{S}_1^m), (\hat{a}_2^m, \hat{p}_2^m, \hat{S}_2^m))$, and

$$\Gamma_\varepsilon^* = \varliminf_{m\to+\infty} \Gamma_\varepsilon^m.$$

We deduce from (5.20) that, P-a.e. on Γ_ε^*,

$$(\hat{t}'_m, \hat{x}_m, \hat{x}'_m, (\hat{a}_1^m, \hat{p}_1^m), (\hat{a}_2^m, \hat{p}_2^m)) \to (\hat{t}', \hat{x}, \hat{x}', (\hat{a}_1, \hat{p}_1), (\hat{a}_2, \hat{p}_2)), \quad m \to +\infty. \tag{5.21}$$

Furthermore, if we denote for any $\gamma > 0$, $M \in \mathbb{N}$,

$$\hat{\mathscr{S}}_M^{2(n+1)}(\gamma) \triangleq \{(S_1, S_2) \in (\mathscr{S}^{n+1}(\mathbf{Z}/2^M))^2\colon |S_1|, |S_2| \leqslant C_{\delta_1,\delta_2}(\gamma)\},$$

then $\hat{\mathscr{S}}_M^{2(n+1)}(\gamma)$ is a finite set. Thus, without specifying its cardinality we may label its elements as H_α, $\alpha = 1, 2, \ldots$. For any $\gamma > 0$, we fix $M > 0$ so that the preceding arguments go through. Then, with possible exception only on a P-null set,

$$\Gamma_\varepsilon^* \subseteq \bigcup_{H_\alpha \in \hat{\mathscr{S}}_M^{2(n+1)}(\gamma)} \{(\hat{S}_1^m(\omega), \hat{S}_2^m(\omega)) = H_\alpha\colon \text{ for infinitely many } m \in \mathbb{N}\}.$$

Now for $\alpha = 1, 2, \ldots$ we define $\Omega_\alpha = \{\omega\colon (\hat{\mathscr{S}}_1^m, \hat{\mathscr{S}}_2^m) = H_\alpha\colon \text{ for infinitely many } m \in \mathbb{N}\}$, and $\bar{\Omega}_\alpha = \Omega_\alpha \setminus \bigcup_{l=1}^{\alpha-1} \Omega_l$; and define $(\hat{\mathscr{X}}, \hat{\mathscr{Y}})(\omega) \triangleq \sum_{\alpha=1}^\infty H_\alpha(\omega) \cdot \mathbf{1}_{\bar{\Omega}_\alpha}(\omega)$. Since all $\bar{\Omega}_\alpha$'s are $\mathscr{F}_{\hat{t}}^B$-measurable, $(\hat{\mathscr{X}}, \hat{\mathscr{Y}})$ is an $\mathscr{S}^{n+1} \times \mathscr{S}^{n+1}$-valued $\mathscr{F}_{\hat{t}}^B$-measurable random variable, and P-almost surely, it is a limit point of the sequence $\{(\hat{S}_1^m, \hat{S}_2^m)\}_{m \geqslant 0}$. Further, by its construction, for P-a.e. $\omega \in \Gamma_\varepsilon^*$,

$$((D_t, D_x)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}'), \hat{\mathscr{X}})(\omega) \in \bar{\mathscr{P}}^{2,+} v_1(\omega, \hat{t}(\omega), \hat{x}(\omega)),$$

$$(-(D_s, D_y)\varphi(\hat{t}, \hat{x}, \hat{t}', \hat{x}'), \hat{\mathscr{Y}})(\omega) \in \bar{\mathscr{P}}^{2,-} v_1(\omega, \hat{t}'(\omega), \hat{x}'(\omega)),$$

and, with $\bar{B}$ defined by (5.12) we have

$$-\left(\frac{1}{2}\gamma + \frac{2}{\gamma} + |\bar{B}|\right) I \leqslant \begin{pmatrix} \hat{\mathscr{X}}(\omega) & 0 \\ 0 & -\hat{\mathscr{Y}}(\omega) \end{pmatrix} \leqslant \Lambda(\tilde{B}_{\delta_1,\delta_2} + \gamma(\tilde{B}_{\delta_1,\delta_2}^2 + I))\Lambda^{\mathrm{T}}.$$

Finally, denote the $\mathscr{F}_{\hat{t}}^B$-measurable, $\mathscr{S}^n \times \mathscr{S}^n$-valued random variables $(\mathscr{X}, \mathscr{Y})$ be such that

$$\hat{\mathscr{X}}^\omega = \begin{pmatrix} x_{11} & * \\ * & \mathscr{X}, \end{pmatrix}, \quad \hat{\mathscr{Y}} = \begin{pmatrix} y_{11} & * \\ * & \mathscr{Y} \end{pmatrix}.$$

Then it is easily checked by definitions of super- (sub-)jets that

$$(\hat{a}_1(\omega), \hat{p}_1(\omega), \mathscr{X}(\omega)) \in \bar{\mathscr{P}}^{1,2,+} v_1(\omega, \hat{t}(\omega), \hat{x}(\omega));$$

$$(\hat{a}_2(\omega), \hat{p}_2(\omega), \mathscr{Y}(\omega)) \in \bar{\mathscr{P}}^{1,2,-} v_2(\omega, \hat{t}'(\omega), \hat{x}'(\omega)).$$

Since for any $(x, x') \in (\mathbb{R}^n)^2$ we can write $\bar{x} = (0, x)$, $\bar{x}' = (0, x')$, one has

$$\left\langle \left(B + \gamma B^2 - \begin{pmatrix} \mathscr{X}(\omega) & 0 \\ 0 & -\mathscr{Y}(\omega) \end{pmatrix}\right) \begin{pmatrix} x \\ x' \end{pmatrix}, \begin{pmatrix} x \\ x' \end{pmatrix} \right\rangle$$

$$= \left\langle \left(\bar{B} + \gamma \bar{B}^2 - \begin{pmatrix} \hat{\mathscr{X}}(\omega) & 0 \\ 0 & -\hat{\mathscr{Y}}(\omega)) \end{pmatrix}\right) \begin{pmatrix} \bar{x} \\ \bar{x}', \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} \right\rangle \geqslant 0,$$

the inequality (5.11) remains true with $\bar{\mathscr{X}}^\omega$, $\bar{\mathscr{Y}}^\omega$, and $|\bar{B}|$ being replaced by $\mathscr{X}(\omega)$, $\mathscr{Y}(\omega)$, and C_{δ_1,δ_2}, respectively. The proof is now complete. □

6. Proof of Theorem 3.1

We are now ready to prove Theorem 3.1. To begin with, we first claim that under Assumption (A5) we can assume without loss of generality that there exists a constant $\mu > 0$ such that for all $h > 0$ it holds that

$$\tilde{f}(t,x,y+h,z) - \tilde{f}(t,x,y,z) \leqslant -\mu h \quad \forall (t,x,y,z),\ P\text{-a.e.} \tag{6.1}$$

Indeed, for any $\mu > 0$, define $\mu(t) \triangleq \mu t + \int_0^t \Theta_s \, \mathrm{d}s \ \forall t \in [0,T]$, where Θ is the increasing process in (A5); and denote

$$\hat{f}(t,x,y,z) \triangleq \mathrm{e}^{-\mu(t)} \tilde{f}(t,x,\mathrm{e}^{\mu(t)}y,\mathrm{e}^{\mu(t)}z) - y\frac{\mathrm{d}\mu}{\mathrm{d}t}(t). \tag{6.2}$$

Then it can be shown, thanks to (A5), that $\hat{f}$ satisfies (6.1). Furthermore, assume that v is a stochastic viscosity solution to SPDE($\tilde{f}$,0), which is uniformly stochastically bounded. Define

$$\hat{v}(t,x) \triangleq \mathrm{e}^{-\mu(t)} v(t,x) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n,$$

then one can show that $\hat{v}$ is a stochastic viscosity solution to the SPDE($\hat{f}$,0):

$$-\hat{v}_t(t,x) + \mathscr{L}\hat{v}(t,x) + \hat{f}(t,x,\hat{v}(t,x),\sigma(x)^{\mathrm{T}} D\hat{v}(t,x)) = 0;$$

$$\hat{v}(0,x) = \varphi(x), \tag{6.3}$$

and $\hat{v}$ is uniformly stochastically bounded.

We would like to point out that if (6.1) holds, then the function $\tilde{F}$, defined similar to (1.2) with f being replaced by $\tilde{f}$, satisfies that for all $h > 0$ and $(t,x,y,p,A) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}^n$

$$\tilde{F}(t,x,y+h,p,A) - \tilde{F}(t,x,y,p,A) \leqslant -\mu h, \quad P\text{-a.e..} \tag{6.4}$$

Now recall the set $\Gamma_\varepsilon^* = \Gamma_{\varepsilon,N^*(\varepsilon),\delta^*(\varepsilon)}$ defined at the end of Section 3. We shall prove that a contradiction can be drawn when $\varepsilon > 0$ is small enough.

To this end, let us choose

$$\gamma \triangleq \min\left(\frac{\delta_1}{2(1+\delta_1\delta_2)}, \frac{\delta_2}{(1+\delta_2^2)} \right)$$

and applying Theorem 5.1, we can find for any $\varepsilon > 0$, a sequence $(\hat{t}_m, \hat{x}_m, \hat{t}'_m, \hat{x}'_m, (\hat{a}_1^m, \hat{p}_1^m, \hat{S}_1^m), (\hat{a}_2^m, \hat{p}_2^m, \hat{S}_2^m))$ and a sequence of sets Γ_ε^m satisfying (i′)–(iii′) of Theorem 5.1, i.e., for $\omega \in \Gamma_\varepsilon^m$,

$$\begin{aligned} v_1(\omega,t,x) &\leqslant v_1(\omega,\hat{t}_m(\omega),\hat{x}_m(\omega)) + \hat{a}_1^m(\omega)(t-\hat{t}_m(\omega)) + \langle \hat{p}_1^m(\omega), x-\hat{x}_m(\omega)\rangle \\ &\quad + \tfrac{1}{2}\langle \hat{S}_1^m(\omega)(t-\hat{t}_m(\omega),x-\hat{x}_m(\omega)),(t-\hat{t}_m(\omega),x-\hat{x}_m(\omega))\rangle \\ &\quad + \mathrm{o}(|t-\hat{t}_m(\omega)|^2 + |x-\hat{x}_m(\omega)|^2), \end{aligned} \tag{6.5}$$

as $t \to \hat{t}_m(\omega)$, $x \to \hat{x}_m(\omega)$. Clearly, the set $\hat{\Gamma}_\varepsilon^m \triangleq \{\omega \in \Omega\colon (6.5) \text{ holds}\} \in \mathscr{F}_{\hat{t}_m}^B$. Hence, setting $\hat{t}_m = T$ on $(\hat{\Gamma}_\varepsilon^m)^c$, $\hat{t}_m$ remains an $\boldsymbol{F}^B$-stopping time, and $\Gamma_\varepsilon^m \subseteq \hat{\Gamma}_\varepsilon^m \subseteq \{0 < \hat{t}_m < T\}$. Now, for fixed $m \in \mathbb{N}$ and $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^n$ define

$$\varphi_m(\omega,t,x) \triangleq v_1(\omega,\hat{t}_m(\omega),\hat{x}_m(\omega)) + \langle (\hat{a}_1^m(\omega),\hat{p}_1^m(\omega)),(t-\hat{t}_m(\omega),x-\hat{x}_m(\omega))\rangle$$

$$+\frac{1}{2}\left\langle\left(\hat{S}_1^m(\omega)+\frac{1}{m}I_{2(n+1)}\right)(t-\hat{t}_m(\omega),x-\hat{x}_m(\omega)),\right.$$

$$\left.(t-\hat{t}_m(\omega),x-\hat{x}_m(\omega))\right\rangle.$$

Then $\varphi_m \in C^2(\mathscr{F}^B_{\hat{t}_m};[0,T]\times\mathbb{R}^n)$; and P-a.e. on $\{0<\hat{t}_m<T\}$ it holds that

$$v_1(\omega,t,x)-\varphi_m(\omega,t,x)=\frac{1}{m}(|t-\hat{t}_m(\omega)|^2+|x-\hat{x}_m(\omega)|^2)$$

$$+\mathrm{o}(|t-\hat{t}_m(\omega)|^2+|x-\hat{x}_m(\omega)|^2),$$

as $t\to\hat{t}_m(\omega), x\to\hat{x}_m(\omega)$. Since $\varphi_m(\omega,\hat{t}_m(\omega),\hat{x}_m(\omega))=v_1(\omega,\hat{t}_m(\omega),\hat{x}_m(\omega))$, one has $v_1(\omega,t,x)\geqslant\varphi_m(\omega,t,x)$ for all (t,x) in a neighbourhood of $(\hat{t}_m(\omega),\hat{x}_m(\omega))$, for P-a.e. $\omega\in\{0<\hat{t}_m<T\}$. Therefore, by definition of a stochastic viscosity subsolution (with $g\equiv 0$) we have that

$$\mathscr{A}\varphi_m(\hat{t}_m,\hat{x}_m)+\tilde{f}(\hat{t}_m,\hat{x}_m,\varphi_m(\hat{t}_m,\hat{x}_m),\sigma^*(\hat{x}_m)D\varphi_m(\hat{t}_m,\hat{x}_m))\geqslant D_t\varphi_m(\hat{t}_m,\hat{x}_m),$$

on $\{0<\hat{t}_m<T\}$. That is, P-a.e. on Γ^m_ε,

$$\frac{1}{2}\mathrm{tr}\{\sigma\sigma^*(\hat{x}_m)\hat{S}_1^m\}+\frac{1}{2m}|\sigma(\hat{x}_m)|^2+\langle\beta(\hat{x}_m),\hat{p}_1^m\rangle+\tilde{f}(\hat{t}_m,\hat{x}_m,v_1(\hat{t}_m,\hat{x}_m),\sigma(\hat{x}_m)\hat{p}_1^m)\geqslant\hat{a}_1^m.$$

Now, thanks to Theorem 5.1, for P-a.e. $\omega\in\Gamma^*_\varepsilon=\underline{\lim}_{m\to\infty}\Gamma^m_\varepsilon$, we can let $m\to\infty$ to obtain that

$$\frac{1}{2}\mathrm{tr}\{\sigma\sigma^*(\hat{x})\hat{\mathscr{X}}\}+\langle\beta(\hat{x}),\hat{p}_1\rangle+\tilde{f}(\hat{t},\hat{x},v_1(\hat{t},\hat{x}),\sigma(\hat{x})\hat{p}_1)\geqslant\hat{a}_1,$$

thanks to the continuity of function $\tilde{f}$. Recall the definitions $(\hat{a}_1,\hat{p}_1)$ and $\tilde{F}(t,x,y,p,S)$, we then have

$$\tilde{F}\left(\hat{t},\hat{x},v_1(\hat{t},\hat{x}),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x},\mathscr{X}\right)\geqslant\frac{1}{2\delta_1}D_t\psi_{\delta_1}(\hat{t}-\hat{t}')+\delta_1\frac{1}{(T-\hat{t})^2}. \tag{6.6}$$

Similarly, since v_2 is an ω-wise viscosity super-solution, we derive easily that for any $\omega\in\Gamma^*_\varepsilon$,

$$\tilde{F}\left(\hat{t}',\hat{x}',v_2(\hat{t}',\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}',\mathscr{Y}\right)\leqslant\frac{1}{2\delta_1}D_t\psi_{\delta_1}(\hat{t}-\hat{t}'). \tag{6.7}$$

Combining (6.5)–(6.7), and noting that $\Gamma^*_\varepsilon\subseteq\{0<\hat{t}<T\}$ by Theorem 5.1, we have, P-a.e. on Γ^*_ε,

$$0<\mu\{\zeta^\ell-2\varepsilon\}\leqslant\mu\{v_1(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}')\}$$

$$\leqslant\tilde{F}\left(\hat{t},\hat{x},v_2(\hat{t}',\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x},\mathscr{X}\right)-\tilde{F}\left(\hat{t},\hat{x},v_1(\hat{t},\hat{x}),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x},\mathscr{X}\right)$$

$$\leqslant\tilde{F}\left(\hat{t},\hat{x},v_2(\hat{t}',\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x},\mathscr{X}\right)-\tilde{F}\left(\hat{t}',\hat{x}',v_2(\hat{t}',\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}',\mathscr{Y}\right)$$

$$=\frac{1}{2}\mathrm{tr}\{\sigma(\hat{x})\sigma(\hat{x})^{\mathrm{T}}\mathscr{X}-\sigma(\hat{x}')\sigma(\hat{x}')^{\mathrm{T}}\mathscr{Y}\}$$

$$+\left\{\left\langle\beta(\hat{x}),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x}\right\rangle-\left\langle\beta(\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}'\right\rangle\right\}$$

$$+\left\{\tilde{f}\left(\hat{t},\hat{x},v_2(\hat{t}',\hat{x}'),\left[\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x}\right]\sigma(\hat{x})\right)\right.$$

$$\left.-\tilde{f}\left(\hat{t}',\hat{x}',v_2\left(\hat{t}',\hat{x}',\left[\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}'\right]\sigma(\hat{x}')\right)\right)\right\}$$

$$=I_1+I_2+I_3,$$

where

$$I_1=\mathrm{tr}(\sigma(\hat{x})\sigma(\hat{x})^*\mathscr{X}-\sigma(\hat{x}')\sigma(\hat{x}')^{\mathrm{T}}\mathscr{Y})=\left\langle\begin{pmatrix}\mathscr{X}&0\\0&-\mathscr{Y}\end{pmatrix}\begin{pmatrix}\sigma(\hat{x})\\\sigma(\hat{x}')\end{pmatrix},\begin{pmatrix}\sigma(\hat{x})\\\sigma(\hat{x}')\end{pmatrix}\right\rangle$$

$$\leqslant\frac{2}{\delta_1}|\sigma(\hat{x})-\sigma(\hat{x}')|^2+2\delta_2(|\sigma(\hat{x})|^2+|\sigma(\hat{x}')|^2)$$

$$\leqslant\frac{2L^2}{\delta_1}|\hat{x}-\hat{x}'|^2+\delta_2C_L(1+|\hat{x}|^2+|\hat{x}'|^2);$$

$$I_2=\left(\left\langle\beta(\hat{x}),\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x}\right\rangle-\left\langle\beta(\hat{x}'),\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}'\right\rangle\right)$$

$$\leqslant\frac{L}{\delta_1}|\hat{x}-\hat{x}'|^2+\delta_2c_L(1+|\hat{x}|^2+|\hat{x}'|^2),$$

and

$$I_3=\tilde{f}\left(\hat{t},\hat{x},v_2(\hat{s},\hat{x}'),\left(\frac{1}{\delta_1}(\hat{x}-\hat{x}')+\delta_2\hat{x}\right)\sigma(\hat{x})\right)$$

$$-\tilde{f}\left(\hat{t}',\hat{x}',v_2(\hat{t}',\hat{x}'),\left(\frac{1}{\delta_1}(\hat{x}-\hat{x}')-\delta_2\hat{x}'\right)\sigma(\hat{x}')\right)$$

$$\leqslant L\left(|\hat{t}-\hat{t}'|+|\hat{x}-\hat{x}'|+\frac{1}{\delta_1}|\hat{x}-\hat{x}'|^2\right)+\delta_2C_L(1+|\hat{x}|^2+|\hat{x}'|^2).$$

Since, by definition of ψ_{δ_1} (recall that $\delta_1\leqslant 1$),

$$|\hat{t}-\hat{t}'|\leqslant\delta_1+|(\hat{t}-\hat{t}')-\delta_1|\leqslant\delta_1+\frac{1}{2}\left(\delta_1^2+\left(\frac{\delta_1-(\hat{t}-\hat{t}')}{\delta_1}\right)^2\right)$$

$$\leqslant 2\delta_1-\frac{1}{2}\ln\left(1-\left(\frac{\delta_1-(\hat{t}-\hat{t}')}{\delta_1}\right)^2\right)=2\delta_1+\psi_{\delta_1}(\hat{t}-\hat{t}'),$$

there exists a constant $C_L>0$ depending only on the Lipschitz constant L and $\sigma(0)$, such that, P-a.e on Γ_ε^*,

$$0<\mu(\zeta^\ell-2\varepsilon)\leqslant C_L\left(|\hat{t}-\hat{t}'|+|\hat{x}-\hat{x}'|+\frac{1}{\delta_1}|\hat{x}-\hat{x}'|^2+\delta_2(1+|\hat{x}|^2+|\hat{x}'|^2)\right)$$

$$\leqslant C_L\left(3\delta_1+\psi_{\delta_1}(\hat{t}-\hat{t}')+\frac{2}{\delta_1}|\hat{x}-\hat{x}'|^2+\delta_2(1+|\hat{x}|^2+|\hat{y}|^2)\right).\qquad(6.8)$$

On the other hand, recall the definition of function Ψ_{δ_1,δ_2} (3.9) and the point $(\hat{t},\hat{x},\hat{t}',\hat{x}')$ (4.7), we have

$$\frac{1}{2\delta_1}|\hat{x}-\hat{x}'|^2+\frac{1}{2}\psi_{\delta_1}(\hat{t}-\hat{t}')+\frac{\delta_2}{2}\left(|\hat{x}|^2+|\hat{x}'|^2+\frac{1}{T-\hat{t}}\right)$$
$$=(v_1(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}'))-\Psi_{\delta_1,\delta_2}(\hat{t},\hat{x},\hat{t}',\hat{x}')$$
$$\leqslant([v_1(\hat{t},\hat{x})-v_2(\hat{t},\hat{x})]+[v_2(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}')])-(\zeta^\ell-2\varepsilon). \tag{6.9}$$

Since on Γ_ε^*, it holds that $\Theta_T\leqslant\ell$, by definition (3.3) we have

$$v_1(\omega,\hat{t}(\omega),\hat{x}(\omega))-v_2(\hat{t}(\omega),\hat{x}(\omega))\leqslant\zeta^\ell,\quad \omega\in\Gamma_\varepsilon^*;$$

and (6.9) leads to that

$$\frac{1}{2\delta_1}|\hat{x}-\hat{x}'|^2+\frac{1}{2}\psi_{\delta_1}(\hat{t}-\hat{t}')+\frac{\delta_2}{2}\left(|\hat{x}|^2+|\hat{x}'|^2+\frac{1}{T-\hat{t}}\right)$$
$$\leqslant|v_2(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}')|+2\varepsilon\leqslant2\ell+2\varepsilon. \tag{6.10}$$

Thus, for any fixed $\delta_2\in(0,\varepsilon/6N^2)$ and any $\delta_1\in(0,\delta\wedge\delta^{**})$, we must have

$$|\hat{x}|^2+|\hat{x}'|^2\leqslant\frac{4}{\delta_2}(\ell+\varepsilon),\quad \hat{t}\in\left(0,T-\frac{1}{4\delta_2(\ell+\varepsilon)}\right],$$

and

$$|\hat{x}-\hat{x}'|^2+\frac{(\delta_1-(\hat{t}-\hat{t}'))^2}{\delta_1}\leqslant|\hat{x}-\hat{x}'|^2+\delta_1\psi_{\delta_1}(\hat{t}-\hat{t}')\leqslant4\delta_1(\ell+\varepsilon).$$

Using the pathwise continuity of v_2 we then conclude that $(v_2(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}'))\to0$, as $\delta_1\to0$. Consequently, from the estimates made above we obtain P-a.e. on Γ_ε^*, for $\delta_2\in(0,\varepsilon/6N^2)$,

$$0<\mu(\zeta^\ell-2\varepsilon)\leqslant C_L\left(3\delta_1+\psi_{\delta_1}(\hat{t}-\hat{t}')+\frac{2}{\delta_1}|\hat{x}-\hat{x}'|^2+\delta_2(1+|\hat{x}|^2+|\hat{y}|^2)\right)$$
$$\leqslant 4C_L\left(\delta_1+\delta_2+\left\{\frac{1}{2\delta_1}|\hat{x}-\hat{x}'|^2+\frac{1}{2}\psi_{\delta_1}(\hat{t}-\hat{t}')+\frac{\delta_2}{2}\left(|\hat{x}|^2+|\hat{x}'|^2+\frac{1}{T-\hat{t}}\right)\right\}\right)$$
$$\leqslant 4C_L(\delta_1+\delta_2+2\varepsilon+|v_2(\hat{t},\hat{x})-v_2(\hat{t}',\hat{x}')|)\to4C_L(\delta_2+2\varepsilon)\quad\text{as }\delta_1\to0.$$

Since $P(\Gamma_\varepsilon^*)>0$, for any $\varepsilon>0$ and some $\delta\leqslant\delta(N,\varepsilon)$, $N\geqslant N(\varepsilon)$, and for all $\delta_2<\varepsilon/6N^2$, we deduce from the above estimate that $0<\mu(\zeta^\ell-2\varepsilon)\leqslant8C_L\varepsilon$, $\forall\varepsilon\in(0,\frac{1}{2}\zeta^\ell)$. Letting $\varepsilon\to0$ we obtain a contradiction to the assumption $P\{\zeta>0\}>0$. Thus we must have $v_1(t,x)\leqslant v_2(t,x)$, $\forall(t,x)\in[0,T]\times\mathbb{R}^d$, P-a.e., the proof of Theorem 3.1 is now complete. □

Acknowledgements

We would like to thank the anonymous referee for the careful reading of the manuscript and helpful suggestions. The second author is supported in part by the US ONR grant #N00014-96-1-0262 and NSF grant #DMS-9971720. Part of this work was completed when this author was visiting the Department of Mathematics, University of Bretagne-Occidentale, France, whose hospitality is greatly appreciated.

References

Bertsekas, D.P., Shreve, S.E., 1978. Stochastic Optimal Control, the Discrete Time Case. Academic Press, New York.

Buckdahn, R., Ma, J., 2001. Stochastic viscosity solutions for nonlinear stochastic PDEs (Part I), Stochastic Process. Appl. 93, 181–204.

Crandall, M.G., Ishii, H., Lions, P.L., 1992. User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (NS) 27, 1–67.

Dellacherie, C., Meyer, P., 1978. Probabilities and Potential. North-Holland, Amsterdam.

Lions, P-L., Souganidis, P.E., 1998a. Fully nonlinear stochastic partial differential equations. C.R. Acad. Sci. Paris Sér. 1 326, 1085–1092.

Lions, P-L., Souganidis, P.E., 1998b. Fully nonlinear stochastic partial differential equations: non-smooth equations and applications. C.R. Acad. Sci. Paris Sér. 1 327, 735–741.

Pardoux, E., Peng, S., 1992. Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations, Lecture Notes in Computers and Information Science, Vol. 176, Springer, Berlin, pp. 200–217.

Pardoux, E., Peng, S., 1994. Backward doubly stochastic differential equations and systems of quasilinear SPDEs. Probab. Theory Related Fields 98, 209–227.

Protter, P., 1990. Stochastic Integration and Differential equations, A New Approach. Springer, Berlin.