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Stochastic viscosity solutions for nonlinear stochastic partial differential equations. Part I

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Abstract

This paper, together with the accompanying work (Part II, Stochastic Process. Appl. 93 (2001) 205–228) is an attempt to extend the notion of *viscosity solution* to nonlinear stochastic partial differential equations. We introduce a definition of *stochastic viscosity solution* in the spirit of its deterministic counterpart, with special consideration given to the stochastic integrals. We show that a stochastic PDE can be converted to a PDE with random coefficients via a Doss–Sussmann-type transformation, so that a stochastic viscosity solution can be defined in a "point-wise" manner. Using the recently developed theory on *backward/backward doubly* stochastic differential equations, we prove the existence of the stochastic viscosity solution, and further extend the nonlinear Feynman–Kac formula. Some properties of the stochastic viscosity solution will also be studied in this paper. The uniqueness of the stochastic viscosity solution will be addressed separately in Part II where the relation between the stochastic viscosity solution and the ω-wise, "deterministic" viscosity solution to the PDE with random coefficients will be established.

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1. Introduction

The notion of the *viscosity solution* for a partial differential equation, first introduced in 1983 by Crandall and Lions (1983), has had tremendous impact on the modern theoretical and applied mathematics. Today the theory has become an indispensable tool in many applied fields, especially in optimal control theory and numerous subjects related to it. We refer to the well-known "User's Guide" by Crandall et al. (1992) and the books by Bardi et al. (1997) and Fleming and Soner (1992) for a detailed account for the theory of (deterministic) viscosity solutions.

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Given the importance of the theory, as well as the fact that almost all the deterministic problems in these applied fields have their stochastic counterparts, it has long been desired that the notion of viscosity solution be extended to stochastic partial differential equations; and consistent efforts have been made to prove or disprove such a possibility. The recent articles by Lions and Souganidis (1998a,b) have finally shown an encouraging sign on this subject: in Lions and Souganidis (1998a) the *stochastic viscosity solution* was introduced for the first time; and in Lions and Souganidis (1998b) the applications of such solutions to, among other things, pathwise stochastic control and front propagation and phase transitions in random media were presented. Inspired by the results of Lions and Souganidis (1998a,b) in this paper we consider the following nonlinear stochastic PDE (SPDE), which is slightly different from the one appeared in Lions and Souganidis (1998a,b)

$$\begin{aligned} \mathrm{d}u(t,x) &= \{\mathscr{A}u(t,x) + f(t,x,u(t,x),\sigma^*(x)Du(t,x))\}\mathrm{d}t \\ &\quad + \sum_{i=1}^{d} g_i(t,x,u(t,x)) \circ \mathrm{d}B_t^i, \quad (t,x)\in(0,T)\times\mathbb{R}^n, \\ u(0,x) &= u_0(x), \quad x\in\mathbb{R}^n, \end{aligned} \tag{1.1}$$

where B is a standard d-dimensional Brownian motion; f, g, u_0 are some measurable functions with appropriate dimensions; $\mathscr{A}$ is the second-order differential operator:

$$\mathscr{A} = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{\ell=1}^{k}\sigma_{i\ell}(x)\sigma_{j\ell}(x)\partial^2_{x_ix_j} + \sum_{i=1}^{n}\beta_i(x)\partial_{x_i}; \tag{1.2}$$

in which $\sigma(\cdot)=[\sigma_{ij}(\cdot)]_{i,j=1}^{n,k}$, $\beta=(\beta_1,\ldots,\beta_n)$, and $u_0(\cdot)$ are certain measurable functions; and $\sigma^*(\cdot)$ is the transpose of $\sigma(\cdot)$. Finally, we note that the stochastic differential is in the sense of Stratonovich. Since the functions f and g play the decisive role in our discussion, in the sequel we often refer to (1.1) as SPDE(f,g). It is worth noting that the SPDE (1.1) and the ones considered by Lions and Souganidis (1998a,b) are *nonoverlapping*, that is, they do not contain each other as special cases.

The main difficulty of extending the viscosity solution theory to the stochastic PDEs, in our opinion, can be described as *local vs. global*. This can be seen from two different angles: first, if one wants to use an ω-wise *local* argument, hoping to *translate* the pivotal results from the deterministic theory to the stochastic case ω-*wisely*, then an immediate difficulty would be that almost no stochastic analysis can be used because there is no clear indication as to why all the devices (e.g., sub(super)-differentials) involved will have to have any *global* properties, such as measurabilities, on the variable ω. Second, the characterization of the solution near a certain temporal-spatial point (*local* in another sense) will no longer be appropriate because of the presence of the martingale term, for which some *global* information of the solution path will be needed, by the nature of a stochatic integral. These obstacles can be felt immediately when one tries to even give a sensible definition of a stochastic viscosity solution for which the uniqueness is amendable. For instance, if, in light of the deterministic theory, one would like to define the stochastic viscosity solution by characterizing the behavior of the solution near its local maximum or minimum point, by using a certain *random* *version* of *sub*-(*super*-)*jet* (see Crandall and Lions, 1983) – an important device in the proof of uniqueness – then it would be necessary to assume that such a point, as well as all the *jets*, are ω-dependent. But on the other hand it seems far from clear why these objects will have to possess any measurability in ω, even if one is lucky enough to find that they are well-defined.

One of the key ideas of Lions and Souganidis (1998a,b) is to use the so-called *stochastic characteristics* to *remove* the stochastic integral from the SPDE, so that the stochastic viscosity solution can be studied ω-wisely. Although technically different, our method has the same spirit. To be more precise, our main device in this paper is a nonlinear version of the Doss–Sussmann transformation in the theory of stochastic differential equations (see, for example, Doss (1977); Sussmann (1978) or Karatzas and Shreve, 1988, Chapter 5). In nonlinear filtering theory, such a transformation gives the so-called *Robust form* of a linear (Zakai) SPDE for the unnormalized conditional density (see, e.g., Bensoussan, 1992). More precisely, we show that under such a random transformation, which we call the Doss–Sussmann-type transformation (as well as the robust form) for the obvious reason, the SPDE (1.1) can be converted to an *ordinary* PDE with random coefficients. Although this does not resolve the aforementioned subtle issue on the measurability, we can nevertheless give a sensible definition of the *stochastic viscosity solution*, which will coincide with the deterministic viscosity solution when f is deterministic and $g \equiv 0$ in (1.1).

Having determined the definition, our next goal is naturally to establish the existence and uniqueness of the stochastic viscosity solution to SPDE (1.1). Due to the tedious technical details on both topics, we shall address the existence and uniqueness issues in two separate papers so as to keep each one in a proper length without sacrificing the readability. Thus in this paper we pursue only the existence part, and we leave the uniqueness part independently to Buckdahn and Ma (2001).

The other main observation of this paper is that our stochastic viscosity solution can be constructed from the solution of the so-called *backward doubly SDEs* (BDSDEs, for short) initiated by Pardoux and Peng (1994). Such a relation in a sense could be viewed as an extension of the nonlinear Feynman–Kac formula to stochastic PDEs, which, to our best knowledge, is new. We should note here that while the Doss–Sussmann-type transformation does *remove* the martingale term from SPDE(f, g), the resulting SPDE$(\tilde{f}, 0)$ has some difficulties of its own. The main problem seems to be that the random function $\tilde{f}$ is of quadratic growth with respect to the gradient of the solution (no matter how *nice* the original function f is!), which causes some essential difficulties with a different nature. Finally, we point out that at this point we have not established any relation between our stochastic viscosity solutions and that of Lions and Souganidis (1998a), we hope to be able to address this issue in our future publications.

The rest of this paper is organized as follows. In Section 2 we clarify all the necessary notations, and give the definition of stochastic viscosity solutions. In Section 3 we introduce the Doss–Sussmann-type transformation, and prove that the stochastic viscosity solutions are *transform invariant*. In Section 4 we review the backward doubly stochastic differential equations, and prove a generalized Itô–Ventzell formula which is interesting in its own right. Finally in Section 5 we establish the relation between the BDSDE and the SPDE, from which the existence of the stochastic viscosity solution, the nonlinear Feynman–Kac formula, as well as some basic properties of the stochastic viscosity solutions will follow.

2. Preliminaries and definitions

Let $(\Omega,\mathscr{F},P)$ be a complete probability space on which is defined a d-dimensional Brownian motion $B=(B_t)_{t\geqslant 0}$. Let $\boldsymbol{F}^B \triangleq \{\mathscr{F}_t^B\}_{t\geqslant 0}$ denote the natural filtration generated by B, augmented by the P-null sets of $\mathscr{F}$; and let $\mathscr{F}^B=\mathscr{F}^B_\infty$. By $\mathscr{M}^B_{0,T}$ we will denote all the $\boldsymbol{F}^B$-stopping times τ such that $0\leqslant\tau\leqslant T$, P-a.s., where $T>0$ is some fixed time horizon, and $\mathscr{M}^B_{0,\infty}$ will be all $\boldsymbol{F}^B$-stopping times that are almost surely finite. Throughout this paper we let $\mathbb{E}$ denote a generic Euclidean space; and regardless of its dimension we denote $\langle\cdot,\cdot\rangle$ to be the inner product and $|\cdot|$ the norm in $\mathbb{E}$. In case other Euclidean spaces are needed, we shall label them as $\mathbb{E}_1,\mathbb{E}_2,\ldots$, etc. Further, we denote

- for any sub-σ-field $\mathscr{G}\subseteq\mathscr{F}_T^B$ and real number $p\geqslant 0$, $L^p(\mathscr{G};\mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathscr{G}$-measurable random variables such that $E|\xi|^p<\infty$. When there is no danger of confusion, we often write $\xi\in\mathscr{G}$ whenever $\xi\in L^0(\mathscr{G};\mathbb{E})$ for simplicity;
- for any $q\geqslant 0$, $L^q(\boldsymbol{F}^B,[0,T];\mathbb{E})$ to be all $\mathbb{E}$-valued, $\boldsymbol{F}^B$-progressively measurable processes ψ such that $E\int_0^T|\psi_t|^q\,\mathrm{d}t<\infty$. In particular, $L^0(\boldsymbol{F}^B,[0,T];\mathbb{E})$ denotes all $\mathbb{E}$-valued, $\boldsymbol{F}^B$-progressively measurable processes; and $L^\infty(\boldsymbol{F}^B,[0,T];\mathbb{E})$ denotes those processes in $L^0(\boldsymbol{F}^B,[0,T];\mathbb{E})$ that are uniformly bounded;
- $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ to be the space of all $\mathbb{E}_1$-valued functions defined on $[0,T]\times\mathbb{E}$ which are k-times continuously differentiable in $t\in[0,T]$ and ℓ-times continuously differentiable in $x\in\mathbb{E}$; $C_b^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ to be the subspace of $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ in which all functions have uniformly bounded partial derivatives; and $C_p^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ to be the subspace of $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ in which all the partial derivatives are of at most polynomial growth;
- for any sub-σ-field $\mathscr{G}\subseteq\mathscr{F}_T^B$, $C^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\mathscr{G},[0,T]\times\mathbb{E};\mathbb{E}_1)$) to be the space of all $C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}([0,T]\times\mathbb{E};\mathbb{E}_1)$)-valued random variables that are $\mathscr{G}\otimes\mathscr{B}([0,T]\times\mathbb{E})$-measurable;
- $C^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\boldsymbol{F}^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$) to be the space of all random fields $\varphi\in C^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$, $C_p^{k,\ell}(\mathscr{F}_T^B,[0,T]\times\mathbb{E};\mathbb{E}_1)$), such that for fixed $x\in\mathbb{E}$, the mapping $(t,\omega)\mapsto\varphi(t,x,\omega)$ is $\boldsymbol{F}^B$-progressively measurable.

The following simplification of notations will be frequently used throughout:

- $C^{k,\ell}([0,T]\times\mathbb{E})=C^{k,\ell}([0,T]\times\mathbb{E};\mathbb{R})$,
- $C([0,T]\times\mathbb{E};\mathbb{E}_1)=C^{0,0}([0,T]\times\mathbb{E};\mathbb{E}_1)$,
- $C(\boldsymbol{F}^B,[0,T]\times\mathbb{E})=C^{0,0}(\boldsymbol{F}^B,[0,T]\times\mathbb{E})$.

Furthermore, for $(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}$, we denote $\partial/\partial y=D_y$, $\partial/\partial t=D_t$, $D=D_x=(\partial/\partial x_1,\ldots,\partial/\partial x_n)$, and $D^2=D_{xx}=(\partial^2_{x_ix_j})_{i,j=1}^n$. The meaning of D_{xy}, D_{yy}, etc. should be clear.

Throughout this paper we shall make use of the following *Standing Assumptions*:

(A1) The functions $\sigma:\mathbb{R}^n\to\mathbb{R}^{n\times k}$ and $\beta:\mathbb{R}^n\to\mathbb{R}$ are uniformly Lipschitz continuous, with a common Lipschitz constant $K>0$.

(A2) The function $f:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\mapsto\mathbb{R}$ is a continuous random field such that for fixed (x,y,p), $f(\cdot,\cdot,x,y,\sigma^*(x)p)$ is $\{\mathscr{F}_t^B\}$-progressively measurable; and there exists $K>0$, such that for P-a.e. $\omega\in\Omega$,

$$
\begin{aligned}
&|f(\omega,0,0,0,0)|\leqslant K,\\
&|f(\omega,t,x,y,z)-f(\omega,t',x',y',z')|\leqslant K(|t-t'|+|x-x'|+|y-y'|+|z-z'|);\\
&\quad\forall(t,x,y,z),(t',x',y',z')\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k. \qquad (2.1)
\end{aligned}
$$

(A3) The function $u_0:\mathbb{R}^n\mapsto\mathbb{R}$ is continuous, such that for some constants $K,p>0$,

$$
|u_0(x)|\leqslant K(1+|x|^p),\quad x\in\mathbb{R}^n. \qquad (2.2)
$$

(A4) The function $g\in C_b^{0,2,3}([0,T]\times\mathbb{R}^n\times\mathbb{R};\mathbb{R}^d)$.

Our definition of stochastic viscosity solution will depend heavily on the following stochastic flow $\eta\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n\times\mathbb{R})$, defined as the unique solution of the stochastic differential equation (SDE) in the Stratonovich sense:

$$
\begin{aligned}
\eta(t,x,y)&=y+\sum_{i=1}^d\int_0^t g_i(s,x,\eta(s,x,y))\circ\mathrm{d}B_s^i\\
&\triangleq y+\int_0^t\langle g(s,x,\eta(s,x,y)),\circ\mathrm{d}B_s\rangle,\quad t\geqslant 0. \qquad (2.3)
\end{aligned}
$$

Since $g\in C_b^{0,2,3}([0,T]\times\mathbb{R}^n\times\mathbb{R};\mathbb{R}^d)$ by (A4), applying Itô's formula to $g(t,x,\eta(t,x,y))$ and using the definition of the Stratonovich integral, one shows easily that the Stratonovich SDE (2.3) is equivalent to the following Itô SDE (with parameter):

$$
\eta(t,x,y)=y+\frac{1}{2}\int_0^t\langle g,D_yg\rangle(s,x,\eta(s,x,y))\mathrm{d}s+\int_0^t\langle g(s,x,\eta(s,x,y)),\mathrm{d}B_s\rangle. \qquad (2.4)
$$

We note that under Assumption (A4), as the *flow* of the SDE (2.3), for fixed x the random field $\eta(\cdot,x,\cdot)$ is continuously differentiable in the variable y; and the mapping $y\mapsto\eta(t,x,y,\omega)$ defines a diffeomorphism for all (t,x), P-a.s. (see, e.g., Protter, 1990, Chapter V). Let us denote the y-inverse of $\eta(t,x,y)$ by $\mathscr{E}(t,x,y)$ in the sequel. We claim that $\mathscr{E}(t,x,y)$ is the solution to the following first-order SPDE:

$$
\mathscr{E}(t,x,y)=y-\int_0^t\langle D_y\mathscr{E}(s,x,y)g(s,x,y),\circ\mathrm{d}B_s\rangle,\quad\forall(t,x,y),\ P\text{-a.s.} \qquad (2.5)
$$

Indeed, under Assumption (A4) the *linear* SPDE (2.5) has a unique global solution (cf., e.g., Kunita, 1990, Chapter 6), denote it by $\hat{\mathscr{E}}$. Since the stochastic characteristics of (2.5) are exactly given by the Stratonovich SDE (2.3), we must have

$$
\hat{\mathscr{E}}(t,x,\eta(t,x,y))\equiv\hat{\mathscr{E}}(0,x,\eta(0,x,y))=y\quad\forall(t,x,y),\ P\text{-a.s.}
$$

In other words, $\hat{\mathscr{E}}$ is a y-inverse function of η, thus it must coincide with $\mathscr{E}$.

We now define the notion of *stochastic viscosity solution* for SPDE(f,g) (1.1).

Definition 2.1. A random field $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ is called a stochastic viscosity subsolution (resp. supersolution) of SPDE(f,g), if $u(0,x) \leqslant$ (resp. $\geqslant$) $u_0(x)$, $\forall x \in \mathbb{R}^n$; and if for any $\tau \in \mathscr{M}^B_{0,T}$, $\xi \in L^0(\mathscr{F}^B_\tau; \mathbb{R}^n)$, and any random field $\varphi \in C^{1,2}(\mathscr{F}^B_\tau, [0,T] \times R^n)$ satisfying

$$u(t,x) - \eta(t,x,\varphi(t,x)) \leqslant \text{ (resp. } \geqslant \text{) } 0 = u(\tau,\xi) - \eta(\tau,\xi,\varphi(\tau,\xi)),$$

for all (t,x) in a neighborhood of (τ,ξ), P-a.e. on the set $\{0 < \tau < T\}$, it holds that

$$\mathscr{A}\psi(\tau,\xi) + f(\tau,\xi,\psi(\tau,\xi),\sigma^*(\xi)D\psi(\tau,\xi)) \geqslant \text{(resp. } \leqslant \text{) } D_y\eta(\tau,\xi,\varphi(\tau,\xi))D_t\varphi(\tau,\xi), \tag{2.6}$$

P-a.e. on $\{0 < \tau < T\}$, where $\psi(t,x) \triangleq \eta(t,x,\varphi(t,x))$.

A random field $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ is called a stochastic viscosity solution of SPDE(f,g), if it is both a stochastic viscosity subsolution and a supersolution.

We remark that if in SPDE(f,g) the function $g \equiv 0$, the flow η becomes $\eta(t,x,y) = y$, $\forall (t,x,y)$ and $\psi(t,x) = \varphi(t,x)$. Thus the definition of a stochastic viscosity solution becomes the same as that of a deterministic viscosity solution (see, e.g., Crandall et al., 1992; Fleming and Soner, 1992), for each fixed $\omega \in \{0 < \tau < T\}$, modulo the $\mathscr{F}^B_\tau$-measurability requirement of the test function φ. The following notion of a random viscosity solution will be a bridge linking the stochastic viscosity solution and its deterministic counterpart.

Definition 2.2. A random field $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ is called an ω-wise viscosity (sub-, super-) solution if for P-a.e. $\omega \in \Omega$, $u(\omega,\cdot,\cdot)$ is a (deterministic) viscosity (sub-, super-) solution of the SPDE$(f,0)$.

Remark 2.3. We should note that in the previous definition the random field φ is required to belong to $C^{1,2}(\mathscr{F}^B_\tau, [0,T] \times \mathbb{R}^n)$ instead of $C^{1,2}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$. In other words, φ is not necessarily progressively measurable! However, if we assume that $\varphi \in C^{1,2}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$, and that $g \in C^{0,0,2}([0,T] \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^d)$, then a straightforward computation using the Itô–Ventzell formula shows that the random field $\psi(t,x) = \eta(t,x,\varphi(t,x))$ satisfies

$$\mathrm{d}\psi(t,x) = D_y\eta(t,x,\varphi(t,x))D_t\varphi(t,x)\mathrm{d}t + \langle g(t,x,\psi(t,x)), \circ \mathrm{d}B_t\rangle, \quad t \in [0,T]. \tag{2.7}$$

Since $g(\tau,\xi,\psi(\tau,\xi)) = g(\tau,\xi,u(\tau,\xi))$ by definition, it seems natural to compare $\mathscr{A}\psi(\tau,\xi) + f(\tau,\xi,\psi(\tau,\xi),D\psi(\tau,\xi)\sigma(\xi))$ with $D_y\eta(t,x,\varphi(t,x))D_t\varphi(t,x)$ to characterize a viscosity solution of SPDE(f,g), as we did in (2.6).

3. Doss–Sussmann transformation (the robust form)

In this section we study the Doss–Sussmann transformation, and show that under such a transformation the SPDE(f,g) will be converted to an SPDE$(\tilde{f},0)$, where $\tilde{f}$ is some (progressively measurable) random field.

To begin with, let us recall the stochastic flow $\mathscr{E}(t,x,y) \triangleq [\eta(t,x,.)]^{-1}(y)$ defined in the previous section (that is, η is the solution to the Stratonovich SDE (2.3)). Note that, under Assumption (A4), the random field $\eta \in C^{0,2,2}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n \times \mathbb{R})$, thus so is $\mathscr{E}$. Now for any random field $\psi: [0,T] \times \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$, consider the transformation introduced in Definition 2.1:

$$\varphi(t,x) = \mathscr{E}(t,x,\psi(t,x)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.1}$$

or equivalently, $\psi(t,x) = \eta(t,x,\varphi(t,x))$, $\forall (t,x)$, P-a.s. One can easily check that $\psi \in C^{0,p}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ if and only if $\varphi \in C^{0,p}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$, for $p = 0,1,2$. Moreover, if $\varphi \in C^{0,2}(\mathscr{F}^B, [0,T] \times \mathbb{R}^n)$, then $D_x\psi = D_x\eta + D_y\eta D_x\varphi$; and

$$D_{xx}\psi = D_{xx}\eta + 2(D_{xy}\eta)(D_x\varphi)^* + (D_{yy}\eta)(D_x\varphi)(D_x\varphi)^* + (D_y\eta)(D_{xx}\varphi). \tag{3.2}$$

Furthermore, since $\mathscr{E}(t,x,\eta(t,x,y)) \equiv y$, $\forall (t,x,y)$, P-a.s., differentiating the equation up to the second order we have (suppressing variables), for all (t,x,y) and P-almost surely,

$$\begin{aligned}
&D_x\mathscr{E} + D_y\mathscr{E} D_x\eta = 0, \quad D_y\mathscr{E} D_y\eta = 1,\\
&D_{xx}\mathscr{E} + 2(D_{xy}\mathscr{E})(D_x\eta)^* + (D_{yy}\mathscr{E})(D_x\eta)(D_x\eta)^* + (D_y\mathscr{E})(D_{xx}\eta) = 0,\\
&(D_{xy}\mathscr{E})(D_y\eta) + (D_{yy}\mathscr{E})(D_x\eta)(D_y\eta) + (D_y\mathscr{E})(D_{xy}\eta) = 0,\\
&(D_{yy}\mathscr{E})(D_y\eta)^2 + (D_y\mathscr{E})(D_{yy}\eta) = 0.
\end{aligned} \tag{3.3}$$

We remark that in (3.3) all the partial derivatives of the random field $\mathscr{E}(\cdots)$ should be evaluated at $(t,x,\eta(t,x,y))$; and all those of $\eta(\cdots)$ are evaluated at (t,x,y)!

Now let us define a new random field

$$\begin{aligned}
\tilde{f}(t,x,y,z) \triangleq \frac{1}{D_y\eta(t,x,y)} \Big\{ & f(t,x,\eta(t,x,y), \sigma^*(x)D_x\eta(t,x,y) + D_y\eta(t,x,y)z)\\
& + \mathscr{A}_x\eta(t,x,y) + \langle \sigma^*(x)D_{xy}\eta(t,x,y), z\rangle + \frac{1}{2}D_{yy}\eta(t,x,y)|z|^2 \Big\},
\end{aligned} \tag{3.4}$$

for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k$, P-a.s. Here $\mathscr{A}_x$ is the same as the operator $\mathscr{A}$ in (1.2), with the emphasis that all the partial derivatives are with respect to x. We shall, however, often omit the subscript x in the sequel when there is no danger of confusion. It is clear that $\tilde{f} \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k)$; and a straightforward computation using (3.2) and (3.3) shows that

$$\begin{aligned}
&D_y\mathscr{E}(t,x,\psi(t,x))\{\mathscr{A}\psi(t,x) + f(t,x,\psi(t,x),\sigma^*(x)D\psi(t,x))\}\\
&\quad = \mathscr{A}\varphi(t,x) + \tilde{f}(t,x,\varphi(t,x),\sigma^*(x)D\varphi(t,x)),
\end{aligned} \tag{3.5}$$

for all $(t,x) \in (0,T) \times \mathbb{R}^n$, P-a.e.

We will call SPDE$(\tilde{f},0)$ the robust form of the SPDE(f,g); and the following result tells us why.

Proposition 3.1. *Assume* (A1)–(A4). *A random field u is a stochastic viscosity sub-(resp. super-)solution to SPDE(f,g)* (1.1) *if and only if $v(\cdot,\cdot) = \mathscr{E}(\cdot,\cdot,u(\cdot,\cdot))$ is a stochastic viscosity sub-(resp. super-)solution to SPDE$(\tilde{f},0)$.*

Consequently, u is a stochastic viscosity solution of SPDE(f,g) (1.1) *if and only if $v(\cdot,\cdot) = \mathscr{E}(\cdot,\cdot,u(\cdot,\cdot))$ is a stochastic viscosity solution to SPDE$(\tilde{f},0)$.*

Proof. We need only prove that if $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ is a stochastic viscosity sub-(resp. super-)solution to SPDE(f,g), then $v(\cdot,\cdot) = \mathscr{E}(\cdot,\cdot,u(\cdot,\cdot)) \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$, and it is a stochastic viscosity sub-(resp. super-)solution to SPDE$(\tilde{f},0)$. The remaining part of the proposition can be proved in a very similar way.

To this end we let $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ be a stochastic viscosity subsolution of SPDE(f,g) and let v be defined by (3.1), then $v \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$. In order to show that v is a stochastic viscosity solution of SPDE$(\tilde{f},0)$, we let $\tau \in \mathscr{M}^B_{0,T}$, $\xi \in L^2(\mathscr{F}^B_\tau, \mathbb{R}^n)$ be arbitrarily given, and let $\varphi \in C^{1,2}(\mathscr{F}^B_\tau, [0,T] \times \mathbb{R}^n)$ be such that

$$v(\omega,t,x) - \varphi(\omega,t,x) \leqslant 0 = v(\omega,\tau(\omega),\xi(\omega)) - \varphi(\omega,\tau(\omega),\xi(\omega)),$$

for all (t,x) of some neighborhood $\mathscr{O}(\omega,\tau(\omega),\xi(\omega))$ of $(\tau(\omega),\xi(\omega))$, and for P-a.e $\omega \in \{0 < \tau < T\}$.

Now let $\psi(t,x) = \eta(t,x,\varphi(t,x))$, $\forall (t,x)$, P-a.e. Since $y \mapsto \eta(t,x,y)$ is strictly increasing, we have

$$\begin{aligned} u(t,x) - \psi(t,x) &= \eta(t,x,v(t,x)) - \eta(t,x,\varphi(t,x)) \\ &\leqslant 0 = \eta(\tau,\xi,v(\tau,\xi)) - \eta(\tau,\xi,\varphi(\tau,\xi)) = u(\tau,\xi) - \psi(\tau,\xi), \end{aligned} \tag{3.6}$$

for all $(t,x) \in \mathscr{O}(\tau,\xi)$, P-a.e on $\{0 < \tau < T\}$. Further, since u is a viscosity subsolution of SPDE(f,g), we have, P-a.e. on $\{0 < \tau < T\}$,

$$\mathscr{A}\psi(\tau,\xi) + f(\tau,\xi,\psi(\tau,\xi),D\psi(\tau,\xi)\sigma(\xi)) \geqslant D_y\eta(\tau,\xi,\varphi(\tau,\xi))D_t\varphi(\tau,\xi). \tag{3.7}$$

We thus deduce from (3.3) and (3.5) that

$$\mathscr{A}\varphi(\tau,\xi) + \tilde{f}(\tau,\xi,\varphi(\tau,\xi),\sigma^*(x)D\varphi(\tau,\xi)) \geqslant D_t\varphi(\tau,\xi), \quad P\text{-a.e. on } \{0 < \tau < T\}.$$

That is, v is a stochastic viscosity subsolution of SPDE$(\tilde{f},0)$. □

In the rest of the section we prove a special type of *boundedness* of the random fields η and $\mathscr{E}$, and their derivatives. Such a boundedness will prove to be one of the main characters of our stochastic viscosity solution. We first give the definition.

Definition 3.2. A random field $u \in C(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^n)$ is said to be stochastically uniformly bounded if there exists a positive, increasing process $\Theta \in L^0(\boldsymbol{F}^B, [0,T])$, such that P-almost surely, it holds that

$$|u(t,x)| \leqslant \Theta_t \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{3.8}$$

To study the stochastic boundedness of random fields η and $\mathscr{E}$, we shall impose an extra condition on the functions $g = (g_1,\ldots,g_d)$ which we shall call *compatible condition* in the sequel:

(A4′) g satisfies (A4); and for any $\varepsilon > 0$, there exists a function $G^\varepsilon \in C^{1,2,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R})$, such that

$$\frac{\partial G^\varepsilon}{\partial t}(t,w,x,y) = \varepsilon; \quad \frac{\partial G^\varepsilon}{\partial w^i} = g_i(t,x,G^\varepsilon(t,w,x,y)), \quad i = 1,\ldots,d; \; G^\varepsilon(0,0,x,y) = y. \tag{3.9}$$

Remark 3.3. We note that the existence of the function G^ε cannot be proved by simply solving a Cauchy problem of a system of first-order PDEs. For example, a necessary condition of (3.9) is the following *compatibility* among the g_i's (suppressing variables):

$$\left(\frac{\partial g_j}{\partial y}\right) g_i = \frac{\partial^2 G^\varepsilon}{\partial w^i \partial w^j} = \frac{\partial^2 G^\varepsilon}{\partial w^j \partial w^i} = \left(\frac{\partial g_i}{\partial y}\right) g_j \quad \forall i,j,$$

which is of course not necessarily true in general. However, (A4′) is trivial in the case when $d=1$; and g is independent of t and satisfies (A4). Indeed, in such a case we can choose G to be the solution of the ODE (with parameter x):

$$\frac{\partial G}{\partial w}(w,x,y) = g(x, G(w,x,y)); \quad G(0,x,y) = y, \tag{3.10}$$

and then let $G^\varepsilon(t,w,x,y) = \varepsilon t + G(w,x,y) \quad \forall \varepsilon > 0$.

We have the following result.

Proposition 3.4. *Assume* (A4′). *Let η be the unique solution to SDE* (2.3) *and $\mathscr{E}$ be the y-inverse of η* (*the solution to* (2.5)). *Then there exists a constant $C>0$, depending only on the bound of g and its partial derivatives, such that for $\zeta = \eta, \mathscr{E}$, it holds for all (t,x,y) and P-a.s. that*

$$|\zeta(t,x,y)| \leqslant |y| + C|B_t|,$$

$$|D_x\zeta|, |D_y\zeta|, |D_{xx}\zeta|, |D_{xy}\zeta|, |D_{yy}\zeta| \leqslant C\exp\{C|B_t|\}, \tag{3.11}$$

where all the derivatives are evaluated at (t,x,y).

Consequently, the partial derivatives of the random fields η and $\mathscr{E}$ with respect to x and y, up to the second order, are all stochastically uniformly bounded, with the required increasing process in Definition 3.2 *being $\Theta_t = C\exp\{C|B|_t^*\}$, $t \geqslant 0$, where C is some generic constant and $|B|_t^* \triangleq = \sup_{0\leqslant s\leqslant t}|B_s|$.*

Proof. We first prove the boundedness for η and $\mathscr{E}$. For any $\varepsilon>0$, let G^ε be the function given in (A4′), and define

$$\gamma^\varepsilon(\theta,t,w,x,y) \triangleq G^\varepsilon(t,\theta w,x,y), \quad (\theta,t,w,x,y) \in [0,1]\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}.$$

By a slight abuse of notations we denote $\gamma^\varepsilon(\theta) = \gamma^\varepsilon(\theta,t,w,x,y)$, $\gamma_y^\varepsilon(\theta) = D_y\gamma^\varepsilon(\theta,t,w,x,y)$, and $\gamma_x^\varepsilon(\theta) = D_x\gamma^\varepsilon(\theta,t,w,x,y)$, etc. Further, we denote any constant depending only on g and its derivatives by a generic one $C>0$ which may vary from line to line. Then, using (3.9) it is readily seen that

$$\gamma^\varepsilon(\theta) = y + \int_0^\theta \langle g(t,x,\gamma^\varepsilon(\mu)), w\rangle \,\mathrm{d}\mu, \quad \theta\in[0,1]. \tag{3.12}$$

Thus one must have

$$|G^\varepsilon(t,w,x,y) - y| = |\gamma^\varepsilon(1) - \gamma^\varepsilon(0)| \leqslant C|w| \quad \forall (t,w,x,y). \tag{3.13}$$

Now define a random field $\eta^\varepsilon(t,x,y) \triangleq G^\varepsilon(t,B_t,x,y)$. A simple application of Itô's formula (Stratonovich form), together with (3.9), shows that η^ε satisfies the Stratonovich SDE:

$$\eta^\varepsilon(t,x,y) = y + \varepsilon t + \int_0^t \langle g(s,x,\eta^\varepsilon(s,x,y)), \circ\mathrm{d}B_s\rangle. \tag{3.14}$$

Using the stability of SDEs and the uniqueness of the solutions to SDE (2.3) one shows that $\lim_{\varepsilon\to 0}\eta^\varepsilon(\cdot,x,y)=\eta(\cdot,x,y)$, the solution to SDE (2.3), and the convergence is in the sense of *uniform on compacts in probability* (**ucp** for short, cf. Protter, 1990). Thus replacing w by B_t in (3.13) we have

$$|\eta(t,x,y)-y| = \lim_{\varepsilon\to 0}|G^\varepsilon(t,B_t,x,y)-y| \leqslant C|B_t|, \quad \forall(t,x,y),\ P\text{-a.s.} \tag{3.15}$$

Consequently, using (3.15) we have

$$|\mathscr{E}(t,x,y)-y| = |\mathscr{E}(t,x,y)-\eta(t,x,\mathscr{E}(t,x,y))| \leqslant C|B_t|, \quad \forall(t,x,y),\ P\text{-a.s.} \tag{3.16}$$

This proves that the random fields η and $\mathscr{E}$ are stochastically uniformly bounded.

We now estimate the derivatives of η and $\mathscr{E}$.

(i) $D_y\eta$ and $D_y\mathscr{E}$. We differentiate (3.12) with respect to y. Then it is easily seen that γ^ε_y satisfies the ODE

$$\frac{\mathrm{d}\gamma^\varepsilon_y(\theta)}{\mathrm{d}\theta} = \langle D_y g(t,x,\gamma^\varepsilon(\theta)), w\rangle \gamma^\varepsilon_y(\theta); \quad \gamma^\varepsilon_y(0)=1.$$

By the variation of parameter formula we have $\gamma^\varepsilon_y(1)=\exp\{\int_0^1\langle D_y g(t,x,\gamma^\varepsilon(\theta)),w\rangle\,\mathrm{d}\theta\}$. Thus for some $C>0$,

$$\mathrm{e}^{-C|w|} \leqslant D_y G^\varepsilon(t,w,x,y) = \gamma^\varepsilon_y(1) \leqslant \mathrm{e}^{C|w|}, \quad \forall(t,w,x,y).$$

Note now that $D_y\eta(t,x,y)=\lim_{\varepsilon\to 0}D_yG^\varepsilon(t,B_t,x,y)$, in **ucp**, and $D_y\mathscr{E}=(D_y\eta)^{-1}$, we derive immediately that, for $\zeta=\eta,\mathscr{E}$,

$$\mathrm{e}^{-C|B_t|} \leqslant D_y\zeta(t,x,y) \leqslant \mathrm{e}^{C|B_t|}, \quad \forall(t,x,y),\ P\text{-a.s.} \tag{3.17}$$

(ii) $D_x\eta$ and $D_x\mathscr{E}$: Similar to (i), we now differentiate (3.12) with respect to x to obtain the following (system) of ODEs:

$$\frac{\mathrm{d}\gamma^\varepsilon_x(\theta)}{\mathrm{d}\theta} = D_x g(t,x,\gamma^\varepsilon(\theta))w + \langle D_y g(t,x,\gamma^\varepsilon(\theta)),w\rangle \gamma^\varepsilon_x(\theta),$$

$$\gamma^\varepsilon_x(0)=0.$$

By the variation of parameter formula again one has

$$\gamma^\varepsilon_x(\theta) = \int_0^\theta \exp\left\{\int_\mu^\theta \langle D_y g(t,x,\gamma^\varepsilon(v)),w\rangle\,\mathrm{d}v\right\} D_x g(t,x,\gamma^\varepsilon(\mu))w\,\mathrm{d}\mu. \tag{3.18}$$

Therefore

$$|D_xG^\varepsilon(t,w,x,y)| = |\gamma^\varepsilon_x(1)| \leqslant C|w|\exp\{C|w|\} \leqslant C\exp\{C|w|\}.$$

(Note: the constant C in the last two terms above can be different!). Replacing w by B_t again we have

$$|D_x\eta(t,x,y)| \leqslant C\exp\{C|B_t|\}, \quad (t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R},\ P\text{-a.s.} \tag{3.19}$$

Further, from (3.3) we see that $D_x\mathscr{E}=-(D_y\mathscr{E})(D_x\eta)$, thus combining (3.17) and (3.18) we have

$$|D_x\mathscr{E}(t,x,y)| \leqslant |D_y\mathscr{E}(t,x,y)|C\exp\{C|B_t|\} \leqslant C\exp\{C|B_t|\}, \quad \forall(t,x,y),\ P\text{-a.s.} \tag{3.20}$$

(iii) $D_x^2\eta$, $D_{xy}^2\eta$, and $D_y^2\eta$: The derivation of these estimates are essentially the same. For example, for any $i,j=1,\ldots,n$ we differentiate (3.12) twice to get

$$\begin{aligned}\frac{\mathrm{d}\gamma^{\varepsilon}_{x_ix_j}(\theta)}{\mathrm{d}\theta} &= \langle D^2_{x_ix_j}g(t,x,\gamma^{\varepsilon}(\theta)),w\rangle + \langle D^2_{x_iy}g(t,x,\gamma^{\varepsilon}(\theta)),w\rangle\gamma^{\varepsilon}_{x_j}(\theta)\\ &\quad+\langle D^2_{x_jy}g(t,x,\gamma^{\varepsilon}(\theta)),w\rangle\gamma^{\varepsilon}_{x_i}(\theta)+\langle D_yg(t,x,\gamma^{\varepsilon}(\theta)),w\rangle\gamma^{\varepsilon}_{x_ix_j}(\theta)\\ &\quad+\langle D^2_yg(t,x,\gamma^{\varepsilon}(\theta)),w\rangle\gamma^{\varepsilon}_{x_i}(\theta)\gamma^{\varepsilon}_{x_j}(\theta),\\ \gamma_{x_ix_j}(0)&=0. \end{aligned}\tag{3.21}$$

Note that from (3.18) it is not hard to show that

$$\sup_{0\leqslant\theta\leqslant 1}|\gamma^{\varepsilon}_x(\theta)|\leqslant C\gamma^{\varepsilon}_x(1)\leqslant C|w|\exp\{C|w|\}.$$

Thus from (3.21) we again have, thanks to the variation of parameter formula, that

$$|(D^2G^{\varepsilon})(t,x,y,w)|=|D^2_x\gamma^{\varepsilon}(1)|\leqslant\hat{C}|w|^2\exp\{\hat{C}|w|\}\leqslant C\exp\{C|w|\},$$

which implies that $|D^2_x\eta(t,x,y)|\leqslant C\exp\{C|B_t|\}$, as desired since $\lim_{\varepsilon\to 0}|D^2_x\eta^{\varepsilon}(\cdot,x,y)|=|D^2_x\eta(\cdot,x,y)|$ in **ucp**. The estimates $|D^2\eta(t,x,y)|\leqslant C\exp\{C|B_t|\}$ for $D^2=D^2_{xy}$ and D^2_y can be proved in a similar way.

It remains to prove the estimates for $D^2_x\mathscr{E}$, $D^2_{xy}\mathscr{E}$, and $D^2_y\mathscr{E}$. But this can be done by combining the identities (3.3) and the estimates derived in part (i)–(iii), we leave it to the readers. The proof is now complete. □

Remark 3.5. (i) Proposition 3.4 only shows that the *derivatives* of η and $\mathscr{E}$ are stochastically uniformly bounded but not the random fields themselves (see (3.11)!).

(ii) From the proof of Proposition 3.4 we see that if the function $g\in C_b^{0,k,k+1}(\boldsymbol{F},[0,T]\times\mathbb{R}^n\times\mathbb{R})$ for some $k>1$, and there exists a constant $K>0$ such that all the derivatives of g up to order k are uniformly bounded by K, then the derivatives of η and $\mathscr{E}$ will satisfy

$$|D^{\alpha}\zeta(t,x,y)|\leqslant C|B_t|^{|\alpha|}\exp\{C|B_t|\}\leqslant C\exp\{C|B_t|\},\quad\forall 1\leqslant|\alpha|\leqslant k,$$

where $\zeta=\eta,\mathscr{E}$; $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_n)$ is a multiindex with $D^{\alpha}=D_y^{\alpha_0}D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$ and $|\alpha|=\sum_{i=0}^n\alpha_i$; $C>0$ is some constant depending only on K and k, and is allowed to vary from place to place.

4. A backward doubly SDE (BDSDE)

In this section we introduce the so-called *backward doubly SDE* (BDSDE for short) initiated by Pardoux and Peng (1994). We point out that our version of BDSDE is in fact a time reversal of that considered by Pardoux and Peng (1994), due to the set-up of our problem. We nonetheless use the same name because they have similar nature.

To begin with, let us introduce another complete probability space $(\Omega',\mathscr{F}',P')$ on which is defined a k-dimensional Brownian motion W. We define the following family of σ-fields:

$$\mathscr{F}^W_{t,T}\triangleq\sigma\{W_s-W_T,\ t\leqslant s\leqslant T\}\vee\mathscr{N}',\tag{4.1}$$

where $\mathscr{N}'$ denotes all the P'-null sets in $\mathscr{F}'$. Denote $\boldsymbol{F}_T^W \triangleq \{\mathscr{F}_{t,T}^W\}_{0\leqslant t\leqslant T}$. Next, we consider the product space $(\bar{\Omega},\bar{\mathscr{F}},\bar{P})$ where

$$\bar{\Omega}=\Omega\times\Omega';\quad \bar{\mathscr{F}}=\mathscr{F}\otimes\mathscr{F}';\quad \bar{P}=P\times P'; \tag{4.2}$$

and define $\bar{\mathscr{F}}_t=\mathscr{F}_t^B\otimes\mathscr{F}_{t,T}^W$, for $0\leqslant t\leqslant T$. We should note that $\bar{\boldsymbol{F}}\triangleq\{\bar{\mathscr{F}}_t\}_{0\leqslant t\leqslant T}$ is neither increasing nor decreasing. Therefore it is *not* a filtration! Further, for random variables $\xi(\omega)$, $\omega\in\Omega$, and $\eta(\omega')$, $\omega'\in\Omega'$, we view them as random variables in $\bar{\Omega}$ by the following identification:

$$\xi(\bar{\omega})=\xi(\omega);\qquad \eta(\bar{\omega})=\eta(\omega'),\quad \bar{\omega}\triangleq(\omega,\omega').$$

For $n\geqslant 1$, we let $\mathscr{M}^2(\bar{\boldsymbol{F}},[0,T];\mathbb{R}^n)$ be the set of n-dimensional jointly measurable random processes $h=\{h_t,\ t\in[0,T]\}$ which satisfy

(i) $\bar{E}\{\int_0^T|h_t|^2\,\mathrm{d}t\}<\infty$;

(ii) h_t is $\bar{\mathscr{F}}_t$-measurable, for a.e. $t\in[0,T]$.

Further, we define

$$S^2(\bar{\boldsymbol{F}},[0,T];\mathbb{R}^n)=\left\{h\in C(\bar{\boldsymbol{F}},[0,T];\mathbb{R}^n),\ \text{and}\ \bar{E}\left[\sup_{0\leqslant t\leqslant T}|h_t|^2\right]<+\infty\right\}. \tag{4.3}$$

For $H\in\mathscr{M}^2(\bar{\boldsymbol{F}},[0,T];\mathbb{R}^n)$, we denote the *backward* stochastic integral of H against W^j on an interval $[s,t]$ by $\int_s^t H_r\downarrow\mathrm{d}W_r^j$, $j=1,\dots,k$, and we denote $\int_s^t\langle H_r,\downarrow\mathrm{d}W_r\rangle=\sum_{j=1}^k\int_s^t H_r^j\downarrow\mathrm{d}W_r^j$. Clearly, such an integral can be understood as a Skorohod integral (see, e.g., Nualart and Pardoux, 1988). However, if H is $\boldsymbol{F}^W$-adapted, then we can *reverse* the time and view such an integral as a standard Itô integral from t to s that is adapted to the filtration $\boldsymbol{F}^W$. With such an observation we now consider the following SDE: for $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$X_s^t(x)=x+\int_s^t b(X_r^t(x))\,\mathrm{d}r+\int_s^t\sigma(X_r^t(x))\downarrow\mathrm{d}W_r,\quad 0\leqslant s\leqslant t. \tag{4.4}$$

We note here that due to the direction of the Itô integral, (4.4) should be viewed as going from t to 0 (i.e., $X_0^t(x)$ should be understood as the *terminal value* of the solution X). It is then clear that under standard conditions on the coefficients b and σ, (4.4) will have a strong solution that is $\boldsymbol{F}^W$-adapted for $0\leqslant s\leqslant t$.

The main subject in this section will be the following SDE: for $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$\begin{aligned}Y_s^t(x)&=u_0(X_0^t(x))+\int_0^s f(r,X_r^t(x),Y_r^t(x),Z_r^t(x))\,\mathrm{d}r\\&\quad+\int_0^s\langle g(r,X_r^t(x),Y_r^t(x)),\circ\,\mathrm{d}B_r\rangle-\int_0^s\langle Z_r^t(x),\downarrow\mathrm{d}W_r\rangle,\quad 0\leqslant s\leqslant t,\end{aligned} \tag{4.5}$$

where u_0 satisfies (A.3).

We remark that although the SDE (4.5) looks like a *forward* SDE, it is indeed a *backward* one because a *terminal* condition is given at time $t=0$ ($Y_0^t=u_0(X_0^t(x))$). We remark also that the stochastic integral with respect to $\mathrm{d}B$ is in the Stratonovich form, while that with respect to $\downarrow\mathrm{d}W$ is in an Itô form. The former is used to make the SDE (4.5) compatible to the SPDE(f,g) (1.1), but for the latter, since from the theory of BSDE we know that in general the process $Z^t(x)$ does not have any regularity, much less a semimartingale; hence the Stratonovich integral $\int\langle Z,\downarrow\mathrm{d}W\rangle$ may not even be well-defined. We should note, however, if we rewrite SDE (4.5) in its equivalent Itô form:

$$Y_s = u_0(X_0) + \int_0^s \left[f(r, X_r, Y_r, Z_r) + \frac{1}{2}\langle g, D_y g\rangle(r, X_r, Y_r) \right] \mathrm{d}r - \int_0^s \langle g(r, X_r, Y_r), \mathrm{d}B_r\rangle - \int_0^s \langle Z_r, \downarrow \mathrm{d}W_r\rangle, \quad 0 \leqslant s \leqslant t, \tag{4.6}$$

where $(X, Y, Z) = (X^t(x), Y^t(x), Z^t(x))$, then a change of time $s \mapsto t - s$ shows that (4.6) is essentially the same as the so-called *backward doubly SDE* in the sense of Pardoux and Peng (1994). Therefore we will refer to (4.5) (or (4.6)) as a BDSDE in the sequel. We have the following result:

Lemma 4.1. *Assume* (A1)–(A4). *For each* $(t,x) \in [0, T] \times \mathbb{R}^n$, *the BDSDE* (4.5) *has a unique solution* $(Y^t_\cdot(x), Z^t_\cdot(x)) \in S^2(\bar{\boldsymbol{F}}, [0, t]; \mathbb{R}) \times \mathscr{M}^2(\bar{\boldsymbol{F}}, [0, t]; \mathbb{R}^k)$.

Moreover, let $X^t_\cdot(x)$ *be the solution of* (4.1). *Then*

(i) *For each* $t > 0$, *there exists a version of* $X^t(x) = \{X^t_s(x),\ 0 \leqslant s \leqslant t\}$ *such that* $(s,x) \mapsto X^t_s(x)$ *is locally Hölder-*$C^{\alpha,2\alpha}$, *for some* $\alpha \in (0, 1/2)$;

(ii) *for such a version, it holds that* $X^t_s(x) = X^r_s(X^t_r(x))$, $0 \leqslant s \leqslant r \leqslant t \leqslant T$, $x \in \mathbb{R}^n$; *and*

(iii) *for any* $q \geqslant 2$, *there exists* $M_q > 0$, *such that for* $t \in [0, T]$ *and* $x, x' \in \mathbb{R}^n$,

$$\bar{E}\left\{ \sup_{s \leqslant r \leqslant t} |X^t_r(x) - x|^q \right\} \leqslant M_q (t - s)^{q/2} (1 + |x|^q),$$

$$\bar{E}\left\{ \sup_{s \leqslant r \leqslant t} |(X^t_r(x) - X^t_r(x')) - (x - x')|^q \right\} \leqslant M_q (t - s)^{q/2} (|x - x'|^q); \tag{4.7}$$

(iv) *for any* $0 \leqslant r \leqslant t \leqslant T$ *and* $x \in \mathbb{R}^n$, *one has*

$$Y^t_s(x) = Y^r_s(X^t_r(x)), \qquad Z^t_s(x) = Z^r_s(X^t_r(x)), \quad s \in [0, r],\ \bar{P}\text{-a.s.};$$

(v) *for any* $q \geqslant 2$, *there exists* $C_{p,q}(T) > 0$, *such that for* $(t,x) \in [0, T] \times \mathbb{R}^n$, *it holds that*

$$\bar{E}\left\{ \left[\sup_{0 \leqslant s \leqslant t} |Y^t_s(x)|^2 + \int_0^t |Z^t_s(x)|^2 \, \mathrm{d}s \right]^{q/2} \right\} \leqslant C_{p,q}(T)(1 + |x|^{pq}); \tag{4.8}$$

Proof. The existence and uniqueness of the $\bar{\boldsymbol{F}}$-adapted solution $(Y^t(x), Z^t(x))$ is the direct consequence of Pardoux and Peng (1994) by the time reversal. Conclusions (i) –(iii) are the consequences of the results of Fujiwara and Kunita (1989); and finally the conclusion (iv) and (v) are the analogy of the results in Pardoux and Peng (1994), we omit the proofs. □

Next, we give a generalized version of the Itô–Ventzell formula that combines the generalized Itô formula of Pardoux and Peng (1994) and the Itô–Ventzell formula of Ocone and Pardoux (1989) (regarding all of our stochastic integrals as Skorohod integrals), and will be a main device for our future analysis.

Theorem 4.2 (Generalized Itô–Ventzell Formula). *Suppose that* $F \in C^{0,2}(\bar{\boldsymbol{F}}, [0,T] \times \mathbb{R}^\ell)$ *is a semimartingale with spatial parameter* $x \in \mathbb{R}^\ell$:

$$F(t,x) = F(0,x) + \int_0^t G(s,x)\,\mathrm{d}s + \int_0^t \langle H(s,x),\, \mathrm{d}B_s\rangle + \int_0^t \langle K(s,x),\, \downarrow \mathrm{d}W_s\rangle, \quad t \in [0,T],$$

where $G \in C^{0,2}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^\ell)$, $H \in C^{0,2}(\boldsymbol{F}^B, [0,T] \times \mathbb{R}^\ell; \mathbb{R}^d)$, *and* $K \in C^{0,2}(\boldsymbol{F}^W; [0,T] \times \mathbb{R}^\ell; \mathbb{R}^k)$. *Let* $\alpha \in C(\bar{\boldsymbol{F}}, [0,T]; \mathbb{R}^\ell)$ *be a process of the form*

$$\alpha_t = \alpha_0 + A_t + \int_0^t \gamma_s\,\mathrm{d}B_s + \int_0^t \delta_s \downarrow \mathrm{d}W_s, \quad t \in [0,T],$$

where $\gamma \in \mathscr{M}^2(\bar{\boldsymbol{F}}, [0,T]; \mathbb{R}^{\ell \times d})$, $\delta \in \mathscr{M}^2(\bar{\boldsymbol{F}}, [0,T]; \mathbb{R}^{\ell \times k})$; *and* A *is a continuous,* $\bar{\boldsymbol{F}}$-*adapted process with paths of locally bounded variation.*

Then, P-almost surely, it holds for all $0 \leqslant t \leqslant T$ *that*

$$\begin{aligned} F(t,\alpha_t) &= F(0,\alpha_0) + \int_0^t G(s,\alpha_s)\,\mathrm{d}s + \int_0^t \langle H(s,\alpha_s),\, \mathrm{d}B_s\rangle + \int_0^t \langle K(s,\alpha_s),\, \downarrow \mathrm{d}W_s\rangle \\ &\quad + \int_0^t \langle D_x F(s,\alpha_s),\, \mathrm{d}A_s\rangle + \int_0^t \langle D_x F(s,\alpha_s), \gamma_s\,\mathrm{d}B_s\rangle + \int_0^t \langle D_x F(s,\alpha_s), \delta_s \downarrow \mathrm{d}W_s\rangle \\ &\quad + \frac{1}{2}\int_0^t \mathrm{tr}(D_{xx}F(s,\alpha_s)\gamma_s\gamma_s^*)\,\mathrm{d}s - \frac{1}{2}\int_0^t \mathrm{tr}(D_{xx}F(s,\alpha_s)\delta_s\delta_s^*)\,\mathrm{d}s \\ &\quad + \int_0^t \mathrm{tr}(D_x H(s,\alpha_s)\gamma_s^*)\,\mathrm{d}s - \int_0^t \mathrm{tr}(D_x K(s,\alpha_s)\delta_s^*)\,\mathrm{d}s. \end{aligned} \tag{4.9}$$

Proof. For simplicity let us assume that $F(0,\cdot)=0$, $G=0$ and $A=0$. Let $\phi \in C^\infty(\mathbb{R}^\ell; \mathbb{R}_+)$ with $\mathrm{supp}(\phi) \subseteq B_1^\ell \triangleq \{x \in \mathbb{R}^\ell \colon |x| \leqslant 1\}$, such that $\int_{\mathbb{R}^\ell} \phi(x)\mathrm{d}x = 1$. For $\varepsilon > 0$, define $\phi_\varepsilon(x) \triangleq \varepsilon^{-\ell}\phi(\varepsilon^{-1}x)$ and

$$F_\varepsilon(t,x) \triangleq F(t,\cdot) * \phi_\varepsilon(x) \triangleq \int_{\mathbb{R}^\ell} F(t,\theta)\phi_\varepsilon(x-\theta)\,\mathrm{d}\theta,$$

$$H_\varepsilon(t,x) \triangleq H(t,\cdot) * \phi_\varepsilon(x), \quad K_\varepsilon(t,x) \triangleq K(t,\cdot) * \phi_\varepsilon(x).$$

Applying Lemma 1.3 of Pardoux and Peng (1994) we have for each $\theta \in \mathbb{R}^\ell$ and $t \in [0,T]$ that

$$\begin{aligned} \phi_\varepsilon(\alpha_t - \theta) &= \phi_\varepsilon(\alpha_0 - \theta) + \int_0^t \langle D_x\phi_\varepsilon(\alpha_s - \theta), \gamma_s\,\mathrm{d}B_s\rangle \\ &\quad + \int_0^t \langle D_x\phi_\varepsilon(\alpha_s - \theta), \delta_s \downarrow \mathrm{d}W_s\rangle \\ &\quad + \frac{1}{2}\int_0^t \mathrm{tr}(D_{xx}\phi_\varepsilon(\alpha_s - \theta)\gamma_s\gamma_s^*)\,\mathrm{d}s - \frac{1}{2}\int_0^t \mathrm{tr}(D_{xx}\phi_\varepsilon(\alpha_s - \theta)\delta_s\delta_s^*)\,\mathrm{d}s. \end{aligned}$$

Next, fixing $\theta \in \mathbb{R}^\ell$ and applying Lemma 1.3 of Pardoux and Peng (1994) again to $F(t,\theta)\phi_\varepsilon(\alpha_t-\theta)$ for $0 \leqslant t \leqslant T$ and then integrating with respect to θ we have

$$\begin{aligned}
F_\varepsilon(t,\alpha_t) &= \int_{\mathbb{R}^\ell} F(t,\theta)\phi_\varepsilon(\alpha_t-\theta)\,\mathrm{d}\theta \\
&= \int_{\mathbb{R}^\ell}\left\{\int_0^t \phi_\varepsilon(\alpha_s-\theta)\langle H(s,\theta),\mathrm{d}B_s\rangle + \int_0^t \phi_\varepsilon(\alpha_s-\theta)\langle K(s,\theta),\downarrow\mathrm{d}W_s\rangle\right\}\mathrm{d}\theta \\
&\quad + \int_{\mathbb{R}^\ell}\int_0^t F(s,\theta)\left\{\langle D_x\phi_\varepsilon(\alpha_s-\theta),\gamma_s\,\mathrm{d}B_s\rangle + \frac{1}{2}\mathrm{tr}(D_{xx}\phi_\varepsilon(\alpha_s-\theta)\gamma_s\gamma_s^*)\,\mathrm{d}s\right. \\
&\quad \left. +\langle D_x\phi_\varepsilon(\alpha_s-\theta),\delta_s\downarrow\mathrm{d}W_s\rangle - \frac{1}{2}\mathrm{tr}(D_{xx}\phi_\varepsilon(\alpha_s-\theta)\delta_s\delta_s^*)\,\mathrm{d}s\right\}\mathrm{d}\theta \\
&\quad + \int_{\mathbb{R}^\ell}\left\{\int_0^t \langle H(s,\theta),\gamma_s^*D_x\phi_\varepsilon(\alpha_s-\theta)\rangle\,\mathrm{d}s\right. \\
&\quad \left. - \int_0^t \langle K(s,\theta),\delta_s^*D_x\phi_\varepsilon(\alpha_s-\theta)\rangle\,\mathrm{d}s\right\}\mathrm{d}\theta. \qquad (4.10)
\end{aligned}$$

Clearly, the left side of (4.10) converges to $F(t,\alpha_t)$ P-almost surely, as $\varepsilon \to 0$. We analyze the convergence of the right side. First, applying the Fubini theorem, integration by parts, and the *dominated convergence theorem* one shows that, P-almost surely,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^\ell}\left\{\int_0^t \langle H(s,\theta),\gamma_s^*D_x\phi_\varepsilon(\alpha_s-\theta)\rangle\,\mathrm{d}s\right\}\mathrm{d}\theta \\
&\quad = \lim_{\varepsilon\to 0}\int_0^t\int_{\mathbb{R}^\ell}\mathrm{tr}(D_xH(s,\theta)\gamma_s^*)\phi_\varepsilon(\alpha_s-\theta)\,\mathrm{d}\theta\,\mathrm{d}s = \int_0^t \mathrm{tr}(D_xH(s,\alpha_s)\gamma_s^*)\,\mathrm{d}s, \qquad (4.11)
\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^\ell}\left\{\int_0^t \langle K(s,\theta),\delta_s^*\phi_\varepsilon(\alpha_s-\theta)\rangle\,\mathrm{d}s\right\}\mathrm{d}\theta = \int_0^t \mathrm{tr}(D_xK(s,\alpha_s)\delta_s^*)\,\mathrm{d}s. \qquad (4.12)$$

Next, a simple application of the *dominated convergence theorem* gives that

$$\int_0^T\left|\int_{\mathbb{R}^\ell}\phi_\varepsilon(\alpha_s-\theta)H(s,\theta)\,\mathrm{d}\theta - H(s,\alpha_s)\right|^2\mathrm{d}s \to 0, \quad \text{as } \varepsilon\to 0; \quad P\text{-a.s.},$$

therefore it is standard to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^\ell}\int_0^t \phi_\varepsilon(\alpha_s-\theta)\langle H(s,\theta),\mathrm{d}B_s\rangle\,\mathrm{d}\theta \\
&\quad = \lim_{\varepsilon\to 0}\int_0^t\left\langle\int_{\mathbb{R}^\ell}\phi_\varepsilon(\alpha_s-\theta)H(s,\theta)\mathrm{d}\theta,\mathrm{d}B_s\right\rangle \\
&\quad = \int_0^t\langle H(s,\alpha_s),\mathrm{d}B_s\rangle, \quad t\in[0,T], \qquad (4.13)
\end{aligned}$$

and the convergence is in the sense of **ucp**. Similarly, one can also show that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^\ell}\int_0^t \phi_\varepsilon(\alpha_s-\theta)\langle K(s,\theta),\downarrow\mathrm{d}W_s\rangle\,\mathrm{d}\theta = \int_0^t\langle K(s,\alpha_s),\downarrow\mathrm{d}W_s\rangle, \quad \text{in } \mathbf{ucp}. \qquad (4.14)$$

Now, since $\int_{\mathbb{R}^\ell}\int_0^t |D_x\phi_\varepsilon(\alpha_s-\theta)\gamma_s|^2|F(s,\theta)|^2\mathrm{d}s\,\mathrm{d}\theta<+\infty$, P-a.e., and $\mathrm{supp}(\phi_\varepsilon)\subset\mathbb{R}^\ell$ is compact, we can apply the stochastic Fubini theorem (cf., e.g., Protter, 1990) and then use integration by parts to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^\ell}\int_0^t \langle D_x\phi_\varepsilon(\alpha_s-\theta),\gamma_s\,\mathrm{d}B_s\rangle F(s,\theta)\,\mathrm{d}\theta\\
&\quad=\int_0^t\int_{\mathbb{R}^\ell}\langle F(s,\theta)D_x\phi_\varepsilon(\alpha_s-\theta)\,\mathrm{d}\theta,\gamma_s\,\mathrm{d}B_s\rangle\\
&\quad=\int_0^t\left\langle\int_{\mathbb{R}^\ell}\phi_\varepsilon(\theta)D_xF(s,\alpha_s-\theta)\,\mathrm{d}\theta,\gamma_s\,\mathrm{d}B_s\right\rangle. \qquad (4.15)
\end{aligned}$$

Further, since $(s,\theta)\to D_xF(s,\alpha_s-\theta)$ is continuous on $[0,T]\times\mathbb{R}^\ell$, and $\gamma(\omega)\in L^2([0,T];\mathbb{R}^d)$ for P-a.e. $\omega\in\Omega$, one shows easily that

$$\int_0^T\left|\int_{\mathbb{R}^\ell}\phi_\varepsilon(\theta)D_xF(s,\alpha_s-\theta)\,\mathrm{d}\theta-D_xF(s,\alpha_s)\right|^2|\gamma_s|^2\mathrm{d}s\to 0,\quad P\text{-a.e.},\quad \text{as } \varepsilon\to 0.$$

This, combined with (4.15), leads to that, as $\varepsilon\to 0$, in the sense of **ucp**,

$$\int_{\mathbb{R}^\ell}\int_0^t\langle D_x\phi_\varepsilon(\alpha_s-\theta),\gamma_s\,\mathrm{d}B_s\rangle F(s,\theta)\,\mathrm{d}\theta\to\int_0^t\langle D_xF(s,\alpha_s),\gamma_s\,\mathrm{d}B_s\rangle\,\mathrm{d}s.$$

Using the same arguments, one can show that each term on the right side of (4.10) converges in **ucp** to the counterpart of that on the right side of (4.9) as $\varepsilon\to 0$, proving the theorem. □

To conclude this section we give a comparison theorem for the BDSDEs, with the help of the Itô–Ventzell formula above.

Theorem 4.3 (Comparison theorem for BDSDEs). *Assume* (A.1)–(A.4). *For* $(t,x)\in[0,T]\times\mathbb{R}^n$, *let* $(Y^{t,i}(x),Z^{t,i}(x))$, $i=1,2$, *be solutions to the BDSDEs*:

$$\begin{aligned}
Y_s^{t,i}(x)&=\varphi_i(X_0^t(x))+\int_0^s f_i(r,X_r^t(x),Y_r^{t,i}(x),Z_r^{t,i}(x))\,\mathrm{d}r\\
&\quad+\int_0^s\langle g(r,X_r^t(x),Y_r^{t,i}(x)),\mathrm{d}B_r\rangle-\int_0^s\langle Z_r^{t,i}(x),\downarrow\mathrm{d}W_r\rangle,\quad s\in[0,t],\quad (4.16)
\end{aligned}$$

where f_1,f_2 *satisfy* (A.2); φ_1,φ_2 *satisfy* (A.3) *and* g *satisfies* (A.4). *Suppose that*

(1) $\varphi_1(x)\leqslant\varphi_2(x)\ \forall x\in\mathbb{R}^n$;
(2) *For either* $i=1$ or $i=2$, *it holds that*, *for all* $(t,x)\in[0,T]\times R^d$,

$$\begin{aligned}
&f_1(s,X_s^t(x),Y_s^{t,i}(x),Z_s^{t,i}(x))\leqslant f_2(s,X_s^t(x),Y_s^{t,i}(x),Z_s^{t,i}(x)),\\
&\mathrm{d}s\times\mathrm{d}P\text{-a.e. } \textit{on } [0,t]\times\Omega. \qquad (4.17)
\end{aligned}$$

Then, one has $Y_s^{t,1}(x)\leqslant Y_s^{t,2}(x)$, $\forall 0\leqslant s\leqslant t\leqslant T$, $x\in\mathbb{R}^n$, *P-a.s.*

Proof. Denote $\bar\varphi=\varphi_1-\varphi_2$, $\bar f=f_1-f_2$, $\bar Y^t(x)=Y^{t,1}(x)-Y^{t,2}(x)$, and $\bar Z^t(x)=Z^{t,1}(x)-Z^{t,2}(x)$. For each $\varepsilon>0$, define $\phi_\varepsilon(\rho)\triangleq(1/3\varepsilon)\rho^3\mathbf{1}_{[0,\varepsilon]}(\rho)+[\rho^2-\varepsilon\rho+\frac{1}{3}\varepsilon^2]\mathbf{1}_{(\varepsilon,\infty)}(\rho)$.

Clearly, $\phi_\varepsilon \in C^2(\mathbb{R})$; and as $\varepsilon \to 0$, one has

$$\phi_\varepsilon(\rho) \to [(\rho)^+]^2, \quad \phi'_\varepsilon(\rho) \to 2(\rho)^+, \quad \phi''_\varepsilon(\rho) \to 2 \cdot \mathbf{1}_{(0,\infty)}(\rho) \quad \forall \rho \in \mathbb{R}.$$

Now assume without loss of generality that (4.17) holds for $i = 1$. Then, by first applying the Itô formula (Theorem 4.2 above or Lemma 3.1 of Pardoux and Peng, 1994) to $\phi_\varepsilon(\bar{Y}_s^t(x))$, and then taking the expectation, and finally sending $\varepsilon \to 0$ we obtain that, for some $C > 0$ depending only on those constants in (A.1)–(A.4),

$$\begin{aligned}
&E\{[(\bar{Y}_s^t(x))^+]^2\} + E\left\{\int_0^s |\bar{Z}_r^t(x)|^2 \mathbf{1}_{(0,\infty)}(\bar{Y}_r^t(x))\,\mathrm{d}r\right\} \\
&\quad = E\left\{[(\bar{\varphi}(X_0^t(x))^+]^2 + 2E\left\{\int_0^s (\bar{Y}_r^t(x))^+ \bar{f}(r, X_r^t(x), Y_r^{t,1}(x), Z_r^{t,1}(x))\,\mathrm{d}r\right\}\right\} \\
&\qquad + 2E\left\{\int_0^s (\bar{Y}_r^t(x))^+ [f_2(r, X_r^t(x), Y_r^{t,1}(x), Z_r^{t,1}(x)) \right. \\
&\qquad \left. - f_2(r, X_r^t(x), Y_r^{t,2}(x) Z_r^{t,2}(x))]\,\mathrm{d}r\right\} \\
&\qquad + E\left\{\int_0^s \mathbf{1}_{(0,\infty)}(\bar{Y}_r^t(x))|g(r, X_r^t(x), Y_r^{t,1}(x)) - g(r, X_r^t(x), Y_r^{t,2}(x))|^2 \mathrm{d}r\right\} \\
&\quad \leqslant C\left\{E\left\{\int_0^s (\bar{Y}_r^t(x))^+ [|\bar{Y}_r^t(x)| + |\bar{Z}_r^t(x)|]\,\mathrm{d}r\right\}\right. \\
&\qquad \left. + E\left\{\int_0^s [(Y_r^t(x))^+]^2 \mathrm{d}r\right\}\right\}, \quad 0 \leqslant s \leqslant t.
\end{aligned}$$

Using a standard estimate and applying Gronwall's inequality one then easily shows that $E[|\bar{Y}_s^t(x)^+|^2] = 0 \ \forall 0 \leqslant s \leqslant t \leqslant T$. Since $\bar{Y}^t(x)$ has continuous paths, the theorem follows. □

5. Existence of stochastic viscosity solution

In this section we prove the existence of the stochastic viscosity solution to the SPDE(f, g). Our main idea is to try to apply the *Doss transformation* to the BDSDE (4.5), with the hope that in the resulting BDSDE the stochastic integral against $\mathrm{d}B$ would disappear, just as we saw in the SDE case. It is then natural to conjecture that the BDSDE will become a BSDE with a new generator being exactly $\tilde{f}(!)$. Thus, in light of the known results regarding the BSDEs and PDEs, we could then derive an ω-wise viscosity solution to the SPDE (whence a stochastic viscosity solution), thanks to Proposition 3.1.

We shall now substantiate the above idea. To perform Doss transformation to the BDSDE (4.5), let us define the following two processes:

$$\begin{aligned}
U_s^t(x) &= \mathscr{E}(s, X_s^t(x), Y_s^t(x)), \quad 0 \leqslant s \leqslant t \leqslant T, \ x \in \mathbb{R}^n; \\
V_s^t(x) &= D_y \mathscr{E}(s, X_s^t(x), Y_s^t(x)) Z_s^t(x) + \sigma^*(X_s^t(x)) D_x \mathscr{E}(s, X_s^t(x), Y_s^t(x)).
\end{aligned} \tag{5.1}$$

From Proposition 3.4 we see that $(U^t_\cdot(x), V^t_\cdot(x)) \in S^2(\bar{\boldsymbol{F}};[0,T]\times\mathbb{R}^n)\times\mathscr{M}^2(\bar{\boldsymbol{F}};[0,T]\times\mathbb{R}^n)$. Our main result of the section is the following.

Theorem 5.1. *For each* $(t,x)\in[0,T]\times\mathbb{R}^n$, *the pair* $(U^t_\cdot(x), V^t_\cdot(x))$ *is the unique solution of the following BSDE*: *for* $0\leqslant s\leqslant t$,

$$U^t_s(x) = u_0(X^t_0(x)) + \int_0^s \tilde{f}(r, X^t_r(x), U^t_r(x), V^t_r(x))\,\mathrm{d}r - \int_0^s \langle V^t_r(x), \downarrow \mathrm{d}W_r\rangle, \quad (5.2)$$

where $\tilde{f}:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k\mapsto\mathbb{R}$ *is given by* (3.4).

Remark 5.2. The existence and uniqueness of BDSDE (5.2) does not follow directly from the standard theory, because of the quadratic growth in the variable z and the special $\bar{\boldsymbol{F}}$-*adaptedness* of $\tilde{f}$ (see (3.4)). It should be noted that the BSDEs with quadratic growth in z were studied recently by Kobylansky (1997). However, we will give a direct proof below, without using the results of Kobylansky (1997).

Proof of Theorem 5.1. For notational simplicity, from now on we shall write X, Y, Z, U, V instead of $X^t(x), Y^t(x), Z^t(x), U^t(x), V^t(x)$. It is easily checked that the mapping $(X,Y,Z)\mapsto(X,U,V)$ is 1–1, with the inverse transformation:

$$Y_s = \eta(s, X_s, U_s); \quad Z_s = D_y\eta(s, X_s, U_s)V_s + \sigma^*(X_s)D_x\eta(s, X_s, U_s). \quad (5.3)$$

Consequently, the uniqueness of (5.2) follows from that of BDSDE (4.5), thanks to (5.1) and (5.3). Thus we need only show that (U,V) is a solution of the BSDE (5.2).

To this end, note that $U_0 = Y_0 = u_0(X_0)$. Rewriting (4.5) as its equivalent Itô form (4.6), and applying the generalized Itô–Ventzell formula (Theorem 4.2) to $\mathscr{E}(s, X_s, Y_s)$, one derives, after a little calculation, that for $(s,x)\in[0,t]\times\mathbb{R}^n$

$$\begin{aligned}
U_s = \mathscr{E}(s, X_s, Y_s) &= u_0(X_0) - \int_0^s \langle D_x\mathscr{E}(r, X_r, Y_r), b(X_r)\rangle\,\mathrm{d}r \\
&\quad - \int_0^s \langle D_x\mathscr{E}(r, X_r, Y_r), \sigma(X_r)\downarrow\mathrm{d}W_r\rangle + \int_0^s D_y\mathscr{E}(r, X_r, Y_r) f(r, X_r, Y_r, Z_r)\,\mathrm{d}r \\
&\quad - \int_0^s \langle D_y\mathscr{E}(r, X_r, Y_r)Z_r, \downarrow\mathrm{d}W_r\rangle - \frac{1}{2}\int_0^s \mathrm{tr}\{\sigma(X_r)\sigma^*(X_r)D_{xx}\mathscr{E}(r, X_r, Y_r)\}\,\mathrm{d}r \\
&\quad - \int_0^s \langle \sigma^*(X_r)D_{xy}\mathscr{E}(r, X_r, Y_r), Z_r\rangle\,\mathrm{d}r - \frac{1}{2}\int_0^s D_{yy}\mathscr{E}(r, X_r, Y_r)|Z_r|^2\mathrm{d}r \\
&= u_0(X_0) + \int_0^s \mathscr{H}(r, X_r, Y_r, Z_r)\,\mathrm{d}r - \int_0^s \langle V_r, \downarrow\mathrm{d}W_r\rangle, \qquad (5.4)
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{H}(s,x,y,z) \triangleq &- \langle D_x\mathscr{E}, b(x)\rangle + (D_y\mathscr{E})f(t,x,y,z) - \tfrac{1}{2}(D_{yy}\mathscr{E})|z|^2 \\
&- \tfrac{1}{2}\mathrm{tr}\{\sigma(x)\sigma^*(x)D_{xx}\mathscr{E}\} - \langle\sigma^*(x)D_{xy}\mathscr{E}, z\rangle, \qquad (5.5)
\end{aligned}$$

where $\mathscr{E} = \mathscr{E}(s,x,y)$. Comparing (5.4) with (5.2), it remains to show that

$$\mathscr{H}(s, X_s, Y_s, Z_s) = \tilde{f}(s, X_s, U_s, V_s) \quad \forall s\in[0,t], \quad P\text{-a.s.} \quad (5.6)$$

To this end we analyze $\mathscr{H}(s,x,y,z)$ term by term. Using (3.3) and (5.1) we see that

$$-\langle D_x\mathscr{E}(s,X_s,Y_s),b(X_s)\rangle = D_y\mathscr{E}(s,X_s,Y_s)\langle D_x\eta(s,X_s,U_s),b(X_s)\rangle. \tag{5.7}$$

To further simplify notations from now on we shall suppress the variables inside $\mathscr{E}(\cdots)$ and $\eta(\cdots)$, as well as their partial derivatives, bearing in mind that the $\mathscr{E}$ and its partial derivatives are all evaluated at (s,X_s,Y_s), while η and its partial derivatives are all evaluated at (s,X_s,U_s). Denoting $b(X_s)=b_s$ and $\sigma(X_s)=\sigma_s$, and applying (3.3) and (5.3) we have

$$\begin{aligned}
&(D_y\mathscr{E})f(s,X_s,Y_s,Z_s) = (D_y\mathscr{E})f(s,X_s,\eta,D_y\eta V_s+\sigma_s^*(D_x\eta)),\\
&-\langle\sigma_s^*(D_{xy}\mathscr{E}),Z_s\rangle = -(D_y\eta)\langle\sigma_s^*(D_{xy}\mathscr{E}),V_s\rangle - \langle\sigma_s^*(D_{xy}\mathscr{E}),\sigma_s^*(D_x\eta)\rangle,\\
&-\tfrac12(D_{yy}\mathscr{E})|Z_s|^2 = \tfrac12(D_y\mathscr{E})(D_{yy}\eta)|V_s|^2 + (D_y\mathscr{E})^2(D_{yy}\eta)\langle V_s,\sigma_s^*(D_x\eta)\rangle\\
&\qquad\qquad + \tfrac12(D_{yy}\eta)(D_y\mathscr{E})|\sigma_s^*(D_x\eta)(D_y\mathscr{E})|^2.
\end{aligned}\tag{5.8}$$

Combining (5.7) and (5.8), we have

$$\begin{aligned}
&\mathscr{H}(s,X_s,Y_s,Z_s)\\
&\quad=(D_y\mathscr{E})\langle D_x\eta,b\rangle - \tfrac12\mathrm{tr}\{\sigma\sigma^*D_{xx}\mathscr{E}\} - (D_y\eta)\langle\sigma^*D_{xy}\mathscr{E},V_s\rangle - \langle\sigma^*D_{xy}\mathscr{E},\sigma^*D_x\eta\rangle\\
&\qquad+\tfrac12(D_y\mathscr{E})(D_{yy}\eta)|V_s|^2 + (D_y\mathscr{E})^2(D_{yy}\eta)\langle V_s,\sigma^*D_x\eta\rangle\\
&\qquad+\tfrac12(D_{yy}\eta)(D_y\mathscr{E})|\sigma^*D_x\eta D_y\mathscr{E}|^2 + (D_y\mathscr{E})f(s,X_s,\eta,(D_y\eta)V_s+\sigma^*D_x\eta)\\
&\quad=(D_y\mathscr{E})\left\{\langle D_x\eta,b\rangle + \tfrac12(D_{yy}\eta)|V_s|^2 + f(s,X_s,\eta,(D_y\eta)V_s+\sigma^*D_x\eta)\right\}\\
&\qquad+\left\{\tfrac12(D_y\mathscr{E})(D_{yy}\eta)|\sigma^*D_x\eta D_y\mathscr{E}|^2 - \tfrac12\mathrm{tr}\{\sigma\sigma^*D_{xx}\mathscr{E}\} - \langle\sigma^*D_{xy}\mathscr{E},\sigma^*D_x\eta\rangle\right\}\\
&\qquad+\langle V_s,\sigma^*[D_x\eta(D_y\mathscr{E})^2(D_{yy}\eta) - D_y\eta D_{xy}\mathscr{E}]\rangle.
\end{aligned}\tag{5.9}$$

Now (3.3) tells us that

$$\begin{aligned}
\mathrm{tr}\{\sigma\sigma^*D_{xx}\mathscr{E}\} &= -2\mathrm{tr}\{\sigma\sigma^*D_x\eta(D_{xy}\mathscr{E})^*\} - D_{yy}\mathscr{E}|\sigma^*D_x\eta|^2 - (D_y\mathscr{E})\mathrm{tr}\{\sigma\sigma^*D_{xx}\eta\}\\
&= -2\langle\sigma^*D_{xy}\mathscr{E},\sigma^*D_x\eta\rangle + (D_y\mathscr{E})D_{yy}\eta|\sigma^*D_x\eta D_y\mathscr{E}|^2\\
&\quad - (D_y\mathscr{E})\mathrm{tr}\{\sigma\sigma^*D_{xx}\eta\}
\end{aligned}$$

and $D_{xy}\mathscr{E}D_y\eta - D_x\eta(D_y\mathscr{E})^2(D_{yy}\eta) = D_{xy}\mathscr{E}D_y\eta + D_x\eta D_y\eta D_{yy}\mathscr{E} = -D_y\mathscr{E}D_{xy}\eta$. Thus the second $\{\cdots\}$ in the right side of (5.9) becomes $\frac12(D_y\mathscr{E})\mathrm{tr}\{\sigma\sigma^*D_{xx}\eta\}$; and the last term there becomes $(D_y\mathscr{E})\langle V_s,\sigma^*D_{xy}\eta\rangle$. Now if we compare with the definition of $\tilde f$ (see (3.4)) and note that $D_y\mathscr{E}(s,X_s,Y_s) = (D_y\eta)^{-1}(s,X_s,U_s)$, (5.9) becomes (5.6), proving Theorem 5.1. □

Our next lemma concerns the *stochastically uniform boundedness* property of the solutions to the BDSDE (4.5) and its *Doss transformation*. Such a property will eventually lead to the same type of boundedness of the stochastic viscosity solution, which will be vital when we study the uniqueness in the future. One should recall Remark 3.5(i) again that Proposition 3.4 gives only the stochastically uniform boundedness of the derivatives of the random fields η and $\mathscr{E}$, but not for η and $\mathscr{E}$ themselves. Thus the stochastic boundedness of the processes U or Y are by no means clear. We should point out here that the proof of the lemma borrows some idea of Kobylansky (1997).

Lemma 5.3. *Assume* (A1)–(A3), *and* (A4′). *Assume also that the functions* σ, β *in* (A1) *and* u_0 *in* (A3) *are all bounded, and that the random function* f *in* (A2) *satisfies that* $|f(\omega,t,x,0,0)|\leqslant C$, $\forall(\omega,t,x)$ *for some constant* $C>0$.

Then, there exists an increasing process $\Theta\in L^0(\boldsymbol{F}^B,[0,T])$, *such that for any solution* $(X^t(x),Y^t(x),Z^t(x))$ *of the SDEs* (4.4) *and* (4.5), *it holds that*, $\bar{P}$*-a.s.*,

$$|Y_s^t(x)|\leqslant\Theta_s,\quad |\mathscr{E}(s,X_s^t(x),Y_s^t(x))|\leqslant\Theta_s,\quad 0\leqslant s\leqslant t\leqslant T,\ x\in\mathbb{R}^n. \tag{5.10}$$

Proof. Denote $f_1(\omega,t,x,y,z)=f(\omega,t,x,y,z)-\frac{1}{2}\langle g,D_yg\rangle(t,x,y)$. By Assumption (A2) and the extra condition we made on f, there exists constant $C>0$, such that

$$f_1(\omega,t,x,y,z)\leqslant C(1+|y|+|z|)-\tfrac{1}{2}\langle g,D_yg\rangle(t,x,y)\stackrel{\Delta}{=}f_2(\omega,t,x,y,z),$$
$$\forall(\omega,t,x,y,z)\in\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^k.$$

Now let $(Y^{t,1},Z^{t,1})=(Y^t(x),Z^t(x))$ and $(Y^{t,2}(x),Z^{t,2}(x))$ be the unique solution of BSDE (4.16) with $Y_0^{t,2}=C_0$, where C_0 is the bound for u_0; and generator f_2. Then, applying Theorem 4.3 we see that $Y_s^t(x)\leqslant Y_s^{t,2}(x)$, $\forall s\in[0,t]$, $\bar{P}$-a.s.

Next, let $(U^{t,2}(x),V^{t,2}(x))$ be the Doss transformation of $(Y^{t,2}(x),Z^{t,2}(x))$ via (5.1). Theorem 5.1 tells us that $(U^{t,2}(x),V^{t,2}(x))$ is the solution to the BSDE:

$$U_s^{t,2}(x)=C_0+\int_0^s\tilde{f}_2(r,X_r^t(x),U_r^{t,2}(x),V_r^{t,2}(x))\,\mathrm{d}r-\int_0^s\langle V_r^t(x),\downarrow\mathrm{d}W_r\rangle, \tag{5.11}$$

for $0\leqslant s\leqslant t$, where

$$\begin{aligned}\tilde{f}_2(t,x,y,z)=D_y\eta(t,x,y))^{-1}\{&C(1+|\eta(t,x,y)|+|\sigma^*(x)D_x\eta(t,x,y)\\&+D_y\eta(t,x,y)z|)+\mathscr{A}_x\eta(t,x,y)+\langle\sigma^*(x)D_{xy}\eta(t,x,y),z\rangle\\&+\tfrac{1}{2}D_{yy}\eta(t,x,y)|z|^2\}.\end{aligned}$$

Note that by Proposition 3.4 we can find a constant $L>0$ such that

$$\begin{aligned}&|\tilde{f}_2(t,\omega,x,y,z)|\leqslant L\exp\{L|B_t(\omega)|\}(1+|y|+|z|^2)\leqslant a(t,\omega)(1+|y|)+A(\omega)|z|^2,\\&\forall(t,x,y,z),\quad P\text{-a.e. }\omega\in\Omega,\end{aligned} \tag{5.12}$$

where $a(t,\omega)\stackrel{\Delta}{=}L\exp\{L|B_t(\omega)|\}$; $A(\omega)=L\exp\{L|B|_T^*(\omega)|\}$, $|B|_t^*(\omega)=\sup_{0\leqslant s\leqslant t}|B_s(\omega)|$. Clearly a is an $\boldsymbol{F}^B$-adapted process. Therefore if we define $\theta(t,\omega)$ to be the solution to the following ODE

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=a(t,\omega)(1+\theta);\quad \theta(0)=C_0>0, \tag{5.13}$$

then it is easy to check that θ is an $\boldsymbol{F}^B$-adapted, continuous, positive, increasing process.

Recall that for fixed $\omega \in \Omega$ (with possible exception on a P-null set) and $t \in [0,T]$, the processes $(U_s^{t,2}(x,\omega,\cdot), V_s^{t,2}(x,\omega,\cdot))$ is an $\{\mathscr{F}_{s,t}^W\}$-adapted solution to the BSDE (5.11). In what follows we content ourselves only to the space $(\Omega', \mathscr{F}', P', \{\mathscr{F}_{s,t}^W\}_{0\leqslant s\leqslant t})$ with $\omega \in \Omega$ being fixed. Also, for notational simplicity we denote $U_\cdot(\omega,\cdot) = U_\cdot^{t,2}(x,\omega,\cdot)$, $V_\cdot(\omega,\cdot) = V_\cdot^{t,2}(x,\omega,\cdot)$, and we often suppress ω in $U_s(\omega)$, $V_s(\omega)$, $a(s,\omega)$, $A(\omega)$, and $\theta(s,\omega)$, etc. We are to show that $|U_s| \leqslant \theta(s)$ $\forall s \in [0,t]$, P'-a.s.

To this end, let us define, for $M > 0$, $\tilde{f}_2^M(t,\omega,x,y,z) \stackrel{\Delta}{=} \tilde{f}_2(t,\omega,x,y,z)\varphi_M(y)$, where $\varphi_M \in C^\infty$, such that $0 \leqslant \varphi_M(y) \leqslant 1$ $\forall y \in \mathbb{R}$, $\varphi_M(y) = 0$ for $|y| \geqslant M+1$, and $\varphi_M(y) = 1$ for $|y| \leqslant M$. Clearly, for each M we have

$$|\tilde{f}_2^M(t,\omega,x,y,z)| \leqslant a(t)[1+\varphi_M(y)|y|] + A|z|^2 \leqslant K_M(1+|z|^2).$$

Thus by Kobylansky (1997), there exists a unique solution, denoted by (U^M, V^M), of the BSDE (5.11) with $\tilde{f}_2$ being replaced by $\tilde{f}_2^M$. Furthermore, by the stability result in Kobylansky (1997) we know that as $M \to \infty$, possibly along a subsequence, one has $U^M \to U$, in **ucp** and $V^M \to V$ in L^2, provided that $\{U^M\colon M > 0\}$ is uniformly bounded (i.e., the bound is independent of M). Therefore, it suffices to show that for all $M > 0$ and all $\omega \in \Omega$, it holds that $|U_s^M(\omega)| \leqslant \theta(s,\omega)$ $\forall s \in [0,t]$, P'-a.s., where θ is the solution to (5.13).

To see this we first apply Tanaka's formula to get

$$\begin{aligned} |U_s^M| = C_0 &+ \int_0^s \operatorname{sign}(U_s^M)\varphi_M(U_r^M)\tilde{f}_2(r,\omega,X_r^t(x),U_r^M,V_r^M)\,\mathrm{d}r \\ &- \int_0^s \langle \operatorname{sign}(U_r^M)V_r^M, \downarrow \mathrm{d}W_r\rangle + K_s - K_0, \quad 0 \leqslant s \leqslant t. \end{aligned} \tag{5.14}$$

where K is an $\{\mathscr{F}_{s,t}^W\}_{0\leqslant s\leqslant t}$-adapted, continuous, *local-time-like* process, i.e., K satisfies $K_t = 0$ and $K_s = \int_s^t \mathbf{1}_{\{U_r^M=0\}}\,\mathrm{d}K_r$, $0 \leqslant s \leqslant t$, P'-a.s.

Next, for fixed ω let us consider the following function:

$$\Phi(\rho,\omega) = \begin{cases} \displaystyle\sum_{k=3}^\infty \frac{(2A(\omega))^k \rho^k}{k!} = \mathrm{e}^{2A(\omega)\rho} - 1 - 2A(\omega)\rho - 2A(\omega)^2\rho^2, & \rho > 0; \\ 0 & \rho \leqslant 0. \end{cases} \tag{5.15}$$

Then it is easy to check that $\Phi(\cdot,\omega)$ is C^2, nonnegative and $\Phi(\rho,\omega) > 0$ if and only if $\rho > 0$. Again, we shall suppress ω in Φ if there is no danger of confusion.

Now let us denote $\Delta_s^M = |U_s^M| - \theta(s)$, $s \in [0,t]$, and applying Itô's formula to $\Phi(\Delta_\cdot^M)$ on $[0,s]$ we have (although this is not a consequence of Theorem 4.2, but it might be helpful to check the formula there to understand the signs for each term)

$$\begin{aligned} \Phi(\Delta_s^M) = &\int_0^s \Phi'(\Delta_r^M)\{\operatorname{sign}(U_r^M)\varphi_M(U_r^M)\tilde{f}_2(r,X_r,U_r^M,V_r^M) - a(r)(1+\theta(r))\}\,\mathrm{d}r \\ &+ \int_0^s \langle \operatorname{sign}(U_r^M)V_r^M, \downarrow \mathrm{d}W_r\rangle + \int_0^s \Phi'(\Delta_r^M)\,\mathrm{d}K_r - \frac{1}{2}\int_0^s \Phi''(\Delta_s^M)|V_r^M|^2\,\mathrm{d}r. \end{aligned} \tag{5.16}$$

Since $\Phi'(u) \equiv 0$ for $u < 0$ and $\theta \geqslant 0$, we have

$$\int_0^s \Phi'(\Delta_r^M)\,\mathrm{d}K_r = \int_0^s \Phi'(\Delta_r^M)\mathbf{1}_{\{U_r^M=0\}}\,\mathrm{d}K_r = \int_0^s \Phi'(-\theta^{\omega}(r))\,\mathrm{d}K_r = 0. \tag{5.17}$$

Moreover,

$$\int_0^s \Phi'(\Delta_r^M)\{\mathrm{sign}(U_r^M)\varphi_M(U_r^M)\tilde{f}_2(r, X_r, U_r^M, V_r^M) - a(r)(1+\theta(r))\}\,\mathrm{d}r$$
$$\leqslant \int_0^s \Phi'(\Delta_r^M)\{\varphi_M(U_r^M)|\tilde{f}_2(r, X_r, U_r^M, V_r^M)| - a(r)(1+\theta(r))\}\,\mathrm{d}r$$
$$\leqslant \int_0^s \Phi'(\Delta_r^M)[a(r)\varphi_M(U_r^M)|U_r^M| - \theta(r)) + A|V_r^M|^2]\,\mathrm{d}r. \tag{5.18}$$

Note that $\Phi''(\rho) - 2A\Phi'(\rho) \geqslant 0\ \forall \rho$, and $a(r) \leqslant A\ \forall r$, Combining (5.16)–(5.18) we have

$$\Phi(\Delta_s^M) \leqslant A\int_0^s \Phi'(\Delta_r^M)[\varphi_M(U_r^M)|U_r^M| - \theta(r)]\,\mathrm{d}r + \int_0^s \langle \mathrm{sign}(U_r^M)V_r^M, \downarrow \mathrm{d}W_r\rangle.$$

Hence for every $s \leqslant \tau \leqslant t$ (and fixed ω) it holds that

$$E^{P'}\{\Phi(\Delta_s^M)|\mathscr{F}_{\tau,t}^W\} \leqslant AE^{P'}\left\{\int_0^s \Phi'(\Delta_r^M)[\varphi_M(U_r^M)|U_r^M| - \theta(r)]\,\mathrm{d}r \,\middle|\, \mathscr{F}_{\tau,t}^W\right\}. \tag{5.19}$$

Since by definition (5.15) one shows that $\Phi'(\rho) = 2A[\Phi(\rho) + 2A^2\rho^2]$, and $\Phi'(\rho) = 0$ for $\rho < 0$, we have

$$\Phi'(\Delta_r^M)[\varphi_M(U_r^M)|U_r^M| - \theta(r)] \leqslant 2A\Phi(\Delta_r^M)[\varphi_M(U_r^M)|U_r^M| - \theta(r)] + 4A^3(\Delta_r^M)^3. \tag{5.20}$$

But since $\lim_{\rho\to 0}\rho^3/\Phi(\rho) = 3/4A^3$, and $\lim_{\rho\to\infty}\rho^3/\Phi(\rho) = 0$, there exists a constant $\tilde{A} > 0$ such that $\rho^3 \leqslant \tilde{A}\Phi(\rho)$ for all ρ. Thus, if we denote $\psi_s^{t,\tau} \triangleq E^{P'}\{\Phi(\Delta_s^M)|\mathscr{F}_{\tau,t}^W\}$, $0 \leqslant s \leqslant \tau \leqslant t$, then (5.19) and (5.20) lead to that $\psi_s^{t,\tau} \leqslant K\int_0^s \psi_r^{t,\tau}\,\mathrm{d}r$, where $K = 2A^2(M + \|\theta\|_\infty + 2A\tilde{A})$. Applying Gronwall's inequality we obtain that $\psi_s^{t,\tau} = 0\ \forall s \in [0,\tau]$, P'-a.s. Let $s = \tau$ and note that Δ_τ^M is $\mathscr{F}_{\tau,t}^W$-measurable, we have $\Phi(\Delta_s^M) = 0\ \forall s \in [0,t]$, P'-a.s. But from the construction of Φ we see that one must have $\Delta_r^M \leqslant 0$, P'-a.s. Namely, $|U_s^M| \leqslant \theta(s)$, P'-a.s., proving the claim.

Consequently, note that both η and $\mathscr{E}$ are increasing in y, we have for all (t,x),

$$U_s^t(x) = \mathscr{E}(s, X_s^t(x), Y_s^t(x)) \leqslant \mathscr{E}(s, X_s^t(x), Y_s^{t,2}(x))$$
$$= U_s^{t,2}(x) \leqslant \theta_s \quad \forall s \in [0,t],\ P\text{-a.s.},$$

and hence (recall Proposition 3.4), for all $0 \leqslant s \leqslant t$,

$$Y_s^t(x) = \eta(s, X_s^t(x), U_s^t(x)) \leqslant \eta(s, X_s^t(x), U_s^{t,2}(x)) \leqslant \eta(s, X_s^t(x), \theta_s) \leqslant C|B|_s^* + \theta_s \triangleq \Theta_s.$$

Note that θ is an $\boldsymbol{F}^B$-adapted increasing process, so is Θ. The proof is now complete. □

We are now ready to prove the existence of the stochastic viscosity solutions. Let us introduce for each $(t,x) \in [0,T] \times \mathbb{R}^n$ two random fields

$$u(t,x) = Y_t^t(x); \quad v(t,x) = U_t^t(x), \tag{5.21}$$

where Y, U are the solutions to the SDEs (4.5) and (5.2), respectively. Then by (5.1) and (5.3) we have, for $(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^n$,

$$u(\omega,t,x)=\eta(\omega,t,x,v(\omega,t,x));\quad v(\omega,t,x)=\mathscr{E}(\omega,t,x,u(\omega,t,x)). \tag{5.22}$$

In light of the Proposition 3.1, to prove the existence of the viscosity solution to SPDE (f,g) we need only show that the random field v defined in (5.22) is a stochastic viscosity solution to the SPDE$(\tilde f,0)$. However, since it is easily checked from the definition that an $\boldsymbol{F}^B$-progressively measurable, ω-wise viscosity solution is automatically a stochastic viscosity solution, the following theorem is not surprising.

Theorem 5.4. *Assume* (A1)–(A4). *Then the random field v is a stochastic viscosity solution of SPDE$(\tilde f,0)$; and u is a stochastic viscosity solution to SPDE(f,g), respectively. Furthermore, if in addition we assume that the conditions of Lemma* 5.3 *hold, then the random fields u and v are locally bounded in the following sense: for some $\boldsymbol{F}^B$-adapted increasing process $\Theta=(\Theta_s)_{s\in[0,T]}$, it holds that $|u(t,x)|\leqslant\Theta_t$, $|v(t,x)|\leqslant\Theta_t$ $\forall(t,x)\in[0,T]\times\mathbb{R}^n$, P-a.s.*

Proof. Since the BDSDE (4.5) is a time-reversed version of the one considered by Pardoux and Peng (1994), it follows from the proof of Theorem 2.1 in Pardoux and Peng (1994), together with Theorem 4.2, that there exists a continuous version of the random field $(s,t,x)\mapsto Y_s^t(x)$, for $(s,t,x)\in[0,T]^2\times\mathbb{R}^n$. Here we define $Y_s^t(x)=Y_{s\wedge t}^t(x)$ for $(s,t)\in[0,T]^2$. Taking this continuous version from now on, then the random field $u(t,x)=Y_t^t(x)$ is also continuous (whence jointly measurable) on $\bar\Omega\times[0,T]\times\mathbb{R}^n$.

Next, by Lemma 4.1 we know that $Y_s^t(x)$ is $\mathscr{F}_s^B\otimes\mathscr{F}_{s,t}^W$-measurable. In particular $u(t,x)$ is $\mathscr{F}_t^B\otimes\mathscr{F}_{t,t}^W$ measurable. But since W is a Brownian motion on $(\Omega',\mathscr{F}',P')$, applying Blumenthal 0–1 law we conclude that u is independent of (or a constant with respect to) $\omega'\in\Omega'$. Therefore, we can identify the random field u as one that is defined on $\Omega\times[0,T]\times\mathbb{R}^n$, and is $\mathscr{F}_t^B$-measurable for each $t\in[0,T]$. In other words, $u\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$. Consequently, from (5.1), $v\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$ as well, thanks to the argument preceding Proposition 3.1.

It remains to show that v is a stochastic viscosity solution to SPDE$(\tilde f,0)$. To this end we consider SDE (4.5). For fixed $\omega\in\Omega$ we denote

$$\bar U_s^{\omega}(x)(\omega')=U_s^t(x)(\omega,\omega');\quad \bar V_s^{\omega}(x)(\omega')=V_s^t(x)(\omega,\omega').$$

Since for fixed $\omega\in\Omega$ we can view (4.5) as a time-reversed version of a standard BSDE on the probability space $(\Omega',\mathscr{F}',P')$, with the *generator* $\tilde f(\omega,\ldots)$, and by Theorem 5.1 we see that $(\bar U^{\omega},\bar V^{\omega})$ is the (pathwisely) unique strong solution of this BSDE, following the arguments of Pardoux and Peng (1992) one then shows that $\bar v(\omega,t,x)\stackrel{\Delta}{=}\bar U_t^{\omega}(x)$ is a viscosity solution to a quasilinear PDE with coefficient $\tilde f(\omega,\ldots)$. By Blumenthal 0–1 law again we have $\bar P\{\bar U_t^{\omega}(x)=U_t^t(x)(\omega,\omega')\}=1$; hence $\bar v(t,x)\equiv v(t,x)$ $\forall(t,x)$, P-a.s.

Since $v\in C(\boldsymbol{F}^B,[0,T]\times\mathbb{R}^n)$ and is a viscosity solution to SPDE $(\tilde f,0)$ for each fixed ω, it is by definition an ω-wise viscosity solution. Hence a stochastic viscosity solution to SPDE$(\tilde f,0)$. The first conclusion of the theorem now follows from Proposition 3.1.

The last statement of the theorem follows from Theorem 5.1 and Lemma 5.3. The proof is now complete. □

Remark 5.5. A direct consequence of Proposition 3.4 and Theorem 5.4 is that the stochastic viscosity solution constructed above is stochastically uniformly bounded.

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