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FULLY NONLINEAR STOCHASTIC AND ROUGH PDES: CLASSICAL AND VISCOSITY SOLUTIONS[*]

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We study fully nonlinear second-order (forward) stochastic partial differential equations (SPDEs). They can also be viewed as forward path-dependent PDEs (PPDEs) and will be treated as rough PDEs (RPDEs) under a unified framework. We develop first a local theory of classical solutions and define then viscosity solutions through smooth test functions. Our notion of viscosity solutions is equivalent to the alternative one using semi-jets. Next, we prove basic properties such as consistency, stability, and a partial comparison principle in the general setting. When the diffusion coefficient is semi-linear (but the drift can be fully nonlinear), we establish a complete theory, including global existence and comparison principle. Our methodology relies heavily on the method of characteristics.

1. Introduction. We study the fully nonlinear second order SPDE

$$du(t,x,\omega) = f(t,x,\omega,u,\partial_x u,\partial^2_{xx}u)\,dt + g(t,x,\omega,u,\partial_x u)\circ dB_t, \tag{1.1}$$

with initial condition $u(0,x,\omega) = u_0(x)$, where $(t,x) \in [0,\infty)\times\mathbb{R}$, B is a standard Brownian motion defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, f and g are $\mathbb{F}^B$-progressively measurable random fields, and $\circ$ denotes the Stratonovic integration. Using the pathwise analysis based on Dupire's path derivatives (see [15]), we showed in Buckdahn, Ma, and Zhang [7] that SPDE (1.1) can be rewritten as the (forward) PPDE

$$\partial^\omega_t u(t,x,\omega) = f(t,x,\omega,u,\partial_x u,\partial^2_{xx}u),\ \ \partial_\omega u(t,x,\omega) = g(t,x,\omega,u,\partial_x u). \tag{1.2}$$

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Here, ∂_t^ω and ∂_ω are temporal and spatial path derivatives in the spirit of [15]. On the other hand, in the recent work Keller and Zhang [26], we also investigated SPDE (1.1) in the framework of rough path theory initiated by Lyons [35]. We showed that Gubinelli's derivative (see [24]) for "controlled rough paths" is essentially equivalent to Dupire's path derivatives. Hence, SPDE (1.1) and also PPDE (1.2) can be viewed as the rough PDE

$$du(t,x,\omega) = f(t,x,\omega,u,\partial_x u,\partial^2_{xx} u)\,dt + g(t,x,\omega,u,\partial_x u)\,d\omega_t. \tag{1.3}$$

Here, ω is a geometric rough path corresponding to Stratonovic integration. Our investigation will be built upon these observations.

SPDE (1.1), especially when both f and g are linear or semilinear, has been studied extensively in the literature. We refer to the well-known reference Rozovsky [41] for a fairly complete theory on linear SPDEs, and to Krylov [27] for an L^p-theory of linear and some semi-linear SPDEs. In the case when SPDE (1.1) is fully nonlinear, as often encountered in applications such as stochastic control theory and many other fields (cf. the lecture notes of Souganidis [47]), the situation is quite different. In fact, in such a case one can hardly expect (global) classical solutions, even in the Sobolev sense. Some weaker forms of solutions will have to come into play.

In a series of works, Lions and Souganidis [30, 31, 32, 33] initiated the investigation of fully nonlinear SPDEs and proposed general schemes for finding viscosity solutions, especially when $g = g(\partial_x u)$ or $g = g(u)$. Roughly speaking, they suggested two approaches. The first one is to use the method of *stochastic characteristics* (cf. Kunita [28]) to remove the stochastic integrals of SPDE (1.1), either by defining test functions along the characteristics, or by defining viscosity solutions via transformed ω-wise (i.e., deterministic) PDEs. The second approach is to approximate the Brownian sample paths by smooth functions and define a solution as the limit, if it exists, of the solutions to the approximating equations, which are standard PDEs.

Since the seminal works [30] - [33], there have been many efforts to develop the theory of stochastic viscosity solutions using both approaches. Along the lines of stochastic characteristics, Buckdahn and Ma [4, 5, 6], Buckdahn, Bulla, and Ma [1], and Matoussi, Possamai, and Sabbagh [36] proposed different notions of stochastic viscosity solutions in various situations when g is independent of $\partial_x u$; see also Seeger [44], who investigated the case $g = g(\partial_x u)$ in the rough path framework. The main challenges in this approach seem to be the following:

(i) The stochastic characteristics exist only locally for general SPDEs;

(ii) the transformed ω-wise PDE becomes rather complex and generally falls out of the standard PDE literature (unless g is linear in u and $\partial_x u$).

As a consequence, many fundamental issues, such as a comparison principle for general nonlinear SPDEs, remain open.

On the other hand, as rough path theory gradually took shape in the early 2000s, many works emerged using the second approach. Publications along this line include Caruana, Friz, and Oberhauser [8], Friz and Oberhauser [21, 22], Diehl and Friz [11], Diehl, Friz, and Oberhauser [13], Diehl, Oberhauser, and Riedel [14], Diehl, Friz, and Gassiat [12], Friz, Gassiat, Lions, and Souganidis [19], and Seeger [45, 46], to mention a few. In many of these works g is linear in u and $\partial_x u$, or $g = g(x, u)$ and f is linear in $\partial^2_{xx} u$ (see [11]), or $g = g(\partial_x u)$ (see [19, 45, 46]). With this approach, the uniqueness often comes free, provided the uniqueness for the approximating standard PDEs is established. The main challenge here is the existence of the solution (i.e., the existence of the limit). Note that the two approaches are often combined; e.g., path-approximation was also used in the first approach and stochastic characteristics are often used to prove the existence of the limit.

In this paper, we take the first approach but in a rough path framework, by taking the advantage that SPDE (1.1), PPDE (1.2), and RPDE (1.3) can now be considered as equal. More precisely, we shall investigate the notion of viscosity solution for the general fully nonlinear RPDE (1.3) as an ω-wise viscosity solution for SPDE (1.1). We utilize PPDE (1.2) by requiring that smooth test functions φ satisfy

$$\partial_\omega \varphi(t, x) = g(t, x, \varphi, \partial_x \varphi). \tag{1.4}$$

Involving the coefficient g in the definition of test functions has already been noted in many previous works; e.g., in [5, 6, 7] the notion of "g-jets" was introduced to reflect the intrinsic g-dependence of the terms involving "path derivatives" in the stochastic Taylor expansion. Another reason that the function g will be treated differently than f, as we will do throughout the paper, is that, in general, there is no comparison principle in terms of the diffusion term g as far as stochastic differential equations are concerned.

Using the rough path language, we define viscosity solutions directly for RPDE (1.3) as well as PPDE (1.2) in a completely local manner in all three variables (t, x, ω). We prove that our definition is equivalent to an alternative definition through semi-jets, and show, by using pathwise characteristics, that RPDE (1.3) can be transformed into a standard PDE (with parameter ω) without the $d\omega_t$ term, as expected. When g is semilinear (i.e., linear in $\partial_x u$), it is not hard to see that our notion of viscosity solution to RPDE (1.3) is also equivalent to the viscosity solution to the transformed PDE in the standard sense of Crandall, Ishii, and Lions [9]. In the general case, however, the issue becomes quite subtle due to some intrinsic singularity of the transformed PDE, and thus we are not able to obtain the desired equivalence for viscosity solutions. In fact, at this point it is even not clear to us how to define viscosity solution for the transformed PDE when the initial condition u_0 is not smooth.

Modulo some technical conditions as well as differences of language, our definition is very similar or essentially equivalent to the ones in the literature; e.g., in the case $g = g(\partial_x u)$ and $g = g(u)$, resp., our definition is essentially equivalent to that introduced in [30] and [32], resp. When f does not depend on $\partial^2_{xx} u$ (i.e., in the case of first order RPDEs), our definition is essentially the same as the one in Gubinelli, Tindel, and Torrecilla [25], which also uses the rough path language. They also introduced the alternative definition through semi-jets but left the equivalence of these two definitions open.

We continue with the next goal of this paper. We establish most, if not all, of the important properties of viscosity solutions. These include consistency (with classical solutions), stability, and a (partial) comparison principle between a viscosity semi-solution and a classical semi-solution. We follow the arguments of our previous works on backward PPDEs (e.g., Ekren, Keller, Touzi, and Zhang [16] and Ekren, Touzi, and Zhang [17, 18]). The main difference between the forward case and the backward case is the additional requirement (1.4). This requirement is the main source of subtlety when small perturbations on the test function φ are needed. These subtleties show up in the proofs of stability and the partial comparison principle especially in the case when g depends on u. With the help of local classical solutions for first order RPDEs and higher order pathwise Taylor expansions along the lines of [7], we establish crucial auxiliary results (Lemmas 5.8 and 6.1) to obtain the desired perturbations. The involved techniques have a certain generic efficiency for dealing with the constraint (1.4). In fact, the equivalence of our definitions (test functions vs. semi-jets) also follows similar lines.

As in all studies involving viscosity solutions, the most challenging part is the comparison principle. Our case is no exception. An inevitable outcome of the stochastic characteristics approach is to study a fairly complicated ω-wise PDE, which will (naturally, due to the nature of the transformation) have quadratic growth in $\partial_x u$ and will never satisfy a Lipschitz condition in the variable u, except for some trivial linear cases. The latter deficiency in the coefficient clearly violates the technical requirements in the standard viscosity solution literature as it is no longer *proper* in the sense of [9]. Thus special considerations are in order. Our plan of attack is the following. We establish first a comparison principle on small time intervals. Then we extend our comparison principle to arbitrary duration by using a combination of uniform *a priori* estimates for PDEs and BMO estimates for backward stochastic differential equations with quadratic growth. Such a "cocktail" approach enables us to prove the comparison principle in the general fully nonlinear case under an extra condition, see (6.13). In the case when g is semilinear however, even when f is fully nonlinear (e.g., Hamilton-Jacobi-Bellman type), we verify the extra condition (6.13) and establish a complete theory including existence and comparison principle. Thereby, we extend the result of Diehl and Friz [11], which follows the second approach and studies the case when both g and f are semilinear. However, the verification of (6.13) in general cases is a challenging issue and requires further investigation.

Another main result of the paper is the equivalence between local classical solutions to RPDE (1.3) and those of the corresponding transformed PDE in the general fully nonlinear case. We provide sufficient conditions for the existence of local classical solutions to this PDE. Similar results have been proven in the stochastic setting by Da Prato and Tubaro [10] when g is linear (linear in u and $\partial_x u$). But to our best knowledge, the results for the general fully nonlinear case are new. We emphasize again that our PDE involves some serious singularity issues so that the local existence interval depends on the regularity of the classical solution (which in turn depends on the regularity of u_0). Consequently, our results are only valid for classical solutions. Whether the comparison principle for viscosity solutions of RPDE (1.3) with such generality can be shown directly remains an open problem.

The paper is organized as follows. In Section 2, we review the basic theory of rough paths, rough differential equations (RDEs), and the crucial rough Taylor expansions. Some results go beyond the standard literature and are proven in the supplement [2]. In Section 3, we set up the framework for SPDEs, RPDEs, and PPDEs. In Section 4, we introduce the crucial characteristic equations and transform the RPDE into a PDE. We establish the equivalence of their local classical solutions and provide sufficient conditions for their existence. Sections 5 and 6 are devoted to viscosity solutions in the general case. In Section 7, we establish the complete viscosity theory in the case that g is semilinear. Finally, in Section 8, we summarize our results.

To conclude this section, we remark that since the paper already involves rather heavy notations, to focus on the main idea and to simplify the presentation, we make three simplifications throughout the paper:

1) We restrict our work to a finite time horizon $[0, T]$. Our results can be easily extended to the infinite horizon case except that the generic constant C in various estimates will then depend on T and may explode when $T \to \infty$.

2) We restrict our work to a one dimensional setting. The extension to multidimensional rough paths is non-trivial but is standard in the literature. Most results in the paper hold true in multidimensional settings. We

provide further remarks when the extension to the multidimensional case is crucial or quite tricky. In particular, Proposition 4.1 relies on results for multidimensional RDEs. Throughout the paper, we use the notation:

$$\mathbb{R}_T := [0,T] \times \mathbb{R} \quad \text{and} \quad \mathbb{R}_T^k := [0,T] \times \mathbb{R}^k. \tag{1.5}$$

3) The paper involves higher order derivatives and related norms. Although we may track the proofs and figure out what regularity exactly is needed, we use norms involving all partial derivatives up to the same order. Our estimates suffice for our purpose but they are not necessarily sharp. Also, many results involve higher order regularities, whose orders are denoted with generic constants k without specifying the precise value of k.

2. Preliminary results from rough path theory.

We introduce the framework for rough path theory that is used in this paper. Only those results that are directly needed are considered. We mainly follow Keller and Zhang [26] (see Friz and Hairer [20] and the references therein for the general theory). Some less standard results will be proved in the supplement [2].

First, we introduce general notation. Given normed spaces E and V, put

$$\mathbb{L}^\infty(E;V) := \big\{u : E \to V : \|u\|_\infty := \sup_{x\in E}\|u(x)\|_V < \infty\big\}.$$

When $V = \mathbb{R}$, we omit V and just write $\mathbb{L}^\infty(E)$. For a constant $\alpha > 0$, set

$$C^\alpha(E;V) := \Big\{u \in \mathbb{L}^\infty(E;V) : [u]_\alpha := \sup_{x,y\in E, x\neq y} \frac{\|u(x)-u(y)\|_V}{\|x-y\|_E^\alpha} < \infty\Big\}.$$

Given functions $u : [0,T] \to \mathbb{R}$ and $\underline{u} : [0,T]^2 \to \mathbb{R}$, we write the time variable as subscript, i.e., $u_t = u(t)$ and $\underline{u}_{s,t} = \underline{u}(s,t)$, and we define

$$u_{s,t} := u_t - u_s, \; s,t \in [0,T], \qquad [\underline{u}]_\alpha := \sup_{s,t\in[0,T], s\neq t} |\underline{u}(s,t)|/|s-t|^\alpha. \tag{2.1}$$

Moreover, we shall use C to denote a generic constant in various estimates, which will typically depend on T and possibly on other parameters as well.

Furthermore, we define the standard Hölder spaces and parabolic Hölder spaces (cf. Lunardi [34, Chapter 5]): Given $k \in \mathbb{N}_0$ and $\beta \in (0,1]$, set

$$C_b^{k+\beta}(\mathbb{R}) := \big\{u : \mathbb{R} \to \mathbb{R} : \|u\|_{C_b^{k+\beta}(\mathbb{R})} < \infty\big\},$$
$$C_b^{\beta}(\mathbb{R}_T) := \big\{u \in C^0(\mathbb{R}_T) : \|u\|_{C_b^{\beta}(\mathbb{R}_T)} < \infty\big\},$$
$$C_b^{2+\beta}(\mathbb{R}_T) := \big\{u \in C^{1,2}(\mathbb{R}_T) : \|u\|_{C_b^{2+\beta}(\mathbb{R}_T)} < \infty\big\},$$

where $\|u\|_{C_b^{k+\beta}(\mathbb{R})} := \sum_{j=0}^{k}\|\partial_x^j u\|_\infty + [\partial^k u]_\beta,$

$$\|u\|_{C_b^{\beta}(\mathbb{R}_T)} := \|u\|_\infty + \sup_{t\in[0,T]}[u(t,\cdot)]_\beta + \sup_{x\in\mathbb{R}}[u(\cdot,x)]_{\beta/2},$$

$$\|u\|_{C_b^{2+\beta}(\mathbb{R}_T)} := \sum_{j=0}^{1}\|\partial_x^j u\|_\infty + \|\partial_{xx}^2 u\|_{C_b^{\beta}(\mathbb{R}_T)} + \|\partial_t u\|_{C_b^{\beta}(\mathbb{R}_T)}.$$

2.1. *Rough path differentiation and integration.* From now on, unless otherwise stated, fix two parameters $\alpha \in (1/3, 1/2]$ and $\beta \in (0,1]$ satisfying

$$\alpha(2+\beta) > 1. \tag{2.2}$$

Let $\hat\omega := (\omega, \underline{\omega})$ be a rough path in the following sense: (i) $\omega \in C^\alpha([0,T])$ and $\underline{\omega}_{s,t} := (1/2)|\omega_{s,t}|^2$; (ii) $\hat\omega$ is *truly rough*, i.e., there is a set A such that

$$\overline{\lim_{t\downarrow s}}\frac{|\omega_{s,t}|}{|t-s|^{\alpha(1+\beta)}} = \infty \text{ for all } s\in A \text{ and } A \text{ is dense in } [0,T]. \tag{2.3}$$

Remark 2.1 (i) The second component $\underline{\omega}$ maps $[0,T]^2$ to $\mathbb{R}$ with $[\underline{\omega}]_{2\alpha} < \infty$. Note that $\underline{\omega}_{s,t}$ should not be understood as $\underline{\omega}_t - \underline{\omega}_s$ as in (2.1).

(ii) For a general d-dimensional rough path, $\underline{\omega} : [0,T]^2 \to \mathbb{R}^{d\times d}$ has 2α-regularity in the sense $[\underline{\omega}]_{2\alpha} < \infty$, and it satisfies Chen's relation, i.e., $\underline{\omega}_{s,t} - \underline{\omega}_{s,r} - \underline{\omega}_{r,t} = \omega_{s,r}\,\omega_{r,t}^\top$, $s,\ t,\ r \in [0,T]$, where $^\top$ denotes the transpose. Moreover, $\hat\omega$ is called a geometric rough path if $\underline{\omega}_{s,t} + \underline{\omega}_{s,t}^\top = \omega_{s,t}\,\omega_{s,t}^\top$, $s,\ t \in [0,T]$. In our setting, $(\omega,\underline{\omega})$ is a geometric rough path and the related integration theory corresponds to Stratonovic integration.

(iii) In standard rough path theory, it is typically not required that $\hat\omega$ is truly rough as defined in (2.3). But it is convenient for us because under (2.3) the rough path derivatives we define next will be unique.

Next, we introduce path derivatives with respect to our rough path. To this end, we introduce spaces of multi-indices

$$\mathcal{V}_n := \{0,1\}^n, \quad \|\nu\| := \sum_{i=1}^{n}[2\mathbf{1}_{\{\nu_i=0\}} + \mathbf{1}_{\{\nu_i=1\}}] \text{ for } \nu = (\nu_1,\cdots,\nu_n)\in\mathcal{V}_n.$$

Definition 2.2 *(a) Let $u \in C^\alpha([0,T])$ and $C^0_{\alpha,\beta}([0,T]) := C^{\alpha\beta}([0,T])$.*

(i) A first order spatial derivative of u is a $\partial_\omega u \in C^0_{\alpha,\beta}([0,T])$ that satisfies

$$u_{s,t} = \partial_\omega u_s\,\omega_{s,t} + R^{1,u}_{s,t}, \quad s,t\in[0,T], \quad \textit{with} \quad [R^{1,u}]_{\alpha(1+\beta)} < \infty. \tag{2.4}$$

(ii) Assume that $\partial_\omega u \in C^\alpha([0,T])$ exists and has a derivative $\partial_\omega\partial_\omega u$, then a temporal derivative of u is a $\partial_t^\omega u \in C^0_{\alpha,\beta}([0,T])$ that satisfies

$$\begin{aligned} u_{s,t} = \partial_t^\omega u_s\ [t-s] + \partial_\omega u_s\ \omega_{s,t} + \partial_\omega\partial_\omega u_s\ \underline{\omega}_{s,t} + R^{2,u}_{s,t},\ s,t\in[0,T],\\ \textit{with } [R^{2,u}]_{\alpha(2+\beta)} < \infty. \end{aligned} \tag{2.5}$$

(iii) For $\nu \in \mathcal{V}_n$, $\mathcal{D}_\nu u := \partial_{\nu_1} \cdots \partial_{\nu_n} u$, *where* $\partial_0 := \partial_t^\omega$ *and* $\partial_1 := \partial_\omega$.

(b) For $k \geq 1$, *let* $C^k_{\alpha,\beta}([0,T]) := \{u \in C^\alpha([0,T]) : \mathcal{D}_\nu u$ *exists* $\forall \|\nu\| \leq k\}$.

Remark 2.3 (i) In the rough path literature, a first order spatial derivative $\partial_\omega u$ is typically called Gubinelli derivative and the corresponding function u is called a *controlled rough path*. In our case, the path derivatives defined above are unique due to $\hat{\omega}$ being truly rough (see [20, Proposition 6.4]).

(ii) The derivative $\partial_\omega u$ depends on ω, but not on $\underline{\omega}$. The derivative $\partial_t^\omega u$ depends on $\underline{\omega}$ as well and should be denoted by $\partial_t^{\hat{\omega}} u$. However, in our setting $\underline{\omega}$ is a function of ω and thus we can write $\partial_t^\omega u$ instead.

(iii) When $\partial_\omega u = 0$, it follows from (2.5) and (2.2) that u is differentiable in t and $\partial_t^\omega u = \partial_t u$, the standard derivative with respect to t.

(iv) In the multidimensional case, $\partial_{\omega\omega} u \in \mathbb{R}^{d\times d}$ could be symmetric if u is smooth enough (see [7, Remark 3.3]); i.e., ∂_{ω^i} and ∂_{ω^j} commute for $1 \leq i$, $j \leq d$. However, typically ∂_t^ω and ∂_ω do not commute, even when $d = 1$.

Remark 2.4 Note that in (2.5) the term $t - s$ is the difference of the identity function $t \mapsto t$, which is Lipschitz continuous. For all estimates below, it suffices to assume $\partial_t^\omega u \in C^{\alpha(2+\beta)-1}([0,T])$. However, to make the estimates more homogeneous, we only use the Hölder-2α regularity of t and thus require $\partial_t^\omega u \in C^{\alpha\beta}([0,T])$. For this same reason, all our estimates will actually hold true if we replace t with a Hölder-2α continuous path $\zeta \in C^{2\alpha}([0,T])$. To be more precise, we define a path derivative of u with respect to ζ as a function $\partial_\zeta^\omega u \in C^0_{\alpha,\beta}([0,T])$ that satisfies $[R^{2,u}]_{\alpha(2+\beta)} < \infty$, where

$$u_{s,t} = \partial_\zeta^\omega u_s \; \zeta_{s,t} + \partial_\omega u_s \; \omega_{s,t} + \partial_{\omega\omega} u_s \; \underline{\omega}_{s,t} + R^{2,u}_{s,t}, \tag{2.6}$$

then Lebesgue integration dt should be replaced with Young integration $d\zeta_t$.

Next, we equip $C^k_{\alpha,\beta}([0,T])$ with a norm $\|\cdot\|_k$. Given $u \in C^k_{\alpha,\beta}([0,T])$, put

$$\begin{aligned} &\|u\|_0 := |u_0| + [u]_{\alpha\beta}; \quad \|u\|_1 := \|u\|_0 + \|\partial_\omega u\|_0 + [R^{1,u}]_{\alpha(1+\beta)}; \\ &\|u\|_2 := \|u\|_1 + \|\partial_\omega u\|_1 + \|\partial_t^\omega u\|_0 + [R^{2,u}]_{\alpha(2+\beta)}; \\ &\|u\|_k := \|u\|_{k-1} + \|\partial_\omega u\|_{k-1} + \|\partial_t^\omega u\|_{k-2}, \quad k \geq 3. \end{aligned} \tag{2.7}$$

We emphasize that, besides k, the norms depend on T, ω, α, and β as well. To simplify the notation, we do not indicate these dependencies explicitly. In some places we restrict u to some subinterval $[t_1, t_2] \subset [0,T]$. Corresponding spaces $C^k_{\alpha,\beta}([t_1,t_2])$ are defined in an obvious way. To not further complicate the notation, the corresponding norm is still denoted by $\|\cdot\|_k$. Note that, for $u \in C^1_{\alpha,\beta}([0,T])$ and for a constant C depending on ω,

$$|u_t| \leq |u_0| + |\partial_\omega u_0| [\omega]_\alpha t^\alpha + [R^{1,u}]_{\alpha(1+\beta)} t^{\alpha(1+\beta)} \leq |u_0| + C\|u\|_1 t^\alpha. \tag{2.8}$$

Finally, we define the rough integral of $u \in C^1_{\alpha,\beta}([0,T])$. Let $\pi : 0 = t_0 < \cdots < t_n = T$ be a time partition and $|\pi| := \max_{0\le i\le n-1} |t_{i+1} - t_i|$. By [24],

$$\int_0^t u_s\, d\omega_s := \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \big[u_{t_i}\omega_{t_i\wedge t,\ t_{i+1}\wedge t} + \partial_\omega u_{t_i}\ \underline{\omega}_{t_i\wedge t,\ t_{i+1}\wedge t}\big]. \tag{2.9}$$

exists and defines the rough integral. The integration path $U_t := \int_0^t u_s\, d\omega_s$ belongs to $C^1_{\alpha,\beta}([0,T])$ with $\partial_\omega U_t = u_t$ and we define $\int_s^t u_r\, d\omega_r := U_{s,t}$.

In this context, we define iterated integrals as follows. For $\nu \in \mathcal{V}_n$, set

$$\mathcal{I}^\nu_{s,t} := \int_s^t \int_s^{t_n} \cdots \int_s^{t_2} d_{\nu_1} t_1 \cdots d_{\nu_n} t_n, \text{ where } d_0 t := dt,\ d_1 t = d\omega_t. \tag{2.10}$$

One can check that $\mathcal{I}^\mu_{s,t} = \int_s^t \mathcal{I}^{(\mu_1,\cdots,\mu_n)}_{s,r}\, d_{\mu_{n+1}} r$ for $\mu = (\mu_1,\cdots,\mu_{n+1}) \in \mathcal{V}_{n+1}$. In the multidimensional case, defining iterated integrals is not trivial. Nevertheless, by [35, Theorem 2.2.1], this can be accomplished via uniquely determined (higher-order) extensions of the geometric rough path $\hat\omega = (\omega, \underline{\omega})$.

By (2.5) and (2.2), the following result is obvious and we omit the proof.

Lemma 2.5 *(i) If $u \in C^2_{\alpha,\beta}([0,T])$, then*

$$du_t = \partial^\omega_t u_t\, dt + \partial_\omega u_t\, d\omega_t. \tag{2.11}$$

(ii) Suppose that $u_t = u_0 + \int_0^t a_s\, ds + \int_0^t \eta_s\, d\omega_s$ with $a \in C^0_{\alpha,\beta}([0,T])$ and $\eta \in C^1_{\alpha,\beta}([0,T])$. Then $u \in C^2_{\alpha,\beta}([0,T])$ with $\partial^\omega_t u = a$ and $\partial_\omega u = \eta$. Moreover, $[R^{2,u}]_{\alpha(2+\beta)} \le C(\|a\|_0 + \|\eta\|_1)$.

Finally, we introduce backward rough paths. Fix $t_0 \in (0,T]$. Set

$$\overset{\leftarrow t_0}{\omega}_t := \omega_{t_0} - \omega_{t_0-t}, \quad \overset{\leftarrow t_0}{\underline{\omega}}_{s,t} := \frac{1}{2}|\overset{\leftarrow t_0}{\omega}_{s,t}|^2. \tag{2.12}$$

Then $(\overset{\leftarrow t_0}{\omega}, \overset{\leftarrow t_0}{\underline{\omega}})$ is a rough path on $[0,t_0]$. Moreover, for $u \in C^1_{\alpha,\beta}([0,t_0])$, the function $\overset{\leftarrow t_0}{u}$ defined by $\overset{\leftarrow t_0}{u}_t := u_{t_0-t}$ belongs to $C^1_{\alpha,\beta}([0,t_0])$ with $\hat\omega$ replaced by $(\overset{\leftarrow t_0}{\omega}, \overset{\leftarrow t_0}{\underline{\omega}})$ in Definition 2.2. In this case, $\int_0^{t_0} \overset{\leftarrow t_0}{u}_s\ d\,\overset{\leftarrow t_0}{\omega}_s = \int_0^{t_0} u_s\, d\omega_s$.

2.2. *Rough differential equations.* We start with controlled rough paths with parameter $x \in \mathbb{R}^d$. They serve as solutions to RPDEs and coefficients for RDEs and RPDEs. For this purpose, we have to allow $d > 1$ here. Consider a function $u : \mathbb{R}^d_T \to \mathbb{R}$. If, for fixed $x \in \mathbb{R}^d$, the mapping $t \mapsto u(t,x)$ is a controlled rough path, we use the notations $\partial_\omega u$, $\partial^\omega_t u$, $\mathcal{D}_\nu u$ to denote the path derivatives as in the previous subsection. For fixed t, we use $\partial_x u$, $\partial^2_{xx} u$, etc., to denote the derivatives of $x \mapsto u(t,x)$ with respect to x. Now, we introduce the appropriate spaces, extending Definition 2.2.

Definition 2.6 *Let $[t_1, t_2] \subset [0, T]$, $O \subset \mathbb{R}^d$ be convex, $u \in C^0([t_1, t_2] \times O)$.*

(i) We say $u \in C^{0,loc}_{\alpha,\beta}([t_1, t_2] \times O)$ if the following holds:

- *$x \mapsto u(\cdot, x)$ maps O into $C^0_{\alpha,\beta}([t_1, t_2])$ and is continuous under $\|\cdot\|_0$.*
- *$x \mapsto u(t, x)$ are locally Hölder-β continuous, uniformly in $t \in [t_1, t_2]$.*

(ii) We say $u \in C^{1,loc}_{\alpha,\beta}([t_1, t_2] \times O)$ if the following holds:

- *$x \mapsto u(\cdot, x)$ maps O into $C^1_{\alpha,\beta}([t_1, t_2])$ and is continuous under $\|\cdot\|_1$.*
- *$\partial_\omega u \in C^{0,loc}_{\alpha,\beta}([t_1, t_2] \times O)$ and $\partial_x u \in C^{0,loc}_{\alpha,\beta}([t_1, t_2] \times O; \mathbb{R}^d)$, in the sense that each component $\partial_{x_i} u \in C^{0,loc}_{\alpha,\beta}([t_1, t_2] \times O)$, $i = 1, \ldots, d$.*

(iii) We say $u \in C^{2,loc}_{\alpha,\beta}([t_1, t_2] \times O)$ if the following holds:

- *$x \mapsto u(\cdot, x)$ maps O into $C^2_{\alpha,\beta}([t_1, t_2])$ and is continuous under $\|\cdot\|_2$.*
- *$\partial_\omega u \in C^{1,loc}_{\alpha,\beta}([t_1, t_2] \times O)$, $\partial_x u \in C^{1,loc}_{\alpha,\beta}([t_1, t_2] \times O; \mathbb{R}^d)$, and $\partial^\omega_t u \in C^{0,loc}_{\alpha,\beta}([t_1, t_2] \times O)$; for all $x \in O$, $[\partial_x R^{1,u}(x)]_{\alpha(1+\beta)} < \infty$.*

(iv) For $k \geq 3$, we say $u \in C^{k,loc}_{\alpha,\beta}([t_1, t_2] \times O)$ if u, $\partial_\omega u \in C^{k-1,loc}_{\alpha,\beta}([t_1, t_2] \times O)$, $\partial_x u \in C^{k-1,loc}_{\alpha,\beta}([t_1, t_2] \times O; \mathbb{R}^d)$, and $\partial^\omega_t u \in C^{k-2,loc}_{\alpha,\beta}([t_1, t_2] \times O)$.

Lemma 2.7 *(i) Let $u \in C^{2,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$. Then ∂_ω and ∂_x commute, i.e.,*

$$\partial_\omega \partial_x u = \partial_x \partial_\omega u. \tag{2.13}$$

Assume further that $u \in C^{3,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$. Then ∂^ω_t and ∂_x commute, i.e.,

$$\partial^\omega_t \partial_x \varphi = \partial_x \partial^\omega_t \varphi. \tag{2.14}$$

(ii) If $u \in C^{1,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$, then, for any bounded domain $O \subset \mathbb{R}^d$,

$$\int_s^t \int_O u(r, x)\, dx\, d\omega_r = \int_O \int_s^t u(r, x)\, d\omega_r\, dx. \tag{2.15}$$

The next result is the crucial chain rule (cf. [26, Theorem 3.4]).

Lemma 2.8 *Assume that $\varphi \in C^{1,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$ and $X \in C^1_{\alpha,\beta}([0, T]; \mathbb{R}^d)$. Let $Y_t := \varphi(t, X_t)$. Then $Y \in C^1_{\alpha,\beta}([0, T])$ and it holds*

$$\partial_\omega Y_t = \partial_\omega \varphi(t, X_t) + \partial_x \varphi(t, X_t) \cdot \partial_\omega X_t. \tag{2.16}$$

If $\varphi \in C^{2,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$, $X \in C^2_{\alpha,\beta}([0, T]; \mathbb{R}^d)$, then $Y \in C^2_{\alpha,\beta}([0, T])$ and

$$\partial^\omega_t Y_t = \partial^\omega_t \varphi(t, X_t) + \partial_x \varphi(t, X_t) \cdot \partial^\omega_t X_t. \tag{2.17}$$

Our study relies heavily on the following rough Taylor expansion. The result holds true for multidimensional cases as well and we emphasize that the numbers δ below can be negative.

Lemma 2.9 *Let $u \in C^{k,loc}_{\alpha,\beta}(\mathbb{R}_T)$ and $K \subset \mathbb{R}$ be compact. Then, for every $(t,x) \in \mathbb{R}_T$ and $(\delta,h) \in \mathbb{R}^2$ with $t+\delta \in [0,T]$ and $x+h \in K$, we have $|R^{k,u}_{t,x;\delta,h}| \le C(K,x)\,(|\delta|^\alpha + |h|)^{k+\beta}$, where*

$$(2.18)\quad u(t+\delta, x+h) = \sum_{m=0}^{k} \sum_{\|\nu\| \le k-m} \frac{1}{m!} \mathcal{D}_\nu \partial_x^m u(t,x)\ h^m\ \mathcal{I}^\nu_{t,t+\delta} + R^{k,u}_{t,x;\delta,h}.$$

To study RDEs, uniform properties for the functions in $C^{k,loc}_{\alpha,\beta}([t_1,t_2]\times O)$ are needed. In the next definition, we abuse the notation $\|\cdot\|_k$ from (2.7).

Definition 2.10 *(i) We say $u \in C^k_{\alpha,\beta}([t_1,t_2]\times O) \subset C^{k,loc}_{\alpha,\beta}([t_1,t_2]\times O)$ if*

$$(2.19)\qquad \|u\|_k := \sum_{i=0}^{k} \sup_{x\in O} \|\partial_x^i u(\cdot,x)\|_{k-i} < \infty.$$

(ii) For solutions to standard PDEs (recall Remark 2.3 (iii)), we use

$$(2.20)\quad C^{k,0}_{\alpha,\beta}([t_1,t_2]\times O) := \Big\{u \in C^k_{\alpha,\beta}([t_1,t_2]\times O) : \partial_\omega u = 0\Big\}.$$

We remark that in (i) we do not require $\sup_{t\in[t_1,t_2]}[\partial_x^k u(t,\cdot)]_\beta < \infty$, but restrict ourselves to local Hölder continuity with respect to x (uniformly in t), which suffices for our rough Taylor expansion above.

Although functions in $C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$ are, in general, only at most once differentiable in time, they behave in our rough path framework as if they were k times differentiable in time (cf. also the discussion in [20, Section 13.1]).

Remark 2.11 (i) If $u : [t_1,t_2]\times O \to \mathbb{R}$ satisfies $\|u\|_{k+1} < \infty$ (as in (2.19)), then $u(t,x+h) - u(t,x) = h\int_0^1 \partial_x u(t,x+\lambda h)\,d\lambda$. Thus the mapping $x \mapsto u(\cdot,x)$, $O \to C^k_{\alpha,\beta}([t_1,t_2])$, is continuous under $\|\cdot\|_k$ (as defined in (2.7)) and, for $\|\nu\| = k$, $\mathcal{D}_\nu u(t,\cdot)$ is Hölder-β continuous, uniformly in t. Hence, the continuity required in the definition of $C^{k,loc}_{\alpha,\beta}(\mathbb{R}^d_T)$ is automatic.

(ii) Similarly, if $u \in C^{k+1}_{\alpha,\beta}(\mathbb{R}^d_T \times \mathbb{R}^{d'})$, then $y \mapsto u(\cdot,y)$, $\mathbb{R}^{d'} \to C^{k+1}_{\alpha,\beta}(\mathbb{R}^d_T)$, is continuous under $\|\cdot\|_k$ (as defined in (2.19)).

Now, we study rough differential equations of the form

$$(2.21)\qquad u_t = u_0 + \int_0^t f(s,u_s)\,ds + \int_0^t g(s,u_s)\,d\omega_s.$$

Lemma 2.12 *If $f \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ and $g \in C^{k+1}_{\alpha,\beta}(\mathbb{R}_T)$ for some $k \geq 2$, then RDE* (2.21) *has a unique solution $u \in C^{k+2}_{\alpha,\beta}([0,T])$ and*

$$\|u - u_0\|_{k+2} \leq C(T, \|f\|_k, \|g\|_{k+1}). \tag{2.22}$$

In the following linear case, we have a representation formula for u:

$$u_t = u_0 + \int_0^t [f_0(s) + f_1(s)u_s]\, ds + \int_0^t [g_0(s) + g_1(s)u_s]\, d\omega_s. \tag{2.23}$$

Lemma 2.13 *If f_0, $f_1 \in C^k_{\alpha,\beta}([0,T])$ and g_0, $g_1 \in C^{k+1}_{\alpha,\beta}([0,T])$ for some $k \geq 2$, then RDE* (2.23) *has a unique solution $u \in C^{k+2}_{\alpha,\beta}([0,T])$ given by*

$$u_t = \Gamma_t\big[u_0 + \int_0^t \frac{f_0(s)}{\Gamma_s}\, ds + \int_0^t \frac{g_0(s)}{\Gamma_s}\, d\omega_s\big], \tag{2.24}$$

where $\Gamma_t := \exp\big\{ \int_0^t f_1(s)\, ds + \int_0^t g_1(s)\, d\omega_s \big\}$.

Remark 2.14 This representation holds only true in the one dimensional case. For multidimensional linear RDEs, Keller and Zhang [26] derived a semi-explicit representation formula. Moreover, note that (2.23) actually does not satisfy the technical conditions in Lemma 2.12 (f and g are not bounded). But nevertheless, due to its special structure, RDE (2.23) is well-posed as shown in this lemma.

Finally, we extend Lemma 2.12 to RDEs with parameters of the form

$$u(t,x) = u_0(x) + \int_0^t f(s,x,u(s,x))\, ds + \int_0^t g(s,x,u(s,x))\, d\omega_s. \tag{2.25}$$

Lemma 2.15 *Assume that $u_0 \in C^{k+\beta}(\mathbb{R})$, $f \in C^{k+1}_{\alpha,\beta}(\mathbb{R}^2_T)$, and $g \in C^{k+1}_{\alpha,\beta}(\mathbb{R}^2_T)$ for some $k \geq 3$. Then $u \in C^{k,loc}_{\alpha,\beta}(\mathbb{R}_T)$ and $\partial_x u$ solves*

$$\begin{aligned}\partial_x u(t,x) = \partial_x u_0(x) &+ \int_0^t \big[\partial_x f(s,x,u(s,x)) + \partial_y f(s,x,u(s,x)) \cdot \partial_x u(s,x)\big]\, ds \\ &+ \int_0^t \big[\partial_x g(s,x,u(s,x)) + \partial_y g(s,x,u(s,x)) \cdot \partial_x u(s,x)\big]\, d\omega_s.\end{aligned} \tag{2.26}$$

If all the related derivatives of u_0 are bounded (but not necessary u_0 itself), then $u - u_0 \in C^k_{\alpha,\beta}(\mathbb{R}_T)$. If u_0 is bounded, then $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$.

3. Stochastic PDEs, rough PDEs, and path-dependent PDEs. Our initial goal is to study the fully nonlinear stochastic PDE

$$du(t,x,B.) = f(t,x,B.,u,\partial_x u,\partial^2_{xx}u) \; + g(t,x,B.,u,\partial_x u)\circ dB_t. \tag{3.1}$$

Here, B is a standard Brownian motion, $\circ$ denotes the Stratonovich integral, and f and g are $\mathbb{F}^B$-progressively measurable.

To transform (3.1) into a rough PDE, we need to introduce some notation. Let $\Omega_0 := \{\omega \in C^0([0,T]) : \omega_0 = 0\}$, B the canonical process on Ω_0, i.e., $B(\omega) := \omega$, $\mathbb{P}_0$ the Wiener measure, and

$$\Omega := \textstyle\bigcup_{1/3<\alpha<1/2}\Omega_\alpha, \quad \Omega_\alpha := \big\{\omega \in \Omega_0 : [\omega]_\alpha < \infty \text{ and (2.3) holds}\big\}. \tag{3.2}$$

Then $\mathbb{P}_0(\Omega) = 1$ (cf. [20, Theorem 6.6]). Moreover, consider the space

$$\mathcal{C}(\Omega) := \bigcup \big\{\mathcal{C}_{\alpha,\beta}(\Omega) : \alpha \in (1/3,1/2),\, \beta \in (0,1], \text{ and (2.2) holds}\big\}, \tag{3.3}$$

where $\mathcal{C}_{\alpha,\beta}(\Omega)$ is the set of all $\mathbb{F}$-progressively measurable real processes $(u_t)_{t\in[0,T]}$ on Ω with $\mathbb{E}^{\mathbb{P}_0}[\|u\|^2_{B;1}] < \infty$ and with $u(\omega) \in C^1_{\omega;\alpha,\beta}([0,T])$ for all $\omega \in \Omega$. Here $\|\cdot\|_{\omega;1}$ and $C^1_{\omega;\alpha,\beta}([0,T])$ are defined by (2.7) and Definition 2.2, respectively, with indication of the dependence on ω.

Then, for $u \in \mathcal{C}(\Omega)$, we have

$$\Big(\int_0^t u_s \circ dB_s\Big)(\omega) = \int_0^t u_s(\omega)\, d\omega_s, \quad 0 \le t \le T, \text{ for } \mathbb{P}_0\text{-a.e. } \omega \in \Omega. \tag{3.4}$$

Here, the left hand side is a Stratonovic integral while the right hand side is a rough path integral. In this sense, we may write SPDE (3.1) as the RPDE

$$du(t,x,\omega) = f(t,x,\omega,u,\partial_x u,\partial^2_{xx}u)\, dt + g(t,x,\omega,u,\partial_x u)\, d\omega_t,\ \omega \in \Omega. \tag{3.5}$$

Remark 3.1 (i) If u is a classical solution of (3.1) with $g(\cdot,x,B.,u,\partial_x u) \in \mathcal{C}(\Omega)$ for all $x \in \mathbb{R}$, then, by (3.4), RPDE (3.5) holds true for $\mathbb{P}_0$-a.e. $\omega \in \Omega$.

(ii) In the earlier version of this paper (see arXiv:1501.06978v1), we studied pathwise viscosity solutions of SPDE (3.1) in the a.s. sense. In this version, we study instead the wellposedness of RPDE (3.5) for fixed ω. This is easier and more convenient. Moreover, the rough path framework allows us to prove crucial perturbation results such as Lemma 5.8.

(iii) If we have obtained a solution (in classical or viscosity sense) $u(\cdot,\omega)$ of RPDE (3.5) for each ω, to go back to SPDE (3.1), one needs to verify the measurability and integrability of the mapping $\omega \mapsto u(\cdot,\omega)$. To do so, one can, in principle, apply the strategy in [10, Section 3], which relies on construction of solutions to SDEs via iteration so that adaptedness is preserved. This strategy can be applied in our setting and does not require f and g to be continuous in ω. Another possible approach is to follow the argument in [20, Section 9.1], which is in the direction of stability and norm estimates but requires at least g to be continuous in ω. Since the paper is already too lengthy, we do not pursue these approaches here in detail.

From now on, we shall fix (α,β) and ω as in Subsection 2.1 and omit ω in f, g, and u. To be precise, the goal of this paper is to study the RPDE

$$du(t,x) = f(t,x,u,\partial_x u,\partial^2_{xx}u)\,dt + g(t,x,u,\partial_x u)d\omega_t \tag{3.6}$$

with initial condition $u(0,x) = u_0(x)$. Note that $u(t,x)$ implicitly depends on ω. In particular, $\partial^\omega_t u$ is different from $\partial_t u$ in the standard PDE literature. Moreover, by Lemma 2.5, we may write (3.6) as the path dependent PDE

$$\partial^\omega_t u(t,x) = f(t,x,u,\partial_x u,\partial^2_{xx}u), \quad \partial_\omega u(t,x) = g(t,x,u,\partial_x u). \tag{3.7}$$

with initial condition. The arguments of f and g are implicitly denoted as $f(t,x,y,z,\gamma)$ and $g(t,x,y,z)$. Throughout the paper, the following assumptions shall be employed.

Assumption 3.2 *Let $g \in C^{k_0,loc}_{\alpha,\beta}(\mathbb{R}^3_T)$ for some sufficiently large regularity index $k_0 \in \mathbb{N}$.*

(i) $\partial_y g \in C^{k_0-1}_{\alpha,\beta}(\mathbb{R}^3_T)$, $\partial_z g \in C^{k_0-1}_{\alpha,\beta}(\mathbb{R}^3_T)$.

(ii) For $i = 0, \ldots, k_0$ and $(y,z) \in \mathbb{R}^2$, $\partial^i_x g(\cdot,y,z) \in C^{k_0-i}_{\alpha,\beta}(\mathbb{R}_T)$ with

$$\|\partial^i_x g(\cdot,y,z)\|_{k_0-i} \le C[1+|y|+|z|].$$

Note that, for any bounded set $Q \subset \mathbb{R}^2$, $g \in C^{k_0}_{\alpha,\beta}(\mathbb{R}_T \times Q)$.

Assumption 3.3 *(i) f is nondecreasing in γ.*

(ii) $f \in C^0(\mathbb{R}^4_T)$ and $|f(t,x,0,0,0)| \le K_0$ for all $(t,x) \in \mathbb{R}_T$.

(iii) f is uniformly Lipschitz in (y,z,γ) with Lipschitz constant L_0.

Assumption 3.4 *Let u_0 be uniformly continuous and $\|u_0\|_\infty \le K_0$.*

We remark that there is no comparison principle in terms of g. Hence, a smooth approximation of g does not help for our purpose and thus we require g to be smooth. By more careful arguments, we may figure out the precise value of k_0, but that would make the paper less readable. In the rest of the paper, we use k to denote a generic index for regularity, which may vary from line to line. We assume always that k is large enough so that we can apply all the results in Section 2 freely, and we assume that the regularity index k_0 in Assumption 3.2 is large enough so that we have the desired k-regularity in the related results.

Definition 3.5 *Let $u \in C^{2,loc}_{\alpha,\beta}(\mathbb{R}_T)$. We say u is a classical solution (resp. subsolution, supersolution) of RPDE* (3.6) *if*

$$\begin{aligned} &\partial_\omega u(t,x) = g(t,x,u,\partial_x u), \\ &\mathcal{L}u(t,x) := \partial^\omega_t u(t,x) - f(t,x,u,\partial_x u,\partial^2_{xx} u) = \ (\textit{resp.} \le, \ge)\ 0. \end{aligned} \tag{3.8}$$

Note again that there is no comparison principle in terms of g. So the first line in (3.8) is an equality even for sub/super-solutions.

4. Classical solutions of rough PDEs. We establish wellposedness of classical solutions for RPDE (3.6). To this end, we have to require that the coefficients f, g and the initial value u_0 are sufficiently smooth. For general RPDEs, most results are valid only locally in time. However, this will turn out to be sufficient for our study of viscosity solutions in the next sections.

4.1. *The characteristic equations.* Our main tool is the *method of characteristics* (see Kunita [28] for the stochastic setting). It will be used to get rid of the diffusion term g and to transform the RPDE to a standard PDE. Given $\theta := (x,y,z) \in \mathbb{R}^3$, consider the coupled system of RDEs

$$\begin{aligned} X_t &= x - \int_0^t \partial_z g(s,\Theta_s)\, d\omega_s, \\ Y_t &= y + \int_0^t \big[g(s,\Theta_s) - Z_s \partial_z g(s,\Theta_s)\big]\, d\omega_s, \\ Z_t &= z + \int_0^t \big[\partial_x g(s,\Theta_s) + Z_s \partial_y g(s,\Theta_s)\big]\, d\omega_s. \end{aligned} \tag{4.1}$$

Its solution is denoted by $\Theta_t(\theta) := (X_t(\theta), Y_t(\theta), Z_t(\theta))$. Fix $K_0 > 0$ and put

$$Q := \mathbb{R} \times Q_2, \quad Q_2 := \{(y,z) \in \mathbb{R}^2 : \max\{|y|,|z|\} \le K_0 + 1\} \subset \mathbb{R}^2. \tag{4.2}$$

Proposition 4.1 *Let Assumption 3.2 hold and let $K_0 \ge 0$ be a constant. Then there exist constants $\delta_0 > 0$ and C_0, depending only on K_0 and the k_0-th norm of g (in the sense of Definition 2.10 (i)) on $[0,T] \times Q$, such that for all $\theta \in Q$, the system* (4.1) *has a unique solution $\Theta(\theta)$ such that*

$$\tilde\Theta \in C^{k_0}_{\alpha,\beta}([0,\delta_0] \times Q; \mathbb{R}^3),\ \|\tilde\Theta\|_{k_0} \le C_0,\ \textit{where } \tilde\Theta_t(\theta) := \Theta_t(\theta) - \theta. \tag{4.3}$$

Proof Uniqueness follows directly from an appropriate multi-dimensional extension of Lemma 2.12 for each $\theta \in Q$. To prove existence, we note that the main difficulty here is that some coefficients in (4.1) are not bounded. To deal with this difficulty, we introduce, for each $N > 0$, a smooth truncation function $\iota^N : \mathbb{R} \to \mathbb{R}$ with $\iota^N(z) = z$ for $|z| \le N$ and $\iota^N(z) = 0$ for $|z| > N+1$, and consider $g^N(t,\theta) := g(t,x,\iota^N(y),\iota^N(z))$. Then, by Assumption 3.2, $g^N \in C^{k_0}_{\alpha,\beta}(\mathbb{R}^3_T)$. Next, for each $\theta \in \mathbb{R}^3$, consider the system

$$\begin{aligned}
X^N_t &= x - \textstyle\int_0^t \partial_z g^N(s,\Theta^N_s)\, d\omega_s,\\
Y^N_t &= y + \textstyle\int_0^t \big[g^N(s,\Theta^N_s) - \iota^N(Z^N_s)\partial_z g^N(s,\Theta^N_s)\big]\, d\omega_s,\\
Z^N_t &= z + \textstyle\int_0^t \big[\partial_x g^N(s,\Theta^N_s) + \iota^N(Z^N_s)\partial_y g^N(s,\Theta^N_s)\big]\, d\omega_s.
\end{aligned}$$

Applying Lemma 2.15, but extended to the multidimensional case (using the extended Lemma 2.13 as shown in Remark 2.14), the RDE above has a unique solution $\Theta^N(\theta) = (X^N, Y^N, Z^N)(\theta) \in C^{k_0}_{\alpha,\beta}([0,T];\mathbb{R}^3)$ and satisfies (4.3) with a constant $C_N := C(N,T,\|g^N\|_{k_0})$. Now set $N := K_0 + 1$. For $(t,\theta) \in [0,T] \times Q$, it follows from (2.8) that

$$|Y^N_t(\theta)| \le K_0 + C_N t^\alpha, \quad |Z^N_t(\theta)| \le K_0 + C_N t^\alpha.$$

Set $\delta_0 := C_N^{-1/\alpha} \wedge T$. Then, for $t \le \delta_0$, we have $|Y^N_t(\theta)|$ and $|Z^N_t(\theta)| \le N$. Thus $g^N(\Theta^N_t) = g(\Theta^N_t)$. Therefore, Θ^N solves the original untruncated equation (4.1) on $[0,\delta_0]$. ■

Next, we linearize system (4.1). To this end, put

$$U := [\partial_x X, \partial_y X, \partial_z X],\ V := [\partial_x Y, \partial_y Y, \partial_z Y],\ W := [\partial_x Z, \partial_y Z, \partial_z Z]. \tag{4.4}$$

Then

$$\begin{aligned}
U_t &= [1,0,0] - \int_0^t \Big[\partial_{xz}g U_s + \partial_{yz}g V_s + \partial_{zz}g\, W_s\Big](s,\Theta_s)\, d\omega_s,\\
V_t &= [0,1,0] + \int_0^t \Big[\big[\partial_x g - \partial_{xz}g\, Z_s\big]U_s + \big[\partial_y g - Z_s\partial_{yz}g\big]\, V_s\\
&\qquad\qquad - Z_s\partial_{zz}g\, W_s\Big](s,\Theta_s)\, d\omega_s,\\
W_t &= [0,0,1] + \int_0^t \Big[\big[\partial^2_{xx}g + Z_s\partial_{xy}g\big]\, U_s + [\partial_{xy}g + \partial_{yy}g\, Z_s]V_s\\
&\qquad\qquad + \big[\partial_{xz}g + Z_s\partial_{yz}g + \partial_y g\big]\, W_s\Big](s,\Theta_s)\, d\omega_s.
\end{aligned} \tag{4.5}$$

The next result is due to Peter Baxendale. It is a slight generalization of [28, (14) on p. 291] (which corresponds to (4.15) below).

Lemma 4.2 *Let Assumption 3.2 hold and let K_0, δ_0 be as in Proposition 4.1. For every $(t,\theta) \in [0,\delta_0] \times Q$ with $\theta = (x,y,z)$ and every $h = (h_1,h_2,h_3) \in \mathbb{R}^3$,*

$$V_t(\theta)\cdot h - Z_t U_t(\theta)\cdot h = (h_2 - z\cdot h_1)\exp\Big\{\int_0^t \partial_y g(s,\Theta_s(\theta))\, d\omega_s\Big\}. \tag{4.6}$$

Proof Fix $\theta \in Q$, $h \in \mathbb{R}^3$. Put $\Gamma_t := V_t \cdot h - Z_t U_t \cdot h$. By Lemma 2.8,

$$\begin{aligned}\partial_\omega \Gamma_t &= \Big[\big[\partial_x g - \partial_{xz} g\, Z_t\big] U_t + \Big[\partial_y g - Z_t \partial_{yz} g\Big] V_t - Z_t \partial_{zz} g\, W_t\Big] \cdot h \\ &\quad - [\partial_x g + Z_t \partial_y g] U_t \cdot h + Z_t \Big[\partial_{xz} g U_t + \partial_{yz} g V_t + \partial_{zz} g\, W_t\Big] \cdot h \\ &= \partial_y g V_t \cdot h - Z_t \partial_y g U_t \cdot h = \partial_y g \Gamma_t.\end{aligned}$$

Clearly, $\partial^\omega_t \Gamma_t = 0$ and $\Gamma_0 = h_2 - z h_1$. Then Lemma 2.13 yields (4.6). ■

4.2. *RPDEs and PDEs.* Our goal is to associate RPDE (3.6) with a function v satisfying

$$\partial_\omega v(t,x) = 0, \tag{4.7}$$

which would imply that v solves a standard PDE. To illustrate this idea, let us first derive the PDE for v heuristically. Assume that u is a classical solution of RPDE (3.6) with sufficient regularity. Recall (4.1). We want to find v satisfying (4.7) and

$$u\big(t, X_t(\theta_t(x))\big) = Y_t(\theta_t(x)), \quad \partial_x u\big(t, X_t(\theta_t(x))\big) = Z_t(\theta_t(x)), \tag{4.8}$$
$$\text{where } \theta_t(x) := (x, v(t,x), \partial_x v(t,x)).$$

In fact, recall (4.4) and write

$$\hat{\Phi}_t(x) := \Phi(t, \theta_t(x)) \quad \text{for } \Phi = \Theta,\, U,\, V,\, W. \tag{4.9}$$

Applying the operator ∂^ω_t on both sides of the first equality of (4.8) yields together with Lemma 2.8

$$\begin{aligned}0 = \partial^\omega_t \Big[u(t,\hat{X}_t) - \hat{Y}_t\Big] &= \partial^\omega_t u(t,\hat{X}_t) + \partial_x u(t,\hat{X}_t) \hat{U}_t \cdot \partial_t \theta_t(x) - \hat{V}_t \cdot \partial_t \theta_t(x) \\ &= f(t, \hat{X}_t, u, \partial_x u, \partial^2_{xx} u) - \big[V_t(\theta_t(x)) - Z_t(\theta_t(x)) U_t(\theta_t(x))\big] \cdot \partial_t \theta_t.\end{aligned}$$

By Lemma 4.2 with $h := \partial_t \theta_t(x) = [0, \partial_t v(t,x), \partial_{tx} v(t,x)]$ and $z = \partial_x v(t,x)$,

$$\begin{aligned}&\big[V_t(\theta_t(x)) - Z_t(\theta_t(x)) U_t(\theta_t(x))\big] \cdot \partial_t \theta_t \\ &= [h_2 - z h_1] e^{\int_0^t \partial_y g(s, \Theta_s(\theta_t(x)))\, d\omega_s} = \partial_t v(t,x) e^{\int_0^t \partial_y g(s, \Theta_s(\theta_t(x)))\, d\omega_s}.\end{aligned}$$

We emphasize that the variable $\theta_t(x)$ above is fixed when Lemma 4.2 is applied, while the variable t in V_t is viewed as the running time. In particular, in the last term above $\Theta_s(\theta_t(x))$ involves both times s and t. Then, by (4.2),

$$\partial_t v(t,x) \exp\Big(\int_0^t \partial_y g(s, \Theta_s(\theta_t(x))) d\omega_s\Big) \quad = \quad f(t, \hat{X}_t, u, \partial_x u, \partial^2_{xx} u).$$

By (4.8), $u(t,\hat X_t)$ and $\partial_x u(t,\hat X_t)$ are functions of $(t,\theta_t(x))$. Moreover, by applying the operator ∂_x on both sides of the second equality of (4.8),

$$\partial^2_{xx}u(t,\hat X_t)\hat U_t\cdot\partial_x\theta_t(x)=\hat W_t\cdot\partial_x\theta_t(x).$$

Note that $\partial_x\theta_t(x)=[1,\partial_x v,\partial^2_{xx}v](t,x)$. Then, provided $\hat U_t\cdot\partial_x\theta_t(x)\neq 0$,

$$\partial^2_{xx}u(t,\hat X_t)=\frac{\hat W_t\cdot\partial_x\theta_t(x)}{\hat U_t\cdot\partial_x\theta_t(x)}=\frac{\partial_x Z_t(\theta_t)+\partial_y Z_t(\theta_t)\,\partial_x v+\partial_z Z_t(\theta_t)\,\partial^2_{xx}v}{\partial_x X_t(\theta_t)+\partial_y X_t(\theta_t)\,\partial_x v+\partial_z X_t(\theta_t)\,\partial^2_{xx}v}.$$

Therefore, formally v should satisfy the PDE

$$\partial_t v(t,x)=F(t,x,v(t,x),\partial_x v(t,x),\partial^2_{xx}v(t,x)),\quad v(0,x)=u_0(x),\tag{4.10}$$

where, for $\theta=(x,y,z)$,

$$F(t,\theta,\gamma):=f\Big(t,\Theta_t(\theta),\frac{W_t(\theta)\cdot[1,z,\gamma]}{U_t(\theta)\cdot[1,z,\gamma]}\Big)e^{-\int_0^t\partial_y g(s,\Theta_s(\theta))d\omega_s}.\tag{4.11}$$

Now, we carry out the analysis above rigorously. We start from PDE (4.10) and derive the solution for RPDE (3.6). Recall (2.20) and that k is a generic, sufficiently large regularity index that may vary from line to line.

Lemma 4.3 *Let Assumption 3.2 hold. Let $v\in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$ for some large k. Put $K_0:=\|v\|_\infty\vee\|\partial_x v\|_\infty$. Let δ_0 be determined by Proposition 4.1. Then there exists a constant $\delta\in(0,\delta_0]$ such that the following holds:*

(i) For every $(t,x)\in[0,\delta]\times\mathbb{R}$, $\partial_x\hat X_t(x)=U_t(\theta_t(x))\cdot\partial_x\theta_t(x)\ge 1/2$.

(ii) For every $t\in[0,\delta]$, $\hat X_t:\mathbb{R}\to\mathbb{R}$ is a C^1-diffeomorphism and $\hat S_t(x):=\hat X_t^{-1}(x)$ belongs to $C^{k,loc}_{\alpha,\beta}([0,\delta]\times\mathbb{R})$ (for a possibly different k) and satisfies

$$\hat S_t(x)=x-\int_0^t\frac{[\hat U_s\cdot\partial_t\theta_s](\hat S_s(x))}{(\partial_x\hat X_s)\,(\hat S_s(x))}\,ds+\int_0^t\frac{\partial_z g(s,\hat\Theta_s(\hat S_s(x)))}{(\partial_x\hat X_s)\,(\hat S_s(x))}\,d\omega_s.\tag{4.12}$$

Proof (i) Note that $\theta_t(x)\in Q$ for all $(t,x)\in\mathbb{R}_T$. By (4.3), it is clear that $U\in C^k_{\alpha,\beta}([0,\delta_0]\times Q;\mathbb{R}^3)$. Recall that, by Definition 2.10 (i), the regularity here is uniform in x. Thus, together with the regularity of v, we have

$$\begin{aligned}&|U_t(\theta_t(x))-U_0(\theta_0(x))|\\&\quad\le|U_t(\theta_t(x))-U_t(\theta_0(x))|+|U_t(\theta_0(x))-U_0(\theta_0(x))|\le Ct^\alpha.\end{aligned}\tag{4.13}$$

Since $\partial_x\theta_t(x)=[1,\partial_x v,\partial^2_{xx}v](t,x)$ is bounded,

$$|U_t(\theta_t(x))\cdot\partial_x\theta_t(x)-U_0(\theta_0(x))\cdot\partial_x\theta_0(x)|\le Ct^\alpha.$$

Note that $U_0(\theta_0(x))\cdot\partial_x\theta_0(x) = [1,0,0]\cdot[1,\partial_x v,\partial^2_{xx}v](t,x) = 1$. Hence, there exists a $\delta\le\delta_0$ such that, for every $(t,x)\in[0,\delta]\times\mathbb{R}$, $\partial_x\hat{X}_t(x) = U_t(\theta_t(x))\cdot\partial_x\theta_t(x)\ge 1/2$.

(ii) First, by (i) we see that $\hat{X}_t$ is one to one for $t\in[0,\delta]$. Choose $\iota\in C_b^\infty(\mathbb{R})$ with $\iota(y) = 1/y$ for $y\ge 1/4$ and $\iota(y) = 0$ for $y\le 1/5$. Define functions $\hat{a}$, $\hat{b} : [0,\delta]\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}\hat{a}(t,x) &:= -\iota\big(\partial_x\hat{X}_t(x)\big)\ \hat{U}_t(x)\cdot\partial_t\theta_t(x),\\ \hat{b}(t,x) &:= \iota\big(\partial_x\hat{X}_t(x)\big)\ \partial_z g(t,\hat{\Theta}_t(x)).\end{aligned}\tag{4.14}$$

Note that $\hat{a}$, $\hat{b}\in C^k_{\alpha,\beta}([0,\delta]\times\mathbb{R})$. Then, by Lemma 2.15, the RDE

$$\tilde{S}_t(x) = x + \int_0^t \hat{a}(s,\tilde{S}_s(x))\,ds + \int_0^t \hat{b}(s,\tilde{S}_s(x))\,d\omega_s,\quad (t,x)\in[0,\delta]\times\mathbb{R},$$

has a unique solution $\tilde{S}\in C^{k,loc}_{\alpha,\beta}([0,\delta]\times\mathbb{R})$. Now, by (i), we see that $\tilde{S}$ actually satisfies RDE (4.12).

It remains to verify that $\hat{X}_t\circ\tilde{S}_t = \mathrm{id}$, $t\in[0,\delta]$. Indeed, note that

$$\hat{X}_t\circ\tilde{S}_t = X_t(\theta_t(\tilde{S}_t(x)))\quad\text{and}\quad \partial_\omega v = 0,\quad \partial^\omega_t X = 0.$$

Then, by (4.1) and (4.12),

$$\begin{aligned}\partial_\omega[\hat{X}_t(\tilde{S}_t(x))] &= \partial_\omega X_t(\theta_t(\tilde{S}_t(x))) + U_t(\theta_t(\tilde{S}_t(x)))\cdot\partial_x\theta_t(\tilde{S}_t(x))\partial_\omega\tilde{S}_t(x)\\ &= -\partial_z g(t,\hat{\Theta}_t(\tilde{S}_t(x))) + \partial_x\hat{X}_t(\tilde{S}_t(x))\frac{\partial_z g(t,\hat{\Theta}_t(\tilde{S}_t(x)))}{(\partial_x\hat{X}_t)\,(\tilde{S}_t(x))} = 0;\\ \partial^\omega_t[\hat{X}_t(\tilde{S}_t(x))] &= U_t(\theta_t(\tilde{S}_t(x)))\cdot\Big[\partial_t\theta_t(\tilde{S}_t(x)) + \partial_x\theta_t(\tilde{S}_t(x))\partial^\omega_t\tilde{S}_t(x)\Big]\\ &= \hat{U}_t(\tilde{S}_t(x))\cdot\partial_t\theta_t(\tilde{S}_t(x)) + \partial_x\hat{X}_t(\tilde{S}_t(x))\Big[-\frac{\hat{U}_t(\tilde{S}_t(x))\cdot\partial_t\theta_t(\tilde{S}_t(x))}{(\partial_x\hat{X}_t)\,(\tilde{S}_t(x))}\Big] = 0.\end{aligned}$$

Thus $\hat{X}_t(\tilde{S}_t(x)) = \hat{X}_0(\tilde{S}_0(x)) = x$. This concludes the proof. ■

Theorem 4.4 *Let Assumption 3.2 hold and v and δ be as in Lemma 4.3. Assume further that v is a classical solution (resp. subsolution, supersolution) of PDE* (4.10). *Put $u(t,x) := \hat{Y}_t\circ\hat{X}_t^{-1}(x)$. Then $u\in C^k_{\alpha,\beta}([0,\delta]\times\mathbb{R})$ is a classical solution (resp. subsolution, supersolution) of RPDE* (3.6).

Proof It is clear that $u\in C^{2,loc}_{\alpha,\beta}([0,\delta]\times\mathbb{R})$. To show the uniform properties in terms of x, define first $\check{S}_t(x) := \hat{S}_t(x) - x$, $\check{Y}_t(\theta) := Y_t(\theta) - y$. Then, by

Lemma 2.15, $\check{S} \in C^k_{\alpha,\beta}([0,\delta]\times\mathbb{R})$ and $\check{Y} \in C^k_{\alpha,\beta}([0,\delta]\times\mathbb{R}^3)$. Note that

$$\begin{aligned}u(t,x) &= \hat{Y}_t(\hat{S}_t(x))\\ &= \tilde{Y}_t\big(\check{S}_t(x)+x, v_t(\check{S}_t(x)+x), \partial_x v_t(\check{S}_t(x)+x)\big) + v_t(\check{S}_t(x)+x).\end{aligned}$$

Since $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$, it is clear that $u \in C^k_{\alpha,\beta}([0,\delta]\times\mathbb{R})$.

We prove only the subsolution case. The other statements can be proved similarly. First, note that $u(t,x) = Y_t\big(\theta_t(\hat{S}_t(x))\big)$. Then, denoting $\hat{x} := \hat{S}_t(x)$,

$$\begin{aligned}\partial_\omega u(t,x) &= \partial_\omega Y_t\big(\theta_t(\hat{x})\big) + V_t\big(\theta_t(\hat{x})\big)\cdot\partial_x\theta_t(\hat{x})\ \partial_\omega\hat{S}_t(x)\\ &= \big[g(t,\hat{\Theta}_t(\hat{x}) - \hat{Z}_t(\hat{x})\partial_z g(t,\hat{\Theta}_t(\hat{x}))\big] + V_t\big(\theta_t(\hat{x})\big)\cdot\partial_x\theta_t(\hat{x})\frac{\partial_z g(t,\hat{\Theta}_t(\hat{x}))}{(\partial_x\hat{X}_t)(\hat{x})}\\ &= g(t,\hat{\Theta}_t(\hat{x})) + \frac{\partial_z g(t,\hat{\Theta}_t(\hat{x}))}{(\partial_x\hat{X}_t)(\hat{x})}\big[\hat{V}_t(\hat{x}) - \hat{Z}_t(\hat{x})\hat{U}_t(\hat{x})\big]\cdot\partial_x\theta_t(\hat{x}).\end{aligned}$$

Note that, for $(x,y,z) := \theta_t(x) = [x, v, \partial_x v]$ and $h := \partial_x\theta_t(x) = [1, \partial_x v, \partial^2_{xx} v]$, we have $h_2 - zh_1 = \partial_x v - \partial_x v = 0$. Then, by Lemma 4.2, we have

$$\Big[\hat{V}_t(\hat{x}) - \hat{Z}_t(\hat{x})\hat{U}_t(\hat{x})\Big]\cdot\partial_x\theta_t(\hat{x}) = 0, \tag{4.15}$$

and thus

$$\partial_\omega u(t,x) \quad = \quad g(t,\hat{\Theta}_t(\hat{x})). \tag{4.16}$$

Similarly, note that $\partial_t\theta_t(x) = [0, \partial_t v, \partial_{tx} v]$,

$$\begin{aligned}\partial^\omega_t u(t,x) &= V_t\big(\theta_t(\hat{x})\big)\cdot\Big[\partial_t\theta_t(\hat{x}) + \partial_x\theta_t(\hat{x})\ \partial^\omega_t\hat{S}_t(x)\Big]\\ &= \hat{V}_t\big(\hat{x}\big)\cdot\partial_t\theta_t(\hat{x}) + \hat{Z}_t(\hat{x})(\partial_x\hat{X}_t)(\hat{x})\Big[-\frac{\hat{U}_t(\hat{x})\cdot\partial_t\theta_t(\hat{x})}{(\partial_x\hat{X}_t)(\hat{x})}\Big]\\ &= \big[\hat{V}_t(\hat{x}) - \hat{Z}_t(\hat{x})\hat{U}_t(\hat{x})\big]\cdot\partial_t\theta_t(\hat{x}) = \partial_t v(t,\hat{x})\exp\Big(\int_0^t \partial_y g(s,\hat{\Theta}_s(\hat{x}))d\omega_s\Big).\end{aligned}$$

Since v is a classical subsolution of (4.10)-(4.11), the definition of F yields

$$\partial^\omega_t u(t,x) \le f\Big(t,\hat{\Theta}_t(\hat{x}), \frac{\hat{W}_t(\hat{x})\cdot[1,\partial_x v(t,\hat{x}),\partial^2_{xx}v(t,\hat{x})]}{\hat{U}_t(\hat{x})\cdot[1,\partial_x v(t,\hat{x}),\partial^2_{xx}v(t,\hat{x})]}\Big). \tag{4.17}$$

Now, we identify the functions inside g and f in (4.16) and (4.17). First, by definition

$$\hat{X}_t(\hat{S}_t(x)) = x \text{ and } \hat{Y}_t(\hat{S}_t(x)) = u(t,x). \tag{4.18}$$

Next, differentiating (4.18) with respect to x, we have

$$1 = \partial_x \big[X_t(\theta_t(\hat{x}))\big] = U_t(\theta_t(\hat{x})) \cdot \partial_x \theta_t(\hat{x}) \ \partial_x \hat{S}_t(x),$$
$$\partial_x u(t,x) = \partial_x \big[Y_t(\theta_t(\hat{S}_t(x)))\big] = V_t(\theta_t(\hat{x})) \cdot \partial_x \theta_t(\hat{x}) \ \partial_x \hat{S}_t(x).$$

Thus, by (4.15),

$$\partial_x u(t,x) - \hat{Z}_t(\hat{x}) = \Big[\hat{V}_t(\hat{x}) - \hat{Z}_t(\hat{x})\hat{U}_t(t,\hat{x})\Big] \cdot \partial_x \theta(t,\hat{x}) = 0. \tag{4.19}$$

Moreover,

$$\begin{aligned} \partial^2_{xx} u(t,x) &= \partial_x[\partial_x u(t,x)] = \partial_x \Big[Z_t(\theta_t(\hat{S}_t(x)))\Big] \\ &= W_t(\theta_t(\hat{x})) \cdot \partial_x \theta_t(\hat{x}) \ \partial_x \hat{S}_t(x) = \frac{\hat{W}_t(\hat{x}) \cdot [1, \partial_x v(t,\hat{x}), \partial^2_{xx} v(t,\hat{x})]}{\hat{U}_t(\hat{x}) \cdot [1, \partial_x v(t,\hat{x}), \partial^2_{xx} v(t,\hat{x})]}. \end{aligned} \tag{4.20}$$

Plugging (4.18)-(4.20) into (4.16)-(4.17), we see that u satisfies the desired subsolution properties. ■

Now, we proceed in the opposite direction, namely deriving v from u. Assume that $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ for some large k and define $K_0 := \|u\|_\infty \vee \|\partial_x u\|_\infty$. Let Q_2 and Q be as in (4.2) and δ_0 as in Proposition 4.1. For any fixed $(t,x) \in [0,\delta_0] \times \mathbb{R}$, consider the mapping

$$(y,z) \mapsto [Y - u(t,X), Z - \partial_x u(t,X)]\,(t,x,y,z) \tag{4.21}$$

from Q_2 to $\mathbb{R}^2$. The Jacobi matrix of this mapping is given by

$$J(t,x,y,z) := \begin{bmatrix} \partial_y Y - \partial_x u(t,X)\partial_y X & \partial_y Z - \partial^2_{xx} u(t,X)\partial_y X \\ \partial_z Y - \partial_x u(t,X)\partial_z X & \partial_z Z - \partial^2_{xx} u(t,X)\partial_z X \end{bmatrix} (t,x,y,z).$$

Note that $\det(J(0,x,y,z)) = 1$. Thus, noting also that $\partial_x u$ and $\partial^2_{xx} u$ are bounded, one can see, similarly to (4.13) in [2], that there exists a $\delta \le \delta_0$ such that $\det(J(t,x,y,z)) \ge 1/2$ for all $(t,x,y,z) \in [0,\delta] \times Q$. This implies that the mapping (4.21) is one to one and the inverse mapping has sufficient regularity. Denote by $R(t,x)$ the range of the mapping (4.21). Then

$$R(0,x) = \{(y - u(0,x), z - \partial_x u(0,x)) : (y,z) \in Q_2\} \supset \mathbb{R} \times (-1,1).$$

Thus, by (4.13) in [2], by the boundedness of $\partial_x u, \partial^2_{xx} u$ again, and by choosing a smaller δ if necessary, we may assume that $(0,0) \in R(t,x)$ for all $(t,x) \in [0,\delta] \times \mathbb{R}$. Therefore, for any $(t,x) \in [0,\delta] \times \mathbb{R}$, there exists a unique $(v(t,x), w(t,x)) \in Q_2$ such that, denoting $\tilde{\theta}_t(x) := (x, v(t,x), w(t,x))$,

$$Y_t(\tilde{\theta}_t(x)) = u(t, X_t(\tilde{\theta}_t(x))), \qquad Z_t(\tilde{\theta}_t(x)) = \partial_x u(t, X_t(\tilde{\theta}_t(x))) \tag{4.22}$$

Differentiating the first equation in (4.22) with respect to x and applying the second, we obtain

$$\begin{aligned} 0 &= \partial_x\Big[Y_t(\tilde\theta_t(x)) - u(t, X_t(\tilde\theta_t(x))\Big] \\ &= \Big[V_t(\tilde\theta_t(x)) - \partial_x u(t, X_t(\tilde\theta_t(x)))U_t(\tilde\theta_t(x))\Big]\cdot\partial_x\tilde\theta_t(x) \\ &= \Big[V_t(\tilde\theta_t(x)) - Z_tU_t(\tilde\theta_t(x))\Big]\cdot[1, \partial_x v(t,x), \partial_x w(t,x)] \\ &= \big[\partial_x v(t,x) - w(t,x)\big]\exp\big(\int_0^t \partial_y g(s, \Theta_s(\tilde\theta_t(x)))d\omega_s\big), \end{aligned}$$

where the last equality holds true thanks to Lemma 4.2. Then $w(t,x) = \partial_x v(t,x)$ and thus (4.8) holds. In particular, we may use the notation $\theta_t(x)$ in (4.8) again to replace $\tilde\theta_t(x)$.

We verify now that v indeed satisfies PDE (4.10).

Theorem 4.5 *Let Assumption 3.2 hold, let $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ for some large k, and let δ and v be determined as above. Assume further that u be a classical solution (resp. subsolution, supersolution) of RPDE* (3.6)*. Then, for a possibly smaller $\delta > 0$, we have $U_t(\theta_t(x))\cdot\partial_x\theta_t(x) \ge 1/2$ for all $(t,x) \in [0,\delta]\times\mathbb{R}$ and $v \in C^{k,0}_{\alpha,\beta}([0,\delta]\times\mathbb{R})$ is a classical solution (resp. subsolution, supersolution) of PDE* (4.10) *on* $[0,\delta]\times\mathbb{R}$.

Proof The regularity of v is straightforward. We prove only the case that u is a classical subsolution. The other cases can be proved similarly.

Recall the notations in (4.9). Differentiating the first equality of (4.8) with respect to ω and applying the second equality, we obtain

$$\begin{aligned} 0 = \partial_\omega\Big[Y_t(\theta_t(x)) - u(t, X_t(\theta_t(x)))\Big] &= \partial_\omega Y_t(\theta_t(x)) + \hat V_t\cdot\partial_\omega\theta_t(x) \\ &- \partial_\omega u(t,\hat X_t) - \partial_x u(t,\hat X_t)\big[\partial_\omega X(t,\theta_t(x)) + \hat U_t\cdot\partial_\omega\theta_t(x)\big]. \end{aligned}$$

By (3.8) and (4.8), $\partial_\omega u(t,\hat X_t) = g(t,\hat X_t, u(t,\hat X_t), \partial_x u(t,\hat X_t)) = g(t,\hat\Theta_t)$. Then, by (4.1) and Lemma 4.2,

$$\begin{aligned} 0 &= [g(t,\hat\Theta_t) - \hat Z_t\partial_z g(t,\hat\Theta_t)] + \hat V_t\cdot\partial_\omega\theta_t(x) - g(t,\hat\Theta_t) - \hat Z_t[-\partial_z g(t,\hat\Theta_t)] \\ &\quad - \hat Z_t\hat U_t\cdot\partial_\omega\theta_t(x) \\ &= \big[\hat V_t - \hat Z_t\hat U_t\big]\cdot[0, \partial_\omega v(t,x), \partial_\omega\partial_x v(t,x)] = \partial_\omega v(t,x)e^{\int_0^t \partial_y g(s,\Theta_s(\theta_t(x)))d\omega_s}. \end{aligned}$$

Thus $\partial_\omega v(t,x) = 0$ and Lemma 4.3 can be applied. In particular, for a possibly smaller $\delta > 0$, $U_t(\theta_t(x))\cdot\partial_x\theta_t(x) \ge 1/2$ for all $(t,x) \in [0,\delta]\times\mathbb{R}$.

Finally, following exactly the same arguments as for deriving (4.10), one can complete the proof that v is a classical subsolution of PDE (4.10). ■

Remark 4.6 We shall investigate the case with semilinear g in details in Section 7 below. Here we consider another special case:

$$g = \sigma(z). \tag{4.23}$$

which has received strong attention in the literature. Let σ' and σ'' denote the first and second order derivatives of σ, respectively. In this case the characteristic equations (4.1) becomes

$$X_t = x - \int_0^t \sigma'(Z_s)\, d\omega_s,\ Y_t = y + \int_0^t \big[\sigma(Z_s) - Z_s\sigma'(Z_s)\big]\, d\omega_s,\ Z_t = z,$$

which has the explicit global solution

$$X_t = x - \sigma'(z)\omega_t, \quad Y_t = y + [\sigma(z) - z\sigma'(z)]\omega_t, \quad Z_t = z. \tag{4.24}$$

Moreover, in this case (4.11) becomes

$$F(t,x,y,z,\gamma) := f\big(t, x - \sigma'(z)\omega_t,\ y + [\sigma(z) - z\sigma'(z)]\omega_t,\ z,\ \frac{\gamma}{1 - \sigma''(z)\omega_t\gamma}\big).$$

4.3. *Local wellposedness of PDE* (4.10). To study the wellposedness of PDE (4.10) and hence that of RPDE (3.6), we first establish a PDE result. Let $K_0 > 0$ and, similar to (4.2), consider

$$Q_3 := \{(y,z,\gamma) \in \mathbb{R}^3 : \max\{|y|,|z|,|\gamma|\} \le K_0 + 1\}. \tag{4.25}$$

Lemma 4.7 *Let $k \ge 2$ and $\delta_0 > 0$.*

(i) Suppose that $u_0 \in C_b^{k+1+\beta}(\mathbb{R})$ with $|u_0|$, $|\partial_x u_0|$, $|\partial^2_{xx} u_0| \le K_0$.

(ii) Suppose that $F \in C^{k+1}_{\alpha,\beta}([0,\delta_0] \times \mathbb{R} \times Q_3)$ and $\partial_\gamma F \ge c_0 > 0$.

Then there exists a constant $\delta \le \delta_0$, depending on K_0, c_0, and the norm $\|F\|_2$ on $[0,\delta_0] \times \mathbb{R} \times Q_3$, such that PDE (4.10) has a classical solution $v \in C^{k,0}_{\alpha,\beta}([0,\delta] \times \mathbb{R})$ on $[0,\delta] \times \mathbb{R}$.

Proof It suffices to prove $v \in C_b^{2+\beta}([0,\delta] \times \mathbb{R})$. The further regularity of v when $k \ge 2$ follows from standard bootstrap arguments (cf. Gilbarg and Trudinger [23, Lemma 17.16]) together with Remark 2.11. Since the proof is very similar to that of Lunardi [34, Theorem 8.5.4], which considers a similar boundary-value problem, we shall present only the main ideas for the more involved existence part of the lemma. The first step is to linearize our equation and set up an appropriate fixed point problem. To this end, let $\delta > 0$ and define an operator $A : C_b^{2+\beta}([0,\delta] \times \mathbb{R}) \to C_b^{\beta}([0,\delta] \times \mathbb{R})$ by

$$\begin{aligned}(Av)\,(t,x) := \partial_y F(\hat\theta_0(x))\, v(t,x) + \partial_z F(\hat\theta_0(x))\, \partial_x v(t,x) \\ + \partial_\gamma F(\hat\theta_0(x))\, \partial^2_{xx} v(t,x),\end{aligned} \tag{4.26}$$

where $\hat{\theta}_0(x) := \big(0, x, u_0(x), \partial_x u_0(x), \partial^2_{xx} u_0(x)\big)$. Next, define

$$B_1 := \{v \in C_b^{2+\beta}([0,\delta] \times \mathbb{R}) : v(0,\cdot) = u_0, \|v - u_0\|_{C_b^{2+\beta}} \le 1\}. \tag{4.27}$$

Now given $v \in B_1$, consider the solution w of the linear PDE

$$\partial_t w = Aw + [F(t, x, v, \partial_x v, \partial^2_{xx} v) - Av] \text{ on } [0,\delta] \times \mathbb{R} \tag{4.28}$$

with $w(0,\cdot) = u_0$. Following the arguments in Lunardi [34, Theorem 8.5.4], when $\delta > 0$ is small enough, PDE (4.28) has a unique solution $w \in B_1$. This defines a mapping $\Gamma(v) := w$ for $v \in B_1$. Moreover, when $\delta > 0$ is small enough, Γ is a contraction mapping, and hence there exists a unique fixed point $v \in B_1$. Then $v = w$ and, by (4.28), v solves (4.10) on $[0,\delta] \times \mathbb{R}$. ■

We now turn back to PDE (4.10)-(4.11) and RPDE (3.6).

Theorem 4.8 *Let Assumption 3.2 hold and let $k \ge 2$, $\delta_0 > 0$.*

(i) Suppose that $u_0 \in C_b^{k+1+\beta}(\mathbb{R})$ with $|u_0|$, $|\partial_x u_0|, |\partial^2_{xx} u_0| \le K_0$.

(ii) Suppose that $f \in C^{k+1}_{\alpha,\beta}([0,\delta_0] \times \mathbb{R} \times Q_3)$ and $\partial_\gamma f \ge c_0 > 0$.

Then there exists a constant $\delta \le \delta_0$, depending on K_0, c_0, the regularity of f on $[0,\delta_0] \times Q_3$, and the regularity of g on $[0,\delta_0] \times Q$, such that PDE (4.10)-(4.11) has a classical solution $v \in C^{k,0}_{\alpha,\beta}([0,\delta] \times \mathbb{R})$ on $[0,\delta] \times \mathbb{R}$, and consequently, for a possibly smaller $\delta > 0$, RPDE (3.6) has a classical solution $u \in C^k_{\alpha,\beta}([0,\delta] \times \mathbb{R})$.

Proof Recall (4.11). By the uniform regularity of Θ in Proposition 4.1, one can verify straightforwardly that, for $\delta > 0$ small enough, F satisfies the conditions in Lemma 4.7 (ii). Then, by Lemma 4.7, PDE (4.10)-(4.11) has a classical solution $v \in B_1$ for a possibly smaller δ. Finally, it follows from Theorem 4.4 that RPDE (3.6) has a local classical solution. ■

4.4. *The first order case.* We consider the case f being of first order, i.e.,

$$f = f(t,\theta) = f(t,x,y,z). \tag{4.29}$$

This case is completely degenerate in terms of γ. It is not covered by Theorem 4.8. However, in this case PDE (4.10)-(4.11) is also of first order, i.e.,

$$F(t,\theta) := f\big(t, \Theta_t(\theta)\big) \exp\big(- \int_0^t \partial_y g(s, \Theta_s(\theta)) d\omega_s\big). \tag{4.30}$$

When f is smooth, so is F. Thus we can modify the characteristic equations (4.1) to solve PDE (4.10)-(4.30) explicitly. Put $\tilde{\Theta} = (\tilde{X}, \tilde{Y}, \tilde{Z})$ and consider

$$
\begin{aligned}
\tilde{X}_t &= x - \int_0^t \partial_z F(s, \tilde{\Theta}_s)\, ds, \\
\tilde{Y}_t &= y + \int_0^t \big[F(s, \tilde{\Theta}_s) - \tilde{Z}_s \partial_z F(s, \Theta_s)\big]\, ds, \qquad (4.31)\\
\tilde{Z}_t &= z + \int_0^t \big[\partial_x F(s, \tilde{\Theta}_s) + \tilde{Z}_s \partial_y F(s, \tilde{\Theta}_s)\big]\, ds.
\end{aligned}
$$

Similar to (4.8), let $\tilde{v}$ be determined (locally in time) by

$$
\tilde{v}\big(t, \tilde{X}_t(\tilde{\theta}_t(x))\big) = \tilde{Y}_t(\tilde{\theta}_t(x)),\ \ \partial_x \tilde{v}\big(t, \tilde{X}_t(\tilde{\theta}_t(x))\big) = \tilde{Z}_t(\tilde{\theta}_t(x)),
$$

where $\tilde{\theta}_t(x) := (x, \tilde{v}(t,x), \partial_x \tilde{v}(t,x))$. Then one can see that (4.7) should be replaced with $\partial_t \tilde{v} = 0$, and thus $\tilde{v}(t,x) = u_0(x)$. By similar (actually easier) arguments as in previous subsections, one can prove the following statement.

Theorem 4.9 *Let Assumption 3.2 hold, f take the form* (4.29) *with $f \in C^{k+1}_{\alpha,\beta}([0,T] \times Q)$, and $u_0 \in C_b^{k+1+\beta}(\mathbb{R})$ for some large k with $|u_0|$, $|\partial_x u_0|$, $|\partial^2_{xx} u_0| \le K_0$. Then there is a constant $\delta > 0$ such that the following holds:*

(i) The system of ODEs (4.31) *is wellposed on $[0,\delta]$ for all $\theta \in Q$.*

(ii) For each $t \in [0,\delta]$, the mapping $x \in \mathbb{R} \mapsto \tilde{X}_t(x, u_0(x), \partial_x u_0(x)) \in \mathbb{R}$ is invertible and thus possesses an inverse function, to be denoted by $\tilde{S}_t$.

(iii) The map v defined by $v(t,x) := \tilde{Y}_t(\tilde{\theta}_t(\tilde{S}_t(x)))$ belongs to $C^{k,0}_{\alpha,\beta}([0,\delta] \times \mathbb{R})$ and is a classical solution to PDE (4.10)-(4.30)*. Consequently RPDE* (3.6)-(4.29) *has a classical solution $u \in C^k_{\alpha,\beta}([0,\delta] \times \mathbb{R})$.*

5. Viscosity solutions of rough PDEs: definitions and basic properties. We introduce a notion of viscosity solution for RPDE (3.6) and study its basic properties. For any $(t_0, x_0) \in (0,T] \times \mathbb{R}$ and $\delta \in (0, t_0)$, define

$$
D_\delta(t_0,x_0) := [t_0 - \delta, t_0] \times O_\delta(x_0) := [t_0 - \delta, t_0] \times \{x \in \mathbb{R} : |x - x_0| \le \delta\}.
$$

5.1. *The definition.* For $u \in C(\mathbb{R}_T)$ and $(t_0,x_0) \in (0,T] \times \mathbb{R}$, put

$$
\begin{aligned}
\mathcal{A}^0_g u(t_0,x_0;\delta) &:= \Big\{\varphi \in C^2_{\alpha,\beta}(D_\delta(t_0,x_0)) : \varphi(t_0,x_0) = u(t_0,x_0) \\
&\qquad\quad \text{and} \quad \partial_\omega \varphi = g(\cdot, \varphi, \partial_x \varphi) \text{ on } D_\delta(t_0,x_0)\Big\}, \\
\overline{\mathcal{A}}_g u(t_0,x_0) &:= \bigcup_{0<\delta\le t_0} \Big\{\varphi \in \mathcal{A}^0_g u(t_0,x_0;\delta) : \varphi \le u \text{ on } D_\delta(t_0,x_0)\Big\}, \qquad (5.1)\\
\underline{\mathcal{A}}_g u(t_0,x_0) &:= \bigcup_{0<\delta\le t_0} \Big\{\varphi \in \mathcal{A}^0_g u(t_0,x_0;\delta) : \varphi \ge u \text{ on } D_\delta(t_0,x_0)\Big\}.
\end{aligned}
$$

Definition 5.1 *Let $u \in C(\mathbb{R}_T)$ and recall the operator $\mathcal{L}$ in* (3.8).

(i) We say u is a viscosity supersolution *(resp.* subsolution*) of RPDE* (3.6) *if, for every $(t_0, x_0) \in (0,T] \times \mathbb{R}$ and $\varphi \in \overline{\mathcal{A}}_g u(t_0, x_0)$ (resp. $\underline{\mathcal{A}}_g u(t_0, x_0)$), we have $\mathcal{L}\varphi(t_0, x_0) \geq$ (resp. $\leq$) 0.*

(ii) We say u is a viscosity solution *of RPDE* (3.6) *if it is both a viscosity supersolution and a viscosity subsolution of* (3.6).

We remark that it is possible to consider semi-continuous viscosity solutions as in the standard literature. However, for simplicity, in this paper we restrict ourselves to continuous solutions only.

Proposition 5.2 (Consistency) *Let Assumptions 3.2 and 3.3 hold and let $u \in C^2_{\alpha,\beta}(\mathbb{R}_T)$. Then u is a classical subsolution (resp. classical supersolution) of RPDE* (3.6) *if and only if it is a viscosity subsolution (resp. viscosity supersolution) of* (3.6).

Proof We prove only the subsolution case. The supersolution case can be proved similarly.

First, assume that u is a viscosity subsolution. By choosing u itself as a test function, we can immediately infer that u is a classical subsolution.

Next, assume that u is a classical subsolution. Let $(t,x) \in (0,T] \times \mathbb{R}$ and $\varphi \in \underline{\mathcal{A}}_g u(t,x)$ with corresponding $\delta_0 \in (0,t]$. Then, at (t,x),

$$
\begin{aligned}
&u - \varphi = 0, \quad \partial_x[u-\varphi] = 0, \qquad \partial^2_{xx}[u-\varphi] \leq 0, \\
&\partial_\omega[u-\varphi] = 0; \quad \partial_{x\omega}[u-\varphi] = c\, \partial^2_{xx}[u-\varphi],
\end{aligned}
\tag{5.2}
$$

where $c := \partial_z g(t,x,u,\partial_x u)$. For any $(\delta, h) \in [0,\delta_0] \times O_{\delta_0}(x)$, by Lemma 2.9,

$$
\begin{aligned}
0 \geq &[u-\varphi](t-\delta, x+h) \\
&= -\partial^\omega_t[u-\varphi](t,x)\ \delta + \frac{1}{2}\partial^2_{xx}[u-\varphi](t,x)|h - c\ \omega_{t-\delta,t}|^2 + R^{2,u-\varphi}_{\delta,h},
\end{aligned}
\tag{5.3}
$$

where $R^{2,u-\varphi}_{\delta,h} = O((\delta^\alpha + h)^{2+\beta})$. Fix a number $\delta_1 \in (0,\delta_0]$ such that, for every $\delta \in (0,\delta_1]$, we have $|c\,\omega_{t-\delta,t}| < \delta_0$. From now on, let $\delta \in (0,\delta_1]$. Setting $h := c\ \omega_{t-\delta,t}$ in (5.3) yields

$$
-\partial^\omega_t[u-\varphi](t,x)\,\delta \leq -R^{2,u-\varphi}_{\delta,h} \leq C(\delta^\alpha + |c\omega_{t-\delta,t}|)^{2+\beta} \leq C\,\delta^{\alpha(2+\beta)}.
$$

Recall (2.2). Dividing the inequality above by δ and sending δ to 0, we have $\partial^\omega_t u(t,x) \geq \partial^\omega_t \varphi(t,x)$. By Assumption 3.3 (i) and by (5.2),

$$
\Big[\partial^\omega_t \varphi - f(\cdot, \varphi, \partial_x\varphi, \partial^2_{xx}\varphi)\Big](t,x) \leq \Big[\partial^\omega_t u - f(\cdot, u, \partial_x u, \partial^2_{xx} u)\Big](t,x) \leq 0,
$$

i.e., u is a viscosity subsolution at (t,x). ■

5.2. *Equivalent definition through semi-jets.* As in the standard PDE case (cf. Crandall, Ishii, and Lions [9]), viscosity solutions can also be defined through semi-jets. To see this, we first note that, for $\varphi \in \mathcal{A}_g^0 u(t_0, x_0; \delta)$, our second order Taylor expansion (Lemma 2.9) yields

$$\begin{aligned}\varphi(t,x) &= \varphi(t_0,x_0) + \partial_t^\omega \varphi(t_0,x_0)(t-t_0) + \partial_\omega \varphi(t_0,x_0)\omega_{t_0,t}\\ &\quad + \partial_x \varphi(t_0,x_0)(x-x_0) + \partial_{\omega\omega}^2 \varphi(t_0,x_0)\underline{\omega}_{t_0,t} + \frac{1}{2}\partial_{xx}^2 \varphi(t_0,x_0)|x-x_0|^2\\ &\quad + \partial_{x\omega}\varphi(t_0,x_0)\omega_{t_0,t}(x-x_0) + R(t,x),\end{aligned}$$

where $(t,x) \in D_\delta(t_0,x_0)$. Since $\partial_\omega \varphi(t,x) = g(t,x,\varphi,\partial_x \varphi)$, we have

$$\begin{aligned}\partial_{x\omega}\varphi &= \partial_x g + \partial_y g \partial_x \varphi + \partial_z g \partial_{xx}^2 \varphi,\\ \partial_{\omega\omega}\varphi &= \partial_\omega g + \partial_y g \partial_\omega \varphi + \partial_z g \partial_{\omega x}\varphi\\ &= \partial_\omega g + \partial_y g g + \partial_z g[\partial_x g + \partial_y g \partial_x \varphi + \partial_z g \partial_{xx}^2 \varphi].\end{aligned} \tag{5.4}$$

Motivated by this, we define semijets as follows. Given $u \in C(\mathbb{R}_T)$, $(t_0,x_0) \in (0,T] \times \mathbb{R}$, and $(a,z,\gamma) \in \mathbb{R}^3$, put

$$\begin{aligned}\psi_{g,u,t_0,x_0}^{a,z,\gamma}(t,x) &:= y + a[t-t_0] + b\,\omega_{t_0,t} + z[x-x_0]\\ &\qquad + c\,\underline{\omega}_{t_0,t} + \frac{1}{2}\gamma|x-x_0|^2 + q\,\omega_{t_0,t}[x-x_0], \quad \text{where}\\ y &:= u(t_0,x_0),\ b := g(t_0,x_0,y,z),\ q := [\partial_x g + \partial_y g z + \partial_z g \gamma](t_0,x_0,y,z),\\ c &:= [\partial_\omega g + \partial_y g g + \partial_z g(\partial_x g + \partial_y g z + \partial_z g \gamma)](t_0,x_0,y,z).\end{aligned}$$

We then define the g-superjet $\overline{\mathcal{J}}_g u(t_0,x_0)$ and the g-subjet $\underline{\mathcal{J}}_g u(t_0,x_0)$ by

$$\begin{aligned}\overline{\mathcal{J}}_g u(t_0,x_0) &:= \bigcup_{0<\delta\le t}\Big\{(a,z,\gamma) \in \mathbb{R}^3 : \psi_{g,u,t_0,x_0}^{a,z,\gamma} \le u \text{ on } D_\delta(t_0,x_0)\Big\},\\ \underline{\mathcal{J}}_g u(t_0,x_0) &:= \bigcup_{0<\delta\le t}\Big\{(a,z,\gamma) \in \mathbb{R}^3 : \psi_{g,u,t_0,x_0}^{a,z,\gamma} \ge u \text{ on } D_\delta(t_0,x_0)\Big\}.\end{aligned} \tag{5.5}$$

Notice that $\partial_\omega \psi_{g,u,t_0,x_0}^{a,z,\gamma} = g(\cdot, \psi_{g,u,t_0,x_0}^{a,z,\gamma}, \partial_x \psi_{g,u,t_0,x_0}^{a,z,\gamma})$ holds true only at (t_0,x_0), but not in $D_\delta(t_0,x_0)$, so in general $\psi_{g,u,t_0,x_0}^{a,z,\gamma} \notin \mathcal{A}_g^0 u(t_0,x_0;\delta)$. Nevertheless, we still have the following equivalence.

Proposition 5.3 *Let Assumptions 3.2 and 3.3 be in force and $u \in C(\mathbb{R}_T)$. Then u is a viscosity supersolution (resp. subsolution) of* (3.6) *at $(t_0,x_0) \in (0,T] \times \mathbb{R}$ if and only if, for every $(a,z,\gamma) \in \overline{\mathcal{J}}_g u(t_0,x_0)$ (resp. $\underline{\mathcal{J}}_g u(t_0,x_0)$),*

$$a - f(t_0,x_0,u(t_0,x_0),z,\gamma) \ge 0 \ \textit{(resp. } \le 0\textit{)}. \tag{5.6}$$

Proof We prove only the supersolution case. The subsolution case can be proved similarly.

First, we prove the if part. Assume that (5.6) holds for every $(a, z, \gamma) \in \overline{\mathcal{J}}_g u(t_0, x_0)$. Let $\varphi \in \overline{\mathcal{A}}_g u(t_0, x_0)$. Then there exists a $\delta_0 \in (0, t_0 \wedge 1]$ such that, whenever $0 \le \delta \le \delta_0$, $|h| \le \delta_0$,

$$u(t_0 - \delta, x_0 + h) - u(t_0, x_0) \ge \varphi(t_0 - \delta, x_0 + h) - \varphi(t_0, x_0),$$

By Lemma 2.9, there exists a $C > 0$ such that

$$\varphi(t_0 - \delta, x_0 + h) - \varphi(t_0, x_0) \ge \Big[- \partial_t^\omega \varphi \ \delta + \partial_x \varphi \ h + \frac{1}{2} \partial_{xx}^2 \varphi \ |h|^2 + \partial_\omega \varphi \ \omega_{t_0,t} + \partial_{\omega\omega}^2 \varphi \ \underline{\omega}_{t_0,t} + \partial_{x\omega} \varphi \ h \ \omega_{t_0,t} \Big](t_0, x_0) - C\big[\delta^\alpha + |h|\big]^{(2+\beta)}.$$

For any $\varepsilon > 0$, by (2.2), there exists a $\delta_\varepsilon \in (0, \delta_0)$, such that, for every $0 \le \delta \le \delta_\varepsilon$, $|h| \le \delta_\varepsilon$,

$$u(t_0 - \delta, x_0 + h) - u(t_0, x_0) \ge \Big[- \partial_t^\omega \varphi \ \delta + \partial_x \varphi \ h + \frac{1}{2} \partial_{xx}^2 \varphi \ |h|^2 + \partial_\omega \varphi \ \omega_{t_0,t} + \partial_{\omega\omega}^2 \varphi \ \underline{\omega}_{t_0,t} + \partial_{x\omega} \varphi \ h \ \omega_{t_0,t} \Big](t_0, x_0) - \varepsilon \ \delta - \frac{1}{2} \varepsilon \ |h|^2,$$

By (5.4), the above inequality implies $(\partial_t^\omega \varphi + \varepsilon, \partial_x \varphi, \partial_{xx}^2 \varphi - \varepsilon)(t_0, x_0) \in \overline{\mathcal{J}}_g u(t_0, x_0)$. Thus, by (5.6), $[\partial_t^\omega \varphi + \varepsilon - f(\cdot, \varphi, \partial_x \varphi, \partial_{xx}^2 \varphi - \varepsilon)](t_0, x_0) \ge 0$. Sending $\varepsilon \to 0$ yields $\mathcal{L}\varphi(t_0, x_0) \ge 0$, i.e., u is a viscosity supersol. at (t_0, x_0).

Next, we prove the only if part. Assume that u is a viscosity supersolution at $(t_0, x_0) \in (0, T] \times \mathbb{R}$. Let $(a, z, \gamma) \in \overline{\mathcal{J}}_g u(t_0, x_0)$ and consider the RPDE

$$\varphi(t, x) = u(t_0, x_0) + [a + \varepsilon]\,[t - t_0] + z\,[x - x_0] + \frac{1}{2}[\gamma - \varepsilon]\,|x - x_0|^2 - \int_t^{t_0} g(\cdot, \varphi, \partial_x \varphi)(s, x)\, d\omega_s.$$

By Theorem 4.9, the RPDE above has a classical solution $\varphi \in C^2_{\alpha,\beta}(D_\delta(t_0, x_0))$ for some $\delta \in (0, \delta_0]$. It is clear that $\varphi \in \mathcal{A}_g^0 u(t_0, x_0; \delta)$. Moreover, by using our Taylor expansion (Lemma 2.9), one may easily verify that

$$\varphi = \psi_{g,u,t_0,x_0}^{a+\varepsilon, z, \gamma-\varepsilon} + R \text{ on } D_\delta(t_0, x_0),$$

where $|R(t, x)| \le C[|t - t_0|^\alpha + |x - x_0|]^{2+\beta}$. Then, by choosing $\delta > 0$ small enough, we have $\varphi \le \psi_{g,u,t_0,x_0}^{a,z,\gamma} \le u$ on $D_\delta(t_0, x_0))$, where the second inequality is due to the assumption $(a, z, \gamma) \in \overline{\mathcal{J}}_g u(t_0, x_0)$. This implies $\varphi \in \overline{\mathcal{A}}_g u(t_0, x_0)$. Thus $0 \le \mathcal{L}\varphi(t_0, x_0) = a + \varepsilon - f(t_0, x_0, u(t_0, x_0), z, \gamma - \varepsilon)$. Sending $\varepsilon \to 0$ yields (5.6). ■

Remark 5.4 By Proposition 5.3 and its proof, we can see that, depending on the regularity order k_0 of g as specified in Assumption 3.2, it is equivalent to use test functions of class $C^k_{\alpha,\beta}(D_\delta(t_0,x_0))$ for any k between 2 and k_0. This is crucial for Theorem 5.9 below.

5.3. *Change of variables formula.* Let $\lambda \in C([0,T])$ and $n \geq 2$ be an even integer. For any $u : \mathbb{R}_T \to \mathbb{R}$, define

$$\tilde{u}(t,x) := \frac{e^{\eta_t}}{1+x^n} u(t,x), \text{ where } \eta_t := \int_0^t \lambda_s\, ds. \tag{5.7}$$

If $u \in C^2_{\alpha,\beta}(\mathbb{R}_T)$, then

$$\begin{aligned} u = e^{-\eta_t}(1+x^n)\tilde{u}, \quad \partial_x u = e^{-\eta_t}\big[(1+x^n)\partial_x\tilde{u} + nx^{n-1}\tilde{u}\big],\\ \partial^2_{xx}u = e^{-\eta_t}\big[(1+x^n)\partial^2_{xx}\tilde{u} + 2nx^{n-1}\partial_x\tilde{u} + n(n-1)x^{n-2}\tilde{u}\big],\\ \partial_\omega u = e^{-\eta_t}(1+x^n)\partial_\omega\tilde{u}, \quad \partial^\omega_t u = e^{-\eta_t}(1+x^n)[\partial^\omega_t\tilde{u} - \lambda\tilde{u}]. \end{aligned} \tag{5.8}$$

Define $\tilde{f} : \mathbb{R}^4_T \to \mathbb{R}$ and $\tilde{g} : \mathbb{R}^3_T \to \mathbb{R}$ by

$$\begin{aligned} \tilde{f}(t,x,y,z,\gamma) &:= \lambda_t\, y + \frac{e^{\eta_t}}{1+x^n}\, f\Big(t, x, \frac{(1+x^n)y}{e^{\eta_t}},\\ &\qquad \frac{(1+x^n)z + nx^{n-1}y}{e^{\eta_t}}, \frac{(1+x^n)\gamma + 2nx^{n-1}z + n(n-1)x^{n-2}y}{e^{\eta_t}}\Big),\\ \tilde{g}(t,x,y,z) &:= \frac{e^{\eta_t}}{1+x^n}\, g\Big(t, x, \frac{(1+x^n)y}{e^{\eta_t}}, \frac{(1+x^n)z + nx^{n-1}y}{e^{\eta_t}}\Big). \end{aligned} \tag{5.9}$$

Clearly, $\tilde{f}$ and $\tilde{g}$ inherit the regularity of f and g. Whenever they are smooth,

$$\begin{aligned} &\partial_y\tilde{f} = \lambda + \partial_y f + \partial_z f\frac{nx^{n-1}}{1+x^n} + \partial_\gamma f\frac{n(n-1)x^{n-2}}{1+x^n},\\ &\partial_z\tilde{f} = \partial_z f + \partial_\gamma f\frac{2nx^{n-1}}{1+x^n}; \quad \partial_\gamma\tilde{f} = \partial_\gamma f; \quad \partial^2_{\gamma\gamma}\tilde{f} = e^{-\eta_t}(1+x^n)\partial^2_{\gamma\gamma}f. \end{aligned} \tag{5.10}$$

Then it is straightforward to verify that $\tilde{f}$ and $\tilde{g}$ inherit most desired properties of f and g that we utilize later.

Lemma 5.5 *(i) If g is of the form of* (7.1) *or of* (7.26) *in [2], then so is $\tilde{g}$; and if f is of the form of* (7.29) *in [2], then so is $\tilde{f}$.*

(ii) If f is convex in γ, then so is $\tilde{f}$.

(iii) If f is uniformly parabolic, then so is $\tilde{f}$.

(iv) If f is uniformly Lipschitz continuous in y, z, γ, then so is $\tilde{f}$.

(v) If $\|f(\cdot,y,z,\gamma)\|_{C^\alpha(\mathbb{R}_T)} \leq C[1+|y|+|z|+|\gamma|]$, then so is $\tilde{f}$.

In particular, if f and g satisfy Assumptions 3.2 and 3.3, then so do $\tilde f$ and $\tilde g$. However, we remark that $\tilde g$ does not inherit the same form when g is in the form of (4.23) in [2]. Now consider the RPDE for $\tilde u$:

$$\begin{aligned}\tilde u(t,x) = u_0(x) + \int_0^t \tilde f(s,x,\tilde u(s,x),\partial_x \tilde u(s,x),\partial^2_{xx}\tilde u(s,x))\,ds \\ + \int_0^t \tilde g(s,x,\tilde u(s,x),\partial_x\tilde u(s,x))\,d\omega_s, \quad (t,x)\in\mathbb{R}_T.\end{aligned} \tag{5.11}$$

Proposition 5.6 *Let Assumptions 3.2 and 3.3 be in force, $\lambda \in C([0,T])$, $n \geq 2$ even, and $u \in C(\mathbb{R}_T)$. Then u is a viscosity subsolution (resp. classical subsolution) of RPDE* (3.6) *if and only if $\tilde u$ is a viscosity subsolution (resp. classical subsolution) of RPDE* (5.11)*.*

Proof The equivalence of the classical solution properties is straightforward. Regarding the viscosity solution properties, we prove the if part; the only if part can be proved similarly.

Assume that $\tilde u$ is a viscosity subsolution of RPDE (5.11). For any $(t_0,x_0)\in (0,T]\times\mathbb{R}$ and $\varphi \in \underline{\mathcal{A}}_g u(t_0,x_0)$, put $\tilde\varphi(t,x) := \frac{e^{\eta t}}{1+x^n}\varphi(t,x)$. It is straightforward to check that $\tilde\varphi \in \underline{\mathcal{A}}_{\tilde g}\tilde u(t_0,x_0)$. Then, by the viscosity subsolution property of $\tilde u$ at (t_0,x_0),

$$0 \geq \partial^\omega_t \tilde\varphi - \tilde f(t_0,x_0,\tilde\varphi,\partial_x\tilde\varphi,\partial^2_{xx}\tilde\varphi) = \frac{e^{\eta t_0}}{1+x_0^n}\Big[\partial^\omega_t\varphi - f(t_0,x_0,\varphi,\partial_x\varphi,\partial^2_{xx}\varphi)\Big].$$

This implies that u is a viscosity subsolution of RPDE (3.6). ■

Remark 5.7 Let (f,g) satisfy Assumptions 3.2 and 3.3 and let u be a viscosity semi-solution of RPDE (3.6).

(i) If u has polynomial growth, by choosing n large enough, we have

$$\lim_{|x|\to\infty}\sup_{0\leq t\leq T}|\tilde u(t,x)| = 0. \tag{5.12}$$

(ii) If f is uniformly Lipschitz continuous in y, by choosing λ sufficiently large (resp. small), we have

$$\tilde f \text{ is strictly increasing (resp. decreasing) in } y. \tag{5.13}$$

In particular, $\tilde f$ will be proper in the sense of Crandall, Ishii, and Lions [9].

5.4. *Stability.* The following technical lemma will be crucial for the stability result. Given $(t_0, x_0) \in [0, T) \times \mathbb{R}$ and $\varepsilon > 0$ small, put

$$D^+_\varepsilon(t_0, x_0) := [t_0, t_0 + \varepsilon^3) \times O_\varepsilon(x_0),$$
$$\partial D^+_\varepsilon(t_0, x_0) := \big\{(t, x) : t \in [t_0, t_0 + \varepsilon^3], |x - x_0| = \varepsilon \text{ or } t = t_0 + \varepsilon^3, |x| \leq \varepsilon\big\}.$$

Lemma 5.8 *Let Assumption 3.2 hold, let $(t_0, x_0) \in [0, T) \times \mathbb{R}$, and let $\delta_0 \in (0, T - t_0]$. Assume that $\varphi \in C^k_{\alpha,\beta}(D^+_{\delta_0^{1/3}}(t_0, x_0))$ for some large k and $\partial_\omega \varphi = g(\cdot, \varphi, \partial_x \varphi)$ in $D^+_{\delta_0^{1/3}}(t_0, x_0)$. Define*

$$g^\varphi(t, x, y, z) := [g(\cdot, \varphi + y, \partial_x \varphi + z) - g(\cdot, \varphi, \partial_x \varphi)](t, x). \tag{5.14}$$

Then there exists an $\varepsilon_0 \in (0, 1]$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, there exists a function $\psi^\varepsilon \in C^4_{\alpha,\beta}(D^+_\varepsilon(t_0, x_0))$ that satisfies the following properties:

$$\partial_\omega \psi^\varepsilon = g^\varphi(\cdot, \psi^\varepsilon, \partial_x \psi^\varepsilon), \quad \partial^\omega_t \psi^\varepsilon = \varepsilon; \tag{5.15}$$
$$|\psi^\varepsilon| + |\partial_x \psi^\varepsilon| + |\partial^2_{xx} \psi^\varepsilon| \leq C\varepsilon^2 \text{ in } D^+_\varepsilon(t_0, x_0); \tag{5.16}$$
$$\psi^\varepsilon(t_0, x_0) < 0 < \inf_{(t,x)\in \partial D^+_\varepsilon(t_0,x_0)} \psi^\varepsilon(t, x). \tag{5.17}$$

Proof Without loss of generality, we let $(t_0, x_0) = (0, 0)$. Since our results are local, without loss of generality, we can assume that $\varphi \in C^k_{\alpha,\beta}(\mathbb{R}_T)$. Let $\iota \in C^\infty(\mathbb{R})$ be such that $\iota(x) = x^4$ for $|x| \leq 1$ and $\iota(x) = 0$ for $|x| \geq 2$. For any $\varepsilon > 0$ small, consider the RPDE

$$\psi^\varepsilon(t, x) = \iota(x) - \varepsilon^5 + \varepsilon t + \int_0^t g^\varphi(s, x, \psi^\varepsilon, \partial_x \psi^\varepsilon) d\omega_s. \tag{5.18}$$

By Theorem 4.9, there exists a $\delta_1 \leq \delta_0$ such that $\psi^\varepsilon \in C^4_{\alpha,\beta}([0, \delta_1] \times \mathbb{R})$ for all $\varepsilon \leq 1$ and

$$\sup_{\varepsilon \leq 1} \|\psi^\varepsilon\|_4 < \infty. \tag{5.19}$$

The equalities in (5.15) are obvious. Now, we verify that ψ^ε satisfies (5.16) and (5.17). Recall the fourth-order Taylor expansion in Lemma 2.9:

$$\psi^\varepsilon(t, x) = \sum_{\ell + |\nu| \leq 4} \frac{1}{\ell!} (\partial^\ell_x \mathcal{D}_\nu \psi^\varepsilon)(0, 0) x^\ell \mathcal{I}^\nu_{0,t} + R_4(t, x). \tag{5.20}$$

We claim that

$$\begin{aligned}
&\partial^4_x \psi^\varepsilon(0, 0) = 24, \quad \partial^{\ell_1}_x \partial^{\ell_2}_\omega \partial_\omega \psi^\varepsilon(0, 0) = O(1) \quad \text{for } \ell_1 + \ell_2 = 3,\\
&\partial^\omega_t \psi^\varepsilon(0, 0) = \varepsilon, \quad \partial^\omega_t \partial_\omega \psi^\varepsilon(0, 0) = O(\varepsilon), \quad \partial^\omega_t \partial^2_\omega \psi^\varepsilon(0, 0) = O(\varepsilon),\\
&\partial^\ell_x \mathcal{D}_\nu \psi^\varepsilon(0, 0) = O(\varepsilon^5) \text{ for all other terms such that } \ell + |\nu| \leq 4.
\end{aligned} \tag{5.21}$$

Note further that, by (5.19), for every $(t,x) \in \overline{D}_\varepsilon^+(0,0)$,

$$|R_4(t,x)| \le C[t^\alpha + |x|]^{4+\beta} \le C[\varepsilon^{3\alpha} + \varepsilon]^{4+\beta} \le C\varepsilon^{4+\beta}. \tag{5.22}$$

Then, for $(t,x) \in \overline{D}_\varepsilon^+(0,0)$, plugging (5.21) and (5.22) into (5.20), we obtain

$$\begin{aligned} |\psi^\varepsilon(t,x)| &\le C\Big[\sum_{\ell_1+\ell_2=4} |x|^{\ell_1} t^{\alpha\ell_2} + \varepsilon[t + t^{1+\alpha} + t^{1+2\alpha}] + \varepsilon^5 + \varepsilon^{4+\beta}\Big] \\ &\le C\Big[\sum_{\ell_1+\ell_2=4} \varepsilon^{\ell_1+3\alpha\ell_2} + \varepsilon^4\Big] \le C\varepsilon^4. \end{aligned}$$

Similarly, applying the third-order Taylor expansion of $\partial_x\psi^\varepsilon$ and the second-order Taylor expansion of $\partial^2_{xx}\psi^\varepsilon$, we obtain

$$|\partial_x\psi^\varepsilon(t,x)| \le C\varepsilon^3, \quad |\partial^2_{xx}\psi^\varepsilon(t,x)| \le C\varepsilon^2, \quad (t,x) \in \overline{D}_\varepsilon^+(0,0).$$

Thus we have proved (5.16).

To prove (5.17), let $(t,x) \in \partial D_\varepsilon^+(0,0)$. By (5.20), (5.21), and (5.22),

$$\psi^\varepsilon(t,x) \ge \varepsilon t + \frac{24}{4!}x^4 - C\big[\sum_{\ell_1+\ell_2=3} |x|^{\ell_1} t^{\alpha(\ell_2+1)} + \varepsilon[t^{1+\alpha} + t^{1+2\alpha}] + \varepsilon^5 + \varepsilon^{4+\beta}\big].$$

Note that in both, the case $t \in [0,\varepsilon^3]$ and $|x| = \varepsilon$ and the case $t = \varepsilon^3$ and $|x| \le \varepsilon$, we have $\varepsilon t + \frac{24}{4!}x^4 \ge \varepsilon^4$. Hence

$$\begin{aligned} \psi^\varepsilon(t,x) &\ge \varepsilon^4 - C\sum_{\ell_1+\ell_2=3} \varepsilon^{\ell_1+3\alpha(\ell_2+1)} - C\varepsilon^{1+3(1+\alpha)} - C\varepsilon^5 - C\varepsilon^{4+\beta} \\ &\ge \varepsilon^4 - C\big[\varepsilon^{3+3\alpha} + \varepsilon^{4+\beta}\big] \ge \frac{1}{2}\varepsilon^4, \end{aligned}$$

when ε is small, thanks to the assumption that $3\alpha > 1$. Moreover, it is clear that $\psi^\varepsilon(0,0) = -\varepsilon^5 < 0$. Thus we have proved (5.17).

It remains to prove (5.21). First, by (5.18) it is clear that

$$\psi^\varepsilon(0,x) = \iota(x) - \varepsilon^5,\ \partial_t^\omega\psi^\varepsilon(t,x) = \varepsilon,\ \partial_\omega\psi^\varepsilon(t,x) = g^\varphi(t,x,\psi^\varepsilon,\partial_x\psi^\varepsilon). \tag{5.23}$$

By the first two equalities of (5.23), one can easily see that

$$\begin{aligned} &\psi^\varepsilon(0,0) = \varepsilon^5 < 0,\ \partial_x^4\psi^\varepsilon(0,0) = 24,\ \partial_x^\ell\psi^\varepsilon(0,0) = 0,\ \text{for } 1 \le \ell \le 3, \\ &\partial_t^\omega\psi^\varepsilon(0,0) = \varepsilon,\ \partial_x^\ell\mathcal{D}_\nu\partial_t^\omega\psi^\varepsilon(0,0) = 0 \text{ for } 1 \le \ell + |\nu| \le 2, \\ &\partial_\omega\psi^\varepsilon(0,0) = g\big(0,0,\varphi(0,0) - \varepsilon^5, \partial_x\varphi(0,0)\big) \\ &\qquad\qquad - g\big(0,0,\varphi(0,0),\partial_x\varphi(0,0)\big) = O(\varepsilon^5). \end{aligned} \tag{5.24}$$

All terms above satisfy (5.21). The remaining derivatives involved in (5.20) take the form

$$\partial_x^\ell \mathcal{D}_\nu \partial_\omega \psi^\varepsilon(0,0) = \partial_x^\ell \mathcal{D}_\nu \big[g^\varphi(\cdot, \psi^\varepsilon, \partial_x \psi^\varepsilon)\big]\big|_{(t,x)=(0,0)}, \ 1 \le \ell + |\nu| \le 3. \tag{5.25}$$

Note that

$$[\partial_x^\ell \mathcal{D}_\nu g^\varphi](\cdot, \psi^\varepsilon, \partial_x \psi^\varepsilon)\big]\big|_{(t,x)=(0,0)} = [\partial_x^\ell \mathcal{D}_\nu g^\varphi](0,0,-\varepsilon^5,0) = O(\varepsilon^5). \tag{5.26}$$

Then (5.25) becomes

$$\begin{aligned}\partial_x^\ell \mathcal{D}_\nu \partial_\omega \psi^\varepsilon(0,0) &= \partial_x^\ell \mathcal{D}_\nu \big[g^\varphi(0,0,\psi^\varepsilon(0,x), \partial_x \psi^\varepsilon(0,x))\big]\big|_{x=0} + O(\varepsilon^5) \\ &= \partial_x^\ell \mathcal{D}_\nu \big[g\big(0,0,\varphi(0,0) + \psi^\varepsilon(0,x), \partial_x \varphi(0,0) + \partial_x \psi^\varepsilon(0,x)\big)\big]\big|_{x=0} + O(\varepsilon^5).\end{aligned} \tag{5.27}$$

Thus the derivatives are combinations of terms involving the derivatives of g with respect to (y,z), which are all bounded by our assumption, and the derivatives of $(\psi^\varepsilon, \partial_x \psi^\varepsilon)$. By a tedious but quite straightforward computation of the derivatives, we obtain from (5.24) and (5.27) with the abbreviation $\eta(0) := \eta(0,0,\varphi(0,0), \partial_x \varphi(0,0))$ for any function η,

$$\begin{gathered}\partial_t^\omega \partial_\omega \psi^\varepsilon(0,0) = \partial_y g(0)\varepsilon + O(\varepsilon^5), \quad \partial_t^\omega \partial_\omega^2 \psi^\varepsilon(0,0) = |\partial_y g(0)|^2 \varepsilon + O(\varepsilon^5), \\ \partial_x^{\ell_1} \partial_\omega^{\ell_2} \partial_\omega \psi^\varepsilon(0,0) = 24[\partial_z g(0)]^{\ell_2+1} + O(\varepsilon^5) \quad \text{for } \ell_1 + \ell_2 = 3,\end{gathered}$$

and all other terms either contain $\partial_x^\ell \psi^\varepsilon(0,0) = 0$ for some $1 \le \ell \le 3$, or $\partial_x^\ell \mathcal{D}_\nu \partial_t^\omega \psi^\varepsilon(0,0) = 0$ for some $1 \le \ell + |\nu| \le 2$, or

$$[\partial_x^\ell \mathcal{D}_\nu g^\varphi](\cdot, \psi^\varepsilon, \partial_x \psi^\varepsilon)\big]\big|_{(t,x)=(0,0)} = O(\varepsilon^5)$$

for some $\ell + |\nu| \le 3$. We thus prove (5.21) for all the cases, and hence complete the proof of the lemma. ■

Theorem 5.9 (Stability) *Let Assumption 3.2 hold and $(f_n)_{n\ge1}$ be a sequence of functions satisfying Assumption 3.3. For each $n \ge 1$, let u_n be a viscosity subsolution of RPDE* (3.6) *with generator (f_n, g). Assume further that, for some functions f and u,*

$$\lim_{n\to\infty} [f_n - f](t,x,y,z,\gamma) = 0 \quad \textit{and} \quad \lim_{n\to\infty} [u_n - u](t,x) = 0 \tag{5.28}$$

locally uniformly in $(t,x,y,z,\gamma) \in \mathbb{R}_T^4$. Then u is a viscosity subsol. of (3.6).

Proof By the locally uniform convergence, f and u are continuous. Let $(t_0,x_0)\in(0,T]\times\mathbb{R}$ and $\varphi\in\underline{\mathcal{A}}_g u(t_0,x_0)$. We apply Lemma 5.8 at (t_0,x_0), but in the left neighborhood

$$D^-_\varepsilon(t_0,x_0):=(t_0-\varepsilon^3,t_0]\times O_\varepsilon(x_0),\tag{5.29}$$
$$\partial D^-_\varepsilon(t_0,x_0):=\big\{(t,x):(t_0-\varepsilon^3,t_0],|x-x_0|=\varepsilon \text{ or } t=t_0-\varepsilon^3,|x|\le\varepsilon\big\}.$$

We emphasize that, while for notational simplicity we established Lemma 5.8 in the right neighborhood $D^+_\varepsilon(t_0,x_0)$, we may easily reformulate it to the left neighborhood by using tbackward rough paths introduced in (2.12). By Remark 5.4, we may assume without loss of generality that $\varphi\in C^k_{\alpha,\beta}(\mathbb{R}_T)$ for some large k. Then, for any $\varepsilon>0$ small, by Lemma 5.8, there exists $\psi^\varepsilon\in C^4_{\alpha,\beta}(D^-_\varepsilon(t_0,x_0))$ such that the following holds:

$$\begin{gathered}\partial_\omega\psi^\varepsilon=g^\varphi(\cdot,\psi^\varepsilon,\partial_x\psi^\varepsilon),\quad \partial^\omega_t\psi^\varepsilon=\varepsilon;\\ |\psi^\varepsilon|+|\partial_x\psi^\varepsilon|+|\partial^2_{xx}\psi^\varepsilon|\le C\varepsilon^2 \text{ in } D^-_\varepsilon(t_0,x_0);\\ \psi^\varepsilon(t_0,x_0)<0<\inf_{(t,x)\in\partial D^-_\varepsilon(t_0,x_0)}\psi^\varepsilon(t,x).\end{gathered}\tag{5.30}$$

This together with setting $\varphi^\varepsilon:=\varphi+\psi^\varepsilon$ yields

$$\sup_{(t,x)\in\partial D^-_\varepsilon(t_0,x_0)}\big[[u-\varphi^\varepsilon](t,x)\big]<0<[u-\varphi^\varepsilon](t_0,x_0).\tag{5.31}$$

Since u^n converges to u locally uniformly, we have, for $n=n(\varepsilon)$ large enough,

$$\sup_{(t,x)\in\partial D^-_\varepsilon(t_0,x_0)}\big[[u_n-\varphi^\varepsilon](t,x)\big]<0<[u_n-\varphi^\varepsilon](t_0,x_0).$$

Then there exists $(t_\varepsilon,x_\varepsilon)=(t^n_\varepsilon,x^n_\varepsilon)\in D^-_\varepsilon(t_0,x_0)$ such that

$$[u_n-\varphi^\varepsilon](t_\varepsilon,x_\varepsilon)=0=\max_{[t_0-\varepsilon^3,t_\varepsilon]\times\bar O_\varepsilon(x_0)}\big[[u_n-\varphi^\varepsilon](t,x)\big].$$

Note that

$$\begin{aligned}\partial_\omega\varphi^\varepsilon&=\partial_\omega\varphi+\partial_\omega\psi^\varepsilon=g(\cdot,\varphi,\partial_x\varphi)\\&\quad+\big[g(\cdot,\varphi+\psi^\varepsilon,\partial_x\varphi+\partial_x\psi^\varepsilon)-g(\cdot,\varphi,\partial_x\varphi)\big]=g(\cdot,\varphi^\varepsilon,\partial_x\varphi^\varepsilon).\end{aligned}$$

Then $\varphi^\varepsilon\in\underline{\mathcal{A}}_g u_n(t_\varepsilon,x_\varepsilon)$. By the viscosity subsolution property of u_n,

$$\partial^\omega_t\varphi^\varepsilon(t_\varepsilon,x_\varepsilon)-f_n(\cdot,\varphi^\varepsilon,\partial_x\varphi^\varepsilon,\partial^2_{xx}\varphi^\varepsilon)(t_\varepsilon,x_\varepsilon)\le 0.$$

Fix n and send $\varepsilon\to0$. Then, by the convergence of ψ^ε and its derivatives,

$$\partial^\omega_t\varphi(t_0,x_0)-f_n(\cdot,\varphi,\partial_x\varphi,\partial^2_{xx}\varphi)(t_0,x_0)\le 0,$$

Now by sending $n\to\infty$ we get $\partial^\omega_t\varphi(t_0,x_0)-f(\cdot,\varphi,\partial_x\varphi,\partial^2_{xx}\varphi)(t_0,x_0)\le0$, i.e., u is a viscosity subsolution of (3.6). ■

6. Viscosity solutions of rough PDEs: comparison principle.

Let

$$(6.1)\quad \begin{array}{l} u_1 \text{ (resp. } u_2\text{) be a viscosity subsol. (resp. supersol.) of RPDE (3.6),} \\ u_1(0,\cdot) \le u_2(0,\cdot), \quad \text{and } u_1, u_2 \text{ have polynomial growth in } x. \end{array}$$

Our goal is to show that $u_1 \le u_2$ on $\mathbb{R}_T$.

When both u_1 and u_2 are smooth, $u := u_1 - u_2$ solve a linear RPDE. Then the function F corresponding to this linear RPDE becomes linear, see (7.30) in [2]. Thus, by using the representation formula (7.31)-(7.32) in [2], one can easily show that $u_1 \le u_2$.

6.1. *Partial comparison principle.* Here, we assume that one of u_1 and u_2 is smooth. We need the following result (cf. Lemma 5.8).

Lemma 6.1 *Let Assumption 3.2 be in force. Let $\varphi \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ for some large k. Let $\partial_\omega \varphi = g(\cdot, \varphi, \partial_x \varphi)$ on $\mathbb{R}_T$. For any $0 \le t_0 < T$, $0 < \delta \le T - t_0$, and $\varepsilon > 0$, recall* (5.14)*, and consider the RPDE*

$$(6.2)\quad \psi^\varepsilon(t,x) = \varepsilon + t - t_0 + \int_{t_0}^t g^\varphi(s, x, \psi^\varepsilon, \partial_x \psi^\varepsilon) d\omega_s, \qquad [t_0, t_0+\delta] \times \mathbb{R}.$$

Then $\psi^\varepsilon \in C^2_{\alpha,\beta}([t_0, t_0+\delta] \times \mathbb{R})$ with $\|\psi^\varepsilon\|_{C^2_{\alpha,\beta}([t_0,t_0+\varepsilon]\times\mathbb{R})} \le C$, where C depends only on g and φ, but not on t_0, ε, and δ. Moreover, ψ^ε satisfies

$$(6.3)\quad \begin{array}{c} \partial_\omega \psi^\varepsilon = g^\varphi(\cdot, \psi^\varepsilon, \partial_x \psi^\varepsilon), \quad \partial^\omega_t \psi^\varepsilon = 1, \\ |\psi^\varepsilon| + |\partial_x \psi^\varepsilon| + |\partial^2_{xx} \psi^\varepsilon| \le C[\varepsilon + \delta^{\alpha\beta}] \text{ in } [t_0, t_0+\delta] \times \mathbb{R}, \\ \inf_{x\in\mathbb{R}} \psi^\varepsilon(t,x) > 0 \quad \text{for all } t \in (t_0, t_0+\delta]. \end{array}$$

Proof The uniform regularity of ψ^ε and the first line of (6.3) are clear. Note that $\psi^\varepsilon(t_0,x) = \varepsilon$, $\partial_x \psi^\varepsilon(t_0,x) = 0$, $\partial^2_{xx} \psi^\varepsilon(t_0,x) = 0$. The second line of (6.3) follows from the Hölder continuity of the functions in terms of t. Moreover, since $g^\varphi(t,x,0,0) = 0$, we may write it as $g^\varphi(t,x,\psi^\varepsilon, \partial_x \psi^\varepsilon) = \sigma(t,x)\psi^\varepsilon + b(t,x)\partial_x \psi^\varepsilon$, where σ and b depend on ψ^ε. Then we may view (6.2) as a linear RPDE with coefficients σ and b. Thus, by (7.31)-(7.32), we have a representation formula for ψ^ε. The uniform regularity of ψ^ε implies the uniform regularity of σ and b, which leads to the third line of (6.3). ■

Theorem 6.2 *Let Assumptions 3.2 and 3.3 and* (6.1) *be in force. If one of u_1 and u_2 is in $C^k_{\alpha,\beta}(\mathbb{R}_T)$ for some large k, then $u_1 \le u_2$.*

Proof For the sake of a contradiction, assume that $[u_1 - u_2](t_0, x_0) > 0$ for some $(t_0, x_0) \in (0,T] \times \mathbb{R}$. By Remark 5.7 (i), without loss of generality we may assume both u_1 and u_2 satisfy (5.12). Put

$$c_\delta := \sup_{(t,x)\in[0,\delta]\times\mathbb{R}} [u_1 - u_2](t,x), \quad \delta_0 := \inf\{\delta \ge 0 : c_\delta > 0\}.$$

Then c_δ is nondecreasing in δ, $c_0 \le 0 < c_{t_0}$, and thus $\delta_0 < t_0$. For any $0 < \delta \le t_0 - \delta_0$, $c_{\delta_0+\delta} > 0$. By (5.12) and since $u(0,\cdot) \le 0$, there exists $(t_\delta, x_\delta) \in (\delta_0, \delta_0 + \delta] \times \mathbb{R}$ such that $(u_1 - u_2)(t_\delta, x_\delta) = c_{\delta_0+\delta}$. Set $\varepsilon := c_{\delta_0+\delta} \wedge \delta^{\alpha\beta}$. Applying Lemma 6.1 with $\varphi := u_2$ on $[\delta_0, t_\delta]$, but again backwardly in time, we have ψ^ε satisfying

$$\psi^\varepsilon(t_\delta, x) = \varepsilon, \quad \partial_t^\omega \psi^\varepsilon = 1, \quad \inf_{x\in\mathbb{R}} \psi^\varepsilon(\delta_0, x) > 0,$$
$$|\psi^\varepsilon| + |\partial_x \psi^\varepsilon| + |\partial_{xx}^2 \psi^\varepsilon| \le C\delta^{\alpha\beta} \text{ in } [t_0, t_\delta] \times \mathbb{R}.$$

Define $\varphi^\varepsilon := u_2 + \psi^\varepsilon$. Note that $[u_1 - \varphi^\varepsilon](t_\delta, x_\delta) \ge 0 > \sup_{x\in\mathbb{R}}[u_1 - \varphi^\varepsilon](\delta_0, x)$. Then there exists $(t_\delta^*, x_\delta^*) \in (\delta_0, t_\delta] \times \mathbb{R}$ such that

$$[u_1 - \varphi^\varepsilon](t_\delta^*, x_\delta^*) = 0 = \sup_{(t,x)\in[\delta_0, t_\delta^*]\times\mathbb{R}} [u_1 - \varphi^\varepsilon](t_\delta^*, x_\delta^*).$$

By the definition of g^φ, it is clear that $\partial_\omega \varphi^\varepsilon = g(\cdot, \varphi^\varepsilon, \partial_x \varphi^\varepsilon)$. Then $\varphi^\varepsilon \in \underline{\mathcal{A}}_g u_1(t_\delta^*, x_\delta^*)$. Thus, by using the classical supersolution property of u_2 and the viscosity subsolution property of u_1 we have

$$\Big[\partial_t^\omega u_2 - f(\cdot, u_2, \partial_x u_2, \partial_{xx}^2 u_2)\Big](t_\delta^*, x_\delta^*)$$
$$\ge 0 \ge \Big[\partial_t^\omega \varphi^\varepsilon - f(\cdot, \varphi^\varepsilon, \partial_x \varphi^\varepsilon, \partial_{xx}^2 \varphi^\varepsilon)\Big](t_\delta^*, x_\delta^*).$$

Now, at (t_δ^*, x_δ^*), we have

$$\begin{aligned} 1 &= \partial_t^\omega \varphi^\varepsilon - \partial_t^\omega u_2 \le f(\cdot, \varphi^\varepsilon, \partial_x \varphi^\varepsilon, \partial_{xx}^2 \varphi^\varepsilon) - f(\cdot, u_2, \partial_x u_2, \partial_{xx}^2 u_2) \\ &\le C\Big[|\psi^\varepsilon| + |\partial_x \psi^\varepsilon| + |\partial_{xx}^2 \psi^\varepsilon|\Big] \le C\delta^{\alpha\beta}, \end{aligned}$$

which is an obvious contradiction when δ is small. ■

Remark 6.3 When g is independent of y, we can prove Proposition 6.2 much easier without invoking Lemma 6.1. In fact, in this case, assuming to the contrary that $(u_1 - u_2)(t_0, x_0) > 0$ for some $(t_0, x_0) \in \mathbb{R}_T$. Then

$$c := \sup_{(t,x)\in[0,t_0]\times\mathbb{R}} [u_1 - u_2](t,x) \ge [u_1 - u_2](t_0, x_0) > 0.$$

By (5.12) and $[u_1 - u_2](0,\cdot) \le 0$, there exists $(t^*, x^*) \in (0, t_0] \times \mathbb{R}$ such that

$$[u_1 - u_2](t^*, x^*) = c := \sup_{(t,x)\in[0,t_0]\times\mathbb{R}} [u_1 - u_2](t,x) \ge [u_1 - u_2)]t_0, x_0) > 0.$$

Define $\varphi = u_2 + c$. Since g is independent of y, we have

$$\partial_\omega \varphi = \partial_\omega u_2 = g(t,x,\partial_x u_2) = g(t,x,\partial_x \varphi).$$

Then one can easily verify that $\varphi \in \underline{\mathcal{A}}_g u_1(t^*,x^*)$. Moreover, by Remark 5.7 (ii), we can assume without loss of generality that f is strictly decreasing in y. Now it follows from the classical supersolution property of u_2 and the viscosity subsolution property of u_1 that, taking values at (t^*,x^*),

$$\begin{aligned}\partial_t^\omega u_2 - f(\cdot,u_2,\partial_x u_2,\partial^2_{xx}u_2) \;\geq\; 0 \;\geq\; \partial_t^\omega \varphi - f(\cdot,\varphi,\partial_x\varphi,\partial^2_{xx}\varphi)\\ = \partial_t^\omega u_2 - f(\cdot,u_2+c,\partial_x u_2,\partial^2_{xx}u_2),\end{aligned}$$

which is the desired contradiction since f is strictly decreasing in y.

The following comparison result follows immediately from Theorem 6.2.

Corollary 6.4 *Let Assumptions 3.2 and 3.3 and* (6.1) *be in force. If RPDE* (3.6) *has a classical solution* $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ *for some large* k *and* $u_1(0,\cdot) \leq u(0,\cdot) \leq u_2(0,\cdot)$*, then* $u_1 \leq u \leq u_2$*. In particular,* u *is the unique solution in the viscosity sense.*

6.2. *Full comparison.* We shall follow the approach in Ekren, Keller, Touzi, and Zhang [16]. For this purpose, we strengthen Assumption 3.2 slightly by imposing some uniform property of g in terms of y.

Assumption 6.5 *The diffusion coefficient* g *belongs* $C^{k_0,loc}_{\alpha,\beta}(\mathbb{R}^3_T)$ *for some* k_0 *large enough, and*

(i) $\partial_z g \in C^{k_0-1}_{\alpha,\beta}(\mathbb{R}^3_T)$,

(ii) for $i = 0,\cdots,k_0$ *and* $z \in \mathbb{R}$, $\partial^i_x g(\cdot,z), \partial^i_y g(\cdot,z) \in C^{k_0-i}_{\alpha,\beta}(\mathbb{R}^2_T)$ *with* $\|\partial^i_x g(\cdot,z)\|_{k_0-i} + \|\partial^i_y g(\cdot,z)\|_{k_0-i} \leq C[1+|z|]$.

We remark that, under Assumption 3.2, all the results in this subsection hold true if we assume instead that T is small enough.

Given an initial condition u_0, motived by the partial comparison, we fix a large k and define

$$\overline{u}(t,x) := \inf\big\{\varphi(t,x) : \varphi \in \overline{\mathcal{U}}\big\}, \quad \underline{u}(t,x) := \sup\big\{\varphi(t,x) : \varphi \in \underline{\mathcal{U}}\big\}, \tag{6.4}$$

where

$$\begin{aligned}
\mathcal{U} := \big\{\varphi \in \mathbb{L}^0(\mathbb{R}_T) &: \text{càglàd in } t, \text{ with polynomial growth in } x,\\
&\varphi(0,\cdot) = u_0, \text{ and } \exists\, 0 = t_0 < \cdots < t_n = T \text{ such that}\\
&\quad \varphi \in C^k_{\alpha,\beta}((t_{i-1}, t_i] \times \mathbb{R}) \text{ for } i = 1, \ldots, n\big\},\\
\overline{\mathcal{U}} := \big\{\varphi \in \mathcal{U} &: \Delta\varphi_{t_i} \geq 0,\ \varphi \text{ is a classical supersolution of}\\
&\quad \text{RPDE (3.6) on each } (t_{i-1}, t_i]\big\},\\
\underline{\mathcal{U}} := \big\{\varphi \in \mathcal{U} &: \Delta\varphi_{t_i} \leq 0,\ \varphi \text{ is a classical subsolution of}\\
&\quad \text{RPDE (3.6) on each } (t_{i-1}, t_i]\big\}.
\end{aligned} \tag{6.5}$$

Lemma 6.6 *Let Assumptions 6.5, 3.3, and 3.4 hold. Then* $\overline{\mathcal{U}}, \underline{\mathcal{U}} \neq \emptyset$.

Proof We prove $\overline{\mathcal{U}} \neq \emptyset$ in several steps. The proof for $\underline{\mathcal{U}}$ is similar.

Step 1. Put $Q_1 := \mathbb{R}^2 \times \{z \in \mathbb{R} : |z| \leq 1\}$. Then $g \in C^{k_0}_{\alpha,\beta}([0,T] \times Q_1)$ and let N_0 denote its k_0-norm. Under our strengthened conditions in Assumption 6.5, it follows from the arguments in Proposition 4.1 that there exist δ_0, C_0, depending only on N_0, such that

$$\Theta^0 \in C^{k_0}_{\alpha,\beta}([0,\delta_0] \times Q_1; \mathbb{R}^3) \text{ with } \|\Theta^0\|_{k_0} \leq C_0, \tag{6.6}$$

where $\Theta^0_t(\theta) := \Theta_t(\theta) - \theta$. Moreover, for a possibly smaller $\delta_0 > 0$, again depending only on N_0, we have

$$\partial_x X_t(x,y,0) \geq 1/2 \quad \text{for all } (t,x,y) \in [0,\delta_0] \times \mathbb{R}^2. \tag{6.7}$$

Step 2. Recall (4.11) and put

$$\begin{aligned}
\overline{F}(t,y) &:= \sup_{x\in\mathbb{R}} F(t,x,y,0,0)\\
&= \sup_{x\in\mathbb{R}} f\Big(t, \Theta_t(x,y,0), \frac{\partial_x Z_t(x,y,0)}{\partial_x X_t(x,y,0)}\Big) \exp\Big(-\int_0^t \partial_y g(s, \Theta_s(x,y,0)) d\omega_s\Big).
\end{aligned}$$

By Assumption 3.3, (6.6) and (6.7) yield, for all $(t,x,y) \in [0,\delta_0] \times \mathbb{R}^2$,

$$\begin{aligned}
|F(t,x,y,0,0)| &\leq C_1\Big[1 + |Y_t(x,y,0)| + |Z_t(x,y,0)| + \Big|\frac{\partial_x Z_t(x,y,0)}{\partial_x X_t(x,y,0)}\Big|\Big]\\
&\leq C_1[1 + |y|],
\end{aligned}$$

where C_1 depends on N_0 and the K_0 and L_0 in Assumption 3.3. Then $|\overline{F}(t,y)| \leq C_1[1 + |y|]$. Moreover, it is clear that $F(t,x,y,0,0)$ is differentiable in y. However, due to the exponential term outside of f, in general $\partial_y F(t,x,y,0,0)$ may not be bounded, and we can only claim that

$|\partial_y F(t,x,y,0,0)| \leq C_1[1+|y|]$. By the regularity of f, it is clear that $\overline{F}$ is continuous in t. Let $\hat{F}$ be a smooth mollifier of $\overline{F}$ such that

$$\overline{F} \leq \hat{F} \leq \overline{F} + 1, \quad |\hat{F}(t,y)| \leq C_1[1+|y|], \quad |\partial_y \hat{F}(t,y)| \leq C_1[1+|y|]. \tag{6.8}$$

Set $K_1 := \|u_0\|_\infty e^{C_1 T}$ for the above C_1. We see that $\hat{F}(t,y)$ is uniformly Lipschitz in y on $[0,\delta_0]\times[-K_1-1, K_1+1]$. Let ι be a smooth truncation function such that $\iota(x) = x$ for $|x| \leq K_1$, $\iota(x) = \operatorname{sign}(x)[K_1+1]$ for $|x| \geq K_1+1$, and $|\iota(x)| \leq |x|$ for all x. Now consider the ODE

$$\overline{\psi}_t = \|u_0\|_\infty + \int_0^t \hat{F}(s, \iota(\overline{\psi}_s))ds, \quad 0 \leq t \leq \delta_0. \tag{6.9}$$

Clearly, (6.9) has a solution $\overline{\psi} \in C^\infty([0,\delta_0])$. Since $|\hat{F}(s,\iota(y))| \leq C_1[1+|\iota(y)|] \leq C_1[1+|y|]$, $|\overline{\psi}_t| \leq \|u_0\|_\infty e^{C_1 t} \leq K_1$ for $t \leq \delta_0$. Thus $\iota(\overline{\psi}_t) = \psi_t$ and

$$\overline{\psi}_t = \|u_0\|_\infty + \int_0^t \hat{F}(s, \overline{\psi}_s)ds, \quad 0 \leq t \leq \delta_0. \tag{6.10}$$

Abusing the notation by letting $\overline{\psi}(t,x) := \overline{\psi}(t)$, we have

$$\partial_\omega \overline{\psi}_t = 0, \quad \partial_x \overline{\psi}_t = 0, \quad \partial^2_{xx}\overline{\psi}_t = 0,$$
$$\partial_t \overline{\psi}(t,x) = \hat{F}(t,\overline{\psi}_t) \geq \overline{F}(t,\overline{\psi}_t) \geq F(t,x,\overline{\psi},\partial_x\overline{\psi},\partial^2_{xx}\overline{\psi}),$$

i.e., $\overline{\psi}$ is a classical supersolution of PDE (4.10)-(4.11). Thus (4.9) becomes

$$\hat{\Theta}_t(x) := \Theta_t(x, \overline{\psi}(t,x), \partial_x\overline{\psi}(t,x)) = \Theta_t(x,\overline{\psi}_t, 0).$$

in this case. By (6.7), for any $t \in [0,\delta_0]$, $x \mapsto \hat{X}_t(x)$ is invertible. Put

$$\overline{\varphi}(0,x) := u_0(x), \quad \overline{\varphi}(t,x) := \hat{Y}_t(\hat{X}_t^{-1}(x)), \ t \in (0,\delta_0].$$

Then $\Delta\overline{\varphi}_0(x) \geq 0$, $\overline{\varphi} \in C^k_{\alpha,\beta}((0,\delta_0]\times\mathbb{R})$, and by Theorem 4.4, $\overline{\varphi}$ is a classical supersolution of PPDE (3.7) on $(0,\delta_0]\times\mathbb{R}$.

Step 3. Let δ_0 be as in Step 1. We emphasize that δ_0 depends only on N_0, in particular not on $\|u_0\|_\infty$. Let $0 = t_0 < \cdots < t_n = T$ be a partition such that $t_i - t_{i-1} \leq \delta_0$, $i = 1, \ldots, n$. By Step 2, we have a desired function $\overline{\varphi}$ on $[t_0,t_1]\times\mathbb{R}$. In particular, $\overline{\varphi}(t_1,\cdot)$ is bounded. Now consider RPDE (3.6) on $[t_1,t_2]$ with initial condition $\overline{\varphi}(t_1,\cdot)$. Following the same arguments, we may extend $\overline{\varphi}$ to $[t_0,t_2]$ such that $\Delta\overline{\varphi}(t_1,\cdot) \geq 0$, $\overline{\varphi} \in C^k_{\alpha,\beta}((t_1,t_2]\times\mathbb{R})$, and $\overline{\varphi}$ is a classical supersolution of RPDE (3.6) on $(t_1,t_2]\times\mathbb{R}$. Repeating the arguments yields the desired $\overline{\varphi}$ on $\mathbb{R}_T$, i.e., $\overline{\varphi} \in \overline{\mathcal{U}}$. ■

Now, by Theorem 6.2 and Proposition 5.2, it is clear that

$$\underline{u} \le \overline{u}. \tag{6.11}$$

We next establish the viscosity solution property of $\overline{u}$ and $\underline{u}$. We shall follow the arguments in Theorem 5.9, which relies on the crucial Lemma 5.8.

Lemma 6.7 *Let Assumptions 6.5, 3.3, and 3.4 hold.*

(i) $\overline{u}$ (resp. $\underline{u}$) is bounded and upper (resp. lower) semi-continuous.

(ii) Moreover, if $\overline{u}$ (resp. $\underline{u}$) is continuous, then $\overline{u}$ (resp. $\underline{u}$) is a viscosity supersolution (resp. viscosity subsolution) of RPDE (3.6).

We remark that it is possible to extend our definition of viscosity supersolutions to lower semicontinuous functions. However, here (i) shows that $\overline{u}$ is upper semicontinuous. So it seems that the continuity of $\overline{u}$ in (ii) is intrinsically required in this approach.

Proof By the proof of Lemma 6.6, $\overline{u}$ is bounded from above. Similarly $\underline{u}$ is bounded from below. Then it follows from (6.11) that $\overline{u}, \underline{u}$ are bounded.

We establish next the upper semicontinuity for $\overline{u}$. The regularity for $\underline{u}$ can be proved similarly. Fix $(\bar{t}, \overline{x}) \in \mathbb{R}_T$. For any $\varepsilon > 0$, there exists $\varphi_\varepsilon \in \overline{\mathcal{U}}$ such that $\varphi_\varepsilon(\bar{t}, \overline{x}) < \overline{u}(\bar{t}, \overline{x}) + \varepsilon$. By the structure of $\overline{\mathcal{U}}$, it is clear that $\varphi_\varepsilon \ge \overline{u}$ on $\mathbb{R}_T$. Assume that $\varphi_\varepsilon \in \mathcal{U}$ corresponds to the partition $0 = t_0 < \cdots < t_n = T$ as in (6.5). We distinguish between two cases.

Case 1. Assume $\bar{t} \in (t_{i-1}, t_i)$ for some $i = 1, \ldots, n$. Since φ_ε is continuous in $(t_{i-1}, t_i) \times \mathbb{R}$, there exists $\delta > 0$ such that $|\varphi_\varepsilon(t,x) - \varphi_\varepsilon(\bar{t}, \overline{x})| \le \varepsilon$ whenever $|t - \bar{t}| + |x - \overline{x}| \le \delta$. Then, for such (t, x),

$$\overline{u}(t,x) \le \varphi_\varepsilon(t,x) \le \varphi_\varepsilon(\bar{t}, \overline{x}) + \varepsilon \le \overline{u}(\bar{t}, \overline{x}) + 2\varepsilon.$$

This implies that $\overline{u}$ is upper semi-continuous at $(\bar{t}, \overline{x})$.

Case 2. Assume $\bar{t} = t_i$ for some $i = 0, \ldots, n$. By the same arguments as in Case 1, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{u}(t,x) \le \overline{u}(\bar{t}, \overline{x}) + 2\varepsilon$ for all $(t,x) \in (t_i - \delta, t_i] \times O_\delta(\overline{x})$. To see the regularity in the right neighborhood, assume for notational simplicity that $\bar{t} = 0$. Let $\hat{F}$ be as in the proof of Lemma 6.6. Consider the following ODE (with parameter x):

$$v(t,x) = \varphi_\varepsilon(0,x) + \int_0^t \hat{F}(s, v(s,x))ds. \tag{6.12}$$

By the arguments in Subsection 4.4, there exists a $\delta_0 > 0$ such that (6.12) has a classical solution $v \in C^{k,0}_{\alpha,\beta}([0,\delta_0] \times \mathbb{R})$, which clearly leads to a classical supersolution $u \in C^k_{\alpha,\beta}([0,\delta_0] \times \mathbb{R})$ of the original RPDE (3.6) with initial condition $\varphi_\varepsilon(0,x)$. Now consider RPDE (3.6) on $[\delta_0, T]$ with initial condition $u(\delta_0,\cdot)$. By the arguments in Lemma 6.6, there exists a $\tilde\varphi_\varepsilon \in \overline{\mathcal{U}}$ such that $\tilde\varphi_\varepsilon(t,x) = u(t,x)$ for $(t,x)\in[0,\delta_0]\times\mathbb{R}$. Then $\overline{u} \le u$ on $[0,\delta_0]\times\mathbb{R}$. Now by the continuity of u, there exists a $\delta \le \delta_0$ such that, whenever $|t|+|x-\overline{x}| \le \delta$,

$$\overline{u}(t,x) \le u(t,x) \le u(0,\overline{x}) + \varepsilon = \varphi_\varepsilon(0,\overline{x}) + \varepsilon \le \overline{u}(0,\overline{x}) + 2\varepsilon,$$

implying the regularity in the right neighborhood, and thus $\overline{u}$ is upper semi-continuous.

We finally show that $\underline{u}$ is a viscosity subsolution, provided its continuity, The viscosity supersolution property of $\overline{u}$ follows similar arguments. Fix $(t_0,x_0)\in(0,T]\times\mathbb{R}$. Let $\varphi \in \underline{\mathcal{A}}_g\underline{u}(t_0,x_0)$. For any $\varepsilon>0$, let $(\partial D^-_\varepsilon(t_0,x_0),\psi^\varepsilon)$ be as in (5.29)-(5.30). By definition, there exists a $u_\varepsilon \in \underline{\mathcal{U}}$ with $\underline{u}(t_0,x_0) - u_\varepsilon(t_0,x_0) \le -\psi^\varepsilon(t_0,x_0)$. Let $\varphi^\varepsilon := \varphi+\psi^\varepsilon$, $\partial D^-_\varepsilon := \partial D^-_\varepsilon(t_0,x_0)$. Then

$$\varphi^\varepsilon(t_0,x_0) = \underline{u}(t_0,x_0) + \psi^\varepsilon(t_0,x_0) \le u_\varepsilon(t_0,x_0 \text{ and}$$
$$\inf_{(t,x)\in\partial D^-_\varepsilon}[\varphi^\varepsilon - u_\varepsilon](t,x) \ge \inf_{(t,x)\in\partial D^-_\varepsilon}[\underline{u}+\psi^\varepsilon - u_\varepsilon](t,x) \ge \inf_{(t,x)\in\partial D^-_\varepsilon}\psi^\varepsilon(t,x) > 0.$$

follow. Thus there exists a $(t_\varepsilon,x_\varepsilon)\in(t_0-\varepsilon^3,t_0]\times O_\varepsilon(x_0)$ such that

$$[\varphi^\varepsilon - u_\varepsilon](t_\varepsilon,x_\varepsilon) = 0 = \inf_{(t,x)\in[t_0-\varepsilon^3,t_\varepsilon]\times O_\varepsilon(x_0)}[\varphi^\varepsilon - u_\varepsilon](t,x).$$

Then $\varphi^\varepsilon \in \underline{\mathcal{A}}_g u_\varepsilon(t_\varepsilon,x_\varepsilon)$. Since u_ε is a classical subsolution, hence a viscosity subsolution, $\mathcal{L}\varphi^\varepsilon(t_\varepsilon,x_\varepsilon)\le 0$. Sending $\varepsilon\to0$ yields $\mathcal{L}\varphi(t_0,x_0)\le 0$, i.e., $\underline{u}$ is a viscosity subsolution. ■

Theorem 6.8 *Let Assumptions 6.5, 3.3, and 3.4 hold. Let* (6.1) *be in force with $u_1(0,\cdot)\le u_0 \le u_2(0,\cdot)$. Assume further that*

$$\overline{u} = \underline{u}. \tag{6.13}$$

Then $u_1 \le \overline{u} = \underline{u} \le u_2$ and $\overline{u}$ is the unique viscosity solution of RPDE (3.6).

Proof By Lemma 6.7 and (6.13), it is clear that $\overline{u}=\underline{u}$ is continuous and is a viscosity solution of RPDE (3.6). By Theorem 6.2 (partial comparison), $u_1 \le \overline{u}$ and $\underline{u}\le u_2$. Thus (6.13) leads to the comparison principle immediately. ■

Remark 6.9 The introduction of $\overline{u}$ and $\underline{u}$ is motivated from Perron's approach in PDE viscosity theory. However, there are several differences.

(i) In Perron's approach, the functions in $\overline{\mathcal{U}}$ are viscosity supersolutions, rather than classical supersolutions. So our $\overline{u}$ is in principle larger than the counterpart in PDE theory. Similarly our $\underline{u}$ is smaller than the the counterpart in PDE theory. Consequently, it is more challenging to verify the condition (6.13).

(ii) The standard Perron's approach is mainly used for the existence of viscosity solution in the case the PDE satisfies the comparison principle. Here we use $\overline{u}$ and $\underline{u}$ to prove both the comparison principle and the existence.

(iii) In the standard Perron's approach, one shows directly that $\overline{u}$ is a viscosity solution, while in Lemma 6.7 we are only able to show $\overline{u}$ is a viscosity supersolution.

The condition (6.13) is in general quite challenging. In the next section, we establish the complete result when the diffusion coefficient g is semilinear.

7. Rough PDEs with semilinear diffusion. We study RPDE (3.6) and PPDE (3.7) in the case that g is semilinear, i.e.,

$$g(t,x,y,z) = \sigma(t,x)\,z + g_0(t,x,y). \tag{7.1}$$

We employ the following assumption.

Assumption 7.1 *$\sigma \in C^{k_0}_{\alpha,\beta}(\mathbb{R}_T)$ and $g_0 \in C^{k_0}_{\alpha,\beta}(\mathbb{R}^2_T)$ for some large k_0.*

Clearly, Assumption 7.1 implies Assumption 6.5. Note that in this section, we obtain global result. Thus we require g_0 and its derivatives are uniformly bounded in y as well.

7.1. *Global equivalence with the PDE.* Here, (4.1) becomes

$$\begin{aligned} X_t(x) &= x - \int_0^t \sigma(s, X_s(x))\,d\omega_s, \\ Y_t(x,y) &= y + \int_0^t g_0\big(s, X_s(x), Y_s(x,y)\big)\,d\omega_s, \end{aligned} \tag{7.2}$$

where X (resp. Y) depends only on x (resp. (x,y)), and

$$\begin{aligned} Z_t(\theta) = z + \int_0^t \Big[& Z_s(\theta)\big[\partial_x\sigma(s, X_s(x)) + \partial_y g_0\big(s, X_s(x), Y_s(x,y)\big)\big] \\ & + \partial_x g_0\big(s, X_s(x), Y_s(x,y)\big)\Big]\,d\omega_s. \end{aligned}$$

where $\theta = (x,y,z)$. By Lemma 2.13, we have, omitting the variable θ,

$$Z_t = \Gamma_t z + \int_0^t \frac{\Gamma_t}{\Gamma_s}\partial_x g_0(s, X_s, Y_s)\,d\omega_s, \tag{7.3}$$

where $\Gamma_t := \exp\big(\int_0^t [\partial_x\sigma(s,X_s) + \partial_y g_0(s,X_s,Y_s)]\,d\omega_s\big)$.

Lemma 7.2 *Let Assumption 7.1 hold.*

(i) RDE (7.2) *has a classical solution* (X, Y) *satisfying*

$$X - x \in C^k_{\alpha,\beta}(\mathbb{R}_T), \quad Y - y \in C^k_{\alpha,\beta}(\mathbb{R}^2_T) \tag{7.4}$$

(ii) There exists a $c > 0$ *such that*

$$\begin{aligned} \partial_x X_t(x) &= \exp\Big(- \int_0^t \partial_x \sigma(s, X_s(x))\, d\omega_s\Big) \ge c, \\ \partial_y Y_t(x,y) &= \exp\Big(\int_0^t \partial_y g_0(s, X_s(x), Y_s(x,y))\, d\omega_s\Big) \ge c. \end{aligned} \tag{7.5}$$

(iii) For each t*, the mapping* $x \mapsto X_t(x)$ *has inverse function* $X_t^{-1}(\cdot)$*; and for each* (t,x)*, the mapping* $y \mapsto Y_t(x,y)$ *has inverse function* $Y_t^{-1}(x,\cdot)$*.*

We remark that the proof below uses (7.5). One can also use the backward rough path in (2.12) to construct the inverse functions directly. This argument works in multidimensional settings as well (see [26]).

Proof (i) follows directly from Lemma 2.15, which also implies

$$\begin{aligned} \partial_x X_t(x) &= 1 - \int_0^t \partial_x \sigma(s, X_s(x)) \partial_x X_s(x)\, d\omega_s; \\ \partial_y Y_t(x,y) &= 1 + \int_0^t \partial_y g_0\big(s, X_s(x), Y_s(x,y)\big) \partial_y Y_s(x,y)\, d\omega_s. \end{aligned}$$

Then the representations in (7.5) follow from Lemma 2.13. Moreover, set $\check X := X - x \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ and $\tilde\sigma(t,x) := \sigma(t, X_t(x)) = \sigma(t, x + \check X_t(x))$. Then by the uniform regularity of σ, $\sup_{x\in\mathbb{R}} \|\tilde\sigma(\cdot, x)\|_k \le C$. This implies that $\int_0^t \partial_x \sigma_s(X_s(x)) \partial_x X_s(x)\, d\omega_s$ is uniformly bounded, uniformly in (t,x). Therefore, we obtain the first estimate for $\partial_x X$ in (7.5). The second estimate for $\partial_y Y$ in (7.5) follows from the similar arguments.

Finally, for each t, the fact $\partial_x X_t(x) \ge c$ implies that $x \mapsto X_t(x)$ is one to one and the range is the whole real line $\mathbb{R}$. Thus $X_t^{-1} : \mathbb{R} \to \mathbb{R}$ exists. Similarly one can show that $Y_t^{-1}(x,\cdot)$ exists. ∎

One can easily check, omitting (x,y,z) in $X_t(x), Y_t(x,y)$, $Z_t(x,y,z)$,

$$\begin{aligned} &\partial_y X_t = 0; \quad \partial_z X_t = 0; \quad Z_t = \tfrac{\partial_y Y_t}{\partial_x X_t} z + \tfrac{\partial_x Y_t}{\partial_x X_t}; \quad \partial_z Z_t = \tfrac{\partial_y Y_t}{\partial_x X_t}; \\ &\partial_y Z_t = \tfrac{\partial_{yy} Y_t}{\partial_x X_t} z + \tfrac{\partial_{xy} Y_t}{\partial_x X_t}, \ \partial_x Z_t = \tfrac{\partial_{xy} Y_t \partial_x X_t - \partial_y Y_t \partial^2_{xx} X_t}{(\partial_x X_t)^2} z + \tfrac{\partial_{xx} Y_t \partial_x X_t - \partial_x Y_t \partial^2_{xx} X_t}{(\partial_x X_t)^2}, \end{aligned}$$

and then (4.11) becomes

$$(7.6)\qquad F(t,x,y,z,\gamma) := \frac{1}{\partial_y Y_t}\ f\Big(t, X_t, Y_t, \frac{\partial_y Y_t}{\partial_x X_t} z + \frac{\partial_x Y_t}{\partial_x X_t}, \frac{\partial_y Y_t}{(\partial_x X_t)^2}\gamma + \frac{\partial_{yy} Y_t}{(\partial_x X_t)^2} z^2 + \frac{2\partial_{xy} Y_t \partial_x X_t - \partial_y Y_t \partial^2_{xx} X_t}{(\partial_x X_t)^3} z + \frac{\partial^2_{xx} Y_t \partial_x X_t - \partial_x Y_t \partial^2_{xx} X_t}{(\partial_x X_t)^3}\Big).$$

Under our conditions, F has typically quadratic growth in z and is not uniformly Lipschitz in y. Moreover, the first equality of (4.8) becomes

$$(7.7)\quad v(t,x) = Y_t^{-1}\big(x, u(t, X_t(x))\big) \text{ or } u(t,x) = Y_t\big(X_t^{-1}(x), v(t, X_t^{-1}(x))\big).$$

By using similar arguments as in Subsection 4.2, we obtain the following result which is global in this semilinear case.

Theorem 7.3 *Let Assumptions 7.1 and 3.3 hold. Assume $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ and $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$ satisfy* (7.7). *Then u is a classical solution (resp. subsolution, supersolution) of RPDE* (3.6)-(7.1) *if and only if v is a classical solution (resp. subsolution, supersolution) of PDE* (4.10)-(7.6).

The next result establishes equivalence in the viscosity sense.

Theorem 7.4 *Let Assumptions 7.1 and 3.3 hold. Assume u, $v \in C(\mathbb{R}_T)$ satisfy* (7.7). *Then u is a viscosity solution (resp. subsol., supersol.) of RPDE* (3.6)-(7.1) *at $(t_0, x_0) \in (0,T] \times \mathbb{R}$ if and only if v is a viscosity solution (resp. subsolution, supersolution) of PDE* (4.10)-(7.6) *at $(t_0, X_{t_0}^{-1}(x_0))$.*

Proof We prove the statement only for supersolutions. First, we prove the if part. Let $\tilde{x}_0 := X_{t_0}^{-1}(x_0)$ and v be a viscosity supersolution of PDE (4.10)-(7.6) at $(t_0, \tilde{x}_0)$. Let $\varphi \in \overline{\mathcal{A}}_g u(t_0, x_0)$ with corresponding δ_0. Define

$$(7.8)\qquad \psi(t,x) := Y_t^{-1}\big(x, \varphi(t, X_t(x))\big),\ \text{i.e., } Y_t(x, \psi(t,x)) = \varphi(t, X_t(x)).$$

It is clear that $\psi(t_0, \tilde{x}_0) = v(t_0, \tilde{x}_0)$. By the continuity of X, there exists a $\delta > 0$ such that $X_t(x) \in O_{\delta_0}(x_0)$ for all $(t,x) \in D_\delta(t_0, \tilde{x}_0)$. By the same arguments as for (4.7), $\partial_\omega \psi = 0$. Moreover, for $(t,x) \in D_\delta(t_0, \tilde{x}_0)$, since $\varphi \in \overline{\mathcal{A}}_g u(t_0, x_0)$, we have $\varphi(t, X_t(x)) \le u(t, X_t(x))$. By Lemma 7.2, the mapping $y \to Y_t(x,y)$ is increasing. Thus Y^{-1} is also increasing and $\psi(t,x) \le v(t,x)$, i.e., ψ is a test function for v at (t_0, x_0) and

$$(7.9)\quad \partial_t \psi(t_0, \tilde{x}_0) \ge F\big(t_0, \tilde{x}_0, \psi(t_0, \tilde{x}_0), \partial_x \psi(t_0, \tilde{x}_0), \partial^2_{xx} \psi(t_0, \tilde{x}_0)\big).$$

By the derivation of F, this implies

$$(7.10)\ \partial^\omega_t \varphi(t_0, x_0) \ge f\big(t_0, x_0, \varphi(t_0, x_0), \partial_x \varphi(t_0, x_0), \partial^2_{xx} \varphi(t_0, x_0)\big),$$

i.e., u is a viscosity supersolution at (t_0, x_0).

For the opposite direction, assume that u is a viscosity supersolution of RPDE (3.6) at (t_0, x_0). For $\psi \in \overline{\mathcal{A}}_0 v(t_0, \tilde{x}_0)$ corresponding to $g = 0$, define $\varphi(t,x) := Y_t\big(X_t^{-1}(x), \psi(t, X_t^{-1}(x))\big)$, which still implies $Y_t(x, \psi(t,x)) = \varphi(t, X_t(x))$. By similar arguments, (7.9) follows from (7.10). ■

Remark 7.5 In the general case, there are two major differences:

(i) The transformation determined by (4.8) involves both u and $\partial_x u$, i.e., to extend Theorem 7.4, one has to assume that the candidate viscosity solution u is differentiable in x.

(ii) The transformation is local, in particular, the δ in Theorem 4.5 depends on $\|\partial^2_{xx} u\|_\infty$, i.e., unless $\partial^2_{xx} u$ is bounded and the solution is essentially classical, we have difficulty to extend Theorem 7.4 to the general case, even just in local sense.

7.2. *Some a priori estimates.* Here, we establish uniform a priori estimates for v that will be crucial for the comparison principle of viscosity solutions in the next subsection. First, we estimate the $\mathbb{L}^\infty$-norm of v.

Proposition 7.6 *Let Assumptions 7.1, 3.3, and 3.4 hold and f be smooth. Assume further that $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$ is a classical solution of PDE* (4.10)-(7.6)*. Then there exists a constant C, which depends only on the constants K_0, L_0 in Assumption 3.3, and the regularity of σ, g_0 in Assumption 7.1, but does not depend on u_0 or the further regularity of f, such that*

$$|v(t,x)| \le e^{Ct}\big[\|u_0\|_\infty + Ct\big]. \tag{7.11}$$

Proof First, we write (4.10)-(7.6) as

$$\begin{aligned}
&\partial_t v = a(t,x)\partial^2_{xx} v + b(t,x)\partial_x v + F(t,x,v,0,0), \quad \text{where}\\
&a(t,x) := \tilde{a}(t,x,v(t,x),\partial_x v(t,x),\partial^2_{xx} v(t,x)),\\
&b(t,x) := \tilde{b}(t,x,v(t,x),\partial_x v(t,x),\partial^2_{xx} v(t,x)),\\
&\tilde{a}(t,x,y,z,\gamma) := \int_0^1 \partial_\gamma F(t,x,y,\lambda z,\lambda\gamma)d\lambda,\\
&\tilde{b}(t,x,y,z,\gamma) := \int_0^1 \partial_z F(t,x,y,\lambda z,\lambda\gamma)d\lambda.
\end{aligned}$$

Since v is a classical solution, a and b are smooth functions. Reversing the time by setting $\hat{\varphi}(t,x) := \varphi(T-t,x)$ for $\varphi = v$, a, b, F, we have

$$\partial_t \hat{v} + \hat{a}(t,x)\partial^2_{xx}\hat{v} + \hat{b}(t,x)\partial_x \hat{v} + \hat{F}(t,x,\hat{v},0,0) = 0, \quad \hat{v}(T,x) = u_0(x).$$

Let B be a standard Brownian motion. Consider the SDE

$$\hat{X}_t = x + \int_0^t \hat{b}(s, \hat{X}_s)ds + \int_0^t \sqrt{2\hat{a}}(s, \hat{X}_s)dB_s. \tag{7.12}$$

Then $\hat{Y}_t := \hat{v}(t, \hat{X}_t)$ solves the BSDE

$$\hat{Y}_t = u_0(\hat{X}_T) + \int_t^T \hat{F}(s, \hat{X}_s, \hat{Y}_s, 0, 0)ds - \int_t^T \hat{Z}_s dB_s.$$

Since $F(t,x,y,0,0) = \frac{1}{\partial_y Y_t} f\big(t, X_t, Y_t, \frac{\partial_x Y_t}{\partial_x X_t}, \frac{\partial^2_{xx} Y_t \partial_x X_t - \partial_x Y_t \partial^2_{xx} X_t}{(\partial_x X_t)^3}\big)$, we have

$$|F(t,x,y,0,0)| \le C[1+|y|], \tag{7.13}$$

following from Lemma 7.2. Then, by standard BSDE estimates,

$$|v(T,x)| = |\hat{v}(0,x)| = |\hat{Y}_0| \le e^{CT}\big[\|u_0\|_\infty + CT\big].$$

This implies (7.11) for $t = T$. Similarly we may prove (7.11) for all $t > 0$. ■

Remark 7.7 (i) We are not able to establish similar a priori estimates for $\partial_x v$. Besides the possible insufficient regularity of u_0, we emphasize that the main difficulty here is not that F has quadratic growth in z, but that F is not uniformly Lipschitz continuous in y. Nevertheless, we obtain some local estimate for $\partial_x v$ in Proposition 7.9, which will be crucial for the comparison principle of viscosity solutions later.

(ii) To overcome the difficulty above and apply standard techniques, [32] imposed technical conditions on f in the case $f = f(z,\gamma)$ (cf. [32, (1.12)]):

$$\gamma\partial_\gamma f + z\partial_z f - f \text{ is either bounded from above or from below,} \tag{7.14}$$

This is essentially satisfied when f is convex or concave in (z,γ). Our f in (7.15) below does not satisfy (7.14), in particular, we do not require f to be convex or concave in z. See also Remark 7.13.

The next result relies on representation of v and BMO estimates for BSDEs with quadratic growth. For this purpose, we restrict f to Bellman-Isaacs type with the Hamiltonian

$$\begin{aligned} f(t,x,y,z,\gamma) = \sup_{e_1\in E_1} \inf_{e_2\in E_2} \Big[&\frac{1}{2}\sigma_f^2(t,x,e)\gamma + b_f(t,x,e)z \\ &+ f_0\big(t,x,y,\sigma_f(t,x,e)z,e\big)\Big], \end{aligned} \tag{7.15}$$

where $E := E_1 \times E_2 \subset \mathbb{R}^2$ is the control set and $e = (e_1, e_2)$.

Assumption 7.8 *(i) σ_f, $b_f \in C^0(\mathbb{R}_T \times E)$ are bounded by K_0, uniformly Lipschitz continuous in x with Lipschitz constant L_0, and $\sigma_f \geq 0$;*

(ii) $f_0 \in C^0(\mathbb{R}^3_T \times E)$ is uniformly Lipschitz continuous in (x, y, z) with Lipschitz constant L_0, and $f_0(t, x, 0, 0, e)$ is bounded by K_0.

Assumption 7.8 obviously implies Assumption 3.3.

Proposition 7.9 *Let Assumptions 7.1, 7.8, and 3.4 hold, and (g, f) take the form* (7.1)-(7.15)*. Assume $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$ is a classical solution of PDE* (4.10)-(7.6)*. Then there exist constants $\delta_0 > 0$ and C_0, which depend only on K_0, L_0 in Assumption 7.8, the regularity of σ, g_0 in Assumption 7.1, and $\|u_0\|_\infty$, but not on the further regularity of f and u_0, such that*

$$|\partial_x v(t,x)| \leq C_0[1 + \|\partial_x u_0\|_\infty] \quad \text{for all } (t,x) \in [0, \delta_0] \times \mathbb{R}. \tag{7.16}$$

Proof Under (7.1) and (7.15), (4.11) and the equivalent (7.6) becomes

$$\begin{aligned} F(t,x,y,z,\gamma) = \sup_{e_1 \in E_1} \inf_{e_2 \in E_2} \Big[& \frac{1}{2}\hat\sigma_f^2(t,x,e)\gamma + \hat b_f(t,x,e) z \\ & + F_0\big(t,x,y,\hat\sigma_f(t,x,e)z,e\big)\Big], \end{aligned} \tag{7.17}$$

where, omitting (x, y) in$X_t(x)$ and $Y_t(x, y)$,

$$\begin{aligned}
&\hat\sigma_f(t,x,e) := \frac{\sigma_f(t,X_t,e)}{\partial_x X_t}, \quad \hat b_f(t,x,e) := \frac{b_f(t,X_t,e)}{\partial_x X_t},\\
&F_0(t,x,y,z,e) := \frac{1}{2}\hat\sigma_f^2(t,x,e)\frac{\partial^2_{yy}Y_t}{\partial_y Y_t}z^2 + \hat\sigma_f^2(t,x,e)[\frac{\partial^2_{xy}Y_t}{\partial_y Y_t} - \frac{\partial^2_{xx}X_t}{2\partial_x X_t}]z\\
&\quad + \frac{1}{\partial_y Y_t} f_0\big(t, X_t, Y_t, \partial_y Y_t z + \hat\sigma_f(t,x,e)\partial_x Y_t, e\big)\\
&\quad + \hat\sigma_f^2(t,x,e)\frac{\partial^2_{xx}Y_t\partial_x X_t - \partial_x Y_t \partial^2_{xx}X_t}{2\partial_y Y_t \partial_x X_t} + \hat b_f(t,x,e)\frac{\partial_x Y_t}{\partial_y Y_t}.
\end{aligned}$$

By (7.5), we have, again omitting (x, y) in $X_t(x), Y_t(x, y)$,

$$\begin{aligned}
&\partial^2_{xx}X_t = -\partial_x X_t \int_0^t \partial^2_{xx}\sigma(s,X_s)\partial_x X_s d\omega_s,\\
&\partial^2_{yy}Y_t = \partial_y Y_t \int_0^t \partial^2_{yy}g_0(s,X_s,Y_s)\partial_y Y_s d\omega_s,\\
&\partial^2_{xy}Y_t = \partial_y Y_t \int_0^t [\partial^2_{xy}g_0(s,X_s,Y_s)\partial_x X_s + \partial^2_{yy}g_0(s,X_s,Y_s)\partial_x Y_s] d\omega_s.
\end{aligned}$$

Then, by (7.4) we can easily verify that

$$\begin{gathered}\hat\sigma_f,\ \hat b_f,\text{ and } F_0(\cdot,0,0,\cdot)\text{ are bounded},\quad |\partial_x\hat\sigma_f|\le C,\quad |\partial_x\hat b_f|\le C,\\ |\partial_z F_0(t,x,y,z)|\le C[1+\rho(t)|z|],\\ |\partial_x F_0(t,x,y,z)|+|\partial_y F_0(t,x,y,z)|\le C[1+|y|+|z|+\rho(t)|z|^2].\end{gathered}\tag{7.18}$$

where $\rho\ge 0$ is a continuous function with $\rho(0)=0$. Here for notational simplicity we are assuming the relevant functions are differentiable, but actually we only need their uniform Lipschitz continuity.

Now, let $\bar B$ be a standard Brownian motion and $\mathcal{E}=\mathcal{E}_1\times\mathcal{E}_2$ be the set of $\mathbb{F}^{\bar B}$-progressively measurable E-valued processes. Fix $\delta>0$ and define $\bar\varphi(t,x,y,z,e):=\varphi(\delta-t,x,y,z,e)$ for $\varphi=\hat\sigma_f,\hat b_f,F_0$. For any $e\in\mathcal{E}$, introduce the following decoupled FBSDE on $[0,\delta]$:

$$\begin{aligned}\mathcal{X}^e_t&=x+\int_0^t\bar b_f(s,\mathcal{X}^e_s,e_s)ds+\int_0^t\bar\sigma_f(s,\mathcal{X}^e_s,e_s)d\bar B_s;\\ \mathcal{Y}^e_t&=u_0(\mathcal{X}^e_\delta)+\int_t^\delta\bar F_0(s,\mathcal{X}^e_s,\mathcal{Y}^e_s,\mathcal{Z}^e_s,e_s)ds-\int_t^\delta\mathcal{Z}^e_s d\bar B_s.\end{aligned}\tag{7.19}$$

By Zhang [48, Theorems 7.2.1, 7.2.3], there exist constants c_0, C_0, depending on $\|u_0\|_\infty$ and $\|f(\cdot,0,0,\cdot)\|_\infty$ (the bound of $|f(t,x,0,0,e)|$), such that

$$\|\mathcal{Y}^e\|_\infty\le C_0,\quad \mathbb{E}\Big[\exp\big(c_0\int_0^\delta|\mathcal{Z}^e_s|^2ds\big)\Big]\le C_0<\infty.\tag{7.20}$$

Differentiating (7.19) with respect to x yields

$$\begin{aligned}\nabla\mathcal{X}^e_t&=1+\int_0^t\partial_x\bar b_f\nabla\mathcal{X}^e_s ds+\int_0^t\partial_x\bar\sigma_f\nabla\mathcal{X}^e_s d\bar B_s;\\ \nabla\mathcal{Y}^e_t&=\partial_x u_0(\mathcal{X}^e_\delta)\nabla\mathcal{X}^e_\delta+\int_t^\delta\Big[\partial_x\bar F_0\nabla\mathcal{X}^e_s+\partial_y\bar F_0\nabla\mathcal{Y}^e_s+\partial_z\bar F_0\mathcal{Z}^e_s\Big]ds\\ &\qquad-\int_t^\delta\nabla\mathcal{Z}^e_s d\bar B_s.\end{aligned}$$

This implies

$$\nabla\mathcal{Y}^e_0=\mathbb{E}\Big[\Gamma^e_\delta\partial_x u_0(\mathcal{X}_\delta)\nabla\mathcal{X}^e_\delta+\int_0^\delta\Gamma^e_t\partial_x\bar F_0\nabla\mathcal{X}^e_t dt\Big]$$

where $\Gamma^e_t := \exp\big(\int_0^t \partial_z \bar F_0 d\bar B_s + \int_0^t [\partial_y \bar F_0 - \frac{1}{2}|\partial_z \bar F_0|^2] ds\big)$. By (7.18),

$$
\begin{aligned}
&\mathbb{E}[|\Gamma^e_t|^4] = \mathbb{E}\Big[\exp\big(4\int_0^t \partial_z \bar F_0 d\bar B_s + \int_0^t [4\partial_y \bar F_0 - 2|\partial_z \bar F_0|^2] ds\big)\Big]\\
&= \mathbb{E}\Big[\exp\big(4\int_0^t \partial_z \bar F_0 d\bar B_s - 16\int_0^t |\partial_z \bar F_0|^2 ds + \int_0^t [4\partial_y \bar F_0 + 14|\partial_z \bar F_0|^2] ds\big)\Big]\\
&\le \Big(\mathbb{E}\Big[\exp\big(8\int_0^t \partial_z \bar F_0 d\bar B_s - 32\int_0^t |\partial_z \bar F_0|^2 ds\big)\Big]\\
&\qquad\qquad \times \mathbb{E}\Big[\exp\big(\int_0^t [8\partial_y \bar F_0 + 28|\partial_z \bar F_0|^2] ds\big)\Big]\Big)^{\frac{1}{2}}\\
&= \Big(\mathbb{E}\Big[\exp\big(\int_0^t [8\partial_y \bar F_0 + 28|\partial_z \bar F_0|^2] ds\big)\Big]\Big)^{\frac{1}{2}}\\
&= \Big(\mathbb{E}\Big[\exp\big(C\int_0^\delta [1 + |\mathcal{Y}^e_s| + |\mathcal{Z}^e_s| + \rho(t)|\mathcal{Z}^e_s|^2 + \rho(t)^2|\mathcal{Z}^e_s|^2] ds\big)\Big]\Big)^{\frac{1}{2}}
\end{aligned}
$$

Set $\delta_0 > 0$ small enough so that $C[\rho(\delta_0) + \rho(\delta_0)^2] \le \frac{c_0}{2}$. Then, for $\delta \le \delta_0$, by (7.20), we obtain $\mathbb{E}\Big[|\Gamma^e_t|^4\Big] \le C_0$, and by the second line of (7.18) it is clear that $\mathbb{E}\Big[\sup_{0\le t\le\delta} |\nabla \mathcal{X}^e_t|^4\Big] \le C_0$. Thus

$$
\begin{aligned}
|\nabla \mathcal{Y}^\varepsilon_0| &\le C_0 \mathbb{E}\Big[\|\partial_x u_0\|_\infty |\Gamma^e_\delta||\nabla \mathcal{X}^e_\delta|\\
&\qquad + \int_0^\delta |\Gamma^e_t||\nabla \mathcal{X}^e_t|[1 + |\mathcal{Y}^e_t| + |\mathcal{Z}^e_t|^2] dt\Big] \le C_0[1 + \|\partial_x u_0\|_\infty].
\end{aligned} \tag{7.21}
$$

Finally, we remark that, since we know a priori that $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$, by the standard truncation arguments, we may assume without loss of generality that F_0 is uniformly Lipschitz in (x, y, z) (with the Lipschitz constant possibly depending on the regularity of v). Then, by Buckdahn and Li [3],

$$
v(\delta, x) = \inf_{\mathcal{S}} \sup_{e_1 \in \mathcal{E}_1} \mathcal{Y}_0^{(e_1, \mathcal{S}(e_1))}, \tag{7.22}
$$

where $\mathcal{S} : \mathcal{E}_1 \to \mathcal{E}_2$ is the so called nonanticipating strategies. This implies

$$
|\partial_x v(\delta, x)| \le \sup_{\mathcal{S}} \sup_{e_1 \in \mathcal{E}_1} |\nabla \mathcal{Y}_0^{(e_1, \mathcal{S}(e_1))}| \le C_0[1 + \|\partial_x u_0\|_\infty].
$$

Since $\delta \le \delta_0$ is arbitrary, the proof is complete. ■

Remark 7.10 (i) We reverse the time in (7.19). Hence, in spirit of the backward rough path in (7.19), $\overline{B}$ and the rough path ω (or the original B

in (3.1)) have opposite directions of time evolvement. Thus (7.19) is in the line of the backward doubly SDEs of Pardoux and Peng [39]. When E_2 is a singleton, [36] provided a representation for the corresponding SPDE (3.1) in the context of second order backward doubly SDEs. We shall remark though, while the wellposedness of backward doubly SDEs holds true for random coefficients, its representation for solutions of SPDEs requires Markovian structure, i.e., the f and g in (3.1) depend only on B_t (instead of the path $B_\cdot$). The stochastic characteristic approach used in this paper does not have this constraint. Note again that our f and g in RPDE (3.6) and PPDE (3.7) are allowed to depend on the (fixed) rough path ω.

(ii) For (7.22), from a game theoretical point of view, it is more natural to use the so-called weak formulation (see Pham and Zhang [40]). However, as we are here mainly concerned about the regularity, the strong formulation in [3] is more convenient.

7.3. *The global comparison principle and existence of viscosity solution.* We need the following PDE result from Safonov [42] (see also [37] for a corresponding statement for bounded domains and [43] for the elliptic case).

Theorem 7.11 *Consider PDE* (4.10). *Assume that, for some* $\beta > 0$,

(i) F *is convex in* γ *and uniformly parabolic, i.e.,* $\partial^2_{\gamma\gamma} F \geq 0$ *and* $\partial_\gamma F \geq c_0 > 0$,

(ii) F *is uniformly Lipschitz continuous in* (y, z, γ),

(iii) $\|F(\cdot, y, z, \gamma)\|_{C^\beta_b(\mathbb{R}_T)} \leq C[1 + |y| + z| + |\gamma|]$,

(iv) $u_0 \in C^{2+\beta}_b(\mathbb{R})$.

Then there exists $\beta_0 \in (0, 1)$ *depending only on* c_0 *such that, whenever* $\beta \in (0, \beta_0]$, *PDE* (4.10) *has a classical solution* $v \in C^{2+\beta}_b(\mathbb{R}_T)$.

Theorem 7.12 *Let Assumptions 7.1, 7.8, and 3.4 hold, and* (g, f) *take the form* (7.1)-(7.15). *Assume further that*

(i) f *is either convex or concave in* γ, *namely either* E_1 *or* E_2 *in* (7.15) *is a singleton,*

(ii) $\sigma_f \geq c_0 > 0$,

(iii) $u_0 \in C^{k+1+\beta}_b(\mathbb{R})$ *and* $f \in C^{k+1,loc}_{\alpha,\beta}(\mathbb{R}^4_T)$.

Then there exists $\delta_0 > 0$, *depending on* K_0, L_0 *in Assumption 7.8, the regularity of* σ, g_0 *in Assumption 7.1, and* $\|u_0\|_\infty$, *but independent of the further regularity of* u_0 *and* f, *such that PDE* (4.10)-(7.6) *has a classical solution* $v \in C^{k,0}_{\alpha,\beta}([0, \delta_0] \times \mathbb{R})$.

Proof We prove only the convex case, i.e., E_2 is a singleton. When f is concave, one can use following the standard transformation: $\tilde{f}(t, x, y, z, \gamma) :=$

$-f(t,x,-y,-z,-\gamma)$ is convex and $\tilde{v}(t,x) := -v(t,x)$ corresponds to $\tilde{f}$. Let δ_0 be determined by Proposition 7.9.

First it is clear that the F in (7.6) (or the equivalent (7.17)) satisfies the requirements in Theorem 7.11 (i). Recall (7.11), (7.16), and the K_0 in Assumption 7.8 (ii). Put

$$C_1 := e^{CT}[\|u_0\|_\infty + CTK_0] + C_0[1 + \|\partial_x u_0\|_\infty].$$

Introduce a truncation function $\iota \in C^\infty(\mathbb{R})$ such that $\iota(x) = x$ for $|x| \le C_1$, and $\iota(x) = 0$ for $|x| \ge C_1 + 1$. Define

$$\tilde{F}(t,x,y,z,\gamma) := F(t,x,\iota(y),\iota(z),\gamma).$$

Then one may verify straightforwardly that $\tilde{F}$ satisfies all the conditions in Theorem 7.11. Thus the following PDE has a classical solution $\tilde{v} \in C_b^{2+(\beta\wedge\beta_0)}(\mathbb{R}_T)$:

$$\partial_t \tilde{v} = \tilde{F}(t,x,\tilde{v},\partial_x\tilde{v},\partial^2_{xx}\tilde{v}), \quad \tilde{v}(0,\cdot) = u_0. \tag{7.23}$$

Applying Propositions 7.6 and 7.9 on the above PDE yields $|\tilde{v}| \le C_1$, $|\partial_x \tilde{v}| \le C_1$ on $[0,\delta_0]\times\mathbb{R}$, i.e., $v := \tilde{v}$ solves PDE (4.10)-(7.6) on $[0,\delta_0]\times\mathbb{R}$.

Finally, the further regularity of v follows from standard bootstrap arguments (cf. [23, Lemma 17.16]) together with Remark 2.11. ■

Remark 7.13 The requirement that f is convex or concave is mainly to ensure the existence of classical solutions for PDE (7.23). Theorem 7.11 holds true for multidimensional case as well. When the dimension of x is 1 or 2, Bellman-Isaacs equations may have classical solution as well, see [29, Theorem 14.24] for $d = 1$ and [40, Lemma 6.5] for $d = 2$ for bounded domains, and also [23, Theorem 17.12] for elliptic equations in bounded domains when $d = 2$. We believe such results can be extended to the whole space and thus the theorem above as well as Theorem 7.14 will hold true when f is indeed Bellman-Isaacs type. However, when the dimension is high, the Bellman-Isaacs equation does, in general, not have a classical solution (see [38] for a counterexample).

Theorem 7.14 *Let* (g,f) *take the form* (7.1)-(7.15). *Let Assumptions 7.1, 7.8, and 3.4 hold. Assume that, for any* $\varepsilon > 0$, *there exist* $\overline{f}^\varepsilon$, $\underline{f}^\varepsilon$ *such that*

(i) $\overline{f}^\varepsilon$, $\underline{f}^\varepsilon$ *satisfy Assumption 7.8 uniformly, i.e., with the same* K_0, L_0 *for all* $\varepsilon > 0$,

(ii) for each $\varepsilon > 0$, $\overline{f}^\varepsilon, \underline{f}^\varepsilon$ *satisfy all the requirements in Theorem 7.12,*

(iii) for each $\varepsilon > 0$, $f - \varepsilon \le \underline{f}^\varepsilon \le f \le \overline{f}^\varepsilon \le f + \varepsilon$.

Then RPDE (3.6) *satisfies the comparison principle and has a unique viscosity solution.*

Proof By Lemma 6.7, $\overline{u}$ and $\underline{u}$ are bounded by some C_0.

Step 1. We prove first (6.13) locally. Let $\delta_0 > 0$ be determined by Proposition 7.9, corresponding to K_0, L_0, but with $\|u_0\|_\infty$ replaced with the global bound C_0 of $\overline{u}$ and $\underline{u}$. For any $\varepsilon > 0$, let $\overline{f}^\varepsilon$, $\underline{f}^\varepsilon$ be as in the assumption of the theorem, and $\overline{F}^\varepsilon$, $\underline{F}^\varepsilon$ correspond to $\overline{f}^\varepsilon$, $\underline{f}^\varepsilon$ as in (7.17). In the spirit of Remark 5.7 (i), we may assume without loss of generality that u_0 is uniformly continuous. Then u_0 has standard smooth mollifiers $\overline{u}_0^\varepsilon$, $\underline{u}_0^\varepsilon$ such that $u_0 - \varepsilon \le \underline{u}_0^\varepsilon \le u_0 \le \overline{u}_0^\varepsilon \le u_0 + \varepsilon$. By Theorem 7.12, let $\overline{v}^\varepsilon$ (resp. $\underline{v}^\varepsilon$) be the classical solution to PDE (4.10) -(7.17) with coefficients $(\overline{F}^\varepsilon, g)$ and initial condition $\overline{u}_0^\varepsilon$ (resp. coefficients $(\underline{F}^\varepsilon, g)$ and initial condition $\underline{u}_0^\varepsilon$) on $[0, \delta_0]$. Then, by (6.4), $\underline{v}^\varepsilon \le \underline{v} \le \overline{v} \le \overline{v}^\varepsilon$, where $\underline{v} := Y_t^{-1}\big(x, \underline{u}(t, X_t(x))\big)$ as in (7.7), and similarly for $\overline{v}$. By (4.11) it is clear that $0 \le \overline{F}^\varepsilon - \underline{F}^\varepsilon \le C\varepsilon$. Define $\Delta v^\varepsilon := \overline{v}^\varepsilon - \underline{v}^\varepsilon$, $\Delta u_0^\varepsilon := \overline{u}_0^\varepsilon - \underline{u}_0^\varepsilon$, $\Delta F^\varepsilon := \overline{F}^\varepsilon - \underline{F}^\varepsilon$. Then

$$\begin{aligned}
&\Delta v^\varepsilon = \partial_t \Delta v^\varepsilon + F^\varepsilon(t, x, \Delta v^\varepsilon, \partial_x \Delta v^\varepsilon, \partial^2_{xx}\Delta v^\varepsilon), \quad \Delta v^\varepsilon(0,\cdot) = \Delta u_0^\varepsilon,\\
&\text{where } F^\varepsilon(t,x,y,z,\gamma) := \overline{F}^\varepsilon(t, x, \underline{v}^\varepsilon + y, \partial_x \underline{v}^\varepsilon + z, \partial^2_{xx}\underline{v}^\varepsilon + \gamma)\\
&\qquad\qquad\qquad\qquad - \underline{F}^\varepsilon(t, x, \underline{v}^\varepsilon, \partial_x \underline{v}^\varepsilon, \partial^2_{xx}\underline{v}^\varepsilon).
\end{aligned}$$

Now following the arguments of Proposition 7.6, we see that there exists a constant C, independent of ε, such that, for every $(t, x) \in [0, \delta_0] \times \mathbb{R}$,

$$\begin{aligned}
\overline{v}(t,x) - \underline{v}(t,x) \le \Delta v^\varepsilon(t,x) &\le C^{Ct}\Big[\|\Delta v_0^\varepsilon\|_\infty + Ct\|F^\varepsilon(\cdot, 0, 0, 0)\|_\infty\Big]\\
&\le C\Big[\|\Delta u_0^\varepsilon\|_\infty + \|\Delta F^\varepsilon(\cdot, \underline{v}^\varepsilon, \partial_x \underline{v}^\varepsilon, \partial^2_{xx}\underline{v}^\varepsilon)\|_\infty\Big] \le C\varepsilon.
\end{aligned}$$

This implies that $\overline{v}(t,x) = \underline{v}(t,x)$. Thus (6.13) holds on $[0, \delta_0]$. Therefore, by Theorem 6.8, $u_1 \le \overline{u} \le u_2$ and $u := \overline{u}$ is the unique viscosity solution of RPDE (3.6)-(7.15) on $[0, \delta_0]$.

Step 2. We prove now the global result. Let $0 = t_0 < \cdots < t_n = T$ be such that $t_i - t_{i-1} \le \delta_0$ for each $i = 1, \ldots, n$. By Step 1, $u_1(t_1, \cdot) \le u(t_1, \cdot) \le u_2(t_1, \cdot)$. Now consider RPDE (3.6)-(7.15) on $[t_1, t_2]$ with initial condition $u(t_1, \cdot)$. Note that $\|u(t_1, \cdot)\|_\infty \le C_0$ for the same global bound C_0. Since δ_0 corresponds to this C_0, following the same arguments, we see that the comparison principle holds on $[t_1, t_2]$. Repeating the arguments establishes the result on the whole interval $[0, T]$. ■

When f is semilinear, i.e., linear in γ, clearly under natural conditions f satisfies the requirements in Theorem 7.14. We provide next a simple fully nonlinear example.

Example 7.15 *Let $\overline{a} > \underline{a} > 0$ be two constants. Then*

$$f(\gamma) := \frac{1}{2}\sup_{\underline{a} \le a \le \overline{a}} [a\gamma] = \frac{1}{2}[\overline{a}\gamma^+ - \underline{a}\gamma^-] \tag{7.24}$$

satisfies the requirements in Theorem 7.14.

Proof Let η be a smooth symmetric density function with support $(-1,1)$. For any $\varepsilon > 0$, introduce a smooth mollifier of f:

$$f_\varepsilon(\gamma) := \int_{-1}^{1} f(\gamma - \varepsilon x)\eta(x)dx = \frac{1}{2}\underline{a}\gamma + \frac{\overline{a}-\underline{a}}{2}\int_{-1}^{1}(\gamma - \varepsilon x)^+\eta(x)dx.$$

It is clear that

$$|f_\varepsilon - f| \le \big[\frac{\overline{a}}{2}\int_{-1}^{1}|x|\eta(x)dx\big]\ \varepsilon =: c\varepsilon.$$

We next consider the Legendre conjugate of f_ε:

$$h_\varepsilon(a) := \sup_{\gamma\in\mathbb{R}}[\frac{1}{2}a\gamma - f_\varepsilon(\gamma)],\ \ a \in [\underline{a},\overline{a}].$$

By straightforward calculation, we have $h_\varepsilon(a) = \infty$ when $a \notin [\underline{a},\overline{a}]$, and

$$h_\varepsilon(a) = \frac{\varepsilon}{2}[\overline{a}-\underline{a}]\int_{-1}^{\Phi^{-1}(\frac{a-\underline{a}}{\overline{a}-\underline{a}})} x\eta(x)dx, \quad a \in [\underline{a},\overline{a}],$$

where $\Phi(x) := \int_{-1}^{x}\eta(y)dy$, $x \in [-1,1]$. Note that $f_\varepsilon(\gamma) = \sup_{\underline{a}\le a\le\overline{a}}[\frac{1}{2}a\gamma - h_\varepsilon(a)]$. Then $\overline{f}_\varepsilon := f_{\frac{\varepsilon}{2c}} + \frac{\varepsilon}{2}$ and $\underline{f}_\varepsilon := f_{\frac{\varepsilon}{2c}} - \frac{\varepsilon}{2}$ are the desired approximations. ∎

Remark 7.16 (i) As pointed out in Remark 7.5, for general $g = g(t,x,y,z)$, the transformation is local and the δ in Theorem 4.5 depends on $\|\partial^2_{xx}u\|_\infty$. Then the connection between RPDE (3.6) and PDE (4.10) exists only for local classical solutions, but is not clear for viscosity solutions. Since our current approach relies heavily on the PDE, we have difficulty to extend Theorem 7.4 to the general case, even just in local sense. We investigate this challenging problem by exploring more approaches in our future research.

(ii) When f is of first order, namely $\sigma_f = 0$ in (7.15), then (7.17) becomes:

$$F(t,x,y,z,\gamma) = \sup_{e_1\in E_1}\inf_{e_2\in E_2}\Big[\hat{b}_f(t,x,e)z + F_0\big(t,x,y,e\big)\Big], \tag{7.25}$$

$$\text{where}\quad \hat{b}_f(t,x,e) := \frac{b_f(t,X_t,e)}{\partial_x X_t},$$

$$F_0(t,x,y,e) := \frac{1}{\partial_y Y_t} f_0\big(t,X_t,Y_t,0,e\big) + \hat{b}_f(t,x,e)\frac{\partial_x Y_t}{\partial_y Y_t}.$$

Under Assumption 7.8, F_0 is uniformly Lipschitz continuous in y, and thus the main difficulty mentioned in Remark 7.7 (i) does not exist here. Then, following similar arguments as in this subsection, we can easily show that the results of Theorems 7.12 and 7.14 still hold true if we replace the uniform nondegeneracy condition $\sigma_f \geq c_0 > 0$ with $\sigma_f = 0$.

7.4. *The case that g is linear.* In this subsection we study the special case that g is linear in (y,z) (by abusing the notation g_0):

$$g(t,x,y,z) = \sigma(t,x)z + h(t,x)y + g_0(t,x). \tag{7.26}$$

We remark that rigorously speaking this case does not satisfy Assumption 7.1, because $g_0(t,x,y) := h(t,x)y + g_0(t,x)$ is not bounded in y. However, similar to the situation in Lemma 2.13, the linear structure allows us to extend all the results in Section 7 to this case.

First, for the X given by (7.2), we have

$$Y_t(x,y) = e^{H_t(x)}\Big[y + \int_0^t e^{-H_s(x)} g_0(s, X_s(x))d\omega_s\Big], \tag{7.27}$$

where $H_t(x) := \int_0^t h(s, X_s(x))d\omega_s$. By straightforward calculation, we see that (7.6) becomes, with omitting (x,y) in (X,Y,H),

$$\begin{aligned} F(t,x,y,z,\gamma) := e^{-H_t} f\Big(&t, X_t, Y_t, \frac{e^{H_t}}{\partial_x X_t} z + \frac{\partial_x Y_t}{\partial_x X_t}, \\ &\frac{e^{H_t}}{(\partial_x X_t)^2}\big[\gamma + [\partial_x H_t - \frac{\partial^2_{xx} X_t}{\partial_x X_t}]z + e^{-H_t}\partial^2_{xx} Y_t - \frac{\partial^2_{xx} X_t}{\partial_x X_t}]\Big). \end{aligned} \tag{7.28}$$

We now provide some sufficient conditions for the existence of classical solutions to PDE (4.10)-(7.28).

Theorem 7.17 *Let all the conditions in Theorem 7.12 hold and let g take the form* (7.26)*. Then PDE* (4.10)-(7.28) *has a classical solution $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$. Consequently, RPDE* (3.6)-(7.26) *has a classical solution $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$.*

Proof As in Theorem 7.12, we shall only prove the convex case. By the regularity of f, g it is clear that F is smooth. Note that, by omitting the variables,

$$\begin{aligned} &\partial_\gamma F = \frac{1}{(\partial_x X)^2}\partial_\gamma f, \quad \partial^2_{\gamma\gamma} F = \frac{e^{H_t}}{(\partial_x X)^4}\partial^2_{\gamma\gamma} f, \\ &\partial_z F = \frac{1}{\partial_x X}\partial_z f + [\frac{\partial_x H}{(\partial_x X)^2} - \frac{\partial^2_{xx} X}{(\partial_x X)^3}]\partial_\gamma f, \\ &\partial_y F = \partial_y f + \partial_z f \frac{\partial_x H}{\partial_x X} + \partial_\gamma f \frac{(\partial_x H)^2 + \partial^2_{xx} H}{(\partial_x X)^2}. \end{aligned}$$

Then one can easily verify that F satisfies all the conditions in Theorem 7.11, thus we obtain $v \in C^{k,0}_{\alpha,\beta}(\mathbb{R}_T)$. Finally the existence of the corresponding function $u \in C^k_{\alpha,\beta}(\mathbb{R}_T)$ follows from Theorem 4.4. ■

We now assume further that f is also linear, i.e.,

$$f(t,x,y,z,\gamma) = a(t,x)\gamma + b(t,x)z + c(t,x)y + f_0(t,x). \tag{7.29}$$

This case is well understood in the literature. By straightforward calculation,

$$F(t,x,y,z,\gamma) = A(t,x)\gamma + B(t,x)z + C(t,x)y + F_0(t,x), \tag{7.30}$$

where, for the H defined by (7.27) and again omitting the variable x,

$$\begin{aligned}
A(t,x) &:= \frac{a(t,X_t)}{(\partial_x X_t)^2},\\
B(t,x) &:= a(t,X_t)\Big[\frac{2\partial_x H_t}{(\partial_x X_t)^2} - \frac{\partial^2_{xx}X_t}{(\partial_x X_t)^3}\Big] + \frac{b(t,X_t)}{\partial_x X_t},\\
C(t,x) &:= a(t,X_t)\frac{(\partial_x H_t)^2 + \partial^2_{xx}H_t}{(\partial_x X_t)^2} + \Big[\frac{b(t,X_t)}{\partial_x X_t} - \frac{a(t,X_t)\partial^2_{xx}X_t}{(\partial_x X_t)^3}\Big]\partial_x H_t\\
&\qquad + c(t,X_t),\\
F_0(t,x) &:= \Big[\frac{a(t,X_t)}{(\partial_x X_t)^2}[(\partial_x H_t)^2 + \partial^2_{xx}H_t]\\
&\quad + \Big[\frac{b(t,X_t)}{\partial_x X_t} - \frac{a(t,X_t)\partial^2_{xx}X_t}{(\partial_x X_t)^3}\Big]\partial_x H_t + c(t,X_t)\Big]\int_0^t e^{-H_s}g_0(s,X_s)d\omega_s\\
&\quad + \Big[2\frac{a(t,X_t)}{(\partial_x X_t)^2}\partial_x H_t + \frac{b(t,X_t)}{\partial_x X_t} - \frac{a(t,X_t)\partial^2_{xx}X_t}{(\partial_x X_t)^3}\Big]\int_0^t \partial_x\Big(\frac{g_0(s,X_s)}{e^{H_s}}\Big)\,d\omega_s\\
&\quad + \frac{a(t,X_t)}{(\partial_x X_t)^2}\int_0^t \partial^2_{xx}(e^{-H_s}g_0(s,X_s))\,d\omega_s + f_0(t,X_t)e^{-H_t}.
\end{aligned} \tag{7.31}$$

Thus PDE (4.10) is linear and we have the representation formula

$$v(t,x) = \mathbb{E}\Big[e^{\int_0^t C(t-r,\mathcal{X}^{t,x}_r)dr}u_0(\mathcal{X}^{t,x}_t) + \int_0^t e^{\int_0^s C(t-r,\mathcal{X}^{t,x}_r)dr}F_0(t-s,\mathcal{X}^{t,x}_s)ds\Big], \tag{7.32}$$

where, for fixed $(t,x) \in \mathbb{R}_T$ and for a Brownian motion $\bar{B}$,

$$\mathcal{X}^{t,x}_s = x + \int_0^s \sqrt{2A(t-r,\mathcal{X}^{t,x}_r)}dr + \int_0^s B(t-r,\mathcal{X}^{t,x}_r)d\bar{B}_s,\ \ 0 \le s \le t.$$

8. Concluding Remarks. In this paper, we established the viscosity theory for general second order fully nonlinear parabolic SPDEs and path-dependent PDEs through a unified framework based on rough path analysis. We allow the diffusion coefficient g to be a general nonlinear first order differential operator, i.e., $g = g(t, \omega, x, u, u_x)$. Assuming that g is smooth enough, we obtained the following general results: 1) **Consistency** (i.e., if a candidate solution is smooth, viscosity solutions are equivalent to classical solutions); 2) **Equivalence** (between the definitions using test functions and by semi-jets, resp.); 3) **Stability**; and 4) **Partial comparison principle**.

Since the generality of the diffusion coefficient g corresponds to a highly convoluted system of characteristics, the complete wellposedness result, especially global existence and the comparison principle for viscosity solutions, is rather challenging. We thus investigated several important cases, summarized as follows.

A. Both f and g are linear. In this case, we have a representation formula, which is of course already well understood in the literature.

B. g is linear. In this case, we proved

- *global existence of classical solutions* when f and u_0 are smooth,
- *global existence and comparison principle* of viscosity solutions when f and u_0 are less smooth.

C. g is semi-linear. In this case, we established

- *global equivalence* of RPDE and associated PDE, in both the classical and the viscosity sense,
- *local existence of classical solutions* when f and u_0 are smooth,
- *global existence and comparison principle* of viscosity solutions when f and u_0 are less smooth but f is convex in $\partial^2_{xx} u$.

D. For the general fully nonlinear case, we proved

- *local equivalence* of RPDE and associated PDE in classical sense (but not in viscosity sense),
- *local existence of classical solutions* when f and u_0 are smooth, where the time interval depends on the regularity of u_0 and f.

We should note that although we finally found a framework on which the viscosity theory can be carried out in a fairly general way, some challenging issues remain, especially in the fully nonlinear case, which calls for further investigations. E.g., when g is semilinear, the current approach relies heavily on the (local) existence of classical solutions of certain mollified PDEs, for which we have to assume that f is *uniformly non-degenerate*. It is desirable to remove both the convexity and the nondegeneracy constraints on f. Also, as we pointed out in Remark 7.5, in the general fully nonlinear case the equivalence between RPDE and PDE in the viscosity sense is by no means clear. We often have to investigate the RPDE directly. Consequently, a direct approach for the comparison principle for RPDE (3.6), which is currently lacking, would help greatly. In particular, it would be interesting to explore the possibility of extending the doubling variable arguments to this situation (cf. Lions and Souganidis [33] for this approach in the case $g = g(\partial_x u)$).

Finally, it would also be interesting to investigate the alternative approach by using rough path approximations as in Caruana, Friz, and Oberhauser [8] and many other papers mentioned in the introduction. I.e., instead of transforming the RPDE into a PDE, one considers the approximating PDE

$$\begin{aligned} &du^n(t,x) = f(t,x,u^n,\partial_x u^n,\partial^2_{xx}u^n)dt + g(t,x,u^n,\partial_x u^n)d\omega^n_t,\\ &u^n(0,x) = u_0(x), \end{aligned} \tag{8.1}$$

where ω^n is a smooth function of t which converges to ω under a rough path norm. The key issue is then to study the convergence of u^n. We hope that we will be able to investigate some of these issues in our future publications.

SUPPLEMENTARY MATERIAL

Supplement A: Supplement to "Fully nonlinear stochastic and rough PDES: Classical and Viscosity solutions"
([doi] TO BE ADDED BY TYPESETTER). This supplement contains most proofs of the results in Section 2.

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