PATHWISE TAYLOR EXPANSIONS FOR ITÔ RANDOM FIELDS

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Abstract. In this paper we study the *pathwise stochastic Taylor expansion*, in the sense of our previous work [3], for a class of Itô-type random fields in which the diffusion part is allowed to contain both the random field itself and its spatial derivatives. Random fields of such an "self-exciting" type particularly contains the fully nonlinear stochastic PDEs of curvature driven diffusion, as well as certain stochastic Hamilton-Jacobi-Bellman equations. We introduce the new notion of "*n*-fold" derivatives of a random field, as a fundamental device to cope with the special self-exciting nature. Unlike our previous work [3], our new expansion can be defined around any random time-space point (τ, ξ), where the temporal component τ does not even have to be a stopping time. Moreover, the exceptional null set is independent of the choice of the random point (τ, ξ). As an application, we show how this new form of pathwise Taylor expansion could lead to a different treatment of the stochastic characteristics for a class of fully nonlinear SPDEs whose diffusion term involves both the solution and its gradient, and hence lead to a definition of the *stochastic viscosity solution* for such SPDEs, which is new in the literature, and potentially of essential importance in stochastic control theory.

1. **Introduction.** In our previous work [3] we studied the so-called *pathwise stochastic Taylor expansion* for a class of Itô-type random fields. The main result can

be briefly described as follows. Suppose that $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^n\}$ is an Itô-type random field of the form

$$u(t,x) = u(0,x) + \int_0^t u_1(s,x)ds + \int_0^t u_2(s,x)dB_s, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \quad (1)$$

where B is a 1-dimensional standard Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. If we denote $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ to be the natural filtration generated by B and augmented by all P-null sets in $\mathcal{F}$, then under reasonable regularity assumptions on the integrands u_1 and u_2, the following stochastic "Taylor expansion" holds: *For any stopping time τ and any $\mathcal{F}_\tau$-measurable, square-integrable random variable ξ, and for any sequence of random variables $\{(\tau_k, \xi_k)\}$ where τ_k's are stopping times such that either $\tau_k > \tau$, $\tau_k \downarrow \tau$; or $\tau_k < \tau$, $\tau_k \uparrow \tau$, and ξ_k's are all $\mathcal{F}_{\tau_k \wedge \tau}$-measurable, square integrable random variables, converging to ξ in L^2, it holds almost surely that*

$$
\begin{aligned}
u(\tau_k, \xi_k) &= u(\tau, \xi) + a(\tau_k - \tau) + b(B_{\tau_k} - B_\tau) + \frac{c}{2}(B_{\tau_k} - B_\tau)^2 && (2)\\
&\quad + \langle p, \xi_k - \xi \rangle + \langle q, \xi_k - \xi \rangle (B_{\tau_k} - B_\tau) + \frac{1}{2} \langle X(\xi_k - \xi), \xi_k - \xi \rangle \\
&\quad + o(|\tau_k - \tau|) + o(|\xi_k - \xi|^2),
\end{aligned}
$$

where (a, b, c, p, q, X) are all $\mathcal{F}_\tau$-measurable random variables, and the remainder $o(\zeta_k)$ are such that $o(\zeta_k)/\zeta_k \to 0$ as $k \to \infty$, in probability. Furthermore, the six-tuple (a, b, c, p, q, X) can be determined explicitly in terms of u_1, u_2 and their derivatives.

By choosing u_1 and u_2 in different forms, we then extended the Taylor expansion to solutions of stochastic differential equations with initial state as parameters, and to solutions of nonlinear stochastic PDEs. In the latter case we further introduced the notion of *stochastic super-(sub-) jets* using the Taylor expansion, from which the definition of stochastic viscosity solution was produced. We should note that in [3] all the SDEs and SPDEs have the diffusion coefficient in the form $g(t,x,u(t,x))$, that is, they only involve the solutions themselves. Such a structure turns out to be essential for the so-called Doss-Sussmann transformation, and in that case the stochastic viscosity solution became natural.

In this paper we are interested in the stochastic Taylor expansion for random fields of the following form: for $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$u(t,x) = u(0,x) + \int_0^t u_1(s,x)ds + \int_0^t \langle g(s,x,Du(s,x)), dB_s \rangle, \quad (3)$$

where u_1 is a random field, and g is a deterministic function. We shall assume that they are "smooth" in the sense that all the desired derivatives exist, and the degree of smoothness will be specified later.

Given such a random field, we again consider the possibility to expand u in the sense of (2), but this time in a more natural way: We shall allow the pair $(\tau, \xi) : \Omega \to [0,T] \times \mathbb{R}^d$ to be arbitrary random points. Furthermore, we should note that although the Taylor expansion (2) holds almost surely, in general the null set may depend on the choice of random point (τ, ξ). In this paper we shall look for a universal expansion, in the sense that there is a subset $\widetilde{\Omega} \subset \Omega$ with full probability measure, on which the stochastic Taylor expansion holds for all choices of random points (τ, ξ). These technical improvements make the stochastic Taylor expansions much more "user-friendly", and more importantly, it will be more effectively used

in our study of stochastic viscosity solution, especially in the proof of uniqueness, as we shall see in our forthcoming publications on that subject.

As one can easily observe from (3) the random field u is actually already a solution to a first order stochastic PDE. An immediate difficulty in deriving the Taylor expansion is the characterization of the derivatives of the random field, as it will increase in a "bootstrap" way, very similar to those that one has often seen in the anticipating calculus. As a consequence, the original Doss-Sussmann type transformation used in our previous works [3, 4, 5] no longer works in this case. To overcome this difficulty we introduce the notion of "*n-fold derivative*" of a random field, which essentially takes $(u, Du, \cdots, D^{n-1}u)$ as a vector-valued random field, and define its derivative in a recursive way. Such a definition turns out to be very close to the idea of converting a higher order ordinary differential equation to a first order system, and is mainly motivated by the "*stochastic characteristics*" of a stochastic PDE (cf. e.g., Kunita [10]). In fact, by combining the definition of stochastic characteristics in [10] and the stochastic Taylor expansion developed in this paper, we are able to rigorously define a stochastic diffeomorphism that relates the Stochastic PDE of the form

$$\begin{cases} du(t, x) = f(x, u, D_x u, D_x^2 u)dt + g(x, D_x u)) \circ dB_t^i, & (t, x) \in (0, T) \times \mathbb{R}^d; \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4}$$

to a PDE without Brownian components in their Taylor expansions around any temporal-spatial point, generalizing the Doss-Sussmann transformation in our previous works to the present case.

We should note that one of the main difficulties in the study of the stochastic viscosity solution can be described as "local" vs. "global". That is, the local nature of the viscosity solution vs. the global nature of the stochastic analysis (e.g., stochastic integrals). Our idea is to "localize" the stochastic integral, or solution to the SPDE via the stochastic Taylor expansion that we established in the previous section. It should be noted that the universal set "Ω'" that we found in these expansions is the essential point here.

Finally, we would like to point out that the topic of stochastic Taylor expansion has been explored by many authors in various forms, and used to provide numerical and other approximation schemes for SDEs and SPDEs or randomized ODEs and PDEs (see, for example, [1, 2, 8, 9], to mention a few). These expansions often use either the Lie-algebraic structure of the path space or the chaotic type expansion of multiple stochastic integrals. As a consequence it is hard to deduce the simultaneous spatial-temporal expansions that we are pursuing in this work, especially when the remainders are estimated in a pathwise manner. We should also note that the pathwise version of stochastic viscosity solutions, suggested by Lions-Souganidis [11, 12, 13], has found an effective framework recently, using the theory of *rough path* (cf. e.g., [6, 16]). However, due to the special nature of the rough path integrals, the arguments seem to depend heavily on the fact that there exist stochastic characteristics in the form of C^3-diffeomorphisms which transform the SPDE to a pathwise PDE. Consequently, the SPDE studied in [6], while fully nonlinear in the drift, seems to be restricted to the cases when the diffusion coefficient depends only (linearly) on Du, the gradient of the solution, so that a chain-rule type of argument could be applied. The generality of the diffusion part in the fully non-linear PDE suggested in this paper, and the stochastic characteristics related to it, does not seem to be an easy consequence of such a method. Furthermore, our Taylor expansions are constructed within a more "elementary" stochastic analysis framework, by

exploiting the properties of Brownian motion without using the advanced algebraic geometric structure of the path spaces, therefore we believe that it provides a more accessible alternative.

This paper is organized as follows. In Section 2 we clarify all the necessary notations, and state the main theorem. In section 3 we give a fundamental estimate of this paper, regarding multiple stochastic integrals. In sections 4 and 5 we study the forward and backward Taylor expansions, respectively. We note that in the stochastic case, the temporal direction of the expansion does affect the outcome. Finally in section 6 we try to apply the Taylor expansion to the solution of a class of nonlinear SPDEs, and propose a possible definition of the stochastic viscosity solution in this case.

2. **Preliminaries and statement of the main theorem.** Throughout this paper we denote $(\Omega, \mathcal{F}, P)$ to be a complete probability space on which is defined an ℓ-dimensional Brownian motion $B = (B_t)_{t \geq 0}$. Let $\mathbf{F}^B \triangleq \{\mathcal{F}_t^B\}_{t \geq 0}$ be the natural filtration generated by B, augmented by the P-null sets of $\mathcal{F}$; and let $\mathcal{F}^B = \mathcal{F}_\infty^B$. We denote $\mathcal{M}_{0,T}^B$ to be the set of all $\mathbf{F}^B$-stopping times τ such that $0 \leq \tau \leq T$, P-a.s., where $T > 0$ is some fixed time horizon; and denote $\mathcal{M}_{0,\infty}^B$ to be all $\mathbf{F}^B$-stopping times that are almost surely finite.

In what follows we write $\mathbb{E}$ (also $\mathbb{E}_1, \cdots$) for a generic Euclidean space, whose inner products and norms will be denoted as the same ones $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively; and write $\mathbb{B}$ for a generic Banach space with norm $\| \cdot \|$. Moreover, we shall denote $\mathcal{G} \subseteq \mathcal{F}^B$ to be a sub-σ-field of $\mathcal{F}^B$, and for any $x \in \mathbb{R}^d$ and constant $r > 0$ we denote $\overline{B}_r(x)$ to be the closed ball with center x and radius r. Furthermore, the following spaces of functions will be frequently used in the sequel. We denote

- $L^p(\mathcal{G}; \mathbb{E})$ to be all $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ, with $E(|\xi|^p) < \infty$. Further, we shall denote $L_-^\infty(\mathcal{G}; \mathbb{E}) \triangleq \cap_{p>1} L^p(\mathcal{G}; \mathbb{E})$.
- $L^q(\mathbf{F}^B, [0, T]; \mathbb{B})$ to be all $\mathbb{B}$-valued, $\mathbf{F}^B$-progressively measurable processes ψ, such that $E \int_0^T \|\psi_t\|^q dt < \infty$. In particular, $q = 0$ stands for all $\mathbb{B}$-valued, $\mathbf{F}^B$-progressively measurable processes; and $q = \infty$ denotes all processes in $L^0(\mathbf{F}^B, [0, T]; \mathbb{B})$ that are uniformly bounded.
- $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ to be the space of all $\mathbb{E}_1$-valued functions defined on $[0, T] \times \mathbb{E}$ which are k-times continuously differentiable in $t \in [0, T]$ and ℓ-times continuously differentiable in $x \in \mathbb{E}$.
- $C_b^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_l^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, and $C_p^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, etc. to be the subspace of $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, where the subscript "$b$" means all functions and their partial derivatives are uniformly bounded; "l" means all functions are of at most linear growth; and "p" means all functions and their partial derivatives are of at most polynomial growth. The subspaces with the combined subscripts of b, l, and p are defined in an obvious way.
- for any sub-σ-field $\mathcal{G} \subseteq \mathcal{F}_T^B$, $C^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{E}_1)$) to be the space of all $C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$)-valued random variables that are $\mathcal{G} \otimes \mathcal{B}([0, T] \times \mathbb{E})$-measurable;
- $C^{k,\ell}(\mathbf{F}^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathbf{F}^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathbf{F}^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$) to be the space of all $\varphi \in C^{k,\ell}(\mathcal{F}_T^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}(\mathcal{F}_T^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}(\mathcal{F}_T^B, [0, T] \times \mathbb{E}; \mathbb{E}_1)$), such that for fixed $x \in \mathbb{E}$, the mapping

$(t, \omega) \mapsto \varphi(t, x, \omega)$ is $\mathbf{F}^B$-progressively measurable, and for P-a.e. ω, the mapping $\phi(\cdot, \cdot, \omega) \in C^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$ (resp. $C_b^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$, $C_p^{k,\ell}([0, T] \times \mathbb{E}; \mathbb{E}_1)$).

If $\mathbb{E}_1 = \mathbb{R}$, we shall drop $\mathbb{E}_1$ from the notation (e.g., $C^{k,\ell}([0, T] \times \mathbb{E})$, and so on); and we write $C^{0,0}([0, T] \times \mathbb{E}; \mathbb{E}_1) = C([0, T] \times \mathbb{E}; \mathbb{E}_1)$, and $C^{0,0}(\mathbf{F}^B, [0, T] \times \mathbb{E}) = C(\mathbf{F}^B, [0, T] \times \mathbb{E})$,..., etc., to simplify notation. Finally, for $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, we denote $D = D_x = \nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)$, $D^2 = D_{xx} = (\partial_{x_i x_j}^2)_{i,j=1}^d$, $D_y = \frac{\partial}{\partial y}$, and $D_t = \frac{\partial}{\partial t}$. The meaning of D_{xy}, D_{yy}, etc., should be clear.

Finally, since the random fields that we are interested in are always of the form of (3), which is an SPDE already, the following definition in [3] is useful. Consider the fully nonlinear second-order SPDE:

$$u(t, x) = u_0(x) + \int_0^t f(s, x, (u, Du, D^2u)(s, x))ds + \int_0^t g(s, x, (u, Du)(s, x))dB_s, \quad (5)$$

where $(t, x) \in [0, T] \times \mathbb{R}^d$, and f and g are functions with appropriate dimensions.

Definition 2.1. *A random field $u = \{u(t, x, \omega) : (t, x, \omega) \in [0, T] \times \mathbb{R}^d \times \Omega\}$ is called a "regular" solution to SPDE (5) if*
(i) $u \in C^{0,2}(\mathbf{F}^B; [0, T] \times \mathbb{R}^d)$;
(ii) u is an Itô-type random field with the form

$$u(t, x) = u_0(x) + \int_0^t u_1(s, x)ds + \int_0^t u_2(s, x)dB_s, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

where

$$u_1(t, x) = f(t, x, (u, Du, D^2u)(t, x)), \quad u_2(t, x) = g(t, x, (u, Du)(t, x)),$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$, P-a.s. $\qquad\qquad\qquad\qquad\square$

In this paper we will consider a special type of "smoothness" for a random field, defined through what we shall call the "n-fold differentiability" below. Such a characterization of differentiability is mainly motivated by the "*stochastic characteristics*" for stochastic PDEs (cf. e.g., [10]), which often take the form of a system of first order Stochastic PDEs. The idea is actually quite similar to the well-known transformation from a higher order ordinary differential equation (ODE) to a first order system of ODEs.

To begin with, we recall that for any multi-index $j = (j_1, \cdots, j_d)$, its "length" is defined by $|j| \stackrel{\triangle}{=} \sum_{k=1}^d j_k$. We have the following definition.

Definition 2.2. *A random field $\zeta \in C^{0,n}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ is called "n-fold" differentiable in the spatial variable x if there exist n smooth random fields $\zeta_i \in C^{0,n}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d; \mathbb{R}^{d_i})$, $2 \le i \le n+1$, with $d_1 = d$ and $d_i \in \mathbb{N}$, $2 \le i \le n+1$, and the functions $F_i, G_i : \mathbb{R}^d \times \mathbb{R}^{d_{i+1}} \to \mathbb{R}^d$, $i = 1, \cdots, n$, such that, denoting $\zeta_1 \stackrel{\triangle}{=} \zeta$, the following properties are satisfied:*

(T1) $F_i, G_i \in C_{\ell,p}^\infty$, $i = 1, \cdots, n$;
(T2) For $1 \le i \le n$, it holds that

$$\zeta_i(t, y) = \zeta_{i,0}(y) + \int_0^t F_i(y, \zeta_{i+1}(s, y))ds + \int_0^t G_i(y, \zeta_{i+1}(s, y))dB_s, \quad (6)$$

for all $(t, y) \in [0, T] \times \mathbb{R}^d$, with $\zeta_{i,0} \in C^2(\mathbb{R}^d; \mathbb{R}^{d_i})$, $1 \le i \le n$.

(T3) For any $1 \le i \le n+1$, $m \ge 1$, and multi-index $j = (j_1, ..., j_d)$ with $|j| = j_1 + ... + j_d \le n$, it holds that

$$\sup\{|D_y^j \zeta_i(t,y)|, t \in [0,T], |y| \le m\} \in L_-^\infty(\Omega, \mathcal{F}, P).$$

We shall call ζ_i, $i = 2, \cdots, n+1$ the "generalized derivatives" of $\zeta = \zeta_1$, with "coefficients" (F_i, G_i), $i = 1, \cdots, n$.

For notational convenience we will often write $F = F_1$ and $G = G_1$ when there is no danger of confusion. We denote the set of all n-fold differentiable random fields by $C_{\mathbf{F}}^{0,(n)}([0,T] \times \mathbb{R}^d)$.

Before we state the main result of this paper, let us note again that the main feature of our stochastic Taylor expansion is that it can be expanded around any random point (τ, ξ), and that the expansion holds almost surely with an exceptional set that is independent of the choice of (τ, ξ). But this amounts to saying that the expansion can be performed around any deterministic point (τ, ξ) with any (deterministic) increments, outside a uniform exceptional set. In other words, the complicated approximating sequences (τ_k, ξ_k) in (2) can be replaced by simple deterministic increments $(t+h, x+k)$, for all (h,k) near zero. We should also note that for the purpose of our application, in this paper we consider only the Taylor expansion up to the second order, and for that purpose the 3-fold differentiability of the random field would suffice. The precise statement of our main result is the following theorem.

Theorem 2.3. *Let $\zeta \in C^{0,(3)}(\mathbf{F}^B, [0,T] \times \mathbb{R}^d)$ be a random field satisfying the standard assumptions (T1)-(T3) with generalized derivatives ζ_i, $i = 2, 3, 4$ and coefficients (F_i, G_i), $i = 1, 2, 3$. Then, for every $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and every $m \in \mathbb{N}$ there exists a subset $\tilde{\Omega}_{\alpha,m} \subset \Omega$ with $P\{\tilde{\Omega}_{\alpha,m}\} = 1$, such that, for all $(t, y, \omega) \in [0,T] \times \overline{B}_m(0) \times \tilde{\Omega}_{\alpha,m}$, we have the following Taylor expansion*

$$\zeta(t+h, y+k) - \zeta(t,y)$$
$$= ah + b(B_{t+h} - B_t) + \frac{c}{2}(B_{t+h} - B_t)^2 + \langle p, k \rangle + \frac{1}{2}\langle Xk, k \rangle \quad (7)$$
$$+ \langle q, k \rangle(B_{t+h} - B_t) + (|h| + |k|^2)^{3\alpha} R_{\alpha,m}(t, t+h, y, y+k),$$

for all $(t+h, y+k) \in [0,T] \times \overline{B}_m(0)$. Here, with $(F,G) = (F_1, G_1)$, one has

$$
\begin{aligned}
a &= F(y, \zeta_2(t,y)) - \frac{1}{2}(D_z G)(y, \zeta_2(t,y)) G_2(y, \zeta_3(t,y)), \\
b &= G(y, \zeta_2(t,y)), \quad c = (D_z G)(y, \zeta_2(t,y)) G_2(y, \zeta_3(t,y)), \\
p &= D_y \zeta(t,y), \quad X = D_y^2 \zeta(t,y), \\
q &= (D_y G)(y, \zeta_2(t,y)) + (D_z G)(y, \zeta_2(t,y)) D_y \zeta_2(t,y).
\end{aligned}
\quad (8)
$$

Furthermore, the remainder of Taylor expansion $R_{\alpha,m} : [0,T]^2 \times (\mathbb{R}^d)^2 \times \Omega \mapsto \mathbb{R}$ is a measurable random field such that

$$\overline{R}_{\alpha,m} \overset{\triangle}{=} \sup_{t,s \in [0,T]; \ y,z \in \overline{B}_m(0)} |R_{\alpha,m}(t,s,y,z)| \in L_-^\infty(\Omega, \mathcal{F}, P). \quad (9)$$

Proof. Since the proof of this theorem is quite lengthy and technical, we shall split it into several cases and carry it out in the following sections. ☐

Remark 2.4. (i) It is worth noting that the "universal" estimate for the remainder $R_{\alpha,m}$ is the main reason why the Taylor expansion can now hold around any random space-time point. In other words, the points (t,y) and (h,k) in Theorem 2.3 can be

replaced by any (τ, ξ), $(\sigma, \eta) \in L^0(\mathcal{F}^B; [0, T]) \times L^0(\mathcal{F}^B; \mathbb{R}^d)$, such that $(\tau + \sigma, \xi + \eta) \in [0, T] \times \overline{B}_m(0)$, P-a.s. on $\widetilde{\Omega}$, except in that case the remainder should read

$$\hat{R}_{\alpha,m} \triangleq R_{\alpha,m}(\tau, \tau + \sigma, \xi, \xi + \eta), \quad \text{where} \quad |\hat{R}_{\alpha,m}| \le \overline{R}_{\alpha,m} \in L^\infty_-(\mathbf{F}^B; P).$$

In what follows we denote $R_{\alpha,m}$ to be a generic term satisfying (9), which is allowed to vary from line to line.

(ii) From the expressions (8) it is clear that the drift term $F(= F_1)$ appears only in the coefficient a, and it is in its original form. In fact, as we shall see in the proof, the exact form of $F(t, y)(= F(y, \zeta_2(t, y)))$ is not important at all. We could simply change it to $F(t, y)$ and the results remains the same.

(iii) Theorem 2.3 remains true if the assumption (T1) is replaced by the weaker assumption:

(T1') $F_j \in C^{5-j}_{\ell,p}$; $G_j \in C^{7-j}_{\ell,p}$; $1 \le j \le 3$.

However, we shall not pursue this generality in this paper due to the length of the paper.

(iv) The reason for the assumption $\alpha \in (\frac{1}{3}, \frac{1}{2})$ will become clearer in the coming sections when the proof proceeds. In particular we would emphasize Proposition 4.1 (forward spatial expansion) as well as Proposition 4.2 and 5.2 (forward and backward temporal expansions), as the consequence of Proposition 3.1. $\qquad\square$

To end this section we give an example, which more or less motivated our study.

Example 2.5. Let $u = \{u(t, x)\}$ be a regular solution of the SPDE (5), and for simplicity we assume that both f and g are "time-homogeneous" (i.e., they are independent of the variable t), and g is independent of x as well. We define $\zeta = \zeta_1 = u$, and $\zeta_{i+1} = (\zeta_i, D\zeta_i, D^2\zeta_i)$, $1 \le i \le 3$. Then, assuming that the coefficients f and g are sufficiently smooth and their derivatives of all order are bounded, one can show that (T1)–(T3) are satisfied. Furthermore, applying Theorem 2.3 we see that on some subset $\widetilde{\Omega}$ of Ω with $P(\widetilde{\Omega}) = 1$, independent of the expansion point $(t, x) \in [0, T] \times \mathbb{R}^d$, the stochastic Taylor expansion (7) holds for u, with

$$
\begin{aligned}
a &= f(x, (u, Du, Du^2)(t, x)) - \frac{1}{2}\Big\{(gD_u g)((u, Du)(t, x)) \\
&\quad + D_u g((u, Du)(t, x))\langle D_p g((u, Du)(t, x)), Du(t, x)\rangle \\
&\quad + \langle D^2 u(t, x) D_p g((u, Du)(t, x)), D_p g((u, Du)(t, x))\rangle\Big\} \\
b &= g((u, Du)(t, x)) \\
c &= (gD_u g)((u, Du)(t, x)) + D_u g((u, Du)(t, x))\langle D_p g((u, Du)(t, x)), Du(t, x)\rangle \\
&\quad + \langle D^2 u(t, x) D_p g((u, Du)(t, x), D_p g((u, Du)(t, x))\rangle \\
p &= Du(t, x); \quad X = D^2 u(t, x) \\
q &= D_u g((u, Du)(t, x) Du(t, x) + D^2 u(t, x) D_p g((u, Du)(t, x)),
\end{aligned}
\tag{10}
$$

for all $h \in \mathbb{R}$ and $k \in \mathbb{R}^d$ such that $t + h \in [0, T]$ and $|k| \le m$, respectively. Here, the coefficients F_2, G_2, F_3, and G_3 are obtained by differentiating (5) with respect to x.

The explicit expressions of the pathwise Taylor expansion in (10) will be important for our study of the stochastic characteristics for SPDE (5), which will be discussed in the last section of this paper. $\qquad\square$

3. **Some fundamental estimates.** Before we prove the theorem, let us first introduce the so-called Wick-square of the Brownian motion, which is originated in the Wiener homogeneous chaos expansion (cf., e.g., [7], [17]). For any $0 \leq s \leq t$, we define the Wick-square of $B_t - B_s$ to be

$$(B_t - B_s)^{\diamond 2} \triangleq (B_t - B_s)^2 - |t - s|.$$

Moreover, let $L^0_{\mathbf{F}^B}(\Omega; W^{0,1}_{2,loc}([0,T] \times \mathbb{R}^d))$ denote the space of all random fields $f \in L^0(\mathbf{F}^B, [0,T] \times \mathbb{R}^d)$ such that $f(\omega, \cdot, \cdot)$ is P-a.s. an element of

$$W^{0,1}_{2,loc}([0,T] \times \mathbb{R}^d) = \{u \in L^2_{loc}([0,T] \times \mathbb{R}^d) : \ D_x u \in L^2_{loc}([0,T] \times \mathbb{R}^d)\},$$

where in this case $D_x u$ denotes the weak partial derivative of u with respect to x.

We begin by proving an important estimate for multiple stochastic integrals. We consider the multiple stochastic integral defined recursively as follows. Let $f_j \in L^0_{\mathbf{F}^B}(\Omega; W^{0,1}_{2,loc}([0,T] \times \mathbb{R}^d))$, for $1 \leq j \leq N$, $N \in \mathbb{N}$. Let $X^0_{t,s}(x) \equiv 1$, and for $j = 1, \cdots, N$,

$$X^j_{t,s}(x) = \int_t^s f_j(s_j, x) X^{j-1}_{t,s_j}(x) dB_{s_j}, \quad s \in [t,T]. \tag{11}$$

It is also useful to define the similar multiple integrals: for $k > 0$, let $X^{k,k-1}_{t,s}(x) \equiv 1$, and for $l \geq k$,

$$X^{k,l}_{t,s}(x) = \int_t^s f_l(u_l, x) X^{k,l-1}_{t,u_l}(x) dB_{u_l}, \quad s \in [t,T]. \tag{12}$$

We note that for $l \geq k$, $X^{k,l}$ is a $l - k + 1$-tuple stochastic integral, and it is clear that $X^{1,j}_{t,s} = X^j_{t,s}$, for $j \geq 1$. We have the following *regularity* result.

Proposition 3.1. *Let* $N \in \mathbb{N}$ *be given. Assume that* $f_j \in L^0_{\mathbf{F}^B}(\Omega; W^{0,1}_{2,loc}([0,T] \times \mathbb{R}^d))$, $1 \leq j \leq N$, *satisfy that*

$$C_{m,p}(f_j) \triangleq E\left[\int_{|x| \leq m} \int_0^T |D_x^i f_j(t,x)|^p dt dx\right] < \infty, \tag{13}$$

for all $m \in \mathbb{N}$, $p \geq 1$, *and* $i = (i_1, \ldots, i_d)$ *with* $|i| = i_1 + \ldots + i_d \leq 1$. *Then it holds that*

$$\zeta_{\beta,m} \triangleq \sup\left\{\frac{|X^N_{t,s}(x)|}{|s-t|^\beta}, \ 0 \leq t < s \leq T, \ |x| \leq m\right\} \in L^\infty_-(\Omega, \mathcal{F}, P) \tag{14}$$

for all $\beta \in \left(0, \frac{N}{2}\right)$, $m \in \mathbb{N}$.

Proof. The proof is based on the Kolmogorov continuity criterion for random fields (cf. e.g., [18, Theorem I.2.1]), combined with an induction argument. We note that throughout the proof we shall use the notations C_m, $C_{m,p}$, $C_{M,k}$, etc., to represent the generic constants depending only on f_i's, T, and the parameters in their subscripts, which are allowed to vary from line to line.

We begin with the case $N = 1$. In this case we write $f_1 = f$ and $X_1 = X$ for simplicity. Let $m \in \mathbb{N}$, $\gamma > 2$, and $0 \leq t < s \leq T$. Applying Sobolev's embedding

theorem, Hölder Inequality, and noting that $X_{t,s}^0 = 1$, we have

$$
E\Big[\sup_{|x|\leq m}|X_{t,s}(x)|^\gamma\Big]
$$

$$
\leq C_m E\Big[\Big(\sum_{|j|\leq 1}\int_{|x|\leq m}|D_x^j X_{t,s}(x)|^{d+1}dx\Big)^{\frac{\gamma}{d+1}}\Big]
$$

$$
\leq C_m \sum_{|j|\leq 1}\Big(\int_{|x|\leq m}E\Big[\Big|\int_t^s D_x^j f(r,x)dB_r\Big|^{\gamma(d+1)}\Big]dx\Big)^{\frac{1}{d+1}}
$$

$$
\leq C_m \sum_{|j|\leq 1}\Big(\int_{|x|\leq m}E\Big[\Big(\int_t^s |D_x^j f(r,x)|^2 dr\Big)^{\frac{\gamma(d+1)}{2}}\Big]dx\Big)^{\frac{1}{d+1}}
$$

$$
\leq C_m \sum_{|j|\leq 1}\Big(\int_{|x|\leq m}E\Big[(s-t)^{\frac{\gamma(d+1)}{2}-1}\int_t^s |D_x^j f(r,x)|^{\gamma(d+1)}dr\Big]dx\Big)^{\frac{1}{d+1}}
$$

$$
\leq C_m(s-t)^{\frac{\gamma}{2}-\frac{1}{d+1}}\sum_{|j|\leq 1}\Big(E\Big[\int_{|x|\leq m}\int_0^T |D_x^j f(r,x)|^{\gamma(d+1)}drdx\Big]\Big)^{\frac{1}{d+1}}.
$$

Consequently,

$$
E\Big[\sup_{|x|\leq m}|X_{t,s}(x)|^\gamma\Big]\leq C_{m,\gamma}(s-t)^{\frac{\gamma}{2}-\frac{1}{d+1}}.
$$

Endowing the space $C(\overline{B}_m)$ of continuous functions defined on the closed m-ball $\overline{B}_m := \{x \in \mathbb{R}^d : |x| \leq m\}$ with the sup-norm, and considering $\{x \mapsto X_{0,s}(x); |x| \leq m\}_{s\in[0,T]}$ as a $C(\overline{B}_m)$-valued process, we see that

$$
\sup_{|x|\leq m}|X_{t,s}(x)| = |X_{0,s}(\cdot) - X_{0,t}(\cdot)|_{C(\overline{B}_m)}.
$$

Applying the Kolmogorov continuity criterion, we conclude that $\zeta_{\beta,m} \in L^\gamma(\Omega,\mathcal{F},P)$ for all $\beta \in \Big[0, \frac{1}{2} - \frac{d+2}{\gamma(d+1)}\Big)$. Since we can choose $\gamma > 2$ arbitrarily large, (14) follows.

We now prove the inductive step. That is, we assume that (14) is true for $N-1$, and show that it is also true for N. To do this, we shall adapt the proof of the Kolmogorov continuity criterion given in [18] to our framework. Let $0 \leq t < s \leq T$, $\gamma > 2$, and $m \in \mathbb{N}$. First note that by a simple application of Burkholder-Davis-Gundy and Hölder inequalities we have

$$
E\Big\{\sup_{r\in[t,s]}|X_{t,r}^N(x)|^p\Big\}
$$

$$
= E\Big\{\sup_{r\in[t,s]}\Big|\int_t^r f_N(s_N,x)X_{t,s_N}^{N-1}(x)dB_{s_N}\Big|^p\Big\}
$$

$$
\leq C_p E\Big\{\Big[\int_t^s |f_N(s_N,x)|^2 |X_{t,s_N}^{N-1}(x)|^2 ds_N\Big]^{p/2}\Big\} \tag{15}
$$

$$
\leq C_p E\Big\{\|f_N(\cdot,x)\|_{L^2([t,s])}^p \sup_{s_N\in[t,s]}|X_{t,s_N}^{N-1}(x)|^p\Big\}
$$

$$
\leq C_p\Big\{E\|f_N(\cdot,x)\|_{L^2([t,s])}^{pN}\Big\}^{1/N}\Big\{E\Big\{\sup_{s_N\in[t,s]}|X_{t,s_N}^{N-1}(x)|^{\frac{pN}{N-1}}\Big\}^{\frac{N-1}{N}}.
$$

By simply iterating the above argument we obtain that

$$
E\Big\{\sup_{r\in[t,s]}|X^N_{t,r}(x)|^p\Big\}^N
$$

$$
\leq \quad C_p E\|f_N(\cdot,x)\|^{pN}_{L^2([t,s])} \cdot E\|f_{N-1}(\cdot,x)\|^{pN}_{L^2([t,s])} \cdot E\Big\{\sup_{s_N\in[t,s]}\Big|X^{N-1}_{t,s_N}(x)\Big|^{\frac{pN}{N-2}}\Big\}^{N-2}
$$

$$
\leq \quad \cdots \leq C_p \prod_{k=1}^N \Big\{E\|f_k(\cdot,x)\|^{pN}_{L^2([t,s])}\Big\}. \tag{16}
$$

Furthermore, with a similar argument as above we can also show that for all $M, k \in \mathbb{N}$, $p > 1$, and multi-index i satisfying $|i| = 1$,

$$
E\Big\{\sup_{t\leq r\leq s}|D^i X^M_{t,r}(x)|^k\Big\}
$$

$$
\leq \quad E\Big\{\sum_{\substack{|i^j|\leq 1\\ j=1,\cdots M}}\sup_{t\leq r\leq s}\Big|\int_t^r D^{i^M}f_M\Big(\int_t^{s_M}\cdots\Big(\int_t^{s_2}D^{i^1}f_1 dB_{s_1}\Big)\cdots dB_{s_{M-1}}\Big)dB_{s_M}\Big|^k\Big\}
$$

$$
\leq \quad C_{M,k}\sum_{\substack{|i^j|\leq 1\\ j=1,\cdots M}}\prod_{j=1}^M E\Big[\|D^{i^j}f_j(s_j,x)\|^{Mk}_{L^2((t,s])}\Big]^{\frac{1}{M}}
$$

$$
\leq \quad C_{M,k}(s-t)^{\frac{Mk(p-1)}{2p}}\sum_{|i^1|,\ldots,|i^M|\leq 1}\prod_{j=1}^M E\Big[\|D^{i^j}f_j(s_j,x))\|^{Mk}_{L^{2p}([t,s])}\Big]^{\frac{1}{M}}.
$$

Here, we applied the Hölder inequality in the last step above. Consequently, if $p \leq \frac{Mk}{2}$, then the assumption (13) implies that, for all $m > 0$,

$$
\int_{|x|\leq m} E\Big[\sup_{t\leq r\leq s}|D^i X^M_{t,r}(x)|^k\Big]dx \leq C_{m,M,k}(s-t)^{\frac{Mk(p-1)}{2p}}. \tag{17}
$$

We can then conclude that, for all $0 \leq t < s \leq T$, $\gamma > 2$,

$$
E\Big[\sup_{|x|\leq m}|X^N_{t,s}(x)|^\gamma\Big]
$$

$$
\leq \quad C_m E\Big[\Big(\sum_{|j|\leq 1}\int_{|x|\leq m}|D^j_x X^N_{t,s}(x)|^{d+1}dx\Big)^{\frac{\gamma}{d+1}}\Big]
$$

$$
\leq \quad C_m E\Big[\Big(\sum_{|i|,|j|\leq 1}\int_{|x|\leq m}\Big|\int_t^s D^i_x X^{N-1}_{t,r}(x)D^j_x f_N(r,x)dB_r\Big|^{\gamma(d+1)}dx\Big)^{\frac{1}{d+1}}\Big]
$$

$$
\leq \quad C_m\Big(\sum_{|i|,|j|\leq 1}\int_{|x|\leq m}E\Big[\Big(\int_t^s|D^i_x X^{N-1}_{t,r}(x)|^2|D^j_x f_N(r,x)|^2 dr\Big)^{\frac{\gamma(d+1)}{2}}\Big]dx\Big)^{\frac{1}{d+1}}
$$

$$
\leq \quad C_m\Big(\sum_{|i|,|j|\leq 1}\int_{|x|\leq m}E\Big[\sup_{t\leq r\leq s}|D^i_x X^{N-1}_{t,r}(x)|^{\gamma(d+1)}\|D^j_x f_N(\cdot,x)\|^{\gamma(d+1)}_{L^2([t,s])}\Big]dx\Big)^{\frac{1}{d+1}}.
$$

Consequently, by Hölder inequality and (17) with $p = \frac{Mk}{2} = (N-1)\gamma(d+1)$, we have

$$
E\Big[\sup_{|x|\leq m} |X^N_{t,s}(x)|^\gamma \Big]
$$

$$
\leq \; C_m \Big\{ \sum_{|i|,|j|\leq 1} \Big[\int_{|x|\leq m} E\Big[\sup_{t\leq r\leq s} |D^i_x X^{N-1}_{t,r}(x)|^{2\gamma(d+1)} \Big] dx \Big]^{\frac{1}{2}} \times
$$

$$
\times \Big(\int_{|x|\leq m} E\Big[\|D^j_x f_N(\cdot,x)\|^{2\gamma(d+1)}_{L^2([t,s])} \Big] dx \Big)^{\frac{1}{2}} \Big\}^{\frac{1}{d+1}}
$$

$$
\leq \; C_{m,\gamma} \sum_{|i|\leq 1,\, j=1,2} \Big((s-t)^{\frac{1}{2}(N-1)\gamma(d+1)-\frac{1}{2}} \times
$$

$$
\times \Big(\int_{|x|\leq m} E\Big[\Big((s-t)^{\frac{\gamma(d+1)-1}{\gamma(d+1)}} \|D^i_x f_j(\cdot,x)\|^2_{L^{2\gamma(d+1)}([t,s])} \Big)^{\gamma(d+1)} \Big] \Big)^{\frac{1}{2}} dx \Big)^{\frac{1}{d+1}}
$$

$$
\leq \; C_{m,\gamma}(s-t)^{\frac{N}{2}\gamma-\frac{1}{d+1}} \sum_{|i|\leq 1,\, j=1,2} \Big(E\Big[\int_{|x|\leq m} \|D^i_x f_j(\cdot,x)\|^{2\gamma(d+1)}_{L^{2\gamma(d+1)}([t,s])} dx \Big]^{\frac{1}{2}} \Big)^{\frac{1}{d+1}}.
$$

In other words we obtained that

$$
E\Big[\sup_{|x|\leq m} |X^N_{t,s}(x)|^\gamma \Big] \leq C_{m,\gamma}(s-t)^{\frac{N}{2}\gamma-\frac{1}{d+1}}, \quad 0\leq t < s \leq T. \tag{18}
$$

Next, recall the multiple integrals $X^{k,l}_{t,s}(x)$ defined by (12). For $0 \leq r_1 \leq r_2 \leq r_3 \leq T$, and $1 \leq n_0 \leq N$, define

$$
Y^N_{r_1,r_2,r_3,n_0}(x) \triangleq X^{N-n_0,N}_{r_2,r_3}(x) X^{1,N-n_0-1}_{r_1,r_2}(x). \tag{19}
$$

An easy calculation shows that

$$
\begin{aligned}
X^{1,N}_{r_1,r_3}(x) &= \int_{r_1}^{r_3} X^{1,N-1}_{r_1,u_N}(x) f_N(u_N,x) dB_{u_N} \\
&= \int_{r_1}^{r_2} X^{1,N-1}_{r_1,u_N}(x) f_N(u_N,x) dB_{u_N} + \int_{r_2}^{r_3} X^{1,N-1}_{r_1,u_N}(x) f_N(u_N,x) dB_{u_N} \\
&= X^{1,N}_{r_1,r_2}(x) + \sum_{n_0=0}^{N-2} Y^N_{r_1,r_2,r_3,n_0}(x) + X^{1,N}_{r_2,r_3}(x).
\end{aligned} \tag{20}
$$

To simplify the further discussion, we now assume without loss of generality that $T = 1$. Let D_n be the set of all $t^n_i := 2^{-n}i$, for some $0 \leq i \leq 2^n$. The set $D = \bigcup_{n\geq 1} D_n$ is then the set of all dyadic numbers in $[0,1]$. Now, let $n \in \mathbb{N}$ and $0 \leq t < s \leq 1$ be arbitrary dyadic numbers such that $s - t \leq 2^{-n}$. Our aim is to estimate $X_{t,s}(x)$ uniformly in $t, s \in D$. To this end, we notice that there exists some $k \geq n$ such that t, s belong to D_k. Moreover, denoting

$$
s_j = \sup\{r \in D_j,\ r \leq s\}, \quad t_j = \sup\{r \in D_j,\ r \leq t\}, \quad n \leq j \leq k,
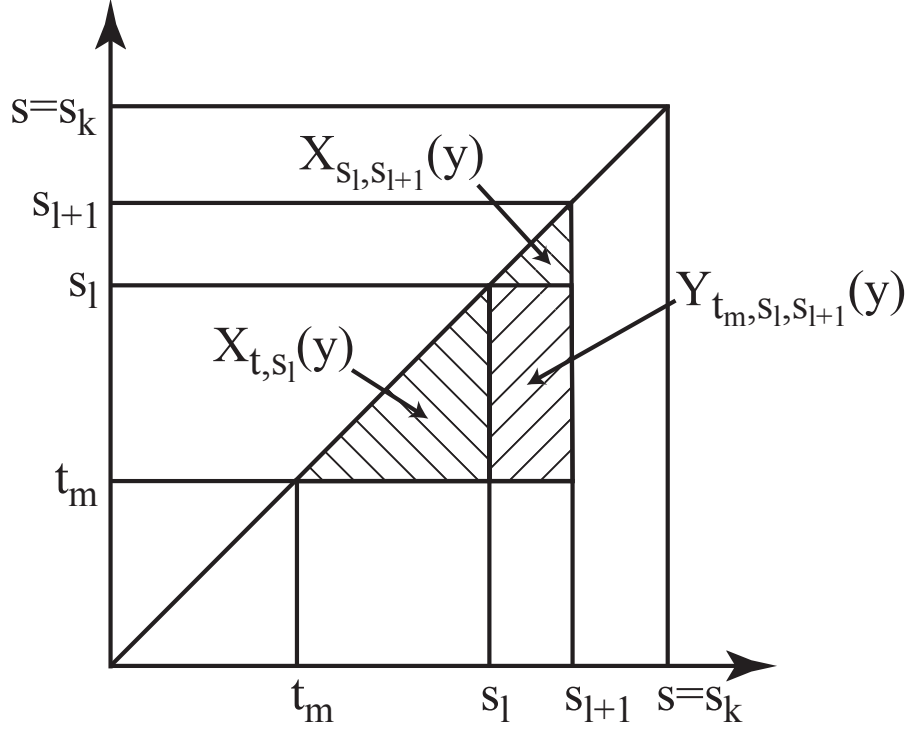$$

FIGURE 1. $X_{s,t} = X_{s,t}^{1,2}$ and $Y_{t,s_1,s_2} = Y_{t,s_1,s_2,n_0}^2$ for $N = 2$ and $n_0 = 1$

we have $s_n \leq s_{n+1} \leq \ldots \leq s_k = s$, $t_n \leq t_{n+1} \leq \ldots \leq t_k = t$. Therefore (see also Figure 1)

$$X_{t_n,s}^{1,N}(x) = X_{t_n,s_n}^{1,N}(x) + \sum_{j=n}^{k-1} \left\{ X_{s_j,s_{j+1}}^{1,N}(x) + \sum_{n_0=0}^{N-2} Y_{t_n,s_j,s_{j+1},n_0}^N(x) \right\}, \quad (21)$$

$$X_{t_n,s}^{1,N}(x) = X_{t,s}^{1,N}(x) + \sum_{j=n}^{k-1} \left\{ X_{t_j,t_{j+1}}^{1,N}(x) + \sum_{n_0=0}^{N-2} Y_{t_j,t_{j+1},s,n_0}^N(x) \right\}. \quad (22)$$

In order to estimate $Y_{t_n,s_j,s_{j+1},n_0}^N(x)$ and $Y_{t_j,t_{j+1},s,n_0}^N(x)$, we notice that

$$s_{j+1} - s_j \in \{0, 2^{-(j+1)}\}, \quad t_{j+1} - t_j \in \{0, 2^{-(j+1)}\}, \qquad n \leq j \leq k,$$

and thus

$$\begin{cases} 0 \leq s_j - t_n = (s_j - s_n) + (s_n - t_n) \leq 2 \cdot 2^{-n}; \\ 0 \leq s - t_{j+1} = (s - t) + (t - t_{j+1}) \leq 2 \cdot 2^{-n}, \end{cases} \quad (23)$$

for $n \leq j \leq k$. Moreover, by using the inductional hypothesis, we know that for every $\alpha \in (0, \frac{1}{2})$, there exists a $\theta_\alpha \in L^\infty(\Omega, \mathcal{F}, P)$ such that for all $l, l' \in \mathbb{N}$ with $0 \leq l' - l + 1 \leq N - 1$, the $l' - l + 1$-tuple integral $X^{l,l'}$ satisfies (14) with

$\beta = (l' - l + 1)\alpha$, that is,

$$P\left\{ \left| X_{t,t+h}^{l,l'} \right| \leq \theta_\alpha h^{(l'-l+1)\alpha}, \ 0 \leq t < t + h \leq T, \ |x| \leq m \right\} = 1.$$

Thus, we have by (23)

$$
\begin{aligned}
\sum_{n_0=0}^{N-2} |Y_{t_n,s_j,s_{j+1},n_0}^N(x)| &\leq \sum_{n_0=0}^{N-2} \theta_\alpha |s_{j+1} - s_j|^{(n_0+1)\alpha} \cdot \theta_\alpha |s_j - t_n|^{(N-n_0-1)\alpha} \\
&\leq \sum_{n_0=0}^{N-2} \theta_\alpha^2 2^{-(n_0+1)(j+1)\alpha} 2^{-(N-n_0-1)(n-1)\alpha} \\
&\leq 2^N \sum_{n_0=0}^{N-2} \theta_\alpha^2 2^{-((n_0+1)j+(N-n_0-1)n)\alpha}.
\end{aligned}
\tag{24}
$$

Similarly, we obtain

$$\sum_{n_0=0}^{N-2} |Y_{t_j,t_{j+1},s,n_0}^N(x)| \leq 2^N \sum_{n_0=0}^{N-2} \theta_\alpha^2 2^{-((n_0+1)j+(N-n_0-1)n)\alpha}. \tag{25}$$

Next, we define $K_j \triangleq \sup\{|X_{t_i^j,t_{i+1}^j}(x)|, \ |x| \leq m, \ 0 \leq i \leq 2^j - 1\}$, $n \leq j \leq k$. Then, combining (21), (22), (25), and (24), we deduce

$$
\begin{aligned}
&|X_{t,s}^N(x)| \\
={}& \left| X_{t_n,s_n}^N(x) + \sum_{j=n}^{k-1} \left\{ X_{s_j,s_{j+1}}^N(x) - X_{t_j,t_{j+1}}^N(x) \right. \right. \\
&\qquad\qquad \left. \left. + \sum_{n_0=0}^{N-2} Y_{t_n,s_j,s_{j+1},n_0}^N(x) - \sum_{n_0=0}^{N-2} Y_{t_j,t_{j+1},s,n_0}^N(x) \right\} \right| \\
\leq{}& K_n + \sum_{j=n}^{k-1} \left(2K_j + 2^{N+1} \sum_{n_0=0}^{N-2} \theta_\alpha^2 2^{-((n_0+1)j+(N-n_0-1)n)\alpha} \right) \\
\leq{}& \sum_{j=n}^{\infty} \left(3K_j + 2^{N+1} \sum_{n_0=0}^{N-2} \theta_\alpha^2 2^{-((n_0+1)j+(N-n_0-1)n)\alpha} \right).
\end{aligned}
$$

Now let $(N-1)\alpha < \beta < N\alpha < \frac{N}{2}$. Define

$$\zeta_{\beta,m}^n \triangleq \sup\left\{ \frac{|X_{t,s}^N(x)|}{|s-t|^\beta}, \ s,t \in D, \ 2^{-(n+1)} < s - t \leq 2^{-n}, \ |x| \leq m \right\},$$

and $M_{\beta,m} \overset{\triangle}{=} \sup_{n\geq 1} \zeta_{\beta,m}^n$. Then,

$$
\begin{aligned}
M_{\beta,m} &\leq \sup_{n\geq 1}\left(2^{(n+1)\beta}\sup\left\{|X_{t,s}^N(x)|,\ s,t\in D,\ 0<s-t\leq 2^{-n},\ |x|\leq m\right\}\right)\\
&\leq \sup_{n\geq 1}\left(2^{(n+1)\beta}\sum_{j=n}^{\infty}\left(3K_j+2^{N+1}\sum_{n_0=0}^{N-2}\theta_\alpha^2 2^{-((n_0+1)j+(N-n_0-1)n)\alpha}\right)\right)\\
&\leq \sup_{n\geq 1}2^{\beta+N+2}\sum_{j=n}^{\infty}\left(2^{n\beta}K_j+\sum_{n_0=0}^{N-2}\theta_\alpha^2 2^{-(n_0+1)\alpha j+(\beta-(N-n_0-1)\alpha)n}\right)\\
&\leq \sup_{n\geq 1}2^{\beta+N+2}\sum_{j=n}^{\infty}\left(2^{n\beta}K_j+\sum_{n_0=0}^{N-2}\theta_\alpha^2 2^{-(n_0+1)\alpha j+(\beta-(N-1)\alpha)n+nn_0\alpha}\right)\\
&\leq 2^{\beta+N+2}\sum_{j=0}^{\infty}\left(2^{j\beta}K_j+(N-1)\theta_\alpha^2 2^{(\beta-N\alpha)j}\right)\\
&= 2^{\beta+N+2}\left(\sum_{j=0}^{\infty}2^{j\beta}K_j+\frac{(N-1)\theta_\alpha^2}{1-2^{-(N\alpha-\beta)}}\right).
\end{aligned}
$$

Noting that by (18)

$$
E[K_j^\gamma]\leq \sum_{i=0}^{2^j-1}E\left[\sup_{|x|\leq m}|X_{t_i^j,t_{i+1}^j}(x)|^\gamma\right]\leq C_{m,\gamma}\frac{2^j}{2^{j(\frac{N}{2}\gamma-1)}}=C_{m,\gamma}2^{-j(\frac{N}{2}\gamma-2)},\quad (26)
$$

we thus obtain for $\gamma>\frac{4}{N-2\beta}$ that

$$
\begin{aligned}
&\left(E\left[M_{\beta,m}^\gamma\right]\right)^{\frac{1}{\gamma}}\\
&\leq 2^{\beta+N+2}\left(\sum_{j=0}^{\infty}2^{j\beta}\left(E\left[K_j^\gamma\right]\right)^{\frac{1}{\gamma}}+\frac{N-1}{1-2^{-(N\alpha-\beta)}}\left(E\left[\theta_\alpha^{2\gamma}\right]\right)^{\frac{1}{\gamma}}\right)\\
&\leq 2^{\beta+N+2}\left(\sum_{j=0}^{\infty}2^{j\beta}C_{m,\gamma}2^{-j\left(\frac{N}{2}-\frac{2}{\gamma}\right)}+\frac{N-1}{1-2^{-(N\alpha-\beta)}}\left(E\left[\theta_\alpha^{2\gamma}\right]\right)^{\frac{1}{\gamma}}\right)\\
&= 2^{\beta+N+2}C_{m,\gamma}\left(\frac{1}{1-2^{-\left(\frac{N}{2}-\frac{2}{\gamma}-\beta\right)}}+\frac{N-1}{1-2^{-(N\alpha-\beta)}}\left(E\left[\theta_\alpha^{2\gamma}\right]\right)^{\frac{1}{\gamma}}\right)<\infty.
\end{aligned}
$$

Consequently, $M_{\beta,m}\in L_-^\infty(\Omega,\mathcal{F},P)$, for all $\beta\in(0,N\alpha)$. But since $\alpha\in(0,\frac{1}{2})$ is arbitrary, we can extend the result to $\beta\in(0,N/2)$. Finally, note that the definitions of $M_{\beta,m}$ and $\{\zeta_{\beta,m}^n\}$, as well as the continuity of the mapping $(t,s)\mapsto X_{t,s}(x)$, imply that

$$
\widehat{\zeta}_{\beta,m}\overset{\triangle}{=}\sup\left\{\frac{|X_{t,s}^N(x)|}{|s-t|^\beta},\ 0<s-t\leq 2^{-1},\ |x|\leq m\right\}\in L_-^\infty(\Omega,\mathcal{F},P),
$$

for all $\beta\in(0,N/2)$ and $m\in\mathbb{N}$. The proposition then follows from the recursive relation (20). $\qquad\square$

The following corollary can be easily obtained by adapting the proof of Proposition 3.1 in an obvious manner.

Corollary 3.2. *The statement of Proposition 3.1 remains valid if, for $1 \leq i \leq N$, dB_{s_i} is replaced by $dA^i_{s_i}$ in (11), where A^i is either the Brownian motion B or $A^i_s = s$, $s \in [0, T]$. Moreover, (14) holds whenever $\beta \in \left(0, l_1 + \frac{l_2}{2}\right)$, $m \in \mathbb{N}$, where l_1 is the number of i, $1 \leq i \leq N$, for which $A^i_s = s$, $s \in [0, T]$, and $l_2 = N - l_1$.*

4. **Forward Taylor expansion.** In this and next section we shall provide a complete proof of Theorem 2.3. The results in each section, however, can also be applied independently, and therefore are of interest in their own right. We should note that, unlike the usual Taylor expansion, in the stochastic case the direction of the time increment makes significant difference in the argument, due to the "progressive measurability" of the random fields. We will thus separate the two cases: in this section we study the *forward* expansion, and leave the *backward* case to the next section.

4.1. **Forward temporal expansion.** We begin with the forward "temporal" expansion, that is, only the time variable has the increment. Let us first introduce the following extra notations. For each $T > 0$ we define the 2-dimensional simplex

$$\triangle_{[0,T]} \overset{\triangle}{=} \{(t, s) \in [0, T] \times \mathbb{R}^+ : 0 \leq t < s \leq T\}.$$

For any real-valued measurable functional θ defined on $\Omega \times \triangle_{[0,T]} \times \mathbb{R}^{d+1}$ and $m \in \mathbb{N}$, we denote

$$\widehat{\theta}_m \overset{\triangle}{=} \sup\{|\theta(t, s, x, z)|, \ (t, s) \in \triangle_{[0,T]}, \ x \in \mathbb{R}^d, \ z \in \mathbb{R} \text{ with } |x|, |z| \leq m\}.$$

Furthermore, by a slight abuse of notations, in what follows we shall denote R_α to be any measurable functional θ_α, indexed by $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, such that $(\widehat{\theta}_\alpha)_m \in L^\infty_-(\Omega, \mathcal{F}, P)$ for all $m \in \mathbb{N}$, and again, it may vary from line to line.

Our main result of this section is the following *Stochastic forward temporal Taylor expansion.*

Proposition 4.1. *Assume that $\zeta \in C^{0,(3)}(\mathbf{F}^B, [0, T] \times \mathbb{R}^d)$ satisfies (T1)-(T3). Then, for all $m \in \mathbb{N}$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ there exists a subset $\widetilde{\Omega} \subset \Omega$ such that $P(\widetilde{\Omega}) = 1$, and that on $\widetilde{\Omega}$, for all $0 \leq t < t + h \leq T$, $x \in \mathbb{R}^d$, the following expansion holds*

$$\zeta(t + h, x) - \zeta(t, x) = ah + b(B_{t+h} - B_t) + \frac{c}{2}(B_{t+h} - B_t)^2 + h^{3\alpha} R_\alpha(t, t+h, x), \ (27)$$

where

$$\begin{aligned} a &= F(x, \zeta_2(t, x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x)), \\ b &= G(x, \zeta_2(t, x)), \\ c &= (D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x)). \end{aligned} \tag{28}$$

Proof. Let $h > 0$ be such that $0 \leq t \leq t + h \leq T$. For any $x \in \mathbb{R}^d$, we write

$$\zeta(t+h, x) - \zeta(t, x) = \int_t^{t+h} F(x, \zeta_2(s, x))ds + \int_t^{t+h} G(x, \zeta_2(s, x))dB_s \overset{\triangle}{=} I^1 + I^2, \ (29)$$

where $I^i = I^i(t, h, x)$, $i = 1, 2$, are the two integrals. We shall study their expansions separately.

We begin by I^1. The argument is very similar to that of [3], we provide a sketch for completeness. Let

$$H^1(x, z_2, z_3) \overset{\triangle}{=} (D_z F)(x, z_2)F_2(x, z_3) + \frac{1}{2}\text{tr}[G_2(G_2)^*(x, z_3)D_z^2 F(x, z_2)]$$

for $(x, z_2, z_3) \in \mathbb{R}^d \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$. Then, applying Itô's formula, we have

$$
\begin{aligned}
F(x, \zeta_2(s, x)) - F(x, \zeta_2(t, x)) \quad = \quad & \int_t^s H^1(x, \zeta_2(r, x), \zeta_3(r, x)) dr \qquad (30) \\
& + \int_t^s \langle (D_z F)(x, \zeta_2(r, x)), G_2(x, \zeta_3(r, x)) \rangle dB_r,
\end{aligned}
$$

for $0 \le t \le s \le T$ and $x \in \mathbb{R}^d$, P-a.s. We now show that there is a universal subset $\Omega' \subseteq \Omega$ with $P(\Omega') = 1$, on which (30) holds for all $0 \le t \le s \le T$. But for this it suffices to prove the random fields

$$
(s, t, x) \mapsto \int_t^s H^1(x, \zeta_2(r, x), \zeta_3(r, x)) dr, \quad \int_t^s \langle (D_z F)(x, \zeta_2(r, x)), G_2(x, \zeta_3(r, x)) \rangle dB_r
$$

have continuous versions. To see this, we try to make use of the Kolmogorov continuity criterion. For notational simplicity let us denote

$$
\Gamma(t, x) \overset{\triangle}{=} (D_z F)(x, \zeta_2(t, x)) G_2(x, \zeta_3(t, x)), \qquad (t, x) \in [0, T] \times \mathbb{R}^d.
$$

Then, we can deduce from (T3) and the Burkholder-Davis-Gundy inequality that, for each $k > 2$,

$$
\begin{aligned}
& E \Big| \int_t^s \Gamma(r, x) dB_r - \int_{t'}^{s'} \Gamma(r, x') dB_r \Big|^k \\
\le \quad & C_k \Big\{ E \Big| \int_t^{t'} |(\Gamma(r, x)|^2 dr \Big|^{\frac{k}{2}} + E \Big| \int_{s'}^s |\Gamma(r, x)|^2 dr \Big|^{\frac{k}{2}} \\
& + E \Big(\int_0^T |\Gamma(r, x) - \Gamma(r, x')|^2 dr \Big)^{\frac{k}{2}} \Big\} \\
\le \quad & C_k \Big(|t' - t|^{\frac{k}{2}} + |s' - s|^{\frac{k}{2}} \Big) E \Big\{ \sup_{\substack{r \in [0, T], \\ |x| \le m}} |\Gamma(r, x)|^k \Big\} \\
& + C_k |x - x'|^k E \Big\{ \sup_{\substack{r \in [0, T], \\ |x| \le m}} |D_x \Gamma(r, x)|^k \Big\} \\
\le \quad & C_k \Big(|t' - t|^{\frac{k}{2}} + |s' - s|^{\frac{k}{2}} + |x - x'|^k \Big)
\end{aligned}
$$

for $0 \le s \le t \le T$, $0 \le s' \le t' \le T$, $|x|, |x'| \le m$, and $k \in \mathbb{N}$. Hence, the Kolmogorov continuity criterion renders that the random field $\Big\{ \int_t^s \Gamma(r, x) dB_r, \ 0 \le t \le s \le T, \ |x| \le m \Big\}$ possesses a version that is continuous in (t, s, x). A similar estimate allows to prove that also the first integral in (30) admits a version continuous in (t, s, x) as well. Hence, we conclude that on some $\Omega' \subset \Omega$ of full probability measure the relation (30) holds for all $0 \le t \le s \le T$ and $|x|, |x'| \le m$.

Consequently, writing the integral I^1 as

$$
\begin{aligned}
I^1(t, h, x) \quad = \quad & F(x, \zeta_2(t, x)) h + \int_t^{t+h} \int_t^s H^1(x, \zeta_2(r, x), \zeta_3(r, x)) dr ds \\
& + \int_t^{t+h} \int_t^s (D_z F)(x, \zeta_2(r, x)) G_2(x, \zeta_3(r, x)) dB_r ds,
\end{aligned}
$$

and using the conclusion above we see that as a function of (t, h, x), I^1 is jointly continuous for all $0 \le t \le t + h \le T$ and $x \in \mathbb{R}^d$, over the universal set Ω'. It is clear that the first integral on the right hand side above is of order h^2. Thus,

applying Corollary 3.2 (with $N = 2$, $l_1 = l_2 = 1$) to the second integral we see that for all $\beta \in (0, \frac{1}{2})$, there exists an $\Omega'' \subseteq \Omega'$ with $P(\Omega'') = 1$ such that for all $0 \le t \le t + h \le T$, $x \in \mathbb{R}^d$, it holds that

$$I^1(t, h, x) = F(x, \zeta_2(t, x)h + h^{1+\beta}R_\beta(t, t + h, x). \tag{31}$$

In what follows we will not distinguish Ω'' from Ω'.

We now turn our attention to I^2. Again, we begin by denoting, for $(x, z_2, z_3) \in \mathbb{R}^d \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$,

$$H^2(x, z_2, z_3) \triangleq (D_z G)(x, z_2)F_2(x, z_3) + \frac{1}{2}\mathrm{tr}[G_2(G_2)^*(x, z_3)D_z^2 G(x, z_2)]. \tag{32}$$

Then, for every $x \in \mathbb{R}^d$, we again apply Itô's formula to get, for $0 \le t \le s \le T$, and P-a.s.,

$$
\begin{aligned}
G(x, \zeta_2(s, x)) - G(x, \zeta_2(t, x)) &= \int_t^s H^2(x, \zeta_2(r, x), \zeta_3(r, x))dr \\
&\quad + \int_t^s (D_z G)(x, \zeta_2(r, x))G_2(x, \zeta_3(r, x))dB_r.
\end{aligned} \tag{33}
$$

Using the similar arguments as before we can find another universal subset, still denoted by $\Omega' \subseteq \Omega$ with $P(\Omega') = 1$, on which (33) holds for all $0 \le t \le s \le T$.

Next, using (33) it is easy to see that I^2 can be written as

$$
\begin{aligned}
I^2(t, h, x) &= G(x, \zeta_2(t, x))(B_{t+h} - B_t) + \int_t^{t+h} \int_t^s H^2(x, \zeta_2(r, x), \zeta_3(r, x))drdB_s \\
&\quad + \int_t^{t+h} \int_t^s (D_z G)(x, \zeta_2(r, x))G_2(x, \zeta_3(r, x))dB_r dB_s \\
&\triangleq G(x, \zeta_2(t, x))(B_{t+h} - B_t) + I^{2,1}(t, h, x) + I^{2,2}(t, h, x),
\end{aligned} \tag{34}
$$

and we can claim as before that (34) holds for all $0 \le t < t + h \le T$, $|x| \le m$, over a universal subset of full probability measure, again denoted by Ω'.

We now analyze $I^{2,1}$ and $I^{2,2}$ separately. Using integration by parts we see that

$$
\begin{aligned}
I^{2,1} &= (B_{t+h} - B_t)\int_t^{t+h} H^2(x, \zeta_2(r, x), \zeta_3(r, x))dr \\
&\quad - \int_t^{t+h} (B_r - B_t)H(x, \zeta_2(r, x), \zeta_3(r, x))dr.
\end{aligned}
$$

Then, following the argument developed in the previous part, we can show that the equality holds for all $0 \le t < t + h \le T$, $x \in \mathbb{R}^d$, over an $\Omega' \subseteq \Omega$ with $P(\Omega') = 1$. Now for $\beta \in (0, \frac{1}{2})$, by Proposition 3.1 one has

$$\sup_{0 \le t < t+h \le T} \{h^{-\beta}|B_{t+h} - B_t|\} \in L_-^\infty(\Omega, \mathcal{F}, P), \tag{35}$$

and it follows that over Ω',

$$I^{2,1}(t, h, x) = h^{1+\beta}R_\beta(t, t + h, x). \tag{36}$$

The estimate for $I^{2,2}$ is slightly more involved. For notational simplicity let us define

$$
\begin{aligned}
\widehat{K}(x, z_2, z_3) &\triangleq (D_z G)(x, z_2) G_2(x, z_3), \quad (x, z_2, z_3) \in \mathbb{R}^d \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}, \\
\widehat{F}(x, z_3, z_4) &\triangleq (F_2(x, z_3), F_3(x, z_4)), \quad (x, z_3, z_4) \in \mathbb{R}^d \times \mathbb{R}^{d_3} \times \mathbb{R}^{d_4}, \\
\widehat{G}(x, z_3, z_4) &\triangleq (G_2(x, z_3), G_3(x, z_4)), \\
\widehat{H}(x, \widehat{z}_2, \widehat{z}_3) &\triangleq D_{\widehat{z}_2} \widehat{K}(x, \widehat{z}_2) \widehat{F}(x, \widehat{z}_3) + \frac{1}{2} \mathrm{tr}[\widehat{G}\widehat{G}^*(x, \widehat{z}_3) D_{\widehat{z}_2}^2 \widehat{K}(x, \widehat{z}_2)],
\end{aligned}
$$

where $\widehat{z}_2 = (z_2, z_3)$ and $\widehat{z}_3 = (z_3, z_4)$. Moreover, we denote

$$
\widehat{\zeta}_i(s, x) = (\zeta_i(s, x), \zeta_{i+1}(s, x)), \qquad i = 2, 3.
$$

Then, applying Itô's formula we have,

$$
\begin{aligned}
\widehat{K}(x, \widehat{\zeta}_2(s, x)) - \widehat{K}(x, \widehat{\zeta}_2(t, x)) &= \int_t^s \widehat{H}(x, \widehat{\zeta}_2(r, x), \widehat{\zeta}_3(r, x)) dr \\
&\quad + \int_t^s D_{\widehat{z}_2} \widehat{K}(x, \widehat{\zeta}_2(r, x)) \widehat{G}(x, \widehat{\zeta}_3(r, x)) dB_r.
\end{aligned}
$$

Again, we assume that the equality holds for all $0 \le s \le t \le T$ and $x \in \mathbb{R}^d$, on Ω'. Therefore

$$
\begin{aligned}
I^{2,2}(t, h, x) &= \int_t^{t+h} \int_t^s \widehat{K}(x, \widehat{\zeta}_2(r, x)) dB_r dB_s \\
&= \frac{1}{2} (D_z G)(x, \zeta_2(t, x)) G_2(x, \zeta_3(t, x)) (B_{t+h} - B_t)^{\diamond 2} \qquad (37) \\
&\quad + \int_t^{t+h} \int_t^s \int_t^r \widehat{H}(x, \widehat{\zeta}_2(v, x), \widehat{\zeta}_3(v, x)) dv dB_r dB_s \\
&\quad + \int_t^{t+h} \int_t^s \int_t^r D_{\widehat{z}_2} \widehat{K}(x, \widehat{\zeta}_2(v, x)) \widehat{G}(x, \widehat{\zeta}_3(v, x)) dB_v dB_r dB_s.
\end{aligned}
$$

Moreover, for $\beta \in (0, \frac{1}{2})$, Proposition 3.1 (with $N = 3$) implies that

$$
h^{-(1+\beta)} \int_t^{t+h} \int_t^s \int_t^r D_{\widehat{z}_2} \widehat{K}(x, \widehat{\zeta}_2(v, x)) \widehat{G}(x, \widehat{\zeta}_3(v, x)) dB_v dB_r dB_s = R_\beta(t, t+h, x);
$$

and Corollary 3.2 (with $\ell_1 = 1$ and $\ell_2 = 2$) implies that

$$
h^{-(1+2\beta)} \int_t^{t+h} \int_t^s \int_t^r \widehat{H}(x, \widehat{\zeta}_2(v, x), \widehat{\zeta}_3(v, x)) dv dB_r dB_s = R_\beta(t, t+h, x).
$$

Consequently, we see that, over Ω', (37) becomes

$$
I^{2,2}(t, h, x) = \frac{1}{2} (D_z G)(x, \zeta_2(t, x)) G_2(x, \zeta_3(t, x)) (B_{t+h} - B_t)^{\diamond 2} + h^{1+\beta} R_\beta(t, t+h, x).
$$
$$(38)$$

Finally, note for each $\beta \in (0, \frac{1}{2})$, we can find $\alpha \in (\frac{1}{3}, \frac{1}{2})$ so that $3\alpha = (1 + \beta)$. Plugging (31), (34), (36), and (38) into (29) and changing R_β to R_α, we obtain (27) and (28) with a universal exceptional null set, proving the proposition. $\square$

4.2. **Forward temporal-spatial Taylor expansion.** Based on the forward temporal Taylor expansion Proposition 4.1, we now add the spatial increment. Our main result of this section is the following proposition.

Proposition 4.2. *Assume that $\zeta \in C^{0,(3)}(\mathbf{F}^B, [0, T], \mathbb{R}^d)$ satisfying (T1)-(T3). Then, for all $m \in \mathbb{N}$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ there exists some subset $\widetilde{\Omega} \subset \Omega$ such that $P(\widetilde{\Omega}) = 1$, and that on $\widetilde{\Omega}$, for all $0 \le t < t + h \le T$ and $x, k \in \mathbb{R}^d$,*

$$
\begin{aligned}
&\zeta(t+h, x+k) - \zeta(t,x) \\
=\ & ah + b(B_{t+h} - B_t) + \frac{c}{2}(B_{t+h} - B_t)^2 + \langle\, p, k\,\rangle + \frac{1}{2}\langle Xk, k \rangle \qquad (39) \\
&+ \langle\, q, k\,\rangle(B_{t+h} - B_t) + (h + |k|^2)^{3\alpha} R_\alpha(t, t+h, x, x+k),
\end{aligned}
$$

where

$$
\begin{aligned}
a &= F(x, \zeta_2(t,x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t,x))G_2(x, \zeta_3(t,x)); \\
b &= G(x, \zeta_2(t,x)); \quad c = (D_z G)(x, \zeta_2(t,x))G_2(x, \zeta_3(t,x)); \qquad (40) \\
p &= (D_x \zeta)(t,x); \quad X = D_x^2 \zeta(t,x); \\
q &= (D_x G)(x, \zeta_2(t,x)) + (D_z G)(x, \zeta_2(t,x))D_x \zeta_2(t,x).
\end{aligned}
$$

Proof. First let us write

$$
\zeta(t+h, x+k) - \zeta(t,x) = [\zeta(t+h, x+k) - \zeta(t+h, x)] + [\zeta(t+h, x) - \zeta(t,x)],
$$

where the second $[\cdots]$ above is the forward temporal expansion studied in the previous subsection. In light of Proposition 4.1, we need only prove the following *stochastic spatial Taylor expansion* for $\alpha \in (\frac{1}{3}, \frac{1}{2})$:

$$
\begin{aligned}
\zeta(t+h, x+k) - \zeta(t+h, x) =\ & D_x \zeta(t,x)k + \frac{1}{2}\langle D_x^2 \zeta(t,x)k, k \rangle + \{(D_x G)(x, \zeta_2(t,x)) \\
&+ (D_z G)(x, \zeta_2(t,x))D_x \zeta_2(t,x)\}k(B_{t+h} - B_t) \\
&+ (h + |k|^2)^{3\alpha} R_\alpha^1(t, t+h, x, x+k). \qquad (41)
\end{aligned}
$$

To this end, we first apply the standard Taylor expansion and use the assumption (T3) to get

$$
\begin{aligned}
\zeta(t+h, x+k) - \zeta(t+h, x) =\ & D_x \zeta(t+h, x)k + \frac{1}{2}\langle D_x^2 \zeta(t+h, x)k, k \rangle \\
&+ |k|^3 R_\alpha^1(t, t+h, x, x+k), \qquad (42)
\end{aligned}
$$

for all $0 \le t < t + h \le T$, $x, k \in \mathbb{R}^d$, P-a.s. Next, differentiating the equation for $\zeta = \zeta_1$ in (6) we have

$$
\begin{aligned}
D_x \zeta(t,x) =\ & D_x \zeta_0(x) + \int_0^t \{(D_x F)(x, \zeta_2(s,x)) + (D_z F)(x, \zeta_2(s,x))D_x \zeta_2(s,x)\}\, ds \\
&+ \int_0^t \{(D_x G)(x, \zeta_2(s,x)) + (D_z G)(x, \zeta_2(s,x))D_x \zeta_2(s,x))\}\, dB_s,
\end{aligned}
$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, P-a.s. Now, applying Proposition 4.1 (to $D_x\zeta$) one can check that, for $\alpha \in (\frac{1}{3}, \frac{1}{2})$,

$$
\begin{aligned}
&D_x\zeta(t+h, x) \\
={}& D_x\zeta(t, x) + \{(D_xF)(x, \zeta_2(t, x)) + (D_zF)(x, \zeta_2(t, x))D_x\zeta_2(t, x)\}h \\
&- \frac{1}{2}\{D_x[(D_zG)(x, \zeta_2(t, x))]G_2(x, \zeta_3) + D_zG(x, \zeta_2(t, x))D_x[G_2(x, \zeta_3)]\}h \\
&+ \{(D_xG)(x, \zeta_2(t, x)) + (D_zG)(x, \zeta_2(t, x))D_x\zeta_2(t, x)\}(B_{t+h} - B_t) \\
&+ \frac{1}{2}\{D_x[(D_zG)(x, \zeta_2(t, x))]G_2(x, \zeta_3) \\
&+ D_zG(x, \zeta_2(t, x))D_x[G_2(x, \zeta_3)]\}(B_{t+h} - B_t)^{\diamond 2} + h^{3\alpha}R_\alpha(t, t+h, x),
\end{aligned}
$$

for all $0 \le t < t + h \le T$, $x \in \mathbb{R}^d$, P-a.s. Consequently, it follows from (35) that

$$
\begin{aligned}
D_x\zeta(t+h, x) = D_x\zeta(t, x) + \{(D_xG)(x, \zeta_2(t, x)) \hspace{4em} (43) \\
+ (D_zG)(x, \zeta_2(t, x))D_x\zeta_2(t, x)\}(B_{t+h} - B_t) + h^{2\alpha}R_\alpha(t, t+h, x),
\end{aligned}
$$

for all $0 \le t < t + h \le T$, $x \in \mathbb{R}^d$, P-a.s. Similarly, one shows that

$$
D_x^2\zeta(t+h, x) = D_x^2\zeta(t, x) + h^\alpha R_\alpha(t, t+h, x), \quad 0 \le t < t + h \le T, \ x \in \mathbb{R}^d. \quad (44)
$$

Combining (42), (43), and (44), we obtain for all $\alpha \in (\frac{1}{3}, \frac{1}{2})$ that

$$
\begin{aligned}
&\zeta(t+h, x+k) - \zeta(t+h, x) \\
={}& D_x\zeta(t+h, x)k + \frac{1}{2}\langle D_x^2\zeta(t+h, x)k, k\rangle + |k|^3 R_\alpha^1(t, t+h, x, x+k) \\
={}& D_x\zeta(t, x)k + \frac{1}{2}\langle D_x^2\zeta(t, x)k, k\rangle \\
&+ \{(D_xG)(x, \zeta_2(t, x)) + (D_zG)(x, \zeta_2(t, x))D_x\zeta_2(t, x)\}k(B_{t+h} - B_t) \\
&+ |k|^3 R_\alpha^1(t, t+h, x, x+k) + h^{2\alpha}R_\alpha(t, t+h, x)k + h^\alpha\langle R_\alpha(t, t+h, x)k, k\rangle,
\end{aligned}
$$

for $0 \le t < t + h \le T$, $x, k \in \mathbb{R}^d$, P-a.s. Finally, noting that

$$
|k|^3 + h^{2\alpha}|k| + h^\alpha|k|^2 \le C_m(h^{3\alpha} + |k|^3) \le C_m(h + |k|^2)^{3\alpha}
$$

for all $h \in [0, T]$, $|k| \le m$, we see that (41) holds, and the proposition follows. $\square$

5. **Backward Taylor expansion.** In this section we treat the backward Taylor expansion, that is, when the temporal increments are negative. As a general belief such an expansion would be more difficult than the forward one, due to the obvious "adaptedness" issue. But we shall see, with our "pathwise" approach, such difficulty is eliminated. We nevertheless would like to separate its proof from the forward case because of the slight difference in the arguments. We again take two steps: first the backward temporal expansion, and then the mixed time-space expansion.

5.1. **Backward temporal expansion.** We have the following analogy of Proposition 4.1.

Proposition 5.1. *Assume that $\zeta \in C^{0,(3)}(\mathbf{F}^B, [0, T], \mathbb{R}^d)$ satisfies (T1)-(T3). Then, for all $m \in \mathbb{N}$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, there exists some subset $\widetilde{\Omega} \subset \Omega$ with $P(\widetilde{\Omega}) = 1$, such that on $\widetilde{\Omega}$, for all $0 \le t - h < t \le T$, $x \in \mathbb{R}^d$, the following expansion holds*

$$
\begin{aligned}
\zeta(t-h, x) - \zeta(t, x) ={}& ah + b(B_{t-h} - B_t) + \frac{c}{2}(B_{t-h} - B_t)^2 \\
&+ h^{1+\alpha}R_\alpha(t-h, t, x) \hspace{6em} (45)
\end{aligned}
$$

where

$$
\begin{aligned}
a &= -\left\{ F(x, \zeta_2(t,x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t,x)) G_2(x, \zeta_3(t,x)) \right\}; \\
b &= G(x, \zeta_2(t,x)); \\
c &= (D_z G)(x, \zeta_2(t,x)) G_2(x, \zeta_3(t,x)).
\end{aligned}
\tag{46}
$$

Proof. Let us fix an arbitrary $\alpha \in (\frac{1}{3}, \frac{1}{2})$. Applying the forward Taylor expansion Proposition 4.1, we have

$$
\begin{aligned}
&\zeta(t,x) - \zeta(t-h,x) \\
&= \left\{ F(x, \zeta_2(t-h,x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t-h,x)) G_2(x, \zeta_3(t-h,x)) \right\} h \\
&\quad + G(x, \zeta_2(t-h,x))(B_t - B_{t-h}) \\
&\quad + \frac{1}{2}(D_z G)(x, \zeta_2(t-h,x)) G_2(x, \zeta_3(t-h,x))(B_t - B_{t-h})^2 \\
&\quad + h^{1+\alpha} R_\alpha(t-h,t,x),
\end{aligned}
\tag{47}
$$

for all $0 \le t - h < t \le T$, $x \in \mathbb{R}^d$, on a full probability set Ω'. Our main task is to replace the temporal variable $t - h$ by t. To do this, we first apply Itô's formula to $G(x, \zeta_2(t,x))$ to obtain

$$
G(x, \zeta_2(t,x)) = G(x, \zeta_2(0,x)) + \int_0^t H(x, \widehat{\zeta}_2(s,x)) ds + \int_0^t L(x, \widehat{\zeta}_2(s,x)) dB_s, \tag{48}
$$

where $\widehat{\zeta}_2(t,x) = (\zeta_2(t,x), \zeta_3(t,x))$, and

$$
\begin{cases}
H(x, (z_2, z_3)) \triangleq (D_z G)(x, z_2) F_2(x, z_3) + \frac{1}{2}\mathrm{tr}[G_2 G_2^*(x, z_3)(D_z^2 G)(x, z_2)], \\
L(x, (z_2, z_3)) \triangleq (D_z G)(x, z_2) G_2(x, z_3).
\end{cases}
\tag{49}
$$

Next, we denote $\widehat{z} = (z_2, z_3)$, and define $\widehat{\zeta}_3(t,x) \triangleq (\zeta_3(t,x), \zeta_4(t,x))$, $\widehat{G}(t,x) \triangleq G(x, \zeta_2(t,x))$ and

$$
\widehat{G}^3(x, (z_3, z_4)) = (G_2(x, z_3), G_3(x, z_4)).
$$

Applying Proposition 4.1 to $G(\cdot, \cdot)$ and using (48), we deduce that

$$
\begin{aligned}
&\widehat{G}(t,x) - \widehat{G}(t-h,x) \\
&= \left\{ H(x, \widehat{\zeta}_2(t-h,x)) - \frac{1}{2} D_{\widehat{z}} L(x, \widehat{\zeta}_2(t-h,x)) \widehat{G}^3(x, \widehat{\zeta}_3(t-h,x)) \right\} h \\
&\quad + L(x, \widehat{\zeta}_2(t-h,x))(B_t - B_{t-h}) \\
&\quad + \frac{1}{2} D_{\widehat{z}} L(x, \widehat{\zeta}_2(t-h,x)) \widehat{G}^3(x, \widehat{\zeta}_3(t-h,x))(B_t - B_{t-h})^2 \\
&\quad + h^{1+\alpha} R_\alpha(t-h,t,x),
\end{aligned}
$$

for all $0 \le t - h < t \le T$, $x \in \mathbb{R}^d$, which also holds on the set Ω'. Consequently, we obtain that

$$
\begin{aligned}
&\widehat{G}(t,x) - \widehat{G}(t-h,x) \\
&= L(x, \widehat{\zeta}_2(t-h,x))(B_t - B_{t-h}) + h^{2\alpha} R_\alpha(t-h,t,x) \\
&= (D_z G)(x, \zeta_2(t-h,x)) G_2(x, \zeta_3(t-h,x))(B_t - B_{t-h}) \\
&\quad + h^{2\alpha} R_\alpha(t-h,t,x).
\end{aligned}
\tag{50}
$$

In particular, by virtue of the Hölder continuity of the Brownian motion (35) and the assumption (T3) we see from (50) that

$$G(x, \zeta_2(t, x)) - G(x, \zeta_2(t-h, x)) = h^\alpha R_\alpha(t-h, t, x). \qquad (51)$$

Similarly, we can also derive the following (recall (49)):

$$\begin{cases} F(x, \zeta_2(t, x)) - F(x, \zeta_2(t-h, x)) = h^\alpha R_\alpha(t-h, t, x); \\ L(x, \widehat{\zeta}_2(t, x)) - L(x, \widehat{\zeta}_2(t-h, x)) = h^\alpha R_\alpha(t-h, t, x), \end{cases} \qquad (52)$$

for all $0 \leq t - h \leq t \leq T$, $|x| \leq m$, on the set Ω'. Now combining (49)—(52), we obtain that, possibly on a different set $\widetilde{\Omega}$, with $P(\widetilde{\Omega}) = 1$,

$$\begin{aligned} & G(x, \zeta_2(t-h, x))(B_t - B_{t-h}) \\ = & G(x, \zeta_2(t, x))(B_t - B_{t-h}) - D_z G(x, \zeta_2(t-h, x))G_2(x, \zeta_3(t-h, x))(B_t - B_{t-h})^2 \\ & + h^{3\alpha} R_\alpha(t-h, t, x) \\ = & G(x, \zeta_2(t, x))(B_t - B_{t-h}) - D_z G(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x))(B_t - B_{t-h})^2 \\ & + h^{3\alpha} R_\alpha(t-h, t, x). \end{aligned}$$

Moreover, rewriting (52) as $F(x, \zeta_2(t-h, x))h = F(x, \zeta_2(t, x))h + h^{1+\alpha} R_\alpha(t-h, t, x)$, and noting from (52) (recall definition (49)) that

$$\begin{aligned} & (D_z G)(x, \zeta_2(t-h, x))G_2(x, \zeta_3(t-h, x))(B_t - B_{t-h})^2 \\ = & (D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x))(B_t - B_{t-h})^2 + h^{3\alpha} R_\alpha(t-h, t, x), \end{aligned}$$

and that

$$\begin{aligned} & (D_z G)(x, \zeta_2(t-h, x))G_2(x, \zeta_3(t-h, x))h \\ = & (D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x))h + h^{1+\alpha} R_\alpha(t-h, t, x), \end{aligned}$$

we obtain from (47) that

$$\begin{aligned} & \zeta(t, x) - \zeta(t-h, x) \\ = & \left\{ F(x, \zeta_2(t, x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x)) \right\}h \\ & + G(x, \zeta_2(t, x))(B_t - B_{t-h}) - \frac{1}{2}(D_z G)(x, \zeta_2(t, x))G_2(x, \zeta_3(t, x))(B_t - B_{t-h})^2 \\ & + h^{3\alpha} R_\alpha(t-h, t, x). \end{aligned}$$

Finally, since $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ is arbitrary, the proposition follows. $\qquad\square$

5.2. Backward temporal-spatial expansion.
We now give the complete statement of the backward temporal-spatial expansion.

Proposition 5.2. *Assume that* $\zeta \in C^{0,(3)}(\mathbf{F}^B, [0, T], \mathbb{R}^d)$ *satisfying (T1)-(T3). Then, for all* $m \in \mathbb{N}$ *and* $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ *there exists some subset* $\widetilde{\Omega} \subset \Omega$ *such that* $P(\widetilde{\Omega}) = 1$, *and that on* $\widetilde{\Omega}$, *for all* $0 \leq t - h < t \leq T$, $x, k \in \mathbb{R}^d$,

$$\begin{aligned} & \zeta(t-h, x+k) - \zeta(t, x) \\ = & ah + b(B_{t-h} - B_t) + \frac{c}{2}(B_{t-h} - B_t)^2 + \langle p, k \rangle + \frac{1}{2} \langle Xk, k \rangle \qquad (53) \\ & + \langle q, k \rangle(B_{t-h} - B_t) + (h + |k|^2)^{3\alpha} R_\alpha(t-h, t, x, x+k), \end{aligned}$$

where

$$
\begin{aligned}
a &= -\left\{ F(x, \zeta_2(t,x)) - \frac{1}{2}(D_z G)(x, \zeta_2(t,x)) G_2(x, \zeta_3(t,x)) \right\}; \\
b &= G(x, \zeta_2(t,x)); \\
c &= (D_z G)(x, \zeta_2(t,x)) G_2(x, \zeta_3(t,x)); \\
p &= (D_x \zeta)(t,x); \\
X &= D_x^2 \zeta(t,x); \\
q &= (D_x G)(x, \zeta_2(t,x)) + (D_z G)(x, \zeta_2(t,x)) D_x \zeta_2(t,x).
\end{aligned}
\tag{54}
$$

Proof. As in the forward expansion case, we need only show that for all $m \in \mathbb{N}$ and $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ there exists some subset $\widetilde{\Omega} \subset \Omega$ of full probability such that on $\widetilde{\Omega}$, for all $0 \le t - h < t \le T$, $x, k \in \mathbb{R}^d$,

$$
\begin{aligned}
&\zeta(t-h, x+k) - \zeta(t-h, x) \\
={}& D_x \zeta(t,x) k + \frac{1}{2} \langle (D_x^2 \zeta(t,x) k, k \rangle \\
&+ \{ (D_x G)(x, \zeta_2(t,x)) + (D_z G)(x, \zeta_2(t,x)) D_x \zeta_2(t,x) \} k (B_{t-h} - B_t) \\
&+ (h + |k|^2)^{3\alpha} R_\alpha(t-h, t, x, x+k).
\end{aligned}
\tag{55}
$$

From the usual Taylor expansion with the remainder in the Lagrange form, we have

$$
\begin{aligned}
&\zeta(t-h, x+k) - \zeta(t-h, x) \\
={}& D_x \zeta(t-h, x) k + \frac{1}{2} \langle (D_x^2 \zeta(t-h, x) k, k \rangle + |k|^3 R_\alpha(t-h, t, x, x+k),
\end{aligned}
$$

for $0 \le t - h < t \le T$, $x, k \in \mathbb{R}^d$, P-a.s. Moreover, from Proposition 5.1 it follows that

$$
\begin{aligned}
&D_x \zeta(t,x) - D_x \zeta(t-h, x) \\
={}& D_x [G(x, \zeta_2(t,x))](B_t - B_{t-h}) + h^{2\alpha} R_\alpha(t-h, t, x) \\
={}& \{ (D_x G)(x, \zeta_2(t,x)) + (D_z G)(x, \zeta_2(t,x)) D_x \zeta_2(t,x) \}(B_t - B_{t-h}) \\
&+ h^{2\alpha} R_\alpha(t-h, t, x)
\end{aligned}
$$

and

$$
D_x^2 \zeta(t,x) - D_x^2 \zeta(t-h, x) = h^\alpha R_\alpha(t-h, t, x).
$$

Consequently,

$$
\begin{aligned}
&\zeta(t-h, x+k) - \zeta(t-h, x) \\
={}& D_x \zeta(t,x) k + \frac{1}{2} \langle (D_x^2 \zeta(t,x) k, k \rangle \\
&+ \{ (D_x G)(x, \zeta_2(t,x)) + (D_z G)(x, \zeta_2(t,x)) D_x \zeta_2(t,x) \} k (B_{t-h} - B_t) \\
&+ |k|^{6\alpha} R_\alpha(t-h, t-h, x, k) + h^{2\alpha} k R_\alpha(t-h, t, x) + h^\alpha \langle R_\alpha(t-h, t, x) k, k \rangle.
\end{aligned}
$$

Thus, by virtue of

$$
|k|^{6\alpha} + h^{2\alpha}|k| + h^\alpha |k|^2 \le C_m (h + |k|^2)^{3\alpha},
$$

for $h \in [0, T]$ and $|k| \le m$, we derive (55). This, combined with (48), leads us to the backward temporal-spatial expansion (53). □

Finally, by combining Proposition 4.2 and Proposition 5.2, we have completed the proof of Theorem 2.3.

6. **Application to stochastic PDEs.** Having tried so hard to develop the various forms of stochastic Taylor expansion, as an application in this section we shall try to use it to study the *stochastic viscosity solutions* for fully nonlinear SPDE, following the idea that we developed in our earlier work [3]. In order not to over-complicate the computation we shall consider the following simpler version of the fully nonlinear SPDE (5): for $(t, x) \in [0, \infty) \times \mathbb{R}^d$,

$$u(t, x) = u_0(x) + \int_0^t f(x, u, Du, D^2u)ds + \int_0^t g(x, Du) \circ dB_s. \tag{56}$$

Compared to SPDE (5), as well as Example 2.5, we see that here the diffusion coefficient g in (56) is independent of u. The general case could be treated in a similar way but with more complicated expressions. Since our purpose here is to outline our idea of a new definition of stochastic viscosity solution, without adding too much technical complexity into this already lengthy paper, we shall leave the study of the general case to a forthcoming paper. We should also note that in (56) we are using the Stratonovich integral instead of the Itô integral for the simplicity of the presentation, the following relation is worth noting:

$$g(x, Du) \circ dB_t = g(x, Du)dB_t + \frac{1}{2}D_zg(x, Du)D_x[g(s, Du)]dt. \tag{57}$$

It is worth pointing out that even in this simplified form, the nonlinearity of the function g on Du already makes it difficult to apply the rough path approach of [6] directly here.

To explain our idea of the definition of stochastic viscosity solution, let us first apply Theorem 2.3 to the regular solution u. Bearing the relation (57) in mind we have:

$$u(t + h, x + k) - u(t, x)$$
$$= ah + b(B_{t+h} - B_t) + \frac{c}{2}(B_{t+h} - B_t)^2 + \langle p, k \rangle + \frac{1}{2}\langle Xk, k \rangle \tag{58}$$
$$+ \langle q, k \rangle(B_{t+h} - B_t) + (|h| + |k|^2)^{3\alpha} R_{\alpha,m}(t, t + h, x, k),$$

where

$$
\begin{aligned}
a &= f(x, (u, Du, Du^2)(t, x)), \\
b &= g(x, Du(t, x)) \\
c &= \langle D_zg(x, Du(t, x)), D_x[g(x, Du(t, x))] \rangle, \\
p &= Du(t, x), \qquad X = D^2u(t, x), \\
q &= D_x[g((x, Du)(t, x)] = D_xg(x, Du) + D^2u(t, x)D_zg(x, Du(t, x)).
\end{aligned}
\tag{59}
$$

It is interesting to note that in this simple case the terms involving D^2u can be written collectively as (suppressing variables):

$$f(\cdots, D^2u) + \langle [D^2u]D_zg(x, Du)(B_{t+h} - B_t), k \rangle$$
$$+ \frac{1}{2}\langle [D^2u]D_zg(x, Du), D_zg(x, Du) \rangle)(B_{t+h} - B_t)^2 + \frac{1}{2}\langle [D^2u]k, k \rangle$$
$$= f(\cdots, D^2u)$$
$$+ \frac{1}{2}\langle [D^2u](D_zg(x, Du)(B_{t+h} - B_t) + k), D_zg(x, Du)(B_{t+h} - B_t) + k \rangle.$$

Compared to the classical deterministic Taylor expansion, and the SPDEs studied in [3] in which g is independent of Du, we can see that in a general case the terms

involving D^2u become much more complicated, and our previous method (via Doss-Sussmann) will face a fundamental challenge, especially in the uniqueness proof. We therefore will try to find a different approach to define the stochastic viscosity solutions, using the stochastic characteristics introduced by Kunita [10], combined with our results on stochastic Taylor expansions. This new method also reflects the basic ideas of the works of Lions and Souganidis [11], [12], [14], and [15], and in a sense includes our previous work [3] as special case. For simplicity we shall now assume all processes involved are real valued.

To motivate our definition of the stochastic viscosity solution let us suppose in a first step that the coefficients of SPDE (4) are sufficiently smooth and that this equation admits a regular solution $u \in C^{0,\infty}(\mathbf{F}^B, [0, T] \times \mathbb{R})$. Under this assumption, we compare the solution u with a smooth test field $\varphi \in C^{0,\infty}(\mathbf{F}^B, [0, T] \times \mathbb{R})$ defined as the unique solution of the equation

$$
\begin{aligned}
d\varphi(t, x) &= \theta(t, x)dt + g(x, D\varphi(t, x)) \circ dB_t, \\
\varphi(0, x) &= \varphi_0(x),
\end{aligned}
\tag{60}
$$

for $t \in [0, T]$, $x \in \mathbb{R}^d$, where $\theta \in C^\infty_{\ell,b}([0, T] \times \mathbb{R})$, $g \in C^\infty_{\ell,b}(\mathbb{R} \times \mathbb{R}; \mathbb{R})$, and $\varphi_0 \in C^\infty_p(\mathbb{R})$.

Let us now fix an $\mathcal{F}^B$-measurable $[0, T]$-valued random variable τ and an $\mathcal{F}^B$-measurable $\mathbb{R}^d$-valued random variable ξ. We say that $u - \varphi$ achieves a *local left-maximum* in (τ, ξ) if for almost all $\omega \in \{\tau < T\}$ there is some $\rho > 0$ (which may depend on ω) such that

$$
(u - \varphi)(\omega, t, x) \le (u - \varphi)(\omega, \tau(\omega), \xi(\omega)),
$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ with $t \in ((\tau(\omega) - \rho)^+, \tau(\omega)]$ and $|x - \xi(\omega)| \le \rho$.

Following the approach by Kunita (see Theorem 6.1.2 in [10]), we formally introduce the following stochastic characteristics. For the moment let us assume that the random time τ is actually deterministic to avoid further complications (keep in mind, however, that the Taylor expansion will hold even for the arbitrary random time τ!).

$$
\begin{aligned}
\phi_t(x, z) &= x - \int_\tau^t D_z g((\phi_s, \chi_s)(x, z)) \circ dB_s, \\
\eta_t(x, y, z) &= y + \int_\tau^t \{g((\phi_s, \chi_s)(x, z)) - \chi_s(x, z) D_z g((\phi_s, \chi_s)(x, z))\} \circ dB_s, \\
\chi_t(x, z) &= z + \int_\tau^t D_x g((\phi_s, \chi_s)(x, z)) \circ dB_s,
\end{aligned}
\tag{61}
$$

where $t \in [0, \tau]$, $(x, y, z) \in \mathbb{R}^3$, and we hope to be able to define a transformation $\psi(t, x)$ by

$$
\varphi(t, \phi_t(x, D\psi(t, x))) = \eta_t(x, \psi(t, x), D\psi(t, x)).
\tag{62}
$$

For notational simplicity let us now denote:

$$
l(x, z) = -D_z g(x, z); \quad h(x, z) = g(x, z) - z D_z g(x, z); \quad k(x, z) = D_x g(x, z).
$$

Remark 6.1. We note that if the function g is linear in Du, then the situation will become much simpler, and the arguments below would become straightforward. To be more precise, let us consider the following two cases:

(i) $g(x, Du) = \langle H, Du \rangle$, where H is a constant vector. Since $D_x g = D_x V(x)$ and $D_z g = H$, the Taylor expansion (59) is drastically simplified. Also, the characteristics (61) becomes almost trivial: $\chi \equiv z$, $\eta \equiv y$, and $\phi \equiv x - H(t - \tau)$.

(ii) $g(x, Du) = V(x)Du$ (see [6]). In this case one has $D_x g(x, z) = D_x V(x)z$, $D_z g(x, z) = V(x)$, thus the Taylor expansion (59) will also become much simpler. Furthermore, the characteristics (61) now become "disentangled" SDEs:

$$
\begin{aligned}
\phi_t(x) &= x - \int_\tau^t V(\phi_s(x)) \circ dB_s, \\
\eta_t(x, y, z) &\equiv y, \\
\chi_t(x, z) &= z + \int_\tau^t D_x V(\phi_s(x)) \chi_s(x, z) \circ dB_s,
\end{aligned}
$$

and (62) takes the special form: $\psi(t, x) = \varphi(t, \phi_t(x))$. In other words, the transformation ψ is globally well-defined by an easy and explicit expression. $\square$

In what follows we shall denote $\Theta \stackrel{\triangle}{=} (l, h, k)$. Also, for any function $\gamma = \gamma(x, y, z)$ we denote $\nabla \gamma = (D_x \gamma, D_y \gamma, D_z \gamma)^T$. Then we can write, for example, the first equation in (61) in the Itô integral from:

$$
\begin{aligned}
\phi_t(x, z) &= x + \int_\tau^t l((\phi_s, \chi_s)(x, z)) \circ dB_s \tag{63} \\
&= x + \int_\tau^t \frac{1}{2} \langle \nabla l, \Theta \rangle ((\phi_s, \chi_s)(x, z)) ds + \int_\tau^t l((\phi_s, \chi_s)(x, z)) dB_s.
\end{aligned}
$$

Now, treating the random fields η and χ in (61) the same way, and applying the stochastic (backward temporal) Taylor expansion to (ϕ, η, χ) we have, for all $\alpha \in (\frac{1}{3}, \frac{1}{2})$,

$$
\begin{cases}
\phi_t(x, z) = x + l(x, z)(B_t - B_\tau) + \dfrac{1}{2} \langle \nabla l, \Theta \rangle (B_t - B_\tau)^2 + |t - \tau|^{3\alpha} R_{\alpha, m}, \\
\eta_t(x, y, z) = y + h(x, z)(B_t - B_\tau) + \dfrac{1}{2} \langle \nabla h, \Theta \rangle (B_t - B_\tau)^2 + |t - \tau|^{3\alpha} R_{\alpha, m}, \quad (64) \\
\chi_t(x, z) = z + k(x, z)(B_t - B_\tau) + \dfrac{1}{2} \langle \nabla k, \Theta \rangle (B_t - B_\tau)^2 + |t - \tau|^{3\alpha} R_{\alpha, m}.
\end{cases}
$$

On the other hand, writing (60) in an Itô integral form we have

$$
\begin{aligned}
\varphi(t, x) &= \varphi_0(x) + \int_\tau^t \{\theta(s, x) + \frac{1}{2} D_z g(x, D\varphi(s, x)) D_x[g(x, D\varphi(s, x))]\} ds \\
&\quad + \int_\tau^t g(x, D\varphi(s, x)) dB_s \tag{65} \\
&\stackrel{\triangle}{=} \varphi_0(x) + \int_\tau^t F(s, x) ds + \int_\tau^t G(x, \zeta_2(s, x)) dB_s,
\end{aligned}
$$

where F and G are defined in an obvious way, and with

$$
\zeta_1(t, x) \stackrel{\triangle}{=} \varphi(t, x), \quad \zeta_2(t, x) \stackrel{\triangle}{=} D\varphi(t, x), \quad \zeta_3(t, x) = (D\varphi, D^2\varphi)(t, x).
$$

Note that differentiating the both sides of (65) we have

$$
dD\varphi(t, x) = D_x F(t, x) dt + \{D_x g(x, D\varphi(t, x)) + D_z g(x, D\varphi(t, x)) D_x^2 \varphi(t, x))\} dB_t.
$$

Applying the backward temporal Taylor expansion (Theorem 2.3) again to the random field φ around any point (τ, ς), with

$$
F_2(t, x) \stackrel{\triangle}{=} D_x F(t, x); \qquad G_2(x, z_1, z_2) \stackrel{\triangle}{=} D_x g(x, z_1) + D_z g(x, z_1) z_2,
$$

we obtain, after some simple cancelations, that

$$
\begin{aligned}
\varphi(t,x) = {}& \varphi(\tau,\varsigma) + \theta(\tau,\varsigma)(t-\tau) + g(\varsigma, D\varphi(\tau,\varsigma))(B_t - B_\tau) \\
& + \frac{1}{2}\Big\{ [D_z g D_x g](\varsigma, D\varphi(\tau,\varsigma)) + (D_z g(\varsigma, D\varphi(\tau,\varsigma)))^2 D_x^2 \varphi(\tau,\varsigma) \Big\}(B_t - B_\tau)^2 \\
& + D\varphi(\tau,\varsigma)(x-\varsigma) + D_x[g(\cdot, D\varphi(\cdot,\cdot))](\tau,\varsigma)(x-\varsigma)(B_t - B_\tau) \qquad (66) \\
& + \frac{1}{2}D_x^2 \varphi(\tau,\varsigma)(x-\varsigma)^2 + (|t-\tau| + |x-\varsigma|^2)^{3\alpha} R_{\alpha,m}.
\end{aligned}
$$

We note that the above holds for all (τ,ς) and all $\omega \in \widetilde{\Omega}_{\alpha,m}$.

Let us now define the desired random field ψ. We should note that our main purpose here is to find such a transformation so as to eliminate the stochastic integral. In other words, we shall look for such ψ that has the following first order Taylor expansion:

$$
\begin{cases}
\psi(t,x) = \varphi(\tau,x) + \partial_t^- \psi(\tau,x)(t-\tau) + |t-\tau|^{3\alpha} R_{\alpha,m}; \\
D\psi(t,x) = D\varphi(\tau,x) + \partial_t^- D\psi(\tau,x)(t-\tau) + |t-\tau|^{3\alpha} R_{\alpha,m}.
\end{cases} \qquad (67)
$$

Here ∂_t^- denotes the left partial derivative with respect to t. To this end, we note that (61) and (62) imply that for $t = \tau$, one has

$$
\varphi(\tau,x) = \varphi(\tau, \phi_\tau(x, D\psi(\tau,x))) = \eta_\tau(x, \psi(\tau,x), D\psi(\tau,x)) = \psi(\tau,x), \ \ x \in \mathbb{R},
$$

and hence $D\varphi(\tau,x) = D\psi(\tau,x)$ holds for all $x \in \mathbb{R}$ as well. Next, we look at the Taylor expansion for both ϕ and η. Recalling (63) and the Taylor expansions (64) we shall first take $\mathbf{x} = (x, \psi(t,x), D\psi(t,x)))$, and then replace x by $\phi_t(x, D\psi(t,x))$ and ς by x in (66). It should be noted that after these substitutions the remainder will look like $R_{\alpha,m}(t,\tau,x,\phi_t(x, D\psi(t,x)))$, a slightly more complicated than the original ones. To make sure the accuracy of the expansion in what follows we shall denote

$$
\overline{R}_{\alpha,m,loc} = \sup_{t,s \in [0,T]; \ x \in \overline{B}_m(0)} |R_{\alpha,m}(t,s,x,\phi_t(x, D\psi(t,x)))|.
$$

Then it is not hard to show that there exists an increasing sequence $\{\Omega_\ell, \ell \geq 1\} \subset \mathcal{F}^B$ with $\lim_{\ell \to +\infty} P(\Omega_\ell) = 1$, such that,

$$
\overline{R}_{\alpha,m,loc} I_{\Omega_\ell} \in L_-^\infty(\Omega, \mathcal{F}, P), \quad \text{for all } \ell \geq 1.
$$

Keeping such a modification in mind, we can now proceed to write down the Taylor expansion:

$$
\begin{aligned}
&\varphi(t, \phi_t(x, D\psi(t, x))) \\
=\ &\varphi(\tau, x) + \theta(\tau, x)(t - \tau) + g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
&+\frac{1}{2}\Big\{[D_z g D_x g](x, D\varphi(\tau, x)) + (D_z g(x, D\varphi(\tau, x))^2 D_x^2 \varphi(\tau, x)\Big\}(B_t - B_\tau)^2 \\
&-D\varphi(\tau, x) D_z g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
&-D\varphi(\tau, x) D_z^2 g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
&+\frac{1}{2} D\varphi(\tau, x)\langle \nabla l, \Theta \rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 \\
&-\frac{1}{2} D_x^2 \varphi(\tau, x)(D_z g(x, D\varphi(\tau, x)))^2 (B_t - B_\tau)^2 \\
&-D_x g(x, D\varphi(\tau, x)) D_z g(x, D\varphi(\tau, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha,m,loc} \\
=\ &\varphi(\tau, x) + \theta(\tau, x)(t - \tau) + g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
&-\frac{1}{2}[D_z g D_x g](x, D\varphi(\tau, x))(B_t - B_\tau)^2 - D\varphi(\tau, x) D_z g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
&+\frac{1}{2} D\varphi(\tau, x)\langle \nabla l, \Theta \rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 \\
&-D\varphi(\tau, x) D_z^2 g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) + |t - \tau|^{3\alpha}\overline{R}_{\alpha,m,loc}.
\end{aligned}
\tag{68}
$$

On the other hand, from (61) we see that

$$
\begin{aligned}
\psi(t, x) =\ &\eta_t(x, \psi(t, x), D\psi(t, x)) - h(x, D\psi(t, x))(B_t - B_\tau) \\
&-\frac{1}{2}\langle \nabla h, \Theta \rangle(x, D\psi(t, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha,m,loc}.
\end{aligned}
$$

Using the form (67) and the smoothness assumptions on all the coefficients, and then noting (68) one can easily rewrite above as

$$
\begin{aligned}
\psi(t, x) =\ &\eta_t(x, \psi(t, x), D\psi(t, x)) - h(x, D\varphi(\tau, x))(B_t - B_\tau) \\
&-D_z h(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
&-\frac{1}{2}\langle \nabla h, \Theta \rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha,m,loc}.
\end{aligned}
$$

Now recalling the relation (62), the fact (68), and the definition of h preceding Remark 6.1, we obtain

$$
\begin{aligned}
&\psi(t, x) \\
=\ & \eta_t(x, \psi(t, x), D\psi(t, x)) - h(x, D\varphi(\tau, x))(B_t - B_\tau) \\
& -\frac{1}{2}\langle\, \nabla h, \Theta\,\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}R_{\alpha, m} \\
=\ & \varphi(\tau, x) + \theta(\tau, x)(t - \tau) + g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
& -\frac{1}{2}[D_z g D_x g](x, D\varphi(\tau, x))(B_t - B_\tau)^2 - D\varphi(\tau, x)D_z g(x, D\varphi(\tau, x))(B_t - B_\tau) \\
& +\frac{1}{2}D\varphi(\tau, x)\langle\, \nabla l, \Theta\,\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 \\
& -D\varphi(\tau, x)D_x^2 g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
& -[g(x, D\varphi(\tau, x)) - D\varphi(\tau, x)D_z g(x, D\varphi(\tau, x))](B_t - B_\tau) \\
& -D_z h(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
& -\frac{1}{2}\langle\, \nabla h, \Theta\,\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha, m, loc} \\
=\ & \varphi(\tau, x) + \theta(\tau, x)(t - \tau) - \frac{1}{2}[D_z g D_x g](x, D\varphi(\tau, x))(B_t - B_\tau)^2 \\
& +\frac{1}{2}D\varphi(\tau, x)\langle\, \nabla l, \Theta\,\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 \\
& -D\varphi(\tau, x)D_x^2 g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
& -D_z h(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) \\
& -\frac{1}{2}\langle\, \nabla h, \Theta\,\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha, m, loc}.
\end{aligned}
\tag{69}
$$

Since $\nabla l = (-D_{xz}g, 0, -D_z^2 g)$ and $\nabla h = (D_x g - z D_{xz}g, 0, -z D_z^2 g)$, one has

$$
-\frac{1}{2}[D_z g D_x g](x, D\varphi(\tau, x)) + \frac{1}{2}D\varphi(\tau, x)\langle\, \nabla l, \Theta\,\rangle(x, D\varphi(\tau, x))
$$
$$
-\frac{1}{2}\langle\, \nabla h, \Theta\,\rangle(x, D\varphi(\tau, x)) = 0,
$$

and

$$
-D\varphi(\tau, x)D_z^2 g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x))(t - \tau)(B_t - B_\tau)
$$
$$
-D_z h(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x)(t - \tau)(B_t - B_\tau) = 0.
$$

In other words, (69) leads to that

$$
\psi(t, x) = \varphi(\tau, x) + \theta(\tau, x)(t - \tau) + |t - \tau|^{3\alpha}\overline{R}_{\alpha, m, loc}.
\tag{70}
$$

Clearly, (70) indicates that

$$
\partial_t^- \psi(\tau, x) = \theta(\tau, x), \qquad \partial_t^- D\psi(\tau, x) = D\theta(\tau, x).
$$

We can now define the notion of stochastic viscosity solution for (4).

Definition 6.2. *A random field $u \in C([0, T] \times \mathbb{R}^d)$ is called a stochastic viscosity subsolution (resp. supersolution) of (4), if $u(0, x) \leq u_0(x)$ $(u(0, x) \geq u_0(x))$ for all $x \in \mathbb{R}^d$ and if for any $\tau \in L^0(\mathcal{F}^B, [0, T])$, $\xi \in L^0(\mathcal{F}^B, \mathbb{R}^d)$, and any (not necessarily adapted) random field $\varphi \in C^{0,2}([0, T] \times \mathbb{R}^d)$ having the expansion (66), it holds that*

$$
\theta(\tau, \xi) \quad \leq (\geq) \quad f(\xi, u(\tau, \xi), D\varphi(\tau, \xi), D_x^2 \varphi(\tau, \xi)).
$$

P-a.s. on the subset of Ω on which $u-\varphi \leq (\geq)(u-\varphi)(\tau,\xi) = 0$ at a left neighborhood of (τ,ξ). A random field $u \in C([0,T] \times \mathbb{R}^d)$ is called a stochastic viscosity solution of (4), if it is both a stochastic viscosity subsolution and a supersolution.

Remark 6.3. We would like to note that in Definition 6.2 the (possibly anticipating) test functions φ can be directly defined by the expansion (66) with $\varphi(\tau,\xi)$ being replaced by $u(\tau,\xi)$ and $(\theta(\tau,\xi), D_x\varphi(\tau,\xi), D_x^2\varphi(\tau,\xi))$ being replaced by a triplet of random variables (β, p, A). The main advantage here is that the random field φ is now defined *globally*, overcoming the essential difficulties in the theory of stochastic viscosity solution thus far, and will significantly facilitate the uniqueness proof. Moreover, on the subset of Ω where $u - \varphi$ achieves the local left-maximum (resp., left-minimum) at the point (τ,ξ), the triplet (β, p, A) can be considered as a stochastic sub- (resp.super-)jet, as it was traditionally done. These issues will be further explored in our forthcoming publications. $\qquad\qquad\square$

In the rest of this section we shall verify that a regular solution must be a stochastic viscosity solution in the sense of Definition 6.2, which will provide a justification for our new definition. To this end, let us assume that the coefficient f is *proper* in the following sense

(H1) *The function $F(t,u,p,X) \overset{\triangle}{=} -f(x,u,p,X)$ is "degenerate elliptic". That is, f is continuous in all variables, and is non-decreasing in the variable X.*

Assume that u is a regular solution to (4), then we have

$$u(t,x) = u_0(x) + \int_0^t F(s,x)ds + \int_0^t g(x, Du(s,x))dB_s, \qquad t \geq 0, \qquad (71)$$

where

$$F(t,x) = f(x, (u, Du, D^2u)(t,x)) + \frac{1}{2}D_z g(x, Du(t,x))D_x\left[g(\cdot, Du(t,\cdot))\right](t,x).$$

For any given pair of random variables (τ,ξ) and an arbitrary test field φ such that $u - \varphi$ attains a local left-maximum at (τ,ξ) on a subset of Ω with positive probability, let ψ be the process associated to φ by (70) (where the remainder can be chosen to be zero!), and ϕ be defined by (64).

We first apply the Taylor expansion on u at point (τ,x), and evaluated at $(t,\phi_t(x, D\psi(t,x)))$ to get

$$
\begin{aligned}
&u(t, \phi_t(x, D\psi(t,x))) \\
=\;& u(\tau,x) + f(x, (u, Du, D^2u)(\tau,x))(t-\tau) + g(x, Du(\tau,x))(B_t - B_\tau) \\
&+ \frac{1}{2}\Big\{ D_z g(x, Du(\tau,x))D_x g(x, Du(\tau,x)) \\
&\qquad + D_z g(x, Du(\tau,x))^2 D_{xx}^2 u(t,x) \Big\}(B_t - B_\tau)^2 \\
&+ D_x u(\tau,x)(\phi_t(x, D\psi(t,x)) - x) + \frac{1}{2}D_{xx}^2 u(\tau,x)(\phi_t(x, D\psi(t,x)) - x)^2 \\
&+ D_x[g(x, Du(\tau,x))](B_t - B_\tau)(\phi_t(x, D\psi(t,x)) - x) \qquad\qquad (72) \\
&+ |t-\tau|^{3\alpha} R_{\alpha,m}(\tau, t, x, \phi_t(x, D\psi(t,x))).
\end{aligned}
$$

Next, we use the expansion of $\phi_t(t, D\psi(t, x))$ (recall (64)) so that the right hand side of (72) becomes

$$u(\tau, x) + f(x, (u, Du, D^2u)(\tau, x)(t - \tau) + g(x, Du(\tau, x))(B_t - B_\tau)$$

$$+\frac{1}{2}\Big\{[D_zgD_xg](x, Du(\tau, x)) + (D_zg(x, Du(\tau, x))^2D_x^2u(\tau, x)\Big\}(B_t - B_\tau)^2 \quad (73)$$

$$-Du(\tau, x)D_zg(x, D\varphi(\tau, x))(B_t - B_\tau)$$

$$-Du(\tau, x)D_z^2g(x, D\varphi(\tau, x))\partial_t^- D\psi(\tau, x))(t - \tau)(B_t - B_\tau)$$

$$+\frac{1}{2}Du(\tau, x)\langle\nabla f, \Theta\rangle(x, D\varphi(\tau, x))(B_t - B_\tau)^2$$

$$+\frac{1}{2}D_x^2u(\tau, x)(D_zg(x, D\varphi(\tau, x)))^2(B_t - B_\tau)^2$$

$$-(D_xg)(x, Du(\tau, x))D_zg(x, D\varphi(\tau, x))(B_t - B_\tau)^2$$

$$-(D_zg)(x, Du(\tau, x))D_zg(x, D\varphi(\tau, x))D_x^2u(\tau, x)(B_t - B_\tau)^2 + |t - \tau|^{3\alpha}\overline{R}_{\alpha, m, loc}.$$

We will now replace x by the random point ξ. To specify the "left-neighbood" required for the viscosity property, we also define for any $\rho > 0$ the following subset of Ω:

$$\Gamma_{\tau, \xi}^{\varphi, \rho} \triangleq \{\omega : (u - \varphi)(\omega, t, x) \le (u - \varphi)(\omega, \tau(\omega), \xi(\omega)), \ (t - \tau(\omega))^- < \rho, \ x \in B_\rho(\xi(\omega))\},$$

To wit, $\Gamma_{\tau, \xi}^{\varphi, \rho}$ is the subset of Ω on which $u - \varphi$ attains a *left local maximum* at (τ, ξ). Now setting $x = \xi$ in (72), and noting that $Du(\tau, \xi) = D\varphi(\tau, \xi)$ and $u(\tau, \xi) = \varphi(\tau, \xi)$, we have

$$u(t, \phi_t(\xi, D\psi(t, \xi)))$$
$$= \ u(\tau, \xi) + f(\xi, (u, Du, D^2u)(\tau, \xi))(t - \tau) + g(\xi, Du(\tau, \xi))(B_t - B_\tau)$$
$$-\frac{1}{2}[D_zgD_xg](\xi, Du(\tau, \xi))(B_t - B_\tau)^2 - Du(\tau, x)D_zg(\xi, Du(\tau, \xi))(B_t - B_\tau)$$
$$+\frac{1}{2}Du(\tau, \xi)\langle\nabla f, \Theta\rangle(x, Du(\tau, \xi))(B_t - B_\tau)^2$$
$$-Du(\tau, \xi)D_z^2g(\xi, Du(\tau, \xi))\partial_t^- D\psi(\tau, \xi))(t - \tau)(B_t - B_\tau) + |t - \tau|^{3\alpha}R_{\alpha, m, loc},$$

almost surely on $\Gamma_{\tau, \xi}^{\varphi, \rho}$. Consequently, from (68) we obtain that, P-a.s. on $\Gamma_{\tau, \xi}^{\varphi}$,

$$0 \ \ge \ \big(u(t, \phi_t(\xi, D\psi(t, \xi))) - u(\tau, \xi)\big) - \big(\varphi(t, \phi_t(\xi, D\psi(t, \xi))) - \varphi(\tau, \xi)\big)$$
$$= \ \{f(\xi, (u, Du, D^2u)(\tau, \xi)) - \theta(\tau, \xi)\}(t - \tau) + |t - \tau|^{3\alpha}\overline{R}_{\alpha, m, loc}.$$

Since $t \le \tau$ on $\Gamma_{\tau, \xi}^{\varphi}$, we deduce that

$$f(\xi, (u, Du, D^2u)(\tau, \xi)) - \theta(\tau, \xi) \ge 0, \quad P\text{-a.s. on } \Gamma_{\tau, \xi}^{\varphi}.$$

Finally, since the mapping $X \mapsto f(x, u, p, X)$ is non-decreasing, thanks to (H1), and since $D^2\varphi(\tau, \xi) \ge D^2u(\tau, \xi)$ on $\Gamma_{\tau, \xi}^{\varphi}$ we have

$$f(\xi, (\varphi, D\varphi, D^2\varphi)(\tau, \xi)) \ge \theta(\tau, \xi) \quad P\text{-a.s. on } \Gamma_{\tau, \xi}^{\varphi}.$$

This proves that the classical solution u is a stochastic viscosity subsolution. That u is also a supersolution can be proved using a similar argument. Therefore, u is a stochastic viscosity solution.

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