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Weak Solutions of Forward–Backward SDE's

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ABSTRACT

In this note we study a class of forward–backward stochastic differential equations (FBSDE for short) with functional-type terminal conditions. In the case when the time duration and the coefficients are "compatible" (e.g., the time duration is small), we prove the existence and uniqueness of the strong adapted solution in the usual sense. In the general case we introduce a notion of *weak solution* for such FBSDEs, as well as two notions of uniqueness. We prove the existence of the weak solution under mild conditions, and we prove that the Yamada–Watanabe Theorem, that is, pathwise uniqueness implies uniqueness in law, as well as the *Principle of Causality* also hold in this context.

Key Words: Backward SDE's; Backward–forward SDE's; Weak solutions.

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1. INTRODUCTION

The theory of backward and forward–backward stochastic differential equations (BSDE/FBSDE) has been studied quite extensively in the past decade and its applications have been found in many areas. We refer the reader to the books of El Karoui-Mazliak[4] and Ma-Yong[15] for an overview on this subject. In recent years, constant effort has been made to weaken the conditions on the coefficients so as to enlarge the class of solvable BSDEs/FBSDEs. For example, the existence and uniqueness of the solutions to BSDEs with non-Lipschitz coefficients was addressed, among others, by Pardoux–Peng,[19] Mao,[17] Hamadene[5] and Lepeltier-San Martin;[12] the same problem for FBSDEs can be found in the works of Hu[6,7] and Hu-Yong.[9]

Besides the regularity of the coefficients, for FBSDEs there is another difficulty, that is, the length of the time duration, often causing restrictions in solvability. It is by now well-known that even a very simple FBSDE may not be solvable over arbitrarily prescribed time duration (see Refs.[1,15] for examples). So far, the solvability of FBSDE over arbitrary duration has been discussed only in some special forms, in particular, the existing methods seem to work only when the terminal condition of the backward equation is a function of the forward component.[13,8,20] Consequently, the solvability of a FBSDE of the following type over arbitrary duration seems yet to be known, no matter how smooth the coefficients are:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T, \\ Y_t = E\left\{ g(X)_T + \int_t^T h(s, X_s, Y_s)ds \middle| \mathcal{F}_t \right\}, \end{cases} \tag{1.1}$$

where $g(X)_T$ is a functional of the whole trajectory of $X = \{X_t;\, 0 \le t \le T\}$. We would like to point out that the terminal conditions of this type have strong motivation in finance theory. Immediate examples include *lookback options, barrier options*, *Asian options*, etc., where the whole or partial history of the underlying securities has to be considered in determining the option price.

We remark here that one of the main advantages of using the special form (1.1) is that, unlike the usual framework in the BSDE/FBSDE literature, one can now consider more general probabilistic set-ups for the BSDEs. To be more precise, an FBSDEs of the form (1.1) could be solvable even without assuming that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion W, or any other Brownian motion. In fact, one of the main objectives of this paper is to explore the possibility of extending the notion of *weak solution* of SDEs to the BSDEs and FBSDEs, which does not have any restriction on the probability set-up *a priori*. We shall propose a definition of weak solution as well as two definitions of uniqueness, i.e., *pathwise uniqueness* and *uniqueness in law*, analogous to the usual ones for forward SDEs. We will also provide an example in the spirit of that of Tanaka, showing that a weak solution which is not a strong solution does exist. In the case when the diffusion coefficient of the forward equation is *decoupled* from the backward components, we show that the existence of the weak solution can be obtained under extremely mild conditions, beyond the reach of any existing results for strong solutions.

We should point out that, as a first attempt on the subject, in this paper we have not been able to solve the problem of (weak) uniqueness completely, which seems to require much deeper tools in analysis. Nevertheless, we shall provide two results that are essential in the theory of weak solutions. First we prove that the Yamada and Watanabe Theorem, that is, the pathwise uniqueness implies the uniqueness in law, still holds in this context; second, we prove that the so-called *Principle of Causality*, that is, the pathwise uniqueness and weak existence imply the existence of the strong (adapted) solution, also holds for FBSDE (1.1), provided that the filtration is "Brownian."

This paper is organized as follows. In section 2 we give the preliminaries, including a discussion of the existence and uniqueness of the adapted solution of FBSDE (1.1) with full generality, within the usual *strong solution* paradigm, and an example showing how the *functional-type* terminal condition causes the essential difference from the FBSDEs studied so far. In section 3 we give the definition of weak solutions and the Tanaka-type example. In section 4 we prove an existence theorem and, finally, in section 5 we discuss the issue of uniqueness of the weak solutions.

2. PRELIMINARIES

Throughout this paper $[0, T]$ is a finite time interval and $(\Omega, \mathcal{F}, P)$ is a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the "usual hypotheses." For any $0 < t \le T$ and $1 \le p \le \infty$, we denote by $\underline{\underline{S}}^p_{[0,t]}$ and $\underline{\underline{H}}^p_{[0,t]}$, the usual spaces of semimartingales defined on $[0, t]$, with their corresponding norms:

$$\begin{cases} \|Y\|_{\underline{\underline{S}}^p_{[0,t]}} \triangleq \|Y^*_t\|_{L^p([0,t]\times\Omega)} = [E(\sup_{0\le s\le t} |Y_s|^p)]^{1/p} \\ \|Y\|_{\underline{\underline{H}}^p_{[0,t]}} \triangleq \inf_{Y=M+A} \{\|[M,M]^{1/2}_t + \int_0^t |dA_s|\|_{L^p(\Omega)}\}, \end{cases} \tag{2.2}$$

where the "inf" is taken over all possible decompositions of the semimartingale Y (see Ref.[21, V.2]). We shall denote $\underline{\underline{S}}^p \triangleq \underline{\underline{S}}^p_{[0,T]}$ and $\underline{\underline{H}}^p \triangleq \underline{\underline{H}}^p_{[0,t]}$ for simplicity.

The following relations between the $\underline{\underline{S}}^p$ and $\underline{\underline{H}}^p$ norms are useful (see Ref.[21, V.2]). For any $1 \le p < \infty$, there exists a constant $c_p > 0$, such that for any semimartingale X with $X_0 = 0$, it holds that $\|X\|_{\underline{\underline{S}}^p} \le c_p \|X\|_{\underline{\underline{H}}^p}$.

Further, for any semimartingale X and any adapted, càglàd process H, taken any $1 \le p, q, r \le \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *Emery's inequality* holds:

$$\left\| \int_0^\infty H_s \, dX_s \right\|_{\underline{\underline{H}}^r} \le \|H\|_{\underline{\underline{S}}^p} \|X\|_{\underline{\underline{H}}^q}. \tag{2.3}$$

Let us now consider the FBSDE (1.1) in the strong solution paradigm. We note that in this case we can treat the FBSDEs with full generality as follows:

$$X_t = x + \int_0^t b(s, X_s, Y_s) dA_s + \int_0^t \sigma(s, X_s, Y_s) dM_s \tag{2.4}$$

$$Y_t = E\left\{ g(X)_T + \int_t^T h(s, X_s, Y_s) dC_s \middle| \mathcal{F}_t \right\}, \tag{2.5}$$

where M is a continuous, square integrable martingale, and A and C are two processes with finite variation paths. For notational simplicity we consider only the one-dimensional case, but our arguments can be extended to higher dimensional cases without substantial difficulties. We shall make use of the following assumptions:

(**H1**) The processes A, C, and M belong to $\underline{\underline{H}}^\infty$;

(**H2**) $b, h, \sigma : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are progressively measurable for any $x, y \in \mathbb{R}$ and there exists a constant $k_1 > 0$ such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le k_1(|x_1 - x_2| + |y_1 - y_2|),$$

P-a.s., for all $t \in [0,T]$, $x_1, y_1, x_2, y_2 \in \mathbb{R}(f = b, \sigma, h)$;
(**H3**) the mapping $g : \Omega \times C([0,T];\mathbb{R}) \to \mathbb{R}$ is $\mathcal{F}_T \times \mathcal{B}(\mathcal{C}([0,T];\mathbb{R}))$-measurable, and for some $k_2 > 0$, for all $u^1, u^2 \in \mathcal{C}([0,T];\mathbb{R})$ it holds that

$$|g(u^2)_T - g(u^1)_T| \le k_2 \sup_{0 \le t \le T} |u_t^2 - u_t^1| \quad P - \text{a.s.};$$

(**H4**) the following integrability conditions hold:

$$E\left\{\int_0^T |b(s,0,0)||dA_s|\right\}^2, \quad E\left\{\int_0^T |h(s,0,0)||dC_s|\right\}^2 < \infty,$$

$$E\left\{\int_0^T |\sigma(s,0,0)|^2 d[M,M]_s\right\} < \infty, \quad E\{|g(0)|_T^2\} < \infty.$$

We note that (H4) may be substituted by a sublinear growth condition on the coefficients. The following theorem is not surprising and since the proof is rather standard, we give only a sketch for completeness.

Theorem 2.1. *Assume* $(H1)$–$(H4)$ *and that the following "compatibility condition" holds:*

$$2k_1[(\|A\|_{\underline{\underline{H}}^\infty}^2 + 4\|M\|_{\underline{\underline{H}}^\infty}^2)(8k_2^2 + 1) + 2\|C\|_{\underline{\underline{H}}^\infty}^2]^{\frac{1}{2}} < 1. \tag{2.6}$$

Then the SDEs (2.4)–(2.5) *has a unique solution in the space* $\underline{\underline{S}}^2 \times \underline{\underline{S}}^2$.

Proof. Let $\mathcal{S} \triangleq \underline{\underline{S}}^2 \times \underline{\underline{S}}^2$. It is known that $\mathcal{S}$ is a Banach space with the norm $\|(X,Y)\|_{\underline{\underline{S}}^2 \times \underline{\underline{S}}^2} \triangleq \sqrt{\|X\|_{\underline{\underline{S}}^2}^2 + \|Y\|_{\underline{\underline{S}}^2}^2}$. For any $(X,Y) \in \mathcal{S}$, define the mapping

$$\mathcal{L}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} F(X,Y)_t \\ G(X,Y)_t \end{pmatrix}$$

$$= \begin{pmatrix} x + \int_0^t b(s,X_s,Y_s)dA_s + \int_0^t \sigma(s,X_s,Y_s)dM_s \\ E\left(g(F(X,Y))_T + \int_t^T h(s,X_s,Y_s)dC_s \,\middle|\, \mathcal{F}_t\right) \end{pmatrix}.$$

We first notice that $\mathcal{L}$ maps $\mathcal{S}$ into itself. Indeed, for any $(X, Y) \in \mathcal{S}$, it is straightforward to verify, by using (H2), Doob's and Emery's inequalities, that for any t in $[0, T]$,

$$\|F(X,Y)\|^2_{\underline{\underline{S}}^2_{[0,t]}} \le 3|x|^2 + 9k_1^2\left(\|A\|^2_{\underline{\underline{H}}^\infty_{[0,t]}} + 36\|M\|^2_{\underline{\underline{H}}^\infty_{[0,t]}}\right)\left(\|X\|^2_{\underline{\underline{S}}^2_{[0,t]}} + \|Y\|^2_{\underline{\underline{S}}^2_{[0,t]}}\right)$$
$$+E\left(9\left[\int_0^T |b(s,0,0)\|dA_s|\right]^2 + 36\int_0^T |\sigma(s,0,0)|^2 d[M,M]_s\right) < \infty.$$

Similarly, applying (H2)–(H4) and Doob's inequality we obtain

$$\|G(X,Y)\|^2_{\underline{\underline{S}}^2} \le 8\left\{\|g(0)\|_2^2 + E\left(\left[\int_0^T |h(s,0,0)\|dC_s|\right]^2\right)\right.$$
$$\left. + k_1^2\|C\|^2_{\underline{\underline{H}}^\infty}(\|X\|^2_{\underline{\underline{S}}^2} + \|Y\|^2_{\underline{\underline{S}}^2}) + k_2^2\|F(X,Y)\|^2_{\underline{\underline{S}}^2}\right\} < \infty.$$

Next, we show that the operator $\mathcal{L}$ is a contraction on $\mathcal{S}$ under assumption (2.6). Indeed, given $(X^1, Y^1), (X^2, Y^2) \in \underline{\underline{S}}^2$, using arguments similar to those before and assumption (H3), one shows that

$$\|F(X^2,Y^2) - F(X^1,Y^1)\|^2_{\underline{\underline{S}}^2_{[0,t]}} \le 4k_1^2(\|A\|^2_{\underline{\underline{H}}^\infty_{[0,t]}} + 4\|M\|^2_{\underline{\underline{H}}^\infty_{[0,t]}})$$
$$\times(\|X^2 - X^1\|^2_{\underline{\underline{S}}^2_{[0,t]}} + \|Y^2 - Y^1\|^2_{\underline{\underline{S}}^2_{[0,t]}})$$
$$\|G(X^2,Y^2) - G(X^1,Y^1)\|^2_{\underline{\underline{S}}^2} \le 8k_2^2\|F(X^2,Y^2) - F(X^1,Y^1)\|^2_{\underline{\underline{S}}^2}$$
$$+ 8k_1^2\|C\|^2_{\underline{\underline{H}}^\infty}(\|X^2 - X^1\|^2_{\underline{\underline{S}}^2} + \|Y^2 - Y^1\|^2_{\underline{\underline{S}}^2}).$$

Therefore we conclude that

$$\|F(X^2, Y^2) - F(X^1, Y^1)\|^2_{\underline{\underline{S}}^2} + \|G(X^2, Y^2) - G(X^1, Y^1)\|^2_{s2}$$
$$\le 4k_1^2\left[(\|A\|^2_{\underline{\underline{H}}^\infty} + 4\|M\|^2_{\underline{\underline{H}}^\infty})(8k_2^2 + 1) + 2\|C\|^2_{\underline{\underline{H}}^\infty}\right]$$
$$\times\left[\|X^2 - X^1\|^2_{\underline{\underline{S}}^2} + \|Y^2 - Y^1\|^2_{\underline{\underline{S}}^2}\right],$$

proving the theorem. □

Remark 2.2. Assumption (2.6) is far from stringent. It only reflects the fact that the Lipschitz constants of the coefficients and the length of the interval have to be "*compatible*", as it is often seen in the FBSDE literature. Using different norms or different techniques, Eq. (2.6) can appear in rather different forms, but they will essentially be of the same nature. We note that if $A_t = C_t = t$ and M is a Brownian motion, then we return to the "classical" form of the FBSDEs. In such a case (2.6) can be simplified to, for example, $2k_1\sqrt{T[(1+T)+2k_2^2(T+4)]} < 1$, from which the "compatibility" nature of this condition is more clearly seen.

To see the necessity of the "compatibility condition", we have the following easy example.

Example 2.3. Consider the following FBSDE with coefficients uniformly Lipschitz with constant 1:

$$\begin{cases} X_t = x + \int_0^t Y_s\,ds + \sigma W_t, \\ Y_t = E\left\{X_T^* + \int_t^T |X_s|\,ds|\mathcal{F}_t\right\} \end{cases},$$

where $x > 0$, σ is a constant, and $X_t^* \triangleq \sup_{0\le s\le t}|X_s|$. It is then clear that Y is positive and so that $Y_t \ge E\{X_T^*|\mathcal{F}_t\}$. On the other hand

$$X_t = X_T - \int_t^T Y_s\,ds - \sigma(W_T - W_t) \le X_T^* - \int_t^T Y_s\,ds - \sigma(W_T - W_t),$$

and taking expectations we obtain

$$\begin{aligned} 0 < x \le x + E\int_t^T Y_s\,ds = E(X_t) \le E(X_T^*) \\ - E\int_t^T Y_s\,ds \le E(X_T^*)(1 - T + t), \end{aligned}$$

for all $0 \le t \le T$, which is possible only if $T < 1$. □

One of the main focuses in the theory of FBSDE has been finding proper conditions under which solvability over arbitrary duration can be established. In the case when the terminal value is of a function type, that is, $Y_T = g(X_T)$ where $g(\cdot)$ is a function, several methods, including the *Four Step Scheme* by Ma-Protter-Yong,[13] the method of continuation by Hu-Peng[8] and Yong,[22] and a variation of this method by Pardoux–Tang,[20] are considered rather effective. However, the situation will change drastically when the terminal value is of the *functional* type, that is, $Y_T = g(X)_T$, where $g : C[0, T] \mapsto C[0, T]$ is a functional. For example, the essence of the Four Step Scheme is that there exists a function $\theta : [0, T] \times R^n \mapsto R^m$, such that $Y_t = \theta(t, X_t)$, $\forall t \in [0, T]$, P-a.s.. The following example shows that even in a "Markovian" case this may not be true.

Example 2.4. Consider the following FBSDE on $[0, 1]$

$$\begin{cases} X_t = x + \int_0^t \sigma(X_s, Y_s) dW_s \\ Y_t = E\left\{ \int_0^1 X_s\, ds + \int_t^1 [Y_s - X_s] ds \middle| \mathcal{F}_t \right\}, \end{cases} \tag{2.7}$$

where $0 < \sigma_0 \le \sigma(x, y) \le C$, for some constants $\sigma_0, C > 0$, for all $x, y \in \mathbb{R}$. We show that for such an FBSDE it does not exist any strong solution (X, Y) for which $Y_t = \theta(t, X_t)$ holds for $t \in [0, T]$, P-a.s., where θ is some smooth deterministic function.

To see this, we first observe that if (X, Y) is a (strong) solution on some $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\})$, with given $\{\mathcal{F}_t\}$ -Brownian motion W, then X must be an $\{\mathcal{F}_t\}$ -martingale and a Markov process. Further, if we define $H_t = \int_0^t X_s\, ds$, $t \in [0, T]$, then the backward equation in Eq. (2.7) can be written as

$$Y_t = H_t + E\left\{ \int_t^1 Y_s\, ds \middle| \mathcal{F}_t \right\}. \tag{2.8}$$

A straightforward calculation using Fubini's theorem verifies that the (unique) solution to Eq. (2.8) is

$$Y_t = H_t + E\left\{ \int_t^1 \mathrm{e}^{(s-t)} H_s\, ds \middle| \mathcal{F}_t \right\}. \tag{2.9}$$

By integration by parts and the martingale property of X, we deduce from (2.9) that

$$\begin{aligned} e^t Y_t &= e^t H_t + E\left\{ eH_1 - e^t H_t - \int_t^1 e^s\, dH_s \middle| \mathcal{F}_t \right\} \\ &= E\left\{ e \int_0^1 X_s\, ds - \int_t^1 e^s X_s\, ds \middle| \mathcal{F}_t \right\} \\ &= e\left[\int_0^t X_s\, ds + X_t(1-t) \right] - X_t \int_t^1 e^s\, ds = (e^t - et)X_t + e\int_0^t X_s\, ds. \end{aligned}$$

In other words,

$$e^t[dY_t + Y_t\,dt] = d(e^tY_t) = d[(e^t - et)X_t] + eX_t\,dt$$
$$= (e^t - et)dX_t + e^tX_t\,dt,$$

or equivalently,

$$dY_t = (1 - te^{1-t})dX_t + (X_t - Y_t)dt$$
$$= (1 - te^{1-t})\sigma(X_t, Y_t)dW_t + [X_t - Y_t]dt.$$

If $Y_t = \theta(t, X_t)$ for some smooth function θ, then the above becomes

$$dY_t = (1 - te^{1-t})\sigma(X_t, \theta(t, X_t))dW_t + [X_t - \theta(t, X_t)]dt. \tag{2.10}$$

On the other hand, by Itô's formula we would have

$$dY_t = \{\theta_t(t, X_t) + \frac{\sigma^2(X_t, \theta(t, X_t))}{2}\theta_{xx}(t, X_t)\}dt$$
$$+ \sigma(X_t, \theta(t, X_t))\theta_x(t, X_t)dW_t.$$

Comparing this with Eq. (2.10), by the Markov property of X and the uniqueness of Doob–Meyer decomposition, we obtain that

$$\begin{cases} \theta_t(t,x) + \frac{1}{2}\sigma^2(x, \theta(t,x))\theta_{xx}(t,x) - x + \theta(t,x) = 0 \\ \qquad\qquad\qquad\qquad\qquad \forall (t,x) \in [0,1] \times \mathbb{R} \\ \sigma(x, \theta(t,x))\theta_x(t,x) = \sigma(x, \theta(t,x))[1 - te^{1-t}], \end{cases} \tag{2.11}$$

Since $\sigma(x, y) \geq \sigma_0 > 0$, the second equation in Eq. (2.11) implies that

$$\theta_x(t,x) = [1 - te^{1-t}], \quad \forall \ (t,x) \in [0,T] \times \mathbb{R}.$$

Thus $\theta(t,x) = [1 - te^{1-t}]x + \phi(t)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, for some smooth function ϕ. Substitute such a function θ into the first PDE in Eq. (2.11) and do a little computation, we see that ϕ has to satisfy the differential equation $\phi'(t) + \phi(t) - e^{1-t}x = 0$, for any $x \in \mathbb{R}$. This is impossible, hence a contradiction. □

3. WEAK SOLUTIONS: DEFINITIONS AND EXAMPLES

In this section we propose a notion of weak solution for BFSDE's, analogous to the concept in the theory of forward SDE's (see Ref.[10]).

To begin with, let us make some remarks on the probabilistic set-ups. In what follows we call a quintuple $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]}, W)$ a *standard set-up* if $(\Omega,\mathcal{F},P)$ is a complete probability space, $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the *usual hypotheses* and W is an $\{\mathcal{F}_t\}$ -Brownian motion. In particular, if $\mathcal{F}_t$ is $\mathcal{F}_t^W$, the natural filtration generated by the Brownian motion W, augmented by all the P -null sets of $\mathcal{F}$, we say that the set-up is *Brownian*.

Fix a standard set-up, let us consider the following FBSDE (in vector form):

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s \\ Y_t = E\left\{ g(X)_T + \int_t^T h(s, X_s, Y_s)ds \middle| \mathcal{F}_t \right\}, \end{cases} \tag{3.1}$$

where $x \in R^n$ and

$$b : [0, T] \times R^n \times R^m \to R^n \qquad \sigma : [0, T] \times R^n \times R^m \to R^{n\times d}$$

$$h : [0, T] \times R^n \times R^m \to R^m \qquad g : \mathcal{C}([0, T]; R^n) \to R^m$$

are jointly measurable with respect to all of their variables. We remind the reader that if the set-up is Brownian, then the FBSDE (1.1) can be rewritten as

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s, X_s, Y_s)ds - \int_t^T Z_s\, dW_{s.} \end{cases} \tag{3.2}$$

We first give the following definitions.

Definition 3.1. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, W)$ be a standard set-up. The pair of processes (X, Y), defined on this set-up, is called a strong solution of Eq. (3.1) if

(i) the set-up is Brownian,

(ii) (X, Y) are both $\{\mathcal{F}_t\}$-adapted, continuous processes such that

$$E\left\{ \sup_{t\in[0,T]} |X_t|^2 + \sup_{t\in[0,T]} |Y_t|^2 \right\} < \infty, \text{ and } E\left(\int_0^T \sigma^2(X_s, Y_s)ds \right) < \infty$$

(iii) (X, Y) satisfies Eq. (3.1) P-a.s.

Definition 3.2. A pair of processes (X, Y), together with a standard set-up $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]}, W)$ on which X and Y are defined, is called a weak solution of Eq. (3.1) if

(i) the process X is continuous, Y is càdlàg and both are $\{\mathcal{F}_t\}$-adapted;
(ii) $P\Big\{\int_0^T[|b| + |\sigma|^2 + |h|^2](s, X_s, Y_s)ds + |g(X)_T|^2 < \infty\Big\} = 1$
(iii) X and Y verify Eq. (3.1) P-a.s.

The notion of weak solution can be extended to more general FBSDEs often met in the classical theory.

Definition 3.3. A triple of processes (X, Y, Z), together with a standard set-up $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, W)$ where X, Y, Z are defined, is called a weak solution of

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \end{cases} \tag{3.3}$$

where b, h, and σ are deterministic functions of (t, x, y, z), if

(i) the process X is continuous, Y is càdlàg and X, Y, Z are all $\mathcal{F}_t$- adapted;
(ii) $P\Big(\int_0^T[(|b| + |\sigma|^2 + |h|^2)(s, X_s, Y_s, Z_s) + |Z_s|^2]ds + |g(X)_T|^2 < \infty\Big) = 1$
(iii) (X, Y, Z) verifies Eq. (3.3) P-a.s.

Next we give an example showing that our definition of weak solution is not fictitious. Namely, we show that there exist FBSDEs that admit only weak solutions but not strong solutions. Of course, in a completely decoupled setting, the well-known Tanaka example (see, e.g., Ref.[11, Example 5.3.5]) would serve the purpose, but the following one seems slightly non-trivial.

Example 3.4. Consider the coupled FBSDE:

$$X_t = \int_0^t \operatorname{sgn}(Y_s)dW_s \tag{3.4}$$

$$Y_t = E\Big\{X_T + \int_t^T (X_s - Y_s)ds \Big| \mathcal{F}_t\Big\}, \tag{3.5}$$

where $\operatorname{sgn}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\le 0\}}$. We first show that the weak solution of Eq. (3.4) and (3.5) exists. Indeed, let X be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$. Setting $\mathcal{F}_t = \mathcal{F}_t^X,\ t \in [0, T]$ we can solve the linear backward SDE (3.5) (or simply by observation) to obtain that $Y_t = X_t, \forall t \in [0, T], P -$ a.s. Thus the forward equation (3.4) becomes

$$X_t = \int_0^t \operatorname{sgn}(X_s) dW_s. \tag{3.6}$$

Therefore, defining $W_t = \int_0^t \operatorname{sgn}(X_s) dX_s$ we see that $\{(X, Y), (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}), W\}$ is a weak solution to Eqs. (3.4) and (3.5).

Next we show that Eqs. (3.4) and (3.5) do not admit any strong solution. Suppose the contrary and let (X, Y) be a strong solution defined on a given set-up $\{(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}), W\}$ such that $\mathcal{F}_t = \mathcal{F}_t^W,\ t \in [0, T]$. It is easy to see that X is an $\{\mathcal{F}_t^W\}$ -Brownian motion and one can still deduce from Eq. (3.5) that $Y_t = X_t,\ \forall t \in [0, T],\ P$ -a.s.. Consequently, Eq. (3.6) still holds. But this leads to the absurd inclusion: $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{|X|}$, $\forall t \in [0, T]$, as we saw in Tanaka's example (see, again, Ref.[11, pp. 302]), a contradiction. Indeed Eq. (3.5) can be written as a more explicit backward SDE driven by W, even if $\mathcal{F}_t = \mathcal{F}_t^X \supseteq \mathcal{F}_t^W$, $t \in [0, T]$. Since X is an $\{\mathcal{F}_t^X\}$ -Brownian motion, we can apply the Martingale Representation Theorem to conclude that there exists an $\mathcal{F}^X$ -predictable process $\hat{Z}$ such that

$$\begin{aligned} Y_t &= X_T + \int_t^T (X_s - Y_s) ds - \int_t^T \hat{Z}\, dX_s \\ &= X_T + \int_t^T (X_s - Y_s) ds - \int_t^T \hat{Z} \operatorname{sgn}(X_s) dW_s \\ &= X_T + \int_t^T (X_s - Y_s) ds - \int_t^T Z_s\, dW_s. \end{aligned}$$

The main difference here (as opposed to the usual framework) is that we cannot expect Y and Z to be $\mathcal{F}_t^W$ -adapted in general. □

To end this section we turn our attention to the uniqueness of weak solutions of FBSDE's. In section 5, we consider only FBSDE's of the form (3.1) rather than (3.3). In light of the standard (forward) SDE's theory, taking into account the path regularity requirement in Definition 3.2, we introduce the following *canonical space* which will be useful in our future discussion.

$$\begin{aligned} &\mathcal{C}^k = \mathcal{C}[0, T]; R^k) \quad \text{for } k = n, d, \ \ \mathcal{D}^m = D_{[0,T]}(R^m) \\ &\Theta = \mathcal{C}^d \times \mathcal{C}^n \times \mathcal{D}^m, \\ &\mathcal{B}(\Theta) = \mathcal{B}(\mathcal{C}^d) \otimes \mathcal{B}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{D}^m). \end{aligned} \tag{3.7}$$

where by $D_{[0,T]}(R^m)$ we denote the space of R^m -valued, càdlàg functions on $[0,T]$. We remark that $\mathcal{D}^m$ is a complete separable space, under the Skorohod topology. Furthermore, if the processes are in fact continuous, the Skorohod topology reduces to the sup-norm topology. We denote by $\theta = (w,x,y)$ the generic element of Θ.

Definition 3.5. If (X,Y) and $(\tilde{X},\tilde{Y})$ are two weak solutions of Eq. (3.1) defined on the same set-up $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\},W)$, we say that pathwise uniqueness holds if

$$P\{(X_t,Y_t) = (\tilde{X}_t,\tilde{Y}_t),\ \forall t \in [0,T]\} = 1.$$

If $\{(X,Y),(\Omega,\mathcal{F},P,\{\mathcal{F}_t\},W)\}$ and $\{(\tilde{X},\tilde{Y}),(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P},\{\tilde{\mathcal{F}}_t\},\tilde{W})\}$ are two weak solutions of Eq. (3.1), we say that uniqueness in the sense of probability laws (or uniqueness in law) holds if (X,Y) and $(\tilde{X},\tilde{Y})$ have the same probability distribution on the space $(\mathcal{C}^d \times \mathcal{D}^m, \mathcal{B}(\mathcal{C}^d \times \mathcal{D}^m))$.

The same definition can be naturally extended to the case of Eq. (3.3)

4. EXISTENCE OF WEAK SOLUTIONS

As in the forward SDE literature, one of the main differences between weak and strong solutions is that a weak solution does not require that the probability space be fixed a priori. In fact, we do not even require that the filtration be Brownian, a key assumption in the standard theory of BSDEs. The following result shows how such a relaxation can drastically reduce the difficulty in proving the existence of the solution, compared to the usual results for strongly coupled FBSDEs.

Let us consider the FBSDE of the form (3.3), that is

$$\begin{cases} X_t = x + \int_0^t b(s,X_s,Y_s,Z_s)ds + \int_0^t \sigma(s,X_s,Y_s,Z_s)dW_s \\ Y_t = g(X)_T + \int_t^T h(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{4.1}$$

For simplicity let us assume that $d = n$ and $m = 1$. We shall make the following extra assumptions.

(H5) (i) b and σ are bounded and continuous. Further, σ is independent of y and z and for some constant $c_0 > 0$, $\sigma^2(s,x) \geq c_0$, for all $(s,x) \in [0,T] \times R^n$;

(ii) h is continuous and there exists a constant $C > 0$ such that $|h(t, x, y, z)| \leq C(1 + |y| + |z|)$.

We have the following theorem.

Theorem 4.1. *Suppose that the coefficients of Eq.* (4.1) *satisfy* (H5). *Then there exists at least one weak solution of FBSDE* (4.1).

Remark 4.2. (i) Since the FBSDE is strongly coupled in every coefficient except for σ and the terminal condition is of a functional form, to our best knowledge, there has been no results concerning the existence and uniqueness of the strong solution of Eq. (4.1) under such mild conditions. In fact, it seems quite unlikely that any of the existing techniques would work;

(ii) From Definitions 3.1and 3.2, it is easily seen that Theorem 4.1 covers also the existence of the weak solution of Eq. (3.1).

Proof. First, consider the forward SDE

$$X_t = x + \int_0^t \sigma(s, X_s) dB_s. \tag{4.2}$$

By (H5) (i), it is well-known that Eq. (4.2) has a (unique) weak solution. Thus, there exists a standard set-up $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, B)$ on which a process X is defined so that Eq. (4.2)) holds P -a.s.. We point out that by virtue of the conditions on σ and the fact that the initial state of X is deterministic, one can show that $\mathcal{F}_t^X = \mathcal{F}_t^B$, $\forall t \in [0, T]$ (see, e.g., Ref.[11, Theorem 5.4.22] for the construction of the weak solution and note that the inclusion $\mathcal{F}_t^B \subseteq \mathcal{F}_t^X$ is trivial due to the nondegeneracy of σ). So, we may assume without loss of generality that the set-up is indeed Brownian and replace $\{\mathcal{F}_t\}$ with $\{\mathcal{F}_t^B\}$.
Next, let us define the function

$$\tilde{h}(t, x, y, z) \triangleq h(t, x, y, z) - z\sigma^{-1}(t, x) b(t, x, y, z), \quad \forall (t, x, y, z).$$

Clearly, $\tilde{h}$ is continuous, and satisfies (ii) of (H5) as well.
For the given set-up $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^B\}, B)$ we consider the following BSDE:

$$Y_t = g(X)_T + \int_t^T \tilde{h}(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s \, dB_s, \tag{4.3}$$

where the process X is the weak solution of Eq. (4.2). Since $\mathcal{F}^X = \mathcal{F}^B$, $\xi \stackrel{\Delta}{=} g(X)_T$ is $\mathcal{F}_T^B$ -measurable. Thus, using the result of Ref.[12] we know that Eq. (4.3) has a strong adapted solution (Y, Z) on the given set-up $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^B\}, B)$.

The rest of the proof is obvious. We define

$$M_t = \exp\left\{\int_0^t \langle \Theta_s, dB_s\rangle - \frac{1}{2}\int_0^t \|\Theta_s\|^2\, ds\right\}, \quad t \in [0, T],$$

where $\Theta_t \stackrel{\Delta}{=} \sigma^{-1}(t, X_t) b(t, X_t, Y_t, Z_t)$, $t \in [0, T]$. Since Θ is a bounded $\{\mathcal{F}_t^B\}$ -adapted process, M is an $\{\mathcal{F}_t^B\}$ -martingale, and by Girsanov's theorem we have that under the new probability measure $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = M_T$, the process $W_t \stackrel{\Delta}{=} B_t - \int_0^t \Theta_s\, ds$ is an $\{\mathcal{F}_t^B\}$ -Brownian motion and the triple (X, Y, Z) satisfies the FBSDE

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s) dW_s \\ Y_t = g(X)_T + \int_t^T (\tilde{h}(s, X_s, Y_s, Z_s) + Z_s \Theta_s) ds - \int_t^T Z_s dW_s \\ \quad = g(X)_T + \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s\, dW_s, \end{cases} \tag{4.4}$$

proving the theorem. □

5. UNIQUENESS OF WEAK SOLUTIONS

In this section we discuss the issues concerning the uniqueness of weak solutions. We first prove a result that is well known in the forward SDE case, that is, pathwise uniqueness implies uniqueness in law. We should point out that the situation here is a little more delicate than the standard one, because the law of the backward component might call for an evaluation of the distributions of the martingale part, indeed expressed by means of the conditional expectation. We have the following extension of Yamada–Watanabe Theorem.

Theorem 5.1. *Assume* (H5)*, then pathwise uniqueness implies uniqueness in law.*

Proof. Let $\{(X^i, Y^i), (\Omega^i, \mathcal{F}^i, \mu^i, \{\mathcal{F}_t^i\}, W^i)\}$, $i = 1, 2$ be two weak solutions of Eq. (3.1). Our first step is to construct an appropriate space to contain both set-ups.

Let Θ be the canonical path space defined by Eq. (3.7) and let P_i, $i = 1, 2$, be the distributions of $\{(W^i, X^i, Y^i)$ on $(\Theta, \mathcal{B}(\Theta))$ respectively, that is

$$P_i(A) = \mu_i\{(W^i, X^i, Y^i) \in A\} \quad \forall A \in \mathcal{B}(\Theta), \quad i = 1, 2.$$

Since the marginal distribution of both P_1 and P_2 on the w coordinate is the Wiener measure, denoted by P_*, let us denote the regular conditional probability of P_i given $W^i = w$ on $(\Theta, \mathcal{B}(\Theta))$ by

$$Q_i(w; A) \triangleq P_i\{A|W_i = w\}, \quad \forall w \in \Theta.$$

Now let us restrict $Q_i(w; \cdot)$ to the following "cylindrical" sets

$$\mathcal{A} = \left\{\tilde{F} = \mathcal{C}^d \times F : F \in \mathcal{B}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{D}^m)\right\} \in \mathcal{B}(\Theta).$$

Then it is easily checked, by definition of the regular conditional probability, that for all $F \in \mathcal{B}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{D}^m)$ and $G \in \mathcal{B}(\mathcal{C}^d)$, it holds that

$$P_i(G \times F) = \int_G Q_i(w; F) P_*(dw).$$

Next, we slightly enlarge the *canonical* space defined by Eq. (3.7): let

$$\Omega = \Theta \times \mathcal{C}^n \times \mathcal{D}^m, \quad \mathcal{F} = \mathcal{B}(\Theta) \otimes \mathcal{F}(\mathcal{C}^n) \otimes \mathcal{B}(\mathcal{D}^n), \tag{5.1}$$

with $\omega = (w, x_1, y_1, x_2, y_2)$ denoting the generic element of Ω. We define a probability on $(\Omega, \mathcal{F})$ by

$$P(d\omega) = Q_1(w; (dx_1 dy_1)) Q_2(w; (dx_2 dy_2)) P_*(dw). \tag{5.2}$$

For each $t \in [0, T]$, let $\mathcal{G}_t = \sigma(\{(w, x_1, y_1, x_2, y_2)(s);\ 0 \leq s \leq t\})$ and $\{\mathcal{F}_t\}$ be the usual P -augmentation of $\{\mathcal{G}_t\}$. The filtration $\{\mathcal{F}_t\}$ clearly satisfies the usual hypotheses and the coordinate process $W_t(\omega) \triangleq w(t)$, $t \in [0, T]$ is an $\{\mathcal{F}_t\}$ -Brownian motion.
By construction, for all $A \in \mathcal{B}(\Theta)$, one has

$$\begin{aligned} P\{\omega \in \Omega(w, x_1, y_1) \in A\} &= \int_A \int_{\mathcal{C}^n} \int_{\mathcal{D}^m} Q_2(w; dx_2 dy_2) Q_1(w; dx_1 dy_1) P_*(dw) \\ &= \int_A \left(\int_{\mathcal{C}^n \times \mathcal{D}^m} Q_2(w; dx_2 dy_2) \right) Q_1(w; dx_1 dy_1) P_*(dw) \\ &= \int_A Q_1(w; dx_1 dy_1) P_*(dw) = P_1(A) = \mu_1\{(W^1, X^1, Y^1) \in A\}. \end{aligned} \tag{5.3}$$

Similarly, for any $A \in \mathcal{B}(\Theta)$ we also have

$$\mu_2\{(W^2, X^2, Y^2) \in A\} = P\{\omega \in \Omega : (w, x_2, y_2) \in A\}. \tag{5.4}$$

Consequently, if the pathwise uniqueness holds, then we have $P\{\omega \in \Omega : (x_1, y_1) = (x_2, y_2)\} = 1$. Applying Eqs. (5.3) and (5.4), we conclude immediately that

$$\mu_1\{(W^1, X^1, Y^1) \in A\} = \mu_2\{(W^2, X^2, Y^2) \in A\}, \quad \text{for all} \quad A \in \mathcal{B}(\Theta).$$

That is, the two weak solutions have the same distribution under P. □

In the rest of this section we shall prove the so-called *Principle of Causality*, namely that "weak existence" and "pathwise uniqueness" imply strong existence of the solution. To do this we need the following lemma.

Lemma 5.2. *Let H be an R^m -valued, adapted process with càdlàg paths, defined on some probabilitic set-up* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\tilde{\mathcal{F}}_t\})$, *and assume that the filtration $\{\tilde{\mathcal{F}}_t\}$ is Brownian. Then there exists a process h defined on the canonical space* $(\Omega, \mathcal{F}, P)$, *given by Eqs.* (5.1) *and* (5.2), *such that h is $\{\mathcal{F}_t\}$ -adapted, has càdlàg paths and has the same distribution as H.*

Proof. First, we note that $\tilde{\mathcal{F}}$ is Brownian by assumption, we may assume without loss of generality that $\tilde{\mathcal{F}}_t = \mathcal{F}_t^{\tilde{W}}$, where $\tilde{W}$ is an $\{\mathcal{F}_t\}$ -Brownian motion. Next, let $t \geq 0$ be fixed. Since H_t is $\mathcal{F}_t^{\tilde{W}}$ -measurable, and $\mathcal{F}_t^{\tilde{W}}$ is generated by the random variables of the form $W_{\bullet \wedge t}$, one can apply the classical result for σ to show (see, e.g., Ref.[2]) that there exists a Borel measurable function $\Psi_t : \mathcal{C}^d \longrightarrow R^m$ such that

$$H_t(\omega) = \Psi_t(\tilde{W}_\bullet(\omega)) = \Psi_t(\tilde{W}_{\bullet \wedge t}(\omega)) \quad \text{for } \tilde{P} - \text{a.e.}\omega.$$

Let us now denote $N_t \stackrel{\Delta}{=} \{\omega \in \Omega : H_t(\omega) \neq \Psi_t(\tilde{W}_{\bullet \wedge t}(\omega))\}$ and $N \stackrel{\Delta}{=} \bigcup_{r \in [0,T] \cap \mathbf{Q}} N_r$. Then obviously $\tilde{P}(N) = 0$.

Now on the canonical space $(\Omega, \mathcal{F}, P)$, we define a process $h_t(w, x_1, y_1, x_2, y_2) = \Psi_t(w_{\bullet \wedge t})$, for $0 \leq t \leq T$ and $w \in \mathcal{C}^d$. We are to show that h is what we want. To see this we first note that

$$\begin{aligned} &\tilde{P}\left(\left\{\exists t \in [0,T] \cap \mathbf{Q} : H_t(\omega) \neq h_t(\tilde{W}_\bullet(\omega))\right\}\right) \\ &\quad = \tilde{P}\left(\left\{\exists t \in [0,T] \cap \mathbf{Q} : H_t(\omega) \neq \Psi_t(\tilde{W}_\bullet(\omega))\right\}\right) = \tilde{P}(N) = 0, \end{aligned}$$

that is, $H_t(\cdot) = h_t(\tilde{W}(\cdot))$, for all $t \in \mathbf{Q}$, $\tilde{P}$- a.e. Consequently, for any $n \in \mathbf{N}$, any partition $0 \le t_1 < \cdots < t_n \le T$, where $t_i \in \mathbf{Q}$, and any n bounded continuous function $f_i : R^m \times R^d \longrightarrow R$, the following holds

$$E_{\tilde{P}}\left(\prod_{i=1}^{n} f_i(H_{t_i}, \tilde{W}_{t_i})\right) = E_{P_*}\left(\prod_{i=1}^{n} f_i(\Psi_{t_i}(w_{\bullet\wedge t}), w_{t_i})\right)$$

$$= E_P\left(\prod_{i=1}^{n} f_i(h_{t_i}(\bullet), w_{t_i}(\bullet))\right) \tag{5.5}$$

where P is defined by Eq. (5.2).

To show that h and H has the same distribution (in fact $(h, \tilde{W})$ and $(H, \tilde{W})$ have the same joint distribution), we need to extend Eq. (5.5) to all partitions with arbitrary partition points. To this end we first show that the process h is in fact càdlàg.

To see this we first note that the set $A_0 = \{\omega \in \tilde{\Omega} : H_\bullet(\omega)$ is not càdlàg$\}$ in $(\tilde{\Omega}, \mathcal{F}, \tilde{P})$ is a $\tilde{P}$ -null set. Therefore if we denote

$$A = \{(w, x_1, y_1, x_2, y_2) \in \Omega : h_\bullet(w, x_1, y_1, x_2, y_2) \text{ is not càdlàg}\}$$

$$= \{(w, x_1, y_1, x_2, y_2) \in \Omega : \Psi_\bullet(w) \text{ is not càdlàg}\},$$

and $A_1 = \{w \in \mathcal{C}^d : \Psi_\bullet(w) \text{ is not càdlàg}\}$, then we see, from the definition of Ψ and h that

$$P(A) = P_*(A_1) = \tilde{P}\circ\tilde{W}^{-1}(A_1) = \tilde{P}(A_0) = 0.$$

This last equality amounts to saying that h has càdlàg paths P -a.s.

Since h is jointly measurable in (t, w) and $\{\mathcal{F}_t^w\}$ -adapted, taking limits in Eq. (5.5), we have that the same equality holds for any choice of $0 \le t_1 < \cdots < t_n \le T$ (not necessarily rational), hence h and H have the same distribution, proving the lemma. □

We now prove the main result of this section.

Theorem 5.3. *Assume* (H5)*, and that there exists a weak solution of FBSDE* (4.1) $((X, Y, Z), (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}), W)$, *such that*

(i) $\mathcal{F}_t = \mathcal{F}_t^W$, $t \in [0, T]$;
(ii) *there exists a version of the process Z that has càdlàg paths.*

Assume further that the pathwise uniqueness hold. Then FBSDE (4.1) *admits a unique strong solution. Consequently, the same result holds for the FBSDE* (3.1).

Remark 5.4. The assumption that Z has a càdlàg version is by no means a stringent condition. In fact, many sufficient condtions, most of them quite minor, have been obtained recently, even in the case when the terminal values are of functional form. We refer the readers to Ref.[16] and Ref.[14] for details.

Proof. Since we are in the case of Brownian filtrations, Y^1 and Y^2 have necessarily continuous paths. Thus we restrict the canonical space to $\mathcal{C}^d \times \mathcal{C}^n \times \mathcal{C}^m$.

For each $t \in [0, T]$ we define a *truncation mapping* $\varphi : \Xi \mapsto \Xi$, where $\Xi = \mathcal{C}^d, \mathcal{C}^n, \mathcal{C}^m$, by

$$\varphi_t(z)(\cdot) \stackrel{\Delta}{=} z(\cdot \wedge t),$$

and we introduce the σ -algebra

$$\mathcal{B}_t(\Xi) = \sigma(z(s), s \leq t; z \in \Xi) = \phi_t^{-1}(\Xi).$$

Proceeding as before, we may define

$$(\Theta, \mathcal{B}_t(\Theta)) = (\mathcal{C}^d \times \mathcal{C}^n \times \mathcal{C}^m, \mathcal{B}_t(\mathcal{C}^d) \otimes \mathcal{B}_t(\mathcal{C}^n) \otimes \mathcal{B}_t(\mathcal{C}^m))$$

with probabilities $P_i^t = \mu_i((W^i, X^i, Y^i) \in A)$, for $A \in \mathcal{B}_t(\Theta)$ and $i = 1, 2$.

We now let $Q_i^t(w; F)$ be the regular conditional probabilities of P_i^t given $\phi_t W^i = \phi_t w$. Then for any $F \in \mathcal{B}_t(\mathcal{C}^n) \otimes \mathcal{B}_t(\mathcal{C}^m)$, it holds that

$$P_i(G \times F) = \int_G Q_i^t(w; F) P_*(dw) \tag{5.6}$$

for all $G \in \mathcal{B}_t(\mathcal{C}^d)$, where P is the Wiener measure on $\mathcal{C}^d$. We are to show that Eq. (5.6) holds for any $G \in \mathcal{B}(\mathcal{C}^d)$.

To this end, we choose G in the family

$$\mathcal{H} = \{G = \phi_t^{-1} G_1 \cap \sigma_t^{-1} G_2, G_1, G_2 \in \mathcal{B}(\mathcal{C}^d)\},$$

where $\sigma_t w(s) = w(t + s) - w(s)$, which is a Dynkin system generating the Borel σ -algebra $\mathcal{B}(\mathcal{C}^d)$. For $i = 1, 2$, since the coordinate process $W_t(w) = w(t)$ is a Brownian motion under P_*, we see that $\phi_t^{-1} G_1$ and $\sigma_t^{-1} G_2$ are independent under P, therefore,

$$\begin{aligned}\int_G Q_i^t(w;F)P^*(dw) &= \int_\Omega Q_i^t(w;F)\mathbf{1}_{\phi_t^{-1}G_1}\mathbf{1}_{\sigma_t^{-1}G_2}P_*(dw)\\ &= \int_\Omega E_*(Q_i^t(w;F)\mathbf{1}_{\phi_t^{-1}G_1}\mathbf{1}_{\sigma_t^{-1}G_2}|\mathcal{B}_t(\mathcal{C}^d))P_*(dw)\\ &= \int_\Omega Q_i^t(w;F)\mathbf{1}_{\phi_t^{-1}G_1}E_*(\mathbf{1}_{\sigma_t^{-1}G_2}|\mathcal{B}_t(\mathcal{C}^d))P_*(dw)\\ &= \int_\Omega Q_i^t(w;F)\mathbf{1}_{\phi_t^{-1}G_1}P_*(\sigma_t^{-1}G_2)P_*(dw)\\ &= P_*(\sigma_t^{-1}G_2)P_i(\phi_t^{-1}G_1\times F).\end{aligned}$$

On the other hand, since $P_i(\phi_t^{-1}G_1\times F)=\mu_i\{\phi_t W^i\in G_2,(X^i,Y^i)\in F\}$, and $P_*(\sigma_t^{-1}G_2)=P_i\{(w,x,y)\in\Theta;\sigma_t w\in G_2\}=\mu_i\{\sigma_t W^i\in G_2\}$, we conclude that

$$\begin{aligned}\int_G Q_i^t(w;F)P^*(dw) &= \mu_i\{\phi_t W^i\in G_2,(X^i,Y^i)\in F\}\mu_i\{\sigma_t W^i\in G_2\}\\ &= \mu_i\{W^i\in G,(X^i,Y^i)\in F\}=P_i(G\times F).\end{aligned}$$

Since we can extend the previous inequality to any set $G\in\mathcal{B}(\mathcal{C}^d)$, for $F\in\mathcal{B}_t(\mathcal{C}^n)\otimes\mathcal{B}_t(\mathcal{C}^m)$ we have that $Q_i^t(w;F)=Q_i(w;F)$, P -a.s.

Now let us define the probability

$$Q(w;dx_1dy_1;dx_2dy_2)=Q_1(w;dx_1dy_1)Q_2(w;dx_2dy_2)$$

By definition, we know that $P(G\times B)=\int_G Q(w;B)P_*(dw)$ for any $G\in\mathcal{B}(\mathcal{C}^d)$ and $B\in(\mathcal{B}(\mathcal{C}^n)\otimes\mathcal{B}(\mathcal{C}^m))^{\otimes 2}$. In particular, if we choose the set

$$G\times B=\mathcal{C}^d\times\{(x_1,y_1),(x_2,y_2)\in(\mathcal{C}^n\times\mathcal{C}^m)^2:(x_1,y_1)=(x_2,y_2)\},$$

the definition of P and the pathwise uniqueness allows to conclude that

$$1=P(G\times B)=\int_G Q(w;B)P_*(dw),$$

which holds only if there exists a set N such that $P_*(N)=0$ and $Q(w;B)=1$ for any $w\notin N$, consequently

$$1=\int_{\mathcal{C}^n\times\mathcal{C}^m}Q_1(w;\{(x_1,y_1\})Q_2(w;dx_1\times dy_1). \tag{5.7}$$

Note that the last equality is possible only if there exists a point, possibly depending on w, where the two measures concentrate; in other words if there exists $(x, y) = (h_1(w), h_2(w))$, so that $Q_i(w; (h_1(w), h_2(w)) = 1$. This amounts to saying that there exists a solution to our equation that is of the form $(X, Y) = (h_1(W), h_2(W))$. It is then easy to check that this is the strong solution we are looking for. □

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