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A Dynamic Principal-Agent Problem with One-Sided Commitment

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Abstract. In this paper, we consider a principal-agent problem where the agent is allowed to quit by incurring a cost. When the current agent quits the job, the principal will hire a new one, possibly with a different type. We characterize the principal's dynamic value function, which could be discontinuous at the boundary, as the (unique) minimal solution of an infinite dimensional system of Hamilton-Jacobi-Bellman equations, parameterized by the agent's type. This dynamic problem is time consistent in a certain sense. Some interesting findings are worth mentioning. First, self-enforcing contracts are typically suboptimal. The principal would rather let the agent quit and hire a new one. Next, the standard contract for a committed agent may also be suboptimal because of the presence of different types of agents in our model. The principal may prefer no commitment from the agent; then, the principal can hire a cheaper one from the market at a later time by designing the contract to induce the current agent to quit. Moreover, because of the cost incurred to the agent, the principal will see only finitely many quittings.

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Keywords: principal-agent problems • contract theory • one-sided commitment • self-enforcing contracts • time inconsistency

1. Introduction

The contract theory involves two parties: the principal (the principal) hires an agent (the agent) by offering a contract, which provides incentives for the agent to work for her. Such theory has had a great impact on organizational economics and corporate finance, as well as many related fields. One typical example that motivates our work is employer (principal) versus employee (agent) in labor markets. Given a contract, the agent's problem is to choose efforts to maximize his expected utility, depending on both the contract and the efforts. Then, the principal's problem is to design an optimal contract to maximize her own expected utility by anticipating that the agent would always choose his corresponding optimal efforts. There is a constraint on the admissible contracts: the agent's optimal utility should satisfy a certain *individual rationality* (IR) constraint, which can be interpreted as the market value of the agent. The continuous-time principal-agent problems were first studied in the seminal paper by Holmström and Milgrom [18], followed by Schättler and Sung [32] and Sung [35, 36]. There have been numerous publications on the subject in the past three decades. We refer to the monograph by Cvitanić and Zhang [6] and the references therein. In particular, we refer to Cvitanić et al. [7, 8] and Sannikov [31] for the dynamic programming approach, which will be used in this paper.

All the above literature consider the principal-agent problem on a fixed time horizon $[0,T]$ (or $[0,\infty]$). There is a crucial time inconsistency issue that is the underlying reason for a possible renegotiation later but does not seem to receive much attention in the literature: the optimal contract typically does not remain optimal (or even not admissible anymore) if the two parties reconsider the contract at a later time t.[1] When both parties are fully committed, this issue is irrelevant. However, in many situations, for example, in a typical labor contract, the agent is allowed to quit before T. Then, the principal will need to either renegotiate with the current agent or hire a new one at the quitting time with a new contract that is different from the original one exactly because of the time inconsistency. Thus, being aware of the agent's possible quitting, the principal would design the contract differently in the beginning.

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There have been very serious efforts on principal-agent problems with one-sided commitment. For example, Grochulski and Zhang [13], Jeon et al. [20], Niu et al. [28], Phelan [29], Ray [30], and Zhang [40] studied models where the agent is not committed. They were restricted to the self-enforcing contracts; that is, the agent's continuation utility satisfies the IR constraint at any time. This is in the spirit of the renegotiation-proof equilibria (see, for example, Farrell and Maskin [11] and Strulovici [34]), and under such a contract, the agent actually will never quit. We also note that these works mainly consider risk sharing in a full-information setting without moral hazard and, in particular, without the incentive compatibility constraint. The works of Gottlieb and Zhang [12] and Karaivanov and Martin [21] require incentive compatibility but also restrict the contracts to self-enforcing ones.[2] The works of Ai and Li [1], Miao and Zhang [27], and Thomas and Worrall [37] are also in the realm of self-enforcing contracts, but with two-sided limited commitment; namely, both the principal and the agent can terminate the contract early. In a different direction, Krueger and Uhlig [22] considered a model with multiple principals and agents where the agents are not committed to the contract and can switch to another principal. Their main focus is to determine endogenously the outside option of the agent, which amounts to the IR constraint, through competition of the principals. However, there is no discussion of the principal's behavior after the agent quits, which is the main focus of our paper. The work of Ai et al. [2] also endogenizes the outside options through assortative matching. However, their stopping times for terminating the contract are exogenously given, and before that, the contracts are still required to be self-enforcing. Another related work is Hu et al. [19], where one agent can switch among multiple principals. Their main focus is to find the equilibrium (and mean field equilibrium) contract among the principals. We also refer to Capponi and Frei [4], Hajjej et al. [14], He et al. [16], and Lin et al. [24], where the principal and/or the agent can choose a stopping time to terminate the contract,[3] but the principal does not hire a new agent. We should note that, even when restricting to self-enforcing contracts so that there is no desire for renegotiation from the agent's side, the problem may still be time inconsistent.[4]

We remark that while disincentivizing the agent from quitting, the self-enforcing contracts are typically suboptimal for the principal, or, say, such a contract is too expensive for the principal. The principal would rather offer a standard contract, which satisfies the IR constraint only at the initial time, and be prepared for a possible quitting of the agent. In this paper, we consider a continuous-time principal-agent problem with moral hazard where the agent is allowed to quit. The main feature of our model is that the principal would hire a new agent when the current agent quits the job, and we shall investigate the time consistency issue from the principal's perspective. One particular application of our model is the dynamic firm-worker relationships (cf. Harris and Holmström [15]). There is one risk-neutral principal and a family of risk-averse agents with different types, parameterized by θ. Unlike the adverse selection problems that involve only one agent but with unknown (to the principal) type, here, the principal knows the types of all agents but hires only one at any particular time. The principal would pay the agent continuously in time, without lump sum payments. The current agent with type θ would quit the job at time t if the agent prefers the new opportunity from the market, quantified by the dynamic individual rationality R_t^θ. If the agent quits at t, the agent will incur a cost c_t^θ, and the principal will also suffer a loss c_t^P. The principal will hire a new agent, possibly with a different type $\tilde{\theta}$, and the new agent is also allowed to quit before the expiration date T. For simplicity, in this paper, we assume $R_t^\theta, c_t^\theta, c_t^P$ are exogenously given and deterministic.

Given a contract, the agent can choose his efforts and the quitting time, which is a stopping time. We note that we do not keep track of the agent's status once the agent quits the current job. This agent's problem is a mixed optimal control and stopping problem and, as standard, can be solved through a reflected backward stochastic differential equation (RBSDE) (cf. Zhang [41]). We follow the idea of Cvitanić et al. [7, 8] and Sannikov [31] to rewrite the agent's continuation (optimal) utility process as a forward process; see also Ma and Yong [26] for a similar idea in the contexts of forward-backward SDEs. Then, the principal's problem becomes a constrained stochastic control problem, where the agent's continuation utility becomes the state process, and the controls are the contract and the target effort, with a constraint that the agent's terminal utility is zero (because there is no lump sum payment). The agent's optimal quitting time is an intrinsic hitting time when the agent's continuation utility process hits the lower barrier $\underline{L}_t^\theta := R_t^\theta - c_t^\theta$ (for type θ agent). This enables us to use the Hamilton-Jacobi-Bellman (HJB) equation approach to solve the principal's problem.

We now discuss the most crucial component of the problem: the boundary condition of the HJB equation on the lower barrier $\underline{L}^\theta$. When the current type θ agent's continuation utility process hits $\underline{L}_t^\theta$ at t, the agent would quit, and then the principal would hire a new agent. First, the principal will choose a different type $\tilde{\theta}$, optimal for the principal's utility. Moreover, the new agent is still allowed to quit, and thus, the boundary condition at $\underline{L}_t^\theta$ should be the principal's optimal utility at $R_t^{\tilde{\theta}}$ (minus the cost c_t^P), which is exactly the solution we are looking for. So we are considering an infinite dimensional system of HJB equations, parameterized by θ, where the

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interaction is through the boundary condition. The last feature, that the boundary condition involves the solution itself, actually reflects the time-consistent nature of our formulation. However, this adds to the difficulty of the problem because the boundary condition is not exogenously given.

We establish the well-posedness of this HJB system by recursive approximation. Let $V_n(\theta;\cdot)$ denote the principal's optimal value function when the current agent is type θ and the principal allows at most n quittings in the remaining time period; that is, after seeing n quittings, the principal will offer only self-enforcing contracts for the remaining period. Then, the boundary condition for $V_n(\theta;\cdot)$ at $\underline{L}_t^\theta$ will be $\sup_{\tilde\theta} V_{n-1}(\tilde\theta;t,R_t^{\tilde\theta}) - c_t^P$, which is exogenously given in a recursive way. Under some technical conditions, we show that V_n converges uniformly to the unique solution of the HJB system. Another subtle point is that at the quitting time t, whereas the agent is indifferent to quitting or staying, and thus, the agent's optimal utility is continuous at t, the principal's utility may have a jump because of the quitting, and thus, the principal's value function is typically discontinuous at the boundary $\underline{L}_t^\theta$. Therefore, the boundary condition is actually an inequality, and our value function is characterized as the minimal-viscosity solution of the HJB system.

Some findings implied from our results are interesting from practical perspectives. First, self-enforcing contracts are typically only suboptimal for the principal. That is, given that the agent is not committed to the contract, it is not desirable for the principal to offer self-enforcing contracts to disincentivize the agent from quitting; the principal would rather let the agent quit and rehire a new one when that happens. This observation is more or less obvious and is strongly confirmed by our results. Next, the standard optimal contract (satisfying the IR constraint only at the initial time) may also be only suboptimal. That is, in some markets, the principal may prefer no commitment from the agent so that the principal can hire a cheaper one from the market at a later time by designing the contract to induce the current agent to quit. The reason is that R^θ (and c^θ, c^P) evolves along the time. Whereas it is optimal for the principal to hire an agent θ at time 0, at some later time t, another agent $\tilde\theta$ may become much cheaper, and thus, the principal would be happy to see agent θ quit so that the principal could hire agent $\tilde\theta$ to replace him.[5] Moreover, by assuming a uniform lower bound of the cost c^θ, the agents won't quit too frequently, and actually, the principal will only see finitely many quittings. This is, of course, consistent with what we observe in practice.

We shall mention another direction in the literature on the time inconsistency issue; see, for example, Balbus et al. [3], Cetemen et al. [5],[6] Djehiche and Helgesson [10], Gottlieb and Zhang [12], Hernández and Possamaï [17], Li et al. [23], Liu et al. [25], Wang et al. [38], and Yilmaz [39], to mention a few. Here, the preference of the agent (and/or the principal) is time inconsistent, for example, because of nonexponential discounting. We refer to the survey paper by Strotz [33] for time-inconsistent stochastic control problems; in particular, one may consider precommitted, naive, and sophisticated agents. The principal knows what type of agent the principal is hiring and would find the optimal contract (or, possibly, the suboptimal one when the principal's preference is also time inconsistent) by anticipating that the agent would always choose his optimal or suboptimal efforts in a corresponding way. We emphasize again that the time inconsistency in these works is because of the individual player's preference. This has a completely different nature from the time inconsistency we investigate in this paper because of the interaction between the principal and the agents.

Finally, we clarify the time consistency we obtain in this paper. First, our agent's problem is always time consistent for the period the agent stays with the job, and we do not keep track of the agent once the agent leaves the current principal. Our principal's problem is time consistent at the agents' optimal quitting times; see Theorem 4. That is, the principal would never regret when an agent quits, and thus, the principal could offer a new contract to another agent. We should note, though, that if the principal reconsiders the contract within the time period an agent is staying, the principal may find it not optimal. However, because the principal is required to commit to the contract, the principal is not allowed to fire the agent and sign a better contract during those time periods, and thus, such time inconsistency for the principal is irrelevant for practical considerations.

The rest of the paper is organized as follows. In Section 2, we introduce our model and consider self-enforcing contracts. In Section 3, we consider the model where only one quitting of the agent is allowed, and the principal's dynamic value function in this model is analyzed in Section 4. In Section 5, we extend these results to the case with at most n quittings, and in Section 6, we remove the restriction on the number of quittings. In Section 7, we reinvestigate the problem when the benchmark model is the standard principal-agent problem with full commitment. Finally, in Section 8, we summarize our findings, and in the appendix, we complete some technical proofs.

2. Our Model and Self-Enforcing Contracts

In this section, we introduce our model on the fixed time horizon $[0,T]$. In particular, we shall consider self-enforcing contracts so that the agent has no incentive to quit.

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Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space on $[0,T]$, B a standard one-dimensional Brownian motion, and $\mathbb{F} = \mathbb{F}^B$. There are two players: a principal (the principal) and an agent (the agent). The agent has a parameter $\theta \in \Theta$, which can be viewed as the type of the agent. In this section, θ is fixed, whereas in the rest of the paper, we will consider different types of agents.

The principal's control, namely, the contract, is a continuous-time payment $\eta = \{\eta_t\}_{0\le t\le T}$. The agent's control, namely, his action or effort, is a process $\alpha = \{\alpha_t\}_{0\le t\le T}$. In particular, there are no terminal payments, which are not convenient to model when an agent quits. The admissible controls satisfy the following conditions, required for technical convenience.

Definition 1.

i. Let $\mathcal{A}^P$ denote the set of $\mathbb{F}$-progressively measurable processes η on $[0,T]$ such that $-C(\eta) \le \eta \le C_0$. That is, η is bounded with a uniform upper bound $C_0 > 0$, but the lower bound $-C(\eta)$ may depend on η.

ii. Let $\mathcal{A}^A$ denote the set of $\mathbb{F}$-progressively measurable processes α on $[0,T]$ such that

$$\mathbb{E}_t^{\mathbb{P}}\left[\int_t^T |\alpha_s|^2 ds\right] \le C(\alpha), \quad 0 \le t \le T, \quad \mathbb{P}\text{-a.s.}, \tag{1}$$

where the constant $C(\alpha) > 0$ may depend on α. That is, $\int_0^t \alpha_s dB_s$ is a BMO martingale. (See definition 7.2.2 in Zhang [41, section 7.2])

We consider moral hazard models, and as in the standard literature, we use the weak formulation as follows (see, e.g., Cvitanić et al. [9]). The state process is

$$X_t \equiv B_t. \tag{2}$$

The agent controls the law of X, denoted as $\mathbb{P}^\alpha$, through the Girsanov theorem:

$$B_t^\alpha := B_t - \int_0^t \alpha_s ds, \quad M_t^\alpha := \exp\left(\int_0^t \alpha_s dB_s - \frac{1}{2}\int_0^t |\alpha_s|^2 ds\right), \quad d\mathbb{P}^\alpha := M_T^\alpha d\mathbb{P}. \tag{3}$$

Note that under (1), M^α is a true martingale under $\mathbb{P}$, and hence, $\mathbb{P}^\alpha$ is another probability measure. Moreover, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that (cf. Zhang [41, section 7.2])

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\varepsilon \int_0^T |\alpha_s|^2 ds\right) + |M_T^\alpha|^{1+\varepsilon}\right] < \infty. \tag{4}$$

We shall also introduce the following notations:

$$\mathbb{E} := \mathbb{E}^{\mathbb{P}}, \quad \mathbb{E}^\alpha := \mathbb{E}^{\mathbb{P}^\alpha}, \quad \mathbb{E}_t^\alpha := \mathbb{E}^\alpha[\cdot|\mathcal{F}_t]. \tag{5}$$

2.1. The Agent's Problem

Given $\eta \in \mathcal{A}^P$, the agent's optimal expected utility at time 0 is

$$V_0^A(\theta; 0, \eta) := \sup_{\alpha\in\mathcal{A}^A} \mathbb{E}^\alpha\left[\int_0^T f(\theta; s, \eta_s) ds - \int_0^T h(\theta; s, \alpha_s) ds\right]. \tag{6}$$

Here, the agent's running utility function f and running cost function h satisfy Assumption 1 below. Moreover, the subscript $_0$ in V_0^A indicates that the agent is allowed to quit zero times, namely, not allowed to quit.

Assumption 1. *There exists a constant $\Lambda_0 \ge 1$ satisfying the following:*

i. *f is twice differentiable in η with*

$$f \le -\frac{1}{\Lambda_0} < 0, \quad \partial_\eta f \ge \frac{1}{\Lambda_0} > 0, \quad \partial_{\eta\eta} f \le -\frac{1}{\Lambda_0} < 0, \quad \text{for all } \eta \le C_0; \quad \text{and} \quad f(\theta; s, C_0) \ge -\Lambda_0. \tag{7}$$

ii. *h is twice differentiable in α with*

$$h \ge 0 \ \text{ with } \ h(\theta; s, 0) = 0, \quad \frac{1}{\Lambda_0} \le \partial_{\alpha\alpha} h \le \Lambda_0. \tag{8}$$

iii. *f, h, and $\partial_\alpha h$ are uniformly continuous in s, with a common modulus of continuity function ρ, and we assume, without loss of generality, that $\rho(\delta) \ge \delta$.*

Basically, we require the utility function f to be increasing and concave in the payment η and the cost function h to be convex in the effort α, which are standard, and the rest of the assumption is for technical convenience.

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Clearly, in this section, we may allow Λ_0 and ρ to depend on θ. However, for the purpose of later sections, we require them to be common for all θ. Throughout the paper, we shall denote by C a generic constant that may depend on T, C_0, and Λ_0, but not on ρ or θ. One typical example satisfying Assumption 1 is: for some appropriate $\lambda_1(\theta), \lambda_2(\theta) > 0$,

$$f(\theta;s,\eta) = -\lambda_1(\theta)e^{-\lambda_1(\theta)\eta}, \quad h(\theta;s,\alpha) = \frac{1}{2\lambda_2(\theta)}|\alpha|^2. \tag{9}$$

Under (7) and (8), it is clear that $f(\theta;s,\cdot)$ and $\partial_\alpha h(\theta;s,\cdot)$ have inverse functions, denoted as $I^f(\theta;s,\cdot)$ and $I^h(\theta;s,\cdot)$, respectively. That is,

$$f(\theta;s,I^f(\theta;s,y)) = y,\ y \le f(\theta;s,C_0); \quad \partial_\alpha h(\theta;s,I^h(\theta;s,z)) = z,\ z \in \mathbb{R}. \tag{10}$$

Moreover, we introduce the following functions:

$$H(\theta;s,z) := \sup_{\alpha\in\mathbb{R}}[\alpha z - h(\theta;s,\alpha)], \quad \tilde{H}(\theta;s,z) := -h(s,\theta,I^h(\theta;s,z)). \tag{11}$$

Then, I^h is the maximum argument of H: $H(\theta;s,z) = zI^h(\theta;s,z) - h(s,\theta,I^h(\theta;s,z))$, and it is straightforward to verify the following: for some constant $C(\Lambda_0) > 0$,

$$\begin{gathered}
0 < \partial_y I^f \le \Lambda_0, \quad |\eta|^2 \le C(\Lambda_0)[1 - f(\theta;s,\eta)], \quad \text{for all} \quad \eta \le C_0;\\
I^h(\theta;s,0) = 0, \quad 0 < \frac{1}{\Lambda_0} \le \partial_z I^h(\theta;s,z) \le \Lambda_0 < \infty;\\
I^h \text{ is uniformly continuous in } s \text{ with modulus of continuity function } \Lambda_0\rho;\\
\tilde{H}(s,\theta,z) \le \tilde{H}(s,\theta,0) = 0 = H(s,\theta,0) \le H(s,\theta,z).\\
|\partial_z H(\theta;s,z)|, |\partial_z \tilde{H}(\theta;s,z)| \le \Lambda_0|z|, \quad |z|^2 \le -C(\Lambda_0)\tilde{H}(\theta;s,z).
\end{gathered} \tag{12}$$

The solution to the agent's problem is standard; see, for example, Cvitanić et al. [7], which provides both the agent's optimal utility and his optimal action through the following BSDE: given $\eta \in \mathcal{A}^P$,

$$Y_t^{\theta,\eta} = \int_t^T [f(\theta;s,\eta) + H(\theta;s,Z_s^{\theta,\eta})]ds - \int_t^T Z_s^{\theta,\eta}dB_s, \quad t \in [0,T]. \tag{13}$$

Proposition 1. *Let Assumption 1 hold, and $\eta \in \mathcal{A}^P$. Then, the agent has optimal utility $V_0^A(\theta;0,\eta) = Y_0^{\theta,\eta}$ and has the unique optimal control $\alpha^* = I^h(\theta;\cdot,Z^{\theta,\eta}) \in \mathcal{A}^A$.*

Following Ma and Young [26] and Sannikov [31], we shall rewrite the BSDE (13) in the forward form, which will be crucial in the next section. For any $t \in [0,T)$, $x \in \mathbb{R}$, and $(\eta, Z) \in \mathcal{A}_t^P \times \mathcal{A}_t^A$, denote

$$\begin{aligned}
\mathcal{X}_s^{\theta;t,x,\eta,Z} &:= x - \int_t^s [f(\theta;r,\eta_r) + H(\theta;r,Z_r)]dr + \int_t^s Z_r dB_r\\
&= x - \int_t^s [f(\theta;r,\eta_r) + \tilde{H}(\theta;r,Z_r)]dr + \int_t^s Z_r dB_r^{\theta,Z},\ t \le s \le T,
\end{aligned} \tag{14}$$

and set, recalling $Y_T^{\theta,\eta} = 0$,

$$\mathcal{A}_{t,x}^\theta := \{(\eta,Z) \in \mathcal{A}_t^P \times \mathcal{A}_t^A : \mathcal{X}_T^{\theta;t,x,\eta,Z} = 0\}. \tag{15}$$

Remark 1. (i) We emphasize that $\mathcal{X}^{\theta;t,x,\eta,Z}$ is the forward form of $Y^{\theta,\eta}$, which stands for the continuation utility of the agent, rather than the state process X in (2). In particular, x is the agent's optimal utility at initial time t for the contract η.

(ii) In Cvitanić et al. [7, 8], the principal has terminal payment ξ; then, ξ has one-to-one correspondence with the corresponding (x, Z), and thus, one may view (x, Z) as the principal's controls. Alternatively (although less natural in this case), we may view (ξ, x, Z) altogether as the principal's control but then require a constraint $\xi = \mathcal{X}_T^{t,x,Z}$ for some appropriate process $\mathcal{X}^{t,x,Z}$ in the spirit of (14). In our situation, η and (x, Z) also have one-to-one correspondence; however, it is not easy to express η directly in terms of (x, Z). So, we take the alternative way by viewing the triple (η, x, Z) as the principal's controls, but we require the constraint $\mathcal{X}_T^{\theta;t,x,\eta,Z} = 0$ in (15).

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Recall (3) and (5); we denote

$$B^{\theta,Z} := B^{I^h(\theta;\cdot,Z)}, \quad \mathbb{P}^{\theta,Z} := \mathbb{P}^{I^h(\theta;\cdot,Z)}, \quad \mathbb{E}^{\theta,Z} := \mathbb{E}^{I^h(\theta;\cdot,Z)}, \quad \mathbb{E}_t^{\theta,Z} := \mathbb{E}_t^{I^h(\theta;\cdot,Z)}. \tag{16}$$

Moreover, recall Definition 1(i) and introduce

$$\overline{L}_t^\theta := \int_t^T f(\theta;s,C_0)ds, \quad 0 \le t \le T. \tag{17}$$

We note that when $\eta \equiv C_0$, obviously, $(\overline{L}_t^\theta, 0)$ satisfies BSDE (13), so $(Y_t^{\theta,C_0}, Z_t^{\theta,C_0}) = (\overline{L}_t^\theta, 0)$. Because $\eta \le C_0$, by applying the comparison principle for BSDE (13), we have

$$Y_t^{\theta,\eta} \le \overline{L}_t^\theta \text{ for all } \eta \in \mathcal{A}^P, \quad \text{and thus,} \quad \mathcal{A}_{t,x}^\theta \ne \emptyset \text{ if and only if } x \le \overline{L}_t^\theta. \tag{18}$$

The following technical estimate will be used frequently in the paper. We postpone its proof to the appendix.

Lemma 1. *Let Assumption 1 hold. For any $t < T$ and $\eta \in \mathcal{A}^P$, we have*

$$\mathbb{E}_t^{\theta,Z^{\theta,\eta}}\left[\int_t^T [|Z_s^{\theta,\eta}|^2 + C_0 - \eta_s]ds\right] \le C[\overline{L}_t^\theta - Y_t^{\theta,\eta}],$$

$$\mathbb{E}_t^{\theta,Z^{\theta,\eta}}\left[\int_t^T |\eta_s|^2 ds\right] \le C[T - t - Y_t^{\theta,\eta}]. \tag{19}$$

Consequently, for any $t < T$, $x \le \overline{L}_t^\theta$, and $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$, we have

$$\mathbb{E}^{\theta,Z}\left[\int_t^T [|Z_s|^2 + C_0 - \eta_s]ds\right] \le C[\overline{L}_t^\theta - x], \quad \mathbb{E}^{\theta,Z}\left[\int_t^T |\eta_s|^2 ds\right] \le C[T - t - x]. \tag{20}$$

2.2. The Principal's Problem

We consider a risk-neutral principal. Given $\eta \in \mathcal{A}^P$, the principal's expected utility is

$$J_P(\theta;0,\eta) := \mathbb{E}^{\alpha^*}\left[X_T - \int_0^T \eta_s ds\right] = \mathbb{E}^{\theta,Z^{\theta,\eta}}\left[\int_0^T [I^h(\theta;s,Z_s^{\theta,\eta}) - \eta_s]ds\right]. \tag{21}$$

Remark 2. (i) In (3), we assume the drift of the state process is α. This is a simplification without loss of generality. Indeed, if we consider a more general drift $b(\theta;s,\alpha_s)$, we may simply set a new control $\tilde{\alpha}_s := b(\theta;s,\alpha_s)$, and one can easily see that the Agent's Problem (6) is equivalent to the problem by replacing h with $\tilde{h}(\theta;s,\tilde{\alpha}_s) := \inf_{\alpha_s : b(\theta;s,\alpha_s) = \tilde{\alpha}_s} h(\theta;s,\alpha_s)$.

(ii) As in the standard literature for principal-agent problems with one-sided commitment, in (21), we consider a risk-neutral principal. We note that the slightly more general case with a discount factor r can be easily transformed into our model. Indeed, consider

$$J_P(\theta;0,\eta) := \mathbb{E}^{\alpha^*}\left[e^{-rT}X_T - \int_0^T e^{-rs}\eta_s ds\right] = e^{-rT}\mathbb{E}^{\theta,Z^{\theta,\eta}}\left[\int_0^T [I^h(\theta;s,Z_s^{\theta,\eta}) - e^{r(T-s)}\eta_s]ds\right].$$

Denote

$$\tilde{\eta}_s := e^{r(T-s)}\eta_s, \ \tilde{Z}^{\theta;\tilde{\eta}} := Z^{\theta,\tilde{\eta}e^{r(\cdot-T)}}, \ \tilde{J}_P(\theta;0,\tilde{\eta}) := \mathbb{E}^{\theta,\tilde{Z}^{\theta,\tilde{\eta}}}\left[\int_0^T [I^h(\theta;s,\tilde{Z}_s^{\theta,\tilde{\eta}}) - \tilde{\eta}_s]ds\right].$$

Then, the principal's problem is equivalent to optimizing $\tilde{J}_P$, which has no discount factor.

(iii) The case with a risk-averse principal is technically harder. Moreover, in (6), we assume the agent's utility f and cost h are separable. The general nonseparable case in the form $f(\theta;s,\eta,\alpha)$ is also technically harder. We do not pursue such generality in this paper.

Introduce the agent's individual rationality R_t^θ at time t, which can be interpreted as the agent's market value if the agent works for a principal over period $[t,T]$. In this paper, we assume R^θ is deterministic and given exogenously, and we leave more general cases to future research. In light of (18), we assume the following, with the same Λ_0, ρ for simplicity.

Assumption 2.

i. $-\Lambda_0 \le R_s^\theta \le \overline{L}_s^\theta$, $0 \le s \le T$, *and* $R_T^\theta = 0$.

ii. R^θ *is uniformly Lipschitz continuous in s, with modulus of continuity function ρ.*

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In the standard literature, the principal's problem is as follows:

$$V_0^{S,P}(\theta;0) := \sup_{\eta\in\mathcal{A}^P} J_P(\theta;0,\eta), \quad \text{subject to} \quad V_0^A(\theta;0,\eta) \ge R_0^\theta. \tag{22}$$

Here, ^{S} stands for standard. We shall provide more discussions on $V_0^{S,P}(\theta;0)$ in Section 7.1 below. In this paper, we mainly focus on self-enforcing contracts: recalling Proposition 1,

$$V_0^P(\theta;0) := \sup_{\eta\in\mathcal{A}^P} J_P(\theta;0,\eta), \quad \text{subject to} \quad Y_t^{\theta,\eta} \ge R_t^\theta,\ 0\le t\le T,\ \text{a.s.} \tag{23}$$

By (18), it is clear that $\eta \equiv C_0$ satisfies the constraints, and thus,

$$V_0^{S,P}(\theta;0) \ge V_0^P(\theta;0) \ge J_P(\theta;0,C_0) = -C_0(T-t). \tag{24}$$

2.3. The Principal's Dynamic Value

We next extend the above principal-agent problem to the arbitrary time interval $[t,T]$. Denote by $B_s^t := B_s - B_t$ the shifted Brownian motion on $[t,T]$, $\mathbb{F}^t := \mathbb{F}^{B^t}$ and $\mathcal{T}_t$ the set of $\mathbb{F}^t$-stopping times on $[t,T]$. We shall extend all the notations in the previous subsection to the shifted space in an obvious manner. In particular, $\mathcal{A}_t^P, \mathcal{A}_t^A$ will denote the set of $\mathbb{F}^t$-progressively measurable processes η,α, but we shall still denote by $(Y^{\theta,\eta}, Z^{\theta,\eta})$ the solution to the BSDE (13) on $[t,T]$. We then define the principal's dynamic value over self-enforcing contracts as follows:

$$\begin{aligned} V_0^P(\theta;t) &:= \sup_{\eta\in\mathcal{A}_t^P} J_P(\theta;t,\eta), \quad \text{subject to} \quad Y_s^{\theta,\eta} \ge R_s^\theta,\ t\le s\le T,\ \text{a.s.} \\ &\text{where} \quad J_P(\theta;t,\eta) := \mathbb{E}^{\theta,Z^{\theta,\eta}}\left[\int_t^T [I^h(\theta;s,Z_s^{\theta;\eta}) - \eta_s]ds\right]. \end{aligned} \tag{25}$$

The following result states that the principal's optimal utility over self-enforcing contracts is uniformly continuous with respect to the contracting period, more precisely, the initial time of the contract. This is interesting in its own right and will be important technically in the next section. We postpone its proof to the appendix.

Proposition 2. *Under Assumptions 1 and 2, $V_0^P(\theta;\cdot)$ is uniformly continuous in t:*

$$|V_0^P(\theta;s) - V_0^P(\theta;t)| \le C\sqrt{\rho(|s-t|)}, \quad \forall s,t\in[0,T]. \tag{26}$$

3. The Case with at Most One Quitting

We now assume the principal offers a standard contract that meets the individual reservation only at initial time. The agent might quit before T, at a stopping time up to his choice, if the agent finds a better opportunity in the market. Then, the principal will hire another agent with a new contract. In this section, we consider the case that, once the current agent quits the job, the principal will offer only a self-enforcing new contract. That is, the principal will tolerate only one quitting. We assume there is a pool of agents in the market with parameter set Θ. For each agent $\theta\in\Theta$, if the agent quits at time t, the agent will incur a cost c_t^θ, and the principal will bear a cost c_t^P, which (for simplicity) does not depend on θ. Recall (18), and denote further that

$$\underline{L}_t^\theta := R_t^\theta - c_t^\theta, \qquad D_\theta := \{(t,x): 0\le t < T, \underline{L}_t^\theta < x \le \overline{L}_t^\theta\}, \quad T_\delta := T-\delta. \tag{27}$$

In this paper, we assume c^θ, c^P are deterministic and given exogenously, and we leave more general cases to future research. The following technical conditions will be in force.

Assumption 3. *R^θ satisfies Assumption 2, and for the same Λ_0,ρ (for simplicity),*

i. *$c_t^\theta \ge \frac{1}{\Lambda_0} > 0$, $c_t^P \ge 0$, and consequently, $\overline{L}_t^\theta - \underline{L}_t^\theta \ge \frac{1}{\Lambda_0}$, $0\le t\le T$;*
ii. *c^θ, c^P are uniformly continuous in t with modulus of continuity function ρ;*
iii. *c^θ, c^P are bounded by Λ_0, and $|\underline{L}_t^\theta - \underline{L}_s^\theta| \le \Lambda_1|t-s|$ for some $\Lambda_1 > 0$.*

From now on, we allow the generic constant $C>0$ to depend on Λ_1 as well.

3.1. The Agent's Problem

Consider the problem of time interval $[t,T]$ with an agent θ at initial time t and contract $\eta\in\mathcal{A}_t^P$. The agent can choose a stopping time $\tau\in\mathcal{T}_t$ to quit the job. In that case, the agent would receive the payment η from the

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principal over $[t,\tau]$, and get a new job with expected utility R_τ^θ for the remaining period $[\tau,T]$, and in the meantime, the agent would incur the cost c_τ^θ. He may also choose not to quit, and in that case, we set $\tau = T$. Therefore, given the contract $\eta \in \mathcal{A}_t^P$, the agent's optimal utility is

$$V_1^A(\theta;t,\eta) := \sup_{\tau\in\mathcal{T}_t} \sup_{\alpha\in\mathcal{A}_t^A} \mathbb{E}^\alpha \cdot \left[\int_t^\tau [f(\theta;s,\eta_s) - h(\theta;s,\alpha_s)\right] ds + \underline{L}_\tau^\theta \mathbf{1}_{\{\tau<T\}}]. \tag{28}$$

Here, the subscript $_1$ in V_1^A indicates that the agent can quit once. This problem is solved through a reflected BSDE (cf. Zhang [41, chapter 6]):

$$Y_s^{\theta;t,\eta} = \int_s^T [f(\theta;r,\eta_r) + H(\theta;r,Z_r^{\theta;t,\eta})]dr - \int_s^T Z_r^{\theta;t,\eta} dB_r + K_T^{\theta;t,\eta} - K_s^{\theta;t,\eta},$$
$$Y_s^{\theta;t,\eta} \ge \underline{L}_s^\theta, \qquad \int_t^T [Y_s^{\theta;t,\eta} - \underline{L}_s^\theta]\, dK_s^{\theta;t,\eta} = 0. \tag{29}$$

Here, the solution is a triplet of processes $(Y^{\theta;t,\eta}, Z^{\theta;t,\eta}, K^{\theta;t,\eta})$, and $K^{\theta;t,\eta}$ is a nondecreasing continuous process with $K_t^{\theta;t,\eta} = 0$. In particular, we use the subscript ^{t} to denote the solution to RBSDE (29) while using $(Y^{\theta,\eta}, Z^{\theta,\eta})$ for BSDE (13) (even if it is on $[t,T]$). We introduce, for the first time, that $Y^{\theta;t,\eta}$ hits the barrier

$$\tau^{\theta;t,\eta} := \inf\{s \ge t : Y_s^{\theta;t,\eta} = \underline{L}_s^\theta\} \wedge T. \tag{30}$$

We then have the following result, which solves the agent's problem in this case.

Proposition 3. *Let Assumptions 1 and 3 hold true. Then, for any $\theta \in \Theta$ and $t < T$,*

i. *$V_1^A(\theta;t,\eta) = Y_t^{\theta;t,\eta}$, the $\tau^{\theta;t,\eta} \in \mathcal{T}_t$ in (30) is an optimal quitting time, and $\alpha^{\theta;t,\eta} := I^h(\theta;s,Z^{\theta;t,\eta})\mathbf{1}_{[t,\tau^{\theta;t,\eta}]} \in \mathcal{A}_t^A$ is the agent's unique optimal effort (unique up to $\tau^{\theta;t,\eta}$).*

ii. *$\{Y_t^{\theta;t,\eta} : \eta \in \mathcal{A}_t^P\} = [\underline{L}_t^\theta, \overline{L}_t^\theta]$.*

We note that (ii) provides the bounds for the agent's optimal utilities over all possible contracts. As in (14), later on, we will rewrite the agent's continuation utility in a forward form; then, such bounds will be crucial for the analysis.

Remark 3. We remark that the agent's optimal quitting time may not be unique, and among them, $\tau^{\theta;t,\eta}$ is the smallest one. Unfortunately, whereas the agent's optimal utility is invariant on his choices of the optimal quitting time, the principal's optimal utility does depend on this choice. For technical simplicity, in this paper, we take the convention that the agent would always take this smallest one, $\tau^{\theta;t,\eta}$; namely, the agent would choose to quit when the agent is indifferent to quitting or staying.

Proof of Proposition 3 relies on the following simple lemma, which will be used frequently in the paper. We postpone both proofs to the appendix.

Lemma 2. *Let Assumption 1 hold. Fix $t < T$, $\delta > 0$, and recall (14) and (15).*

i. *For any $x \le \overline{L}_t^\theta$, there exists a deterministic $\eta^{\theta;t,x} \in \mathcal{A}_t^P$ such that*

$$(\eta^{\theta;t,x}, 0) \in \mathcal{A}_{t,x}^\theta, \quad \int_t^T [C_0 - \eta_s^{\theta;t,x}]ds \le \delta. \tag{31}$$

More precisely, $\eta^{\theta;t,x} := \eta^n$ can be constructed as follows: for some n sufficiently large,

$$\varepsilon_n := \frac{\overline{L}_t^\theta - x}{n} < T - t, \quad \eta_s^n := I^f(\theta;s,f(\theta;s,C_0) - n)\mathbf{1}_{[t,t+\varepsilon_n)} + C_0 \mathbf{1}_{[t+\varepsilon_n,T]}. \tag{32}$$

ii. *Assume further that Assumption 3 holds and $x > \underline{L}_t^\theta$. Then, we may and will always construct $\eta^{\theta;t,x}$ in a way such that $(Y,Z,K) := (\mathcal{X}^{\theta;t,x,\eta^{\theta;t,x},0}, 0, 0)$ solves RBSDE (29) corresponding to $\eta^{\theta;t,x}$. In particular, $\mathcal{X}_s^{\theta;t,x,\eta^{\theta;t,x},0} > \overline{L}_s^\theta$ for all $s \in [t,T]$, and $\tau^{\theta;t,\eta^{\theta;t,x}} = T$.*

Remark 4. In the rest of the paper, we will often use a random version of Lemma 2 under Assumptions 1 and 3. Fix $\theta \in \Theta$, $(t,x) \in \mathcal{A}_{t,x}^\theta$, $(\eta,Z) \in \mathcal{A}_{t,x}^\theta$, and $\tau \in \mathcal{T}_t$ such that $\tau \le \tau^{\theta;t,\eta} \wedge T_\delta$. Then, we may construct $\eta^{\theta;\tau,\mathcal{X}_\tau^{\theta;t,x,\eta,Z}}$ in a way such that

$$(\tilde{\eta},\tilde{Z}) := (\eta,Z)\mathbf{1}_{[t,\tau)} + (\eta^{\theta;\tau,\mathcal{X}_\tau^{\theta;t,x,\eta,Z}}, 0)\mathbf{1}_{[\tau,T]} \in \mathcal{A}_{t,x}^\theta.$$

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Indeed, from Construction (32), we can easily see that the above $\tilde{\eta}, \tilde{Z}$ are $\mathbb{F}^t$-progressively measurable, $\tilde{Z} \in \mathcal{A}_t^A$, and $\mathcal{X}_T^{\theta;t,x,\tilde{\eta},\tilde{Z}} = 0$. Moreover, because $\tau \le \tau^{\theta;t,\eta}$, then $\mathcal{X}_\tau^{\theta;t,x,\tilde{\eta},\tilde{Z}} \in [\underline{L}_\tau^\theta, \overline{L}_\tau^\theta]$ is bounded. Note further that $T - \tau \ge T - T_\delta = \delta$; then, from Construction (32), we can easily see that $\eta^{\theta;\tau,\mathcal{X}_\tau^{\theta;t,x,\eta,Z}}$ is uniformly bounded, which implies that $\tilde{\eta}$ has a uniform lower bound, and hence, $\tilde{\eta} \in \mathcal{A}_t^P$.

3.2. The Principal's Problem

Assume the principal hires an agent θ at time t with a contract $\eta \in \mathcal{A}_t^P$. At time $\tau^{\theta;t,\eta}$, the agent would quit the job; then, the principal will hire a new agent from the market, with possibly a different type $\tilde{\theta}$. In this section, we assume the principal does not want the new agent to quit, by offering only self-enforcing contracts. Then, by the previous section, the principal's optimal value for the new contract will be $V_0^P(\tilde{\theta}; \tau^{\theta;t,\eta})$. The principal will choose $\tilde{\theta}$ to maximize this value, and recall that the principal will incur a cost c^P, so we introduce

$$\overline{u}_0(t) := \sup_{\tilde{\theta} \in \Theta} V_0^P(\tilde{\theta}; t) - c_t^P. \tag{33}$$

Clearly, $\overline{u}_0$ inherits the regularity in Proposition 2, and it is bounded. We collect these results in the following simple lemma and provide a proof in the appendix for completeness.

Lemma 3. *Under Assumptions 1 and 3, $\overline{u}_0$ is uniformly continuous on $[0,T]$ with a modulus of continuity function $\rho_0 := C\sqrt{\rho}$. Moreover, $\overline{u}_0$ is bounded with $\overline{u}_0(T) = -c_T^P$.*

The principal's problem in this case is

$$V_1^P(\theta; t) := \sup_{\eta \in \mathcal{A}_t^P} \mathbb{E}^{\theta, Z^{\theta;t,\eta}} \left[\int_t^{\tau^{\theta;t,\eta}} [I^h(\theta; s, Z^{\theta;t,\eta}) - \eta_s] ds + \overline{u}_0(\tau^{\theta;t,\eta}) \mathbf{1}_{\{\tau^{\theta;t,\eta} < T\}} \right],$$
$$\text{subject to} \quad V_1^A(\theta; t, \eta) \ge R_t^\theta. \tag{34}$$

We remark that, here, we consider the principal's utility for a given θ. In practice, of course, the principal would choose an optimal θ in the beginning, which amounts to computing

$$V_1^P(t) := \sup_{\theta \in \Theta} V_1^P(\theta; t).$$

Remark 5. We emphasize that in our model, there is a family of agents whose types are indexed by θ. The principal knows the value θ of each agent and can choose the optimal θ. This is completely different from the adverse selection problems, which involve only one agent, but the principal does not know the agent's type θ and cannot choose θ (cf. Cvitanić et al. [7]).

Similar to (14), we can rewrite RBSDE (29) in a forward form, which leads to the following dynamic utility function:

$$\begin{aligned} u_1(\theta; t, x) &:= \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^\theta} J_1(\theta; t, x, \eta, Z), \quad (t, x) \in D_\theta, \\ \text{where} \quad \tau^{\theta;t,x,\eta,Z} &:= \inf\{s \ge t : \mathcal{X}_s^{\theta;t,x,\eta,Z} = \underline{L}_s^\theta\} \wedge T, \\ J_1(\theta; t, x, \eta, Z) &:= \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta;t,x,\eta,Z}} [I^h(\theta; s, Z_s) - \eta_s] ds + \overline{u}_0(\tau^{\theta;t,x,\eta,Z}) \mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z} < T\}} \right]. \end{aligned} \tag{35}$$

The following result establishes the connection between V_1^P and u_1 and thus enables us to use the partial differential equation (PDE) approach for u_1 to study V_1^P; see the next section.

Proposition 4. *Under Assumptions 1 and 3, for any $\theta \in \Theta$ and $t < T$, we have*

$$V_1^P(\theta; t) = \sup_{R_t^\theta \le x \le \overline{L}_t^\theta} u_1(\theta; t, x). \tag{36}$$

Proof. Let $\tilde{J}_1(\theta; t, \eta)$ denote the expectation on the right side of (34) for a fixed η. For any $\eta \in \mathcal{A}_t^P$ satisfying the constraint in (34), by Proposition 3, we have $x := Y_t^{\theta;t,\eta} = V_1^A(\theta; t, \eta) \in [R_t^\theta, \overline{L}_t^\theta]$. Fix $\delta > 0$ small and set

$$\tau_\delta := \tau^{\theta;t,\eta} \wedge T_\delta, \ (\eta^\delta, Z^\delta)_s := (\eta, Z^{\theta;t,\eta})_s \mathbf{1}_{[t,\tau_\delta)}(s) + (\eta_s^{\theta;\tau_\delta, Y_{\tau_\delta}^{\theta;t,\eta}}, 0) \mathbf{1}_{[\tau_\delta, T]}(s), \tag{37}$$

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where $\eta^{\theta;\tau_\delta,Y^{\theta;t,\eta}_{\tau_\delta}}$ is constructed in (32). By Remark 4, we have $(\eta^\delta, Z^\delta) \in \mathcal{A}^\theta_{t,x}$. By (30), we have $Y^{\theta;t,\eta}_s > \underline{L}^\theta_s$, and hence, $K^{\theta;t,\eta}_s = 0$ for $s \in [t, \tau^{\theta;t,\eta})$. Then,

$$\mathcal{X}^{\theta;t,x,\eta^\delta,Z^\delta}_s = Y^{\theta;t,\eta}_s > \underline{L}^\theta_s,\ s \in [t,\tau_\delta), \quad \text{and} \quad \tau^{\theta;t,x,\eta^\delta,Z^\delta} = \tau^{\theta;t,\eta} \text{ on } \{\tau^{\theta;t,\eta} \le T_\delta\}.$$

Moreover, by Lemma 2(ii), we have $\tau^{\theta;t,x,\eta^\delta,Z^\delta} = T$ on $\{\tau^{\theta;t,\eta} > T_\delta\}$. Then, because $\eta^{\theta;T_\delta,Y^{\theta;t,\eta}_{T_\delta}}_s$ is $\mathcal{F}_{T_\delta}$-measurable, we have

$$J_1(\theta;t,x,\eta^\delta,Z^\delta) = \mathbb{E}^{\theta,Z^\delta}[\Xi] = \mathbb{E}^{\theta,Z^{\theta;t,\eta}}[\Xi], \quad \text{where}$$

$$\Xi := \int_t^{\tau_\delta} [I^h(\theta;s,Z^{\theta;t,\eta}_s) - \eta_s]ds + \mathbf{1}_{\{\tau^{\theta;t,\eta} \le T_\delta\}} \overline{u}_0(\tau^{\theta;t,\eta}) - \mathbf{1}_{\{\tau^{\theta;t,\eta} > T_\delta\}} \int_{T_\delta}^T \eta^{\theta;T_\delta,Y^{\theta;t,\eta}_{T_\delta}}_s ds.$$

Thus, by omitting the superscripts $^{\theta;t,\eta}$, namely, denoting $(\tau, Y, Z) := (\tau^{\theta;t,\eta}, Y^{\theta;t,\eta}, Z^{\theta;t,\eta})$,

$$\begin{aligned}
\tilde{J}_1(\theta;t,\eta) - u_1(\theta;t,x) &\le \tilde{J}_1(\theta;t,\eta) - J_1(\theta;t,x,\eta^\delta,Z^\delta) \\
&= \mathbb{E}^{\theta,Z}\left[\mathbf{1}_{\{\tau>T_\delta\}}\left[\int_{T_\delta}^{\tau}[I^h(\theta;s,Z_s) - \eta_s]ds + \int_{T_\delta}^T \eta^{\theta;T_\delta,Y_{T_\delta}}_s ds + \overline{u}_0(\tau)\mathbf{1}_{\{\tau<T\}}\right]\right] \\
&\le \mathbb{E}^{\theta,Z}\left[\int_{T_\delta}^T [|I^h(\theta;s,Z_s)| + |\eta_s|]ds + C\mathbf{1}_{\{T_\delta<\tau<T\}}\right] + C_0\delta,
\end{aligned}$$

where the last inequality is because of Lemma 3 and the fact $\eta^{\theta;T_\delta,Y_{T_\delta}}_s \le C_0$. Send $\delta \to 0$; we obtain $\tilde{J}_1(\theta;t,\eta) \le u_1(\theta;t,x)$. Because η is arbitrary, we prove the "$\le$" part of the proposition.

To see the opposite direction, for any $x \in [R^\theta_t, \overline{L}^\theta_t]$, $(\eta, Z) \in \mathcal{A}^\theta_{t,x}$, and $\delta > 0$ small, set

$$\tilde{\tau}_\delta := \tau^{\theta;t,x,\eta,Z} \wedge T_\delta, \quad (\tilde{\eta}^\delta, \tilde{Z}^\delta)_s := (\eta, Z)_s \mathbf{1}_{[t,\tilde{\tau}_\delta)}(s) + (\eta^{\theta;\tilde{\tau}_\delta,\mathcal{X}^{\theta;t,x,\eta,Z}_{\tilde{\tau}_\delta}}_s, 0)\mathbf{1}_{[\tilde{\tau}_\delta,T]}(s).$$

Following similar arguments as above, by sending $\delta \to 0$, one can show that $J_1(\theta;t,x,\eta,Z) \le V^p_1(\theta;t)$. Then, because x and (η, Z) are arbitrary, we obtain the desired inequality. □

Remark 6. In this section, we assume the principal will offer self-enforcing contracts to hire a new agent when the current one quits. There can be other alternatives; for example, the principal may close the business or run the firm by herself. In those cases, we need to first model the principal's outside option, denoted as $\overline{u}'_0(t)$, and then study the principal's problem by replacing the $\overline{u}_0$ in (34) with $\overline{u}'_0$. Provided that $\overline{u}'_0$ satisfies the technical properties in Lemma 3, the main results in this paper will remain true.

4. The Dynamic Value Function u_1

In this section, we shall characterize the function u_1 as the minimal solution of an HJB equation. For this purpose, we shall first establish its monotonicity, regularity, and boundary conditions. Their proofs involve very technical constructions. Because they are the main mathematical components of the paper, we report them at the end of this section. However, readers who are not interested in the technical part can skip reading these subsections, which will not affect their understanding of the economic messages of the paper.

We first show that u_1 is decreasing in x. That is, the principal's optimal utility would suffer when the agent's continuation utility x increases. Equivalently, the principal will gain less utility when the agent becomes more expensive, which, of course, is natural in practice.

Proposition 5. *Let Assumptions 1 and 3 hold. For any $\theta \in \Theta$ and $t < T$, the value function $u_1(\theta;t,x)$ is decreasing in $x \in (\underline{L}^\theta_t, \overline{L}^\theta_t]$. Consequently,*

$$V^p_1(\theta;t) = u_1(\theta;t,R^\theta_t). \tag{38}$$

We next show that u_1 is uniformly continuous in (t, x). Such regularity is important in its own right and is crucial for the PDE characterization of u_1.

Proposition 6. *Let Assumptions 1 and 3 hold.*

i. *For any $\theta \in \Theta$ and $t < T$, the value function $u_1(\theta;t,x)$ is uniformly continuous in $x \in (\underline{L}^\theta_t, \overline{L}^\theta_t]$. That is, for any $\underline{L}^\theta_t < x_1, x_2 \le \overline{L}^\theta_t$, we have*

$$|u_1(\theta;t,x_1) - u_1(\theta;t,x_2)| \le C\rho_0(|x_1 - x_2|), \tag{39}$$

where ρ_0 is the modulus of continuity function of $\overline{u}_0$ specified in Lemma 3.

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ii. *For any* $(t_i, x_i) \in D_\theta$, *we have*

$$|u_1(\theta; t_1, x_1) - u_1(\theta; t_2, x_2)| \le C\rho_0(\rho(|t_1 - t_2|) + |x_1 - x_2|). \tag{40}$$

Note that $u_1(\theta; t, x)$ involves the hitting time $\tau^{\theta;t,x,\eta,Z}$, which is, in general, not continuous in (t, x). This can be circumvented as follows. Assume we want to estimate $u_1(\theta; t_1, x_1) - u_1(\theta; t_2, x_2)$ from above. For any given $(\eta^2, Z^2) \in \mathcal{A}^\theta_{t_2,x_2}$, we want to construct $(\eta^1, Z^1) \in \mathcal{A}^\theta_{t_1,x_1}$ so that $\tau^{\theta;t_1,x_1,\eta^1,Z^1}$ is close to $\tau^{\theta;t_2,x_2,\eta^2,Z^2}$ (in addition to other small errors). However, because we have both upper and lower bounds for $\mathcal{X}$ and because the setting is random, such construction is highly nontrivial and will rely heavily on Lemma 2. In particular, we emphasize that the upper bound $\overline{L}^\theta$ and lower bound $\underline{L}^\theta$ play very different roles in the problem; consequently, the constructions are quite different for the two cases $x_1 < x_2$ and $x_1 > x_2$ and even more different for the two cases $t_1 < t_2$ and $t_1 > t_2$. We refer to Section 4.2 for details.

Remark 7. In general, the monotonicity property and the continuity of u_1 may fail at the boundary point $x = \underline{L}^\theta_t$. For example, for any $x \in (\underline{L}^\theta_t, \overline{L}^\theta_t]$, construct η^n as in (32), and send $n \to \infty$; one can easily see that under these contracts, the agent never quits, and $u_1(\theta; t, x) \ge -C_0(T - t)$. However, for $x = \underline{L}^\theta_t$, the agent quits immediately at t, and thus, $u_1(t, \underline{L}^\theta_t) = \overline{u}_0(t) = \sup_{\tilde\theta} V_0^P(\tilde\theta; t) - c_t^P$. Clearly, it is possible that $-C_0(T - t) > \overline{u}_0(t)$ when c_t^P is sufficiently large.

Finally, u_1 satisfies the following boundary conditions:

Proposition 7. *Under Assumptions 1 and 3,* $u_1(\theta; \cdot, \cdot)$ *has boundary conditions*

$$u_1(\theta; t, \overline{L}^\theta_t) = -C_0(T - t), \quad \forall t \in [0, T); \qquad u_1(\theta; T-, x) = 0, \quad \forall x \in (\underline{L}^\theta_T, 0);$$
$$u_1(\theta; t, \underline{L}^\theta_t+) \ge \overline{u}_0(t), \quad \forall t \in [0, T). \tag{41}$$

We are now ready to characterize u_1 through an HJB equation. Introduce

$$\mathcal{L}^\theta u(t, x) := \partial_t u + \sup_{\eta \le C_0, z \in \mathbb{R}} \left[\frac{1}{2}|z|^2 \partial_{xx} u - [f(\theta; t, \eta) + \tilde{H}(\theta; t, z)]\partial_x u + I^h(\theta; t, z) - \eta\right]. \tag{42}$$

Theorem 1. *Under Assumptions 1 and 3, the value function* $u_1(\theta; \cdot, \cdot)$ *is the minimal continuous viscosity solution of the following HJB equation with Boundary Conditions* (41)*:*

$$\mathcal{L}^\theta u(t, x) = 0, \quad (t, x) \in D_\theta. \tag{43}$$

Proof. First, by the regularity in Proposition 6(i), one may follow rather standard arguments to obtain the dynamic programming principle: for any $(t, x) \in D_\theta$ and $\tau \in \mathcal{T}_t$,

$$u_1(\theta; t, x) := \sup_{(\eta, Z) \in A^\theta_{t,x}} \mathbb{E}^{\theta,Z}\left[\int_t^{\tau \wedge \tau^{\theta;t,x,\eta,Z}} [I^h(\theta; r, Z_r) - \eta_r]dr + u_1(\theta; \tau, X_\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau < \tau^{\theta;t,x,\eta,Z}\}}\right.$$
$$\left. + \overline{u}_0(\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z} \le \tau,\ \tau^{\theta;t,x,\eta,Z} < T\}}\right]. \tag{44}$$

Then, by (14), it is clear that u_1 is a viscosity solution of (43). Moreover, by Proposition 7, it satisfies the Boundary Conditions (41).

We next show that u_1 is minimal. Assume u is an arbitrary continuous viscosity solution of (43) satisfying (41). In particular, $u(t, \underline{L}^\theta_t) \ge \overline{u}_0(t)$. For each $n \ge 1$, introduce

$$u^{(n)}(t, x) := \sup_{(\eta, Z) \in A_t^P \times A_t^A,\ -n \le \eta \le C_0} \mathbb{E}^{\theta,Z}\left[\int_t^\tau [I^h(\theta; r, Z_r) - \eta_r]dr + u(\tau, X_\tau^{\theta;t,x,\eta,Z})\right],$$
$$\text{where} \quad \tau := \underline{\tau} \wedge \overline{\tau}, \quad \underline{\tau} := \tau^{\theta;t,x,\eta,Z}, \quad \overline{\tau} := \inf\{s \ge t : X_s^{\theta;t,x,\eta,Z} = \overline{L}^\theta_s)\} \wedge T. \tag{45}$$

Because η is uniformly bounded here, one can easily see that $u^{(n)}$ is continuous on $\overline{D}_\theta$, the closure of D_θ. Then, $u^{(n)}$ is the unique viscosity solution of the truncated HJB equation:

$$\partial_t u^{(n)} + \sup_{-n \le \eta \le C_0, z \in \mathbb{R}} \left[\frac{1}{2}|z|^2 \partial_{xx} u^{(n)} - [f(\theta; t, \eta) + \tilde{H}(\theta; t, z)]\partial_x u^{(n)} + I^h(\theta; t, z) - \eta\right] = 0;$$
$$u^{(n)}(t, \overline{L}^\theta_t) = -C_0(T - t), \quad u^{(n)}(T, x) = 0, \quad u^{(n)}(t, \underline{L}^\theta_t) = u(t, \underline{L}^\theta_t). \tag{46}$$

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Note that u is a viscosity supersolution of the above HJB equation with the same boundary conditions; then, by the comparison principle for (46), we have $u \geq u^{(n)}$.

Now for any $(t,x) \in D_\theta$ and $(\eta, Z) \in \mathcal{A}^\theta_{t,x}$, recall Definition 1, and let $n \geq C(\eta)$; then, (η, Z) satisfy the requirements in (45). Let $\tau, \underline{\tau}, \overline{\tau}$ be defined in (45). Note that

$$u_1(\theta; \overline{\tau}, \overline{L}^\theta_{\overline{\tau}}) = -C_0(T - \overline{\tau}) = u(\overline{\tau}, \overline{L}^\theta_{\overline{\tau}}), \quad \overline{u}_0(\underline{\tau}) \leq u(\underline{\tau}, \underline{L}^\theta_{\underline{\tau}}) \text{ on } \{\underline{\tau} < T\}.$$

Similarly to the Dynamic Programming Principle (44), we have

$$\begin{aligned}
&J_1(\theta; t, x, \eta, Z)\\
&\leq \mathbb{E}^{\theta,Z}\left[\int_t^{\overline{\tau}\wedge\underline{\tau}} |[I^h(\theta; r, Z_r) - \eta_r]dr + \overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}\leq\overline{\tau}\}} + u_1(\theta; \overline{\tau}, X^{\theta;t,x,\eta,Z}_{\overline{\tau}})\mathbf{1}_{\{\underline{\tau}>\overline{\tau}\}}\right]\\
&= \mathbb{E}^{\theta,Z}\left[\int_t^{\tau} [I^h(\theta; r, Z_r) - \eta_r]dr + \overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}\leq\overline{\tau}\}} + u_1(\theta; \overline{\tau}, \overline{L}^\theta_{\overline{\tau}})\mathbf{1}_{\{\underline{\tau}>\overline{\tau}\}}\right]\\
&\leq \mathbb{E}^{\theta,Z}\left[\int_t^{\tau} [I^h(\theta; r, Z_r) - \eta_r]dr + u(\underline{\tau}, \underline{L}^\theta_{\underline{\tau}})\mathbf{1}_{\{\underline{\tau}\leq\overline{\tau}\}} + u(\overline{\tau}, \overline{L}^\theta_{\overline{\tau}})\mathbf{1}_{\{\underline{\tau}>\overline{\tau}\}}\right]\\
&= \mathbb{E}^{\theta,Z}\left[\int_t^{\tau} [I^h(\theta; r, Z_r) - \eta_r]dr + u(\tau, X^{\theta;t,x,\eta,Z}_{\tau})\right] \leq u^{(n)}(t,x) \leq u(t,x).
\end{aligned}$$

Because $(\eta, Z) \in \mathcal{A}^\theta_{t,x}$ is arbitrary, we obtain $u_1(\theta; t, x) \leq u(t,x)$. □

Remark 8. (i) As explained in Remark 7, in general, it is possible that $u_1(\theta; t, \underline{L}^\theta_t+) > \overline{u}_0(t)$. One can show that $u_1(\theta; \cdot, \cdot)$ is the unique viscosity solution to (43) if we replace the boundary condition at $x = \underline{L}^\theta_t$ with $u_1(\theta; t, \underline{L}^\theta_t+)$. However, we are not able to compute $u_1(\theta; t, \underline{L}^\theta_t+)$ explicitly, so we characterize u_1 as the minimal solution in Theorem 1.

(ii) The differential operator $\mathcal{L}^\theta$ in (42) is the same as the case without constraint on (η, Z). The constraint $(\eta, Z) \in \mathcal{A}^\theta_{t,x}$ in (35) is reflected in the Boundary Conditions (41).

4.1. The Monotonicity: Proof of Proposition 5

Proof of Proposition 5. First, (38) follows directly from Proposition 4 and the monotonicity of u_1. Without loss of generality, we prove the monotonicity only at $t = 0$. Fix $\underline{L}^\theta_0 < x < x_\delta := x + \delta \leq \overline{L}^\theta_0$, and assume $\delta > 0$ is sufficiently small. For any $(\eta, Z) \in A^\theta_{0,x_\delta}$ and $n \geq 1$ such that $\frac{\delta}{n} < T$, our goal is to construct $(\eta^n, Z^n) \in \mathcal{A}^\theta_{0,x}$ such that

$$J_1(\theta; 0, x_\delta, \eta, Z) - u_1(\theta; 0, x) \leq J_1(\theta; 0, x_\delta, \eta, Z) - J_1(\theta; 0, x, \eta^n, Z^n) \to 0, \text{ as } n \to \infty. \tag{47}$$

Then, by the arbitrariness of (η, Z), we obtain $u_1(\theta; 0, x_\delta) \leq u_1(\theta; 0, x)$.

To see this, similarly to (32) and (37), we construct

$$\begin{aligned}
&t_n := \frac{\delta}{n}, \qquad \eta^n_s := I^f(\theta; s, f(\theta; s, \eta_s) - n), \quad Z^n_s := Z_s, \quad 0 \leq s \leq t_n;\\
&\tau_n := \inf\{s \geq 0 : \mathcal{X}^{\theta;0,x,\eta^n,Z^n}_s = \underline{L}^\theta_s\} \wedge t_n, \quad \tau^\delta := \tau^{\theta;0,x_\delta,\eta,Z};\\
&(\eta^n_s, Z^n_s) := (\eta^{\theta;\tau_n,\underline{L}^\theta_{\tau_n}}_s, 0)\mathbf{1}_{\{\tau_n<t_n\}} + (\eta_s, Z_s)\mathbf{1}_{\{\tau_n=t_n\}}, \quad \tau_n < s \leq T,
\end{aligned} \tag{48}$$

where $\eta^{\theta;\tau_n,\underline{L}^\theta_{\tau_n}}$ is as in Lemma 2 (with an arbitrary δ in (31)) and Remark 4. Denote $\mathcal{X}^n := \mathcal{X}^{\theta;0,x,\eta^n,Z^n}$. It is clear that $(\eta^n, Z^n) \in \mathcal{A}^P_0 \times \mathcal{A}^A_0$, and by Lemma 2, we see that $\mathcal{X}^n_T = 0$ on $\{\tau_n < t_n\}$. Note further that $f(\theta; s, \eta^n_s) = f(\theta; s, \eta_s) - n$; then,

$$\mathcal{X}^n_s - \mathcal{X}^{\theta;0,x_\delta,\eta,Z}_s = ns - \delta < 0 \text{ for } s \in [0, t_n), \quad \text{and} \quad \mathcal{X}^n_{t_n} - \mathcal{X}^{\theta;0,x_\delta,\eta,Z}_{t_n} = 0. \tag{49}$$

Then, because $(\eta^n, Z^n) = (\eta, Z)$ on $[t_n, T]$, we have $\mathcal{X}^n_T = \mathcal{X}^{\theta;0,x_\delta,\eta,Z}_T = 0$ on $\{\tau_n = t_n\}$. Therefore, $\mathcal{X}^n_T = 0$ in all the cases, and thus, $(\eta^n, Z^n) \in \mathcal{A}^\theta_{0,x}$.

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Moreover, note that (49) implies that

$$\tau^{\theta;0,x,\eta^n,Z^n} = \tau_n \mathbf{1}_{\{\tau_n<t_n\}} + \tau^\delta \mathbf{1}_{\{\tau_n=t_n\}} \le \tau^\delta, \quad Z^n_s = Z_s,\ 0 \le s \le \tau^{\theta;0,x,\eta^n,Z^n}.$$

Then,

$$J_1(\theta;0,x,\eta^n,Z^n) = \mathbb{E}^{\theta,Z}\left[\mathbf{1}_{\{\tau_n<t_n\}}\left[\int_t^{\tau_n}[I^h(\theta;s,Z_s)-\eta^n_s]ds+\overline{u}_0(\tau_n)\right]+\mathbf{1}_{\{\tau_n=t_n\}}\left[\int_0^{\tau^\delta}[I^h(\theta;s,Z_s)-\eta^n_s]ds+\overline{u}_0(\tau^\delta)\mathbf{1}_{\{\tau^\delta<T\}}\right]\right],$$

and therefore,

$$\begin{aligned}
&J_1(\theta;0,x_\delta,\eta,Z)-J_1(\theta;0,x,\eta^n,Z^n)\\
&=\mathbb{E}^{\theta,Z}\left[\int_0^{\tau_n}[\eta^n_s-\eta_s]ds+\mathbf{1}_{\{\tau_n<t_n\}}\left[\int_{\tau_n}^{\tau^\delta}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\tau^\delta)\mathbf{1}_{\{\tau^\delta<T\}}-\overline{u}_0(\tau_n)\right]\right]\\
&\le\mathbb{E}^{\theta,Z}\left[\mathbf{1}_{\{\tau_n<t_n\}}\left[\int_{\tau_n}^{T}[|Z_s|+C_0-\eta_s]ds+C\right]\right],
\end{aligned}$$

where the last inequality is thanks to (12), Lemma 3, and the fact $\eta^n \le \eta$. Moreover, noting that D_θ is bounded, by (20), we have

$$\mathbb{E}^{\theta,Z}_{\mathcal{F}_{\tau_n}}\left[\int_{\tau_n}^{T}[|Z_s|+C_0-\eta_s]ds\right] \le C\sqrt{\overline{L}^\theta_{\tau_n}-X^\delta_{\tau_n}}+C(\overline{L}^\theta_{\tau_n}-X^\delta_{\tau_n}) \le C.$$

Then,

$$J_1(\theta;0,x_\delta,\eta,Z)-u_1(\theta;0,x) \le C\mathbb{P}^{\theta,Z}(\tau_n<t_n).$$

Recall (14), and note that $f<0, \tilde{H}\le 0$. Then, by Assumption 3(iii), we have

$$\mathcal{X}^{\theta;0,x,\eta^n,Z^n}_s-\underline{L}^\theta_s \ge x+\int_0^s Z_r dB^{\theta,Z}_r-\underline{L}^\theta_0-\Lambda_1 t_n, \quad 0\le s\le t_n.$$

Thus, because $x-\underline{L}^\theta_0>0$, for n large enough such that $\Lambda_1 t_n < (x-\underline{L}^\theta_0)/2$,

$$\begin{aligned}
\mathbb{P}^{\theta,Z}(\tau_n<t_n) &\le \mathbb{P}^{\theta,Z}\left(\inf_{0\le s\le t_n}[\mathcal{X}^{\theta;0,x,\eta^n,Z^n}_s-\underline{L}^\theta_s]\le 0\right)\\
&\le \mathbb{P}^{\theta,Z}\left(\inf_{0\le s\le t_n}\int_0^s Z_r dB^{\theta,Z}_r \le -\frac{x-\underline{L}^\theta_0}{2}\right)\to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

This implies (47), and hence, $u_1(\theta;0,x_\delta)\le u_1(\theta;0,x)$. □

4.2. The Regularity: Proof of Proposition 6

We first need the following technical lemma, whose proof is postponed to the appendix.

Lemma 4. *Let Assumptions 1 and 3 hold. Let $\delta>0$ be small, $t\le T_{2\delta}$, and $\underline{L}^\theta_t < x \le \underline{L}^\theta_t+\delta$. Set $\underline{\tau}_\delta := \tau^{\theta;t,x,C_0,1}\wedge T_\delta$, and*

$$(\underline{\eta}^{\theta;t,x}_s,\underline{Z}^{\theta;t,x}_s) := \mathbf{1}_{[t,\underline{\tau}_\delta)}(s)(C_0,1)+\mathbf{1}_{[\underline{\tau}_\delta,T)}(s)(\eta^{\theta;\underline{\tau}_\delta}_s,\mathcal{X}^{\theta;t,x,C_0,1}_{\underline{\tau}_\delta},0)]. \tag{50}$$

Then, $(\underline{\eta}^{\theta;t,x},\underline{Z}^{\theta;t,x})\in\mathcal{A}^\theta_{t,x}$, and

$$\overline{u}_0(t)-J_1(\theta;t,x,\underline{\eta}^{\theta;t,x},\underline{Z}^{\theta;t,x}) \le C\rho_0(\delta). \tag{51}$$

We now turn to the proof of the proposition.

Proof of Proposition 6. (i) Proof of (39). Again, without loss of generality, we prove it only at $t=0$. Let $x_1=x$ and $x_2=x+\delta$ for some $\delta>0$ sufficiently small. Then, by the monotonicity in Proposition 5, we have

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$u_1(\theta;t,x+\delta)-u_1(\theta;t,x)\le 0$. So, it suffices to prove the opposite direction. Fix $(\eta,Z)\in\mathcal{A}^\theta_{0,x}$, and in light of (47), construct

$$(\eta^\delta_s,Z^\delta_s):=(\eta_s,Z_s)\mathbf{1}_{[0,\tau)}(s)+\mathbf{1}_{[\tau,T)}(s)$$
$$\times[\mathbf{1}_{\{\tau=\overline{\tau}\}}(C_0,0)+\mathbf{1}_{\{\tau=T_{2\delta}<\overline{\tau}\wedge\underline{\tau}\}}(\eta_t^{\theta;T_{2\delta},\mathcal{X}^{\theta;0,x,\eta,Z}_{T_{2\delta}}+\delta},0)+\mathbf{1}_{\{\tau=\underline{\tau}\}}(\underline{\eta}_s^{\theta;\tau,\underline{L}^\theta_\tau+\delta},\underline{Z}_s^{\theta;\tau,\underline{L}^\theta_\tau+\delta}];$$
$$\text{where}\quad \overline{\tau}:=\inf\{s\ge 0:\mathcal{X}^{\theta;0,x,\eta,Z}_s=\overline{L}^\theta_s-\delta\};\quad \underline{\tau}:=\tau^{\theta;0,x,\eta,Z};\quad \tau:=\overline{\tau}\wedge\underline{\tau}\wedge T_{2\delta}. \tag{52}$$

One can verify straightforwardly that $(\eta^\delta,Z^\delta)\in\mathcal{A}^\theta_{0,x_\delta}$, where the lower boundedness of η^δ is because of (32). Denote

$$\mathcal{X}^x:=\mathcal{X}^{\theta;0,x,\eta,Z};\quad \mathcal{X}^\delta:=\mathcal{X}^{\theta;0,x_\delta,\eta^\delta,Z^\delta},\quad \tau^\delta:=\tau^{\theta;0,x_\delta,\eta^\delta,Z^\delta}.$$

Then, one can easily check that

$$\mathcal{X}^\delta_t=\mathcal{X}_t+\delta,\ 0\le t\le\tau;\quad \mathcal{X}^\delta_\tau=\overline{L}^\theta_\tau \text{ on } \{\tau=\overline{\tau}\};\quad \mathcal{X}^\delta_\tau=\underline{L}^\theta_\tau+\delta \text{ on } \{\tau=\underline{\tau}\};$$
$$\tau^\delta>\tau \text{ on } \{\tau=\underline{\tau}\},\quad \text{and}\quad \tau^\delta=T \text{ on } \{\tau\ne\underline{\tau}\}.$$

Thus,

$$\begin{aligned}
&J_1(\theta;0,x,\eta,Z)-u_1(\theta;0,x_\delta)\le J_1(\theta;0,x,\eta,Z)-J_1(\theta;0,x_\delta,\eta^\delta,Z^\delta)\\
&=\mathbb{E}^{\theta,Z}\left[\int_\tau^{\underline{\tau}}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}\right]\\
&\quad-\mathbb{E}^{\theta,Z^\delta}\left[\int_\tau^{\tau^\delta}[I^h(\theta;s,Z^\delta_s)-\eta^\delta_s]ds+\overline{u}_0(\tau^\delta)\mathbf{1}_{\{\tau^\delta<T\}}\right]\\
&=\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\overline{\tau}\}}I_1+\mathbf{1}_{\{\tau=T_{2\delta}<\overline{\tau}\wedge\underline{\tau}\}}I_2+\mathbf{1}_{\{\tau=\underline{\tau}\}}I_3],
\end{aligned}\tag{53}$$

where

$$I_1:=\mathbb{E}^{\theta,Z}_{\overline{\tau}}\left[\int_{\overline{\tau}}^{\underline{\tau}}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}+C_0(T-\overline{\tau})\right];$$
$$I_2:=\mathbb{E}^{\theta,Z}_{T_{2\delta}}\left[\int_{T_{2\delta}}^{\underline{\tau}}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}+\int_{T_{2\delta}}^{T}\eta_s^{\theta;T_{2\delta},X^\delta_{T_{2\delta}}}ds]\right];$$
$$I_3:=\overline{u}_0(\underline{\tau})-J_1(\theta;\underline{\tau},\underline{L}^\theta_\tau+\delta,\underline{\eta}^{\theta;\underline{\tau},\underline{L}^\theta_\tau+\delta},\underline{Z}^{\theta;\underline{\tau},\underline{L}^\theta_\tau+\delta}).$$

We now estimate I_1,I_2,I_3 separately. First, on $\{\tau=\overline{\tau}\}$, because $\overline{u}_0$ is bounded, we have

$$\begin{aligned}
I_1&=\mathbb{E}^{\theta,Z}_{\overline{\tau}}\left[\int_{\overline{\tau}}^{\underline{\tau}}[I^h(\theta;s,Z_s)+C_0-\eta_s]ds\right]+C\mathbb{P}^{\theta,Z}_{\overline{\tau}}(\underline{\tau}<T)\\
&\le\mathbb{E}^{\theta,Z}_{\overline{\tau}}\left[\int_{\overline{\tau}}^{T}[\Lambda_0|Z_s|+C_0-\eta_s]ds\right]+C\mathbb{P}^{\theta,Z}_{\overline{\tau}}(\underline{\tau}<T)\\
&\le C[\sqrt{\delta}+\delta]+C\mathbb{P}^{\theta,Z}_{\overline{\tau}}(\underline{\tau}<T)\le C\sqrt{\delta}+C\mathbb{P}^{\theta,Z}_{\overline{\tau}}(\underline{\tau}<T),
\end{aligned}$$

where the second inequality is because of (20) and the fact $\overline{L}^\theta_{\overline{\tau}}-\mathcal{X}^x_{\overline{\tau}}=\delta$. Moreover, note that $\mathcal{X}^x_{\underline{\tau}}=\underline{L}^\theta_{\underline{\tau}}\mathbf{1}_{\{\underline{\tau}<T\}}$ and $\mathcal{X}^x_T=\overline{L}^\theta_T=0$; then, by Assumption 3(i), we have

$$\begin{aligned}
0&\le\mathbb{E}^{\theta,Z}_{\overline{\tau}}\left[\int_{\overline{\tau}}^{\underline{\tau}}[f(\theta;r,C_0)-f(\theta;r,\eta_r)-\tilde{H}(\theta;r,Z_r)]dr\right]\\
&=\mathbb{E}^{\theta,Z}_{\overline{\tau}}[[\mathcal{X}^x_{\underline{\tau}}-\mathcal{X}^x_{\overline{\tau}}]-[\overline{L}^\theta_{\underline{\tau}}-\overline{L}^\theta_{\overline{\tau}}]]\\
&=\mathbb{E}^{\theta,Z}_{\overline{\tau}}[\mathbf{1}_{\{\underline{\tau}<T\}}[\underline{L}^\theta_{\underline{\tau}}-\overline{L}^\theta_{\underline{\tau}}]+\delta]\le-\frac{1}{\Lambda_0}\mathbb{P}^{\theta,Z}_{\overline{\tau}}(\underline{\tau}<T)+\delta.
\end{aligned}$$

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Thus, $\mathbb{P}_{\overline{\tau}}^{\theta,Z}(\underline{\tau} < T) \le \Lambda_0\delta$, and therefore,

$$I_1 \le C\sqrt{\delta} + \Lambda_0\delta \le C\sqrt{\delta}, \quad \text{on} \quad \{\tau = \overline{\tau}\}. \tag{54}$$

Next, on $\{\tau = T_{2\delta} < \overline{\tau} \wedge \underline{\tau}\}$, by (20) and noting that $\eta_s^{\theta;T_{2\delta},\mathcal{X}^\delta_{T_{2\delta}}} \le C_0$, we have

$$I_2 \le \mathbb{E}^{\theta,Z}_{T_{2\delta}}\left[C\int_{T_{2\delta}}^T [|Z_s| + |\eta_s| + C_0]ds + \overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}\right] \le C\sqrt{\delta} + C\mathbb{E}^{\theta,Z}_{T_{2\delta}}[\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}].$$

Because $\overline{u}_0(t)$ is uniformly continuous and $\overline{u}_0(T) = -c_T^p < 0$, we have

$$\overline{u}_0(\underline{\tau}) \le \underline{u}_0(\underline{\tau}) - \overline{u}_0(T) \le \sup_{T_{2\delta}\le s\le T} |\overline{u}_0(s) - \overline{u}_0(T)| \le \rho_0(2\delta) \le 2\rho_0(\delta).$$

Here, $\rho_0(2\delta) \le 2\rho_0(\delta)$ is without loss of generality. Thus,

$$I_2 \le C\sqrt{\delta} + 2\rho_0(\delta) \le C\rho_0(\delta), \quad \text{on} \quad \{\tau = T_{2\delta} < \overline{\tau} \wedge \underline{\tau}\}. \tag{55}$$

Finally, on $\{\tau = \underline{\tau}\}$, by Lemma 4, we have $I_3 \le C\rho_0(\delta)$. Plug this and (54) and (55), into (53); we get

$$J_1(\theta;0,x,\eta,Z) - u_1(\theta;0,x_\delta) \le C\rho_0(\delta).$$

Because $(\eta,Z) \in \mathcal{A}^\theta_{0,x}$ is arbitrary, we prove the remaining estimate of (39) at $t = 0$.

(ii) Proof of (40). For simplicity, we assume $t_1 = 0$, $x_1 = x$, and $t_2 = \delta$, and denote $\Delta x := x_2 - x_1$. We assume, without loss of generality, that $\delta > 0$ is sufficiently small.

Step 1. First, recall Assumption 3(iii), and denote

$$\overline{t} := \inf\{s \ge 0 : x + \Lambda_1 s = \overline{L}_s^\theta\} \wedge \delta; \quad \eta_s^0 := I^f(\theta;s,-\Lambda_1)\mathbf{1}_{[0,\overline{t}]}(s) + C_0\mathbf{1}_{[(\overline{t},\delta]}(s),\ 0 \le s \le \delta;$$
$$x_\delta := \mathcal{X}_\delta^{\theta;0,x,\eta^0,0} = x + \Lambda_1\overline{t} - \int_{\overline{t}}^\delta f(\theta;s,C_0)ds.$$

Then, it is clear that

$$\underline{L}_s^\theta < \mathcal{X}_s^{\theta;0,x,\eta^0,0} \le \overline{L}_s^\theta,\ s \in [0,\delta]; |x_\delta - x| \le C\delta.$$

Now for any $(\eta,Z) \in \mathcal{A}^\theta_{\delta,x_\delta}$, denote

$$(\eta_s^\delta, Z_s^\delta) := (\eta_s^0, 0)\mathbf{1}_{[0,\delta)}(s) + (\eta_s, Z_s)\mathbf{1}_{[\delta,T)}(s).$$

One can easily see that $\mathcal{X}_s^{\theta;0,x,\eta^\delta,Z^\delta} = \mathcal{X}_s^{\theta;\delta,x_\delta,\eta,Z}$ for $s \in [\delta,T]$, and $\tau^{\theta;0,x,\eta^\delta,Z^\delta} = \tau^{\theta;\delta,x_\delta,\eta,Z}$. Then, $(\eta^\delta,Z^\delta) \in \mathcal{A}^\theta_{0,x}$, and thus,

$$J_1(\theta;\delta,x_\delta,\eta,Z) - u_1(\theta;0,x) \le J_1(\theta;\delta,x_\delta,\eta,Z) - J_1(\theta;0,x,\eta^\delta,Z^\delta) = \int_0^\delta \eta_s^0 ds \le C_0\delta.$$

By the arbitrariness of $(\eta,Z) \in \mathcal{A}^\theta_{\delta,x_\delta}$, we obtain $u_1(\theta;\delta,x_\delta) - u_1(\theta;0,x) \le C_0\delta$. Then, by (39) and noting that $|x_2 - x_\delta| \le |\Delta x| + C\delta$, we have

$$\begin{aligned} u_1(\theta;\delta,x_2) - u_1(\theta;0,x) &= u_1(\theta;\delta,x_2) - u_1(\theta;\delta,x_\delta) + u_1(\theta;\delta,x_\delta) - u_1(\theta;0,x) \\ &\le C\rho_0(|\Delta x| + C\delta) + C_0\delta \le C\rho_0(|\Delta x| + \delta). \end{aligned}$$

This implies immediately one direction of (40) at $t = 0$.

Step 2. To see the opposite direction of (40), we denote

$$\delta' := \Lambda_1\delta + 2T\rho(\delta). \tag{56}$$

We first consider the case that $\overline{L}_0^\theta - x \le \delta'$. Then,

$$0 \le \overline{L}_\delta^\theta - x_2 = \overline{L}_\delta^\theta - \overline{L}_0^\theta + \overline{L}_0^\theta - x + x - x_2 \le C\rho(\delta) + |\Delta x|.$$

Thus, by (39), we have

$$u_1(\theta;0,x) - u_1(\theta;\delta,x_2) \le u_1(\theta;0,\overline{L}_0^\theta) - u_1(\theta;\delta,\overline{L}_\delta^\theta) + C\rho_0(\rho(\delta) + |\Delta x|).$$

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Moreover, one can easily see that $u_1(\theta;t,\overline{L}_t^\theta) = -C_0(T-t)$. Then,

$$u_1(\theta;0,x) - u_1(\theta;\delta,x_2) \le -C_0\delta + C\rho_0(\rho(\delta)+|\Delta x|) \le C\rho_0(\rho(\delta)+|\Delta x|). \tag{57}$$

We now consider the case that $x < \overline{L}_0^\theta - \delta'$. Because $x > \underline{L}_0^\theta$, we can easily see that

$$x'_\delta := x + \delta' \in (\underline{L}_\delta^\theta, \overline{L}_\delta^\theta). \tag{58}$$

For any $(\eta,Z) \in \mathcal{A}_{0,x}^\theta$, we shall construct, in Step 3 below, $(\eta^\delta,Z^\delta) \in \mathcal{A}_{\delta,x'_\delta}^\theta$ such that

$$J_1(\theta;0,x,\eta,Z) - J_1(\theta;\delta,x'_\delta,\eta^\delta,Z^\delta) \le C\rho_0(\rho(\delta)). \tag{59}$$

Then, $J_1(\theta;0,x,\eta,Z) - u_1(\theta;\delta,x'_\delta) \le C\rho_0(\rho(\delta))$. Thus, by the arbitrariness of $(\eta,Z) \in \mathcal{A}_{0,x}^\theta$, we obtain $u_1(\theta;0,x) - u_1(\theta;\delta,x'_\delta) \le C\rho_0(\rho(\delta))$. Note further that $|x'_\delta - x_2| \le |\Delta x| + \delta'$; then,

$$\begin{aligned} u_1(\theta;0,x) - u_1(\theta;\delta,x_2) &= u_1(\theta;0,x) - u_1(\theta;\delta,x'_\delta) + u_1(\theta;\delta,x'_\delta) - u_1(\theta;\delta,x_2) \\ &\le C\rho_0(\rho(\delta)) + C\rho_0(|\Delta x| + \delta') \le C\rho_0(|\Delta x| + \rho(\delta)). \end{aligned}$$

This, together with (57), proves the other direction of (40) in both cases.

Step 3. We now construct $(\eta^\delta,Z^\delta) \in \mathcal{A}_{\delta,x'_\delta}^\theta$ satisfying (59). Denote

$$\hat{\mathcal{X}}_s^{\theta;\delta,x,\eta,Z} := x - \int_0^s [f(\theta;r+\delta,\eta_r) + H(\theta;r+\delta,Z_r)]dr + \int_0^s Z_r dB_r,\ 0 \le s \le T_\delta, \tag{60}$$

and introduce the following processes by shifting the space:

$$B'_s := B^\delta_{s+\delta}, \quad \eta'_s(B^\delta_{[\delta,s]}) := \eta_{s-\delta}(B'_{[0,s-\delta]}), \quad Z'_s(B^\delta_{[\delta,s]}) := Z_{s-\delta}(B'_{[0,s-\delta]}). \tag{61}$$

Denote

$$\begin{aligned} &\mathcal{X}^x := \mathcal{X}^{\theta;0,x,\eta,Z}, \quad \hat{\mathcal{X}}^x := \hat{\mathcal{X}}^{\theta;\delta,x,\eta,Z}, \quad \mathcal{X}'_t := \hat{\mathcal{X}}^x_{t-\delta}(B'); \\ &\overline{\tau} := \inf\{t \ge 0 : \hat{\mathcal{X}}^x_t = \overline{L}^\theta_{t+\delta} - \delta'\}; \quad \underline{\tau} := \inf\{t \ge 0 : \mathcal{X}^x_t = \underline{L}^\theta_t\}; \quad \tau := \overline{\tau} \wedge \underline{\tau} \wedge T_{3\delta}; \\ &\overline{\tau}' := \overline{\tau}(B') + \delta; \quad \underline{\tau}' := \underline{\tau}(B') + \delta; \quad \tau' := \tau(B') + \delta = \overline{\tau}' \wedge \underline{\tau}' \wedge T_{2\delta}. \end{aligned} \tag{62}$$

We see that $(B^\delta_{[\delta,T]}, \eta'_{[\delta,T]}, Z'_{[\delta,T]}, \mathcal{X}'_{[\delta,T]}, \overline{\tau}', \underline{\tau}', \tau')$ and $(B_{[0,T_\delta]}, \eta_{[0,T_\delta]}, Z_{[0,T_\delta]}, \hat{\mathcal{X}}^x_{[0,T_\delta]}, \overline{\tau}+\delta, \underline{\tau}+\delta, \tau+\delta)$ have the same joint distribution. We then construct, similar to (52),

$$\begin{aligned} (\eta^\delta_s, Z^\delta_s) &:= (\eta'_s, Z'_s)\mathbf{1}_{[\delta,\tau')}(s) + \mathbf{1}_{[\tau',T)}(s) \\ &\times [\mathbf{1}_{\{\tau'=\overline{\tau}'\}}(C_0,0) + \mathbf{1}_{\{\tau'=T_{2\delta}<\overline{\tau}'\wedge\underline{\tau}'\}}(\eta_s^{\theta;T_{2\delta},\mathcal{X}'_{T_{2\delta}}+\delta'}, 0) + \mathbf{1}_{\{\tau'=\underline{\tau}'\}}(\underline{\eta}_s^{\theta;\underline{\tau}',L^\theta_{\underline{\tau}'}+\delta'}, \underline{Z}_s^{\theta;\underline{\tau}',L^\theta_{\underline{\tau}'}+\delta'})]. \end{aligned}$$

Denote $\mathcal{X}^\delta := \mathcal{X}^{\theta;\delta,x'_\delta,\eta^\delta,Z^\delta}$ and $\tau^\delta := \tau^{\theta;\delta,x'_\delta,\eta^\delta,Z^\delta}$. It is clear that

$$\mathcal{X}^\delta_s = \mathcal{X}'_s + \delta',\ \delta \le s \le \tau'. \tag{63}$$

Moreover, by Assumption 1(iii) and (12), one can easily see that

$$|\hat{\mathcal{X}}^x_s - \mathcal{X}^x_s| \le 2T\rho(\delta), \quad 0 \le s \le T_\delta. \tag{64}$$

Then, by Assumption 3(iii),

$$\hat{\mathcal{X}}^x_s + \delta' \ge \mathcal{X}^x_s + \delta' - 2T\rho(\delta) > \underline{L}^\theta_s + \Lambda_1\delta \ge \underline{L}^\theta_{s+\delta}, \quad \text{for} \quad 0 \le s < \underline{\tau}.$$

By (63) and recalling that $\hat{\mathcal{X}}^x$ and $\mathcal{X}'$ have the same distribution, this implies $\mathcal{X}^\delta_s > \underline{L}^\theta_s$ for $\delta \le s < \underline{\tau}'$, a.s. Then, one can easily see that $(\eta^\delta,Z^\delta) \in \mathcal{A}_{\delta,x'_\delta}^\theta$, and

$$\begin{aligned} &\mathcal{X}^\delta_{\tau'} = \overline{L}^\theta_{\tau'} \text{ on } \{\tau' = \overline{\tau}'\}; \quad 0 \le \mathcal{X}^\delta_{\tau'} - \underline{L}^\theta_{\tau'-\delta} \le 2\delta' \text{ on } \{\tau' = \underline{\tau}'\}; \\ &\tau^\delta > \tau' \text{ on } \{\tau' = \underline{\tau}'\}, \quad \text{and} \quad \tau^\delta = T \text{ on } \{\tau' \ne \underline{\tau}'\}. \end{aligned}$$

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Note further that

$$\mathbb{E}^{\theta,Z^\delta}\left[\int_\delta^{\tau'}[I^h(\theta;s,Z^\delta_s)-\eta^\delta_s]ds\right]=\mathbb{E}^{\theta,Z'}\left[\int_\delta^{\tau(B')+\delta}[I^h(\theta;s,Z'_s)-\eta'_s]ds\right]$$
$$=\mathbb{E}^{\theta,Z}\left[\int_0^{\tau}[I^h(\theta;s+\delta,Z_s)-\eta_s]ds\right].$$

Then,

$$J_1(\theta;0,x,\eta,Z)-J_1(\theta;\delta,x_{\delta'},\eta^\delta,Z^\delta)=\mathbb{E}^{\theta,Z}\left[\int_0^{\tau}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}\right]$$
$$-\mathbb{E}^{\theta,Z^\delta}\left[\int_\delta^{\tau^\delta}[I^h(\theta;s,Z^\delta)-\eta^\delta_s]ds+\overline{u}_0(\tau^\delta)\mathbf{1}_{\{\tau^\delta<T\}}\right]$$
$$=I_0+I_1+I_2+I_3, \tag{65}$$

where, by abusing the notations slightly with (53),

$$I_0:=\mathbb{E}^{\theta,Z}\left[\int_0^{\tau}[I^h(\theta;s,Z_s)-I^h(\theta;s+\delta,Z_s)]ds\right];$$
$$I_1:=\mathbb{E}^{\theta,Z}\left[\mathbf{1}_{\{\tau=\overline{\tau}\}}\left[\int_{\overline{\tau}}^{\underline{\tau}}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}\right]\right]+C_0\mathbb{E}^{\theta,Z'}[\mathbf{1}_{\{\tau'=\overline{\tau}'\}}(T-\overline{\tau}')];$$
$$I_2:=\mathbb{E}^{\theta,Z}\left[\mathbf{1}_{\{\tau=T_{3\delta}<\overline{\tau}\wedge\underline{\tau}\}}\left[\int_{T_{3\delta}}^{\tau}[I^h(\theta;s,Z_s)-\eta_s]ds+\overline{u}_0(\underline{\tau})\mathbf{1}_{\{\underline{\tau}<T\}}\right]\right]$$
$$+\mathbb{E}^{\theta,Z'}\left[\mathbf{1}_{\{\tau'=T_{2\delta}<\overline{\tau}'\wedge\underline{\tau}'\}}\int_{T_{2\delta}}^{T}\eta_s^{\theta;T_{2\delta},X^\delta_{T_{2\delta}}}ds\right]\Bigg];$$
$$I_3:=\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\underline{\tau}\}}\overline{u}_0(\underline{\tau})]-\mathbb{E}^{\theta,Z'}[\mathbf{1}_{\{\tau'=\underline{\tau}'\}}J_1(\theta;\underline{\tau}',X^\delta_{\underline{\tau}'},\underline{\eta}^{\theta;\underline{\tau}',X^\delta_{\underline{\tau}'}},\underline{Z}^{\theta;\underline{\tau}',X^\delta_{\underline{\tau}'}})].$$

We now estimate I_i. First, by the regularity of h, we have $|I_0|\le C\rho(\delta)$. Next, note that

$$\mathbb{E}^{\theta,Z'}[\mathbf{1}_{\{\tau'=\overline{\tau}'\}}(T-\overline{\tau}')]=\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\overline{\tau}\}}[T-\tau-\delta]]\le\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\overline{\tau}\}}[T-\tau]];$$
$$\overline{L}^\theta_{\overline{\tau}}-\mathcal{X}^x_{\overline{\tau}}\le\overline{L}^\theta_{\overline{\tau}}-\hat{\mathcal{X}}^x_{\overline{\tau}}+\delta'=\overline{L}^\theta_{\overline{\tau}}-\overline{L}^\theta_{\overline{\tau}+\delta}+2\delta'\le 2\delta'.$$

Then, similar to (54) and (55), we have

$$I_1\le C\sqrt{\delta'}\le C\sqrt{\rho(\delta)}\quad\text{and}\quad I_2\le C\rho_0(\delta')\le C\rho_0(\rho(\delta)).$$

Finally, when $\underline{\tau}=\tau\le T_{3\delta}$, by (62) and (64), we have

$$\hat{\mathcal{X}}^x_{\underline{\tau}}-\underline{L}^\theta_{\underline{\tau}}=\hat{\mathcal{X}}^x_{\underline{\tau}}-\mathcal{X}^x_{\underline{\tau}}\le C\rho(\delta).$$

Again, due to the related identical distribution, we have $\mathcal{X}'_{\underline{\tau}'}-\underline{L}^\theta_{\underline{\tau}'-\delta}\le C\rho(\delta)$ a.s. on $\{\underline{\tau}=\underline{\tau}'\}$. Then, by (63) and the regularity of $\underline{L}^\theta$,

$$\mathcal{X}^\delta_{\underline{\tau}'}-\underline{L}^\theta_{\underline{\tau}'}\le\mathcal{X}'_{\underline{\tau}'}+\delta'-\underline{L}^\theta_{\underline{\tau}'-\delta}+\Lambda_1\delta\le C\rho(\delta),\quad\text{a.s.}$$

Now it follows from Lemma 4(iii) that

$$I_3\le\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\underline{\tau}\}}\overline{u}_0(\underline{\tau})]-\mathbb{E}^{\theta,Z'}[\mathbf{1}_{\{\tau'=\underline{\tau}'\}}\overline{u}_0(\underline{\tau}')]+C\rho_0(\rho(\delta))$$
$$=\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\underline{\tau}\}}\overline{u}_0(\underline{\tau})]-\mathbb{E}^{\theta,Z}[\mathbf{1}_{\{\tau=\underline{\tau}\}}\overline{u}_0(\underline{\tau}+\delta)]+C\rho_0(\rho(\delta))\le C\rho_0(\rho(\delta)).$$

Plug all these estimates into (65); we obtain (59). □

4.3. The Boundary Conditions: Proof of Proposition 7

Proof of Proposition 7. Because $\mathcal{A}^\theta_{t,\overline{L}^\theta_t}=\{(C_0,0)\}$, the equality $u_1(\theta;t,\overline{L}^\theta_t)=-C_0(T-t)$ is obvious and actually has

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already been used in Proof of Proposition 6(ii). Next, by using Lemma 2,

$$u_1(\theta;t,x) \geq J_1(\theta;t,x,\eta^{\theta;t,x},0) = -\int_t^T \eta_s^{\theta;t,x} ds \geq -C_0(T-t).$$

Then, $\liminf_{t\uparrow T} u_1(\theta;t,x) \geq 0$. On the other hand, for any $\underline{L}_T^\theta < x < 0$, any $t < T$ large enough such that $\underline{L}_t^\theta < x$ and any $(\eta,Z) \in \mathcal{A}_{t,x}^\theta$, by (20) and Lemma 3, we have

$$\begin{aligned} J_1(\theta;t,x,\eta,Z) &= \mathbb{E}^{\theta,Z}\left[\int_t^T [I^h(\theta;s,Z_s) - \eta_s]ds + \overline{u}_0(\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z}<T\}}\right] \\ &\leq \mathbb{E}^{\theta,Z}\left[\int_t^T [C|Z_s| + |\eta_s|]ds + \mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z}<T\}}[\overline{u}_0(T) + \sup_{t\leq s\leq T}|\overline{u}_0(s) - \overline{u}_0(T)|]\right] \\ &\leq C\sqrt{T-t} + [\rho_0(T-t) - c_T^P]\mathbb{P}(\tau^{\theta;t,x,\eta,Z} < T) \leq C\sqrt{T-t} + \rho_0(T-t). \end{aligned}$$

By the arbitrariness of (η,Z), we have $u_1(\theta;t,x) \leq C\sqrt{T-t} + \rho_0(T-t)$. Send $t \uparrow T$ together with $\liminf_{t\uparrow T} u_1(\theta;t,x) \geq 0$; we obtain $u_1(\theta;T-,x) = 0$.

Finally, fix $t < T$. For any $\delta > 0$ small, by (51), we have $\overline{u}_0(t) \leq u_1(\theta;t,x) + C\rho_0(\delta)$ for all $x \in (\underline{L}_t^\theta, \underline{L}_t^\theta + \delta]$. Send $\delta \downarrow 0$; because u_1 is decreasing, we obtain $u_1(\theta;t,\underline{L}_t^\theta+) \geq \overline{u}_0(t)$. □

5. The Case with at Most *n* Quittings

In this section, we extend the results in the previous section to the case that the principal allows for at most n quittings. That is, if there have been n agents quitting the job, the principal will only offer a self-enforcing contract so that the $(n+1)$-th agent will not quit anymore. We remark that for each agent, the agent's problem remains the same as in the previous section; in particular, we do not keep track of the agent after the agent quits the job. Then, given the setting in the beginning of Section 3, in particular, under (27) and Assumption 3, we introduce the principal's dynamic utility function recursively as in (35):

$$\begin{aligned} u_n(\theta;t,x) &:= \sup_{(\eta,Z)\in\mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta,Z}\left[\int_t^{\tau^{\theta;t,x,\eta,Z}} [I^h(\theta;s,Z_s) - \eta_s]ds + \overline{u}_{n-1}(\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z}<T\}}\right]; \\ \overline{u}_n(t) &:= \sup_{\theta\in\Theta} u_n(\theta;t,R_t^\theta) - c_t^P, \end{aligned} \tag{66}$$

for $n \geq 1$ and $(t,x) \in D_\theta$, and as in (34), the principal's problem is

$$\begin{aligned} V_n^P(\theta;t) := \sup_{\eta\in\mathcal{A}_t^P} \mathbb{E}^{\theta,Z^{\theta;t,\eta}}&\left[\int_t^{\tau^{\theta;t,\eta}} [I^h(\theta;s,Z_s^{\theta;t,\eta}) - \eta_s]ds + \overline{u}_{n-1}(\tau^{\theta;t,\eta})\mathbf{1}_{\{\tau^{\theta;t,\eta}<T\}}\right], \\ &\text{subject to} \quad V_1^A(\theta;t,\eta) \geq R_t^\theta. \end{aligned} \tag{67}$$

By the same arguments as in the previous section, the following results are obvious.

Theorem 2.

i. *$\overline{u}_n$ is bounded and uniformly continuous on $[0,T]$ with certain modulus of continuity function ρ_n, and $\overline{u}_n(T) = -c_T^P$.*

ii. *u_n is decreasing in x. In particular, $V_n^P(\theta;t) = u_n(\theta;t,R_t^\theta)$.*

iii. *For any $(t_i,x_i) \in D_\theta$, $i = 1,2$, with $\Delta t := t_2 - t_1$, $\Delta x := x_2 - x_1$, we have*

$$|u_n(\theta;t_1,x_1) - u_n(\theta;t_2,x_2)| \leq C\rho_{n-1}(\rho(|\Delta t|) + |\Delta x|). \tag{68}$$

iv. *The value function $u_n(\theta;\cdot,\cdot)$ is the minimal continuous viscosity solution of the HJB Equation (43) with the following boundary conditions:*

$$\begin{aligned} u_n(\theta;t,\overline{L}_t^\theta) = -C_0(T-t), \quad \forall t\in[0,T); \qquad u_n(\theta;T-,x) = 0, \quad \forall x \in (\underline{L}_T^\theta, 0); \\ u_n(\theta;t,\underline{L}_t^\theta+) \geq \overline{u}_{n-1}(t), \quad \forall t\in[0,T). \end{aligned} \tag{69}$$

Moreover, we have the following monotonicity result.

Proposition 8. *Under Assumptions 1 and 3, u_n and V_n^P are increasing in n.*

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Proof. For any $\theta \in \Theta$, $t < T$, and $\eta \in \mathcal{A}_t^P$ satisfying the constraint in (25), by setting $\tau = T$ in (28), we have $V_1^A(\theta;t,\eta) \geq Y_t^{\theta;\eta} \geq R_t^\theta$, where $Y^{\theta;\eta}$ solves the BSDE (13). Moreover, by (29) and (30), clearly, $(Y^{\theta;t,\eta}, Z^{\theta;t,\eta}, K^{\theta;t,\eta}) = (Y^{\theta;\eta}, Z^{\theta;\eta}, 0)$, and $\tau^{\theta;t,\eta} = T$. Then, by (25) and (34), we see that $V_1^P(\theta;t) \geq J_P(\theta;t,\eta)$, and thus, $V_1^P(\theta;t) \geq V_0^P(\theta;t)$. Now, by (38), (34), and (67), we see that $u_2 \geq u_1$. Then, by (67) again, we may prove recursively that u_n is increasing in n, which implies further that V_n^P is also increasing. □

Remark 9. (i) One might think that V_n^P should be decreasing in n at first thought because, for a larger n, the principal is giving agents more opportunity to quit, and in the meantime, the principal would incur a cost c^P. This increasing property is because of our setting that the principal offers self-enforcing contracts after n quittings. Indeed, in the formulation of V_n^P, the principal can still allow the $(n+1)$-th agent to quit, but because the contract is self-enforcing, the agent has incentive to quit, whereas for the problem V_{n+1}^P, the principal has more freedom to choose contracts for the $(n+1)$-th agent, and therefore, her optimal utility may increase. Such monotonicity may not hold true if we replace the principal's outside option $\overline{u}_0(t)$ with another $\overline{u}_0'(t)$ as in Remark 6. In fact, the opposite monotonicity is also possible; see Proposition 13 below.

(ii) The above monotonicity holds true even when there is only one type of agent; namely, $\Theta = \{\theta_0\}$ is a singleton, as in the standard literature. Again, this proposition implies that it is indeed not optimal for the principal to restrict to self-enforcing contracts. The principal would rather allow the agents to quit and then hire a new one when the current one quits the job.

5.1. An Alternative Representation Formula for u_n

In this subsection, we provide an alternative representation formula for u_n, which is not recursive and will be used in the next section. To illustrate the idea, consider $u_2(\theta_0;t,x)$ for some $\theta_0 \in \Theta$ and $(t,x) \in D_{\theta_0}$. First, there exists an (approximately) optimal $(\eta^0, Z^0) \in \mathcal{A}_{t,x}^{\theta_0}$. Denote $\tau_1 := \tau^{\theta_0;t,x,\eta^0,Z^0}$. When $\{\tau_1 < T\}$, by (66), we may choose an (approximately) optimal θ_1, which is an $\mathcal{F}_{\tau_1}$-measurable Θ-valued random variable such that

$$\overline{u}_1(\tau_1) \approx u_1(\theta_1;\tau_1, R_{\tau_1}^{\theta_1}) - c_{\tau_1}^P.$$

We may continue to find (approximately) optimal $(\eta^1, Z^1) \in \mathcal{A}_{\tau_1, R_{\tau_1}^{\theta_1}}^{\theta_1}$ for $u_1(\theta_1;\tau_1,R_{\tau_1}^{\theta_1})$. Denote $\tau_2 := \tau^{\theta_1;\tau_1,R_{\tau_1}^{\theta_1},\eta^1,Z^1}$; then, formally, we may rewrite (66) as

$$u_1(\theta_0;t,x) = \sup_{(\eta^0,Z^0)\in\mathcal{A}_{t,x}^{\theta_0}} \sup_{\theta_1} \sup_{(\eta^1,Z^1)\in\mathcal{A}_{\tau_1,R_{\tau_1}^{\theta_1}}^{\theta_1}} \mathbb{E}^{\theta_0,Z^0}\left[\int_t^{\tau_1} [I^h(\theta_0;s,Z_s^0) - \eta_s^0]ds + \mathbb{E}_{F_{\tau_1}}^{\theta_1,Z^1}\left[\int_{\tau_1}^{\tau_2} [I^h(\theta_1;s,Z_s^1) - \eta_s^1]ds + \overline{u}_0(\tau_2)\mathbf{1}_{\{\tau_2<T\}}\right] - c_{\tau_1}^P \mathbf{1}_{\{\tau_1<T\}}\right]. \tag{70}$$

Moreover, denote

$$\zeta_s := \theta_0 \mathbf{1}_{[t,\tau_1)}(s) + \theta_1 \mathbf{1}_{[\tau_1,T)}(s), \quad (\eta_s, Z_s) := (\eta_s^0, Z_s^0)\mathbf{1}_{[t,\tau_1)}(s) + (\eta_s^1, Z_s^1)\mathbf{1}_{[\tau_1,T)}(s), \tag{71}$$

and let $\mathbb{L}^0(\mathbb{F}^t;\Theta)$ denote the set of $\mathbb{F}^t$-progressively measurable Θ-valued processes. Then, the expectation in (70) can be rewritten as

$$\mathbb{E}^{\zeta,Z}\left[\int_t^{\tau_2} [I^h(\zeta_s;s,Z_s) - \eta_s]ds - c_{\tau_1}^P \mathbf{1}_{\{\tau_1<T\}} + \overline{u}_0(\tau_2)\mathbf{1}_{\{\tau_2<T\}}\right]. \tag{72}$$

Recall (1); it is clear that $Z \in \mathcal{A}_t^A$. However, we remark that the lower bound of η^1 may not be uniform and, thus, in general, the above $\eta \notin \mathcal{A}_t^P$.

We now rigorously present the alternative representation formula for u_n. Fix $\theta \in \Theta$ and $(t,x) \in D_\theta$; let $\mathcal{A}_{t,x}^{\infty,\theta}$ denote the set of $(\eta, Z, \zeta) \in \mathcal{A}_t^P \times \mathcal{A}_t^A \times \mathbb{L}^0(\mathbb{F}^t;\Theta)$ such that there exist associated $(\{\tau_i\}_{i\geq 0}, \mathcal{X})$ satisfying

$$\begin{aligned}
&\tau_0 := t, \quad \mathcal{X}_{\tau_0} := x, \quad \zeta_{\tau_0} = \theta, \quad \text{and for } i \geq 0,\\
&\tau_{i+1} := \inf\{s \geq \tau_i : \mathcal{X}_s^{\zeta_{\tau_i};\tau_i,\mathcal{X}_{\tau_i},\eta,Z} = \underline{L}_s^{\zeta_{\tau_i}}\} \wedge T;\\
&\zeta_s = \zeta_{\tau_i}, \quad \mathcal{X}_s := \mathcal{X}_s^{\zeta_{\tau_i};\tau_i,\mathcal{X}_{\tau_i},\eta,Z}, \ \tau_i \leq s < \tau_{i+1};\\
&\mathcal{X}_{\tau_{i+1}} := R_{\tau_{i+1}}^{\zeta_{\tau_{i+1}}} \mathbf{1}_{\{\tau_{i+1}<T\}} + \mathcal{X}_T^{\zeta_{\tau_i};\tau_i,\mathcal{X}_{\tau_i},\eta,Z} \mathbf{1}_{\{\tau_{i+1}=T\}}.
\end{aligned} \tag{73}$$

Here, τ_i stands for the quitting time of the i-th agent, and when the agent quits, the principal would hire a new agent with type ζ_{τ_i}. We note that $\mathcal{X}$ have jumps at τ_i.

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Proposition 9. *Let Assumptions 1 and 3 hold. Then,*

$$u_n(\theta;t,x) = \sup_{(\eta,Z,\zeta)\in A^{\infty,\theta}_{t,x}} J_n(t,x,\eta,Z,\zeta), \quad \text{subject to} \quad X_T = 0 \text{ on } \{\tau_n = T\}, \quad \text{where}$$

$$J_n(t,x,\eta,Z,\zeta) := \mathbb{E}^{\zeta,Z}\left[\int_t^{\tau_n} [I^h(\zeta_s;s,Z_s) - \eta_s]ds - \sum_{i=1}^{n-1} c^P_{\tau_i}\mathbf{1}_{\{\tau_i<T\}} + \overline{u}_0(\tau_n)\mathbf{1}_{\{\tau_n<T\}}\right]. \tag{74}$$

The proof is purely technical and is postponed to the appendix.

6. The General Case with an Arbitrary Number of Quittings

In light of (74), we first introduce the principal's dynamic value function when there is no constraint on the number of agent's quittings: for any $\theta\in\Theta$ and $(t,x)\in D_\theta$,

$$u_\infty(\theta;t,x) = \sup_{(\eta,Z,\zeta)\in A^{\infty,\theta}_{t,x}} J_\infty(t,x,\eta,Z,\zeta), \quad \text{subject to} \quad \lim_{i\to\infty}\tau_i = T, X_{T-} = 0,$$

$$\text{where} \quad J_\infty(t,x,\eta,Z,\zeta) := \mathbb{E}^{\zeta,Z}\left[\int_t^{T} [I^h(\zeta_s;s,Z_s) - \eta_s]ds - \sum_{i=1}^{\infty} c^P_{\tau_i}\mathbf{1}_{\{\tau_i<T\}}\right]. \tag{75}$$

Then, if the principal hires an agent θ at time t and allows the agent, as well as all future agents, to quit as they want, the principal's value at time t will be

$$V^P_\infty(\theta,t) := \sup_{R^\theta_t\le x\le \overline{L}^\theta_t} u_\infty(\theta;t,x), \quad V^P_\infty(t) := \sup_{\theta\in\Theta} V^P_\infty(\theta,t). \tag{76}$$

Here, $V^P_\infty(t)$ will be the principal's real optimal utility if the agent hires an agent at time t, by hiring the best agent for her.

We remark that, in general, it may look possible that agents quit too frequently, and there are already infinitely many agents quitting at some time before T; namely, $\tau_\infty := \lim_{i\to\infty}\tau_i < T$. However, because of our assumption $c^\theta_t \ge \frac{1}{\Lambda_0} > 0$, this can never be the case. Roughly speaking, each new agent is hired with continuation utility R^θ, and the agent will remain in the position until his continuation utility hits the lower barrier $\underline{L}^\theta = R^\theta - c^\theta \le R^\theta - \frac{1}{\Lambda_0}$. Then, by the desired regularity of R^θ and $\underline{L}^\theta$, it will take a while for the agent to reach the lower barrier. That is, the time period $\tau_i - \tau_{i-1}$ during which the i-th agent works for the principal will have a lower bound (in a certain random sense); consequently, within a finite horizon $[0,T]$, there can be only finitely many agents. Indeed, the following result shows that we shall have $\tau_n = T$, for n large enough, a.s. for all $(\eta,Z,\zeta)\in\mathcal{A}^{\infty,\theta}_{t,x}$, and thus, the constraint $\lim_{i\to\infty}\tau_i = T$ in (75) is actually redundant. In particular, there is no self-enforcing contract involved anymore in (75) or (76). The proof is again postponed to the appendix.

Proposition 10. *Let Assumptions 1 and 3 hold. Fix arbitrary $\theta\in\Theta$ and $(t,x)\in D_\theta$.*

i. *For any $(\eta,Z,\zeta)\in\mathcal{A}^{\infty,\theta}_{t,x}$ with corresponding $(\{\tau_i\}_{i\ge0},\mathcal{X})$ defined in (73), we have*

$$\mathbb{P}^{\zeta,Z}(\tau_n < T) \le \frac{C}{n}, \quad \forall n\ge 1, \quad \text{and consequently,} \quad \mathbb{P}^{\zeta,Z}(\tau_n < T, \text{ for all } n) = 0. \tag{77}$$

ii. *u_n converges to u_∞ uniformly:*

$$|u_n(\theta;t,x) - u_\infty(\theta;t,x)| \le \frac{C}{n}. \tag{78}$$

The following result is a direct consequence of Theorem 2 and (78). Define

$$\overline{u}_\infty(t) := \sup_{\theta\in\Theta} u_\infty(\theta;t,R^\theta_t) - c^P_t. \tag{79}$$

Proposition 11. *Let Assumptions 1 and 3 hold. Then,*

i. *u_∞ is decreasing in x. In particular,*

$$V^P_\infty(\theta;t) = u_\infty(\theta;t,R^\theta_t), \quad \text{and thus,} \quad \overline{u}_\infty(t) = V^P_\infty(t) - c^P_t.$$

ii. *u_∞ is uniformly continuous in D_θ, uniformly in θ.*

iii. *$\overline{u}_\infty$ is bounded and uniformly continuous on $[0,T]$ with $\overline{u}_\infty(T) = -c^P_T$.*

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iv. $u_\infty(\theta;t,x)$ *has boundary conditions*

$$u_\infty(\theta;t,\overline{L}_t^\theta) = -C_0(T-t), \quad \forall t\in[0,T); \qquad u_\infty(\theta;T-,x)=0, \quad \forall x\in(\underline{L}_T^\theta,0);$$
$$u_\infty(\theta;t,\underline{L}_t^\theta+) \geq \overline{u}_\infty(t), \quad \forall t\in[0,T). \tag{80}$$

We now extend Theorem 2(iv).

Theorem 3. *The value function* $\{u_\infty(\theta;\cdot,\cdot)\}_{\theta\in\Theta}$ *is the minimal continuous viscosity solution of the HJB System* (43) *for all* $\theta\in\Theta$ *with Boundary Conditions* (80).

Remark 10. In Theorem 2, the boundary condition $\overline{u}_{n-1}$ is given (recursively), and then, $u_n(\theta;\cdot,\cdot)$ can be solved through (43) for fixed θ. The situation here is quite different. By (79), the boundary condition $\overline{u}_\infty$ relies on the solution u_∞ itself, and it involves all θ. So here, the equations for different θ are interacting through the boundary condition, and thus, we are considering an infinite dimensional system of HJB equations.

Proof of Theorem 3. By extending (44) to u_n and sending $n\to\infty$, it follows from the uniform convergence of u_n that u_∞ satisfies the dynamic programming principle

$$u_\infty(\theta;t,x) := \sup_{(\eta,Z)\in\mathcal{A}^\theta_{t,x}} \mathbb{E}^{\theta,Z}\Bigg[\int_t^{\tau\wedge\tau^{\theta;t,x,\eta,Z}} [I^h(\theta;r,Z_r)-\eta_r]dr$$
$$+ u_\infty(\theta;\tau,X_\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau<\tau^{\theta;t,x,\eta,Z}\}} + \overline{u}_\infty(\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z}\leq\tau,\,\tau^{\theta;t,x,\eta,Z}<T\}}\Bigg], \tag{81}$$

for any $\theta\in\Theta$, $(t,x)\in D_\theta$, and $\tau\in\mathcal{T}_t$. Then, it is straightforward to show that $\{u_\infty(\theta;\cdot,\cdot)\}_{\theta\in\Theta}$ is a continuous viscosity solution of the HJB System (43) for all $\theta\in\Theta$ with boundary conditions (80). Thus it remains to prove the minimal property.

Let $\{u(\theta;\cdot,\cdot)\}_{\theta\in\Theta}$ be an arbitrary continuous viscosity solution of the HJB System (43) for all $\theta\in\Theta$ with boundary conditions (80). We emphasize that, here, the boundary condition at $\underline{L}_t^\theta$ is not $\overline{u}_\infty$ but, rather,

$$\overline{u}(t) := \sup_{\theta\in\Theta} u(\theta;t,R_t^\theta) - c_t^P.$$

Fix (θ,t,x) and $(\eta,Z,\zeta)\in\mathcal{A}^{\infty,\theta}_{t,x}$ with corresponding $(\{\tau_i\}_{i\geq0},\mathcal{X})$ satisfying $\mathcal{X}_{T-}=0$. Given $\overline{u}$, recall (35), and introduce

$$\tilde{u}_1(\theta;t,x) := \sup_{(\eta^0,Z^0)\in\mathcal{A}^\theta_{t,x}} \mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau^{\theta;t,x,\eta^0,Z^0}} [I^h(\theta;s,Z_s^0)-\eta_s^0]ds + \overline{u}(\tau^{\theta;t,x,\eta^0,Z^0})\mathbf{1}_{\{\tau^{\theta;t,x,\eta^0,Z^0}<T\}}\right].$$

Note that $u(\theta;\cdot,\cdot)$ is a viscosity solution to (43) with boundary condition $\overline{u}$. Then, by Theorem 1, we see that $\tilde{u}_1(\theta;t,x)\leq u(\theta;t,x)$. Thus, for any $(\eta^0,Z^0)\in\mathcal{A}^\theta_{t,x}$, we have

$$\mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau^{\theta;t,x,\eta^0,Z^0}} [I^h(\theta;s,Z_s^0)-\eta_s^0]ds + \overline{u}(\tau^{\theta;t,x,\eta^0,Z^0})\mathbf{1}_{\{\tau^{\theta;t,x,\eta^0,Z^0}<T\}}\right] \leq u(\theta;t,x).$$

Note that, in general, $(\eta,Z)\notin\mathcal{A}^\theta_{t,x}$, and thus, as above, one cannot set $(\eta^0,Z^0)=(\eta,Z)$. However, following the arguments in Proposition 9, one can still derive from above that

$$\mathbb{E}^{\theta,Z}\left[\int_t^{\tau_1} [I^h(\theta;s,Z_s)-\eta_s]ds + \overline{u}(\tau_1)\mathbf{1}_{\{\tau_1<T\}}\right] \leq u(\theta;t,x).$$

Moreover, by the definition of $\overline{u}$, we have $u(\zeta_{\tau_1};\tau_1,R_{\tau_1}^{\zeta_{\tau_1}}) - c_{\tau_1}^P \leq \overline{u}(\tau_1)$ on $\{\tau_1<T\}$; then,

$$\mathbb{E}^{\theta,Z}\left[\int_t^{\tau_1} [I^h(\theta;s,Z_s)-\eta_s]ds + [u(\zeta_{\tau_1};\tau_1,R_{\tau_1}^{\zeta_{\tau_1}}) - c_{\tau_1}^P]\mathbf{1}_{\{\tau_1<T\}}\right] \leq u(\theta;t,x).$$

Similarly, on $\{\tau_1<T\}$, we have

$$\mathbb{E}^{\zeta_{\tau_1},Z}_{\mathcal{F}_{\tau_1}}\left[\int_{\tau_1}^{\tau_2} [I^h(\zeta_{\tau_1};s,Z_s)-\eta_s]ds + \overline{u}(\tau_2)\mathbf{1}_{\{\tau_2<T\}}\right] \leq u(\zeta_{\tau_1};\tau_1,R_{\tau_1}^{\zeta_{\tau_1}}).$$

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Then,

$$\mathbb{E}^{\zeta,\cdot,Z}\left[\int_t^{\tau_2}[I^h(\zeta_s;s,Z_s)-\eta_s]ds+\overline{u}(\tau_2)\mathbf{1}_{\{\tau_2<T\}}-c^P_{\tau_1}\mathbf{1}_{\{\tau_1<T\}}\right]\le u(\theta;t,x).$$

Repeating the arguments we have, for any $n\ge 1$,

$$\mathbb{E}^{\zeta,Z}\left[\int_t^{\tau_n}[I^h(\zeta_s;s,Z_s)-\eta_s]ds+\overline{u}(\tau_n)\mathbf{1}_{\{\tau_n<T\}}-\sum_{i=1}^{n-1}c^P_{\tau_i}\mathbf{1}_{\{\tau_i<T\}}\right]\le u(\theta;t,x).$$

Send $n\to\infty$; by (77), we have

$$\mathbb{E}^{\zeta,Z}\left[\int_t^{T}[I^h(\zeta_s;s,Z_s)-\eta_s]ds-\sum_{i=1}^{\infty}c^P_{\tau_i}\mathbf{1}_{\{\tau_i<T\}}\right]\le u(\theta;t,x).$$

Now by the arbitrariness of (η,Z,ζ), we prove $u_\infty(\theta;t,x)\le u(\theta;t,x)$. □

We conclude this section with the following result, which shows that the Principal-Agent Problem (76) is time consistent in the following sense:

Theorem 4. *The Principal-Agent Problem (76) satisfies the following dynamic programming principle: for any $\theta\in\Theta$ and $t<T$ and abbreviating the notation $\tau^\theta:=\tau^{\theta;t,R^\theta_t,\eta,Z}$,*

$$V^P_\infty(\theta;t)=\sup_{(\eta,Z)\in A^\theta_{t,R^\theta_t}}\mathbb{E}^{\theta,Z}\left[\int_t^{\tau^\theta}[I^h(\theta;r,Z_r)-\eta_r]dr+\sup_{\tilde\theta\in\Theta}V^P_\infty(\tilde\theta;\tau^\theta)-c^P_{\tau^\theta}\mathbf{1}_{\{\tau^\theta<T\}}\right];$$

$$V^P_\infty(t)=\sup_{\theta\in\Theta}\sup_{(\eta,Z)\in A^\theta_{t,R^\theta_t}}\mathbb{E}^{\theta,Z}\left[\int_t^{\tau^\theta}[I^h(\theta;r,Z_r)-\eta_r]dr+V^P_\infty(\tau^\theta)-c^P_{\tau^\theta}\mathbf{1}_{\{\tau^\theta<T\}}\right]. \tag{82}$$

Moreover, let $(\eta^*,Z^*,\zeta^*)\in\mathcal{A}^{\infty,\theta}_{t,R^\theta_t}$ with corresponding $(\{\tau^*_i\}_{i\ge0},\mathcal{X}^*)$ be an optimal control for $V^P_\infty(\theta;t)=u_\infty(\theta;t,R^\theta_t)$; then, for any $i\ge1$, $(\eta^*,Z^*,\zeta^*)|_{[\tau^*_i,T]}$ remains optimal for the problem $V^P_\infty(\zeta^*_{\tau^*_i};\tau^*_i)$.

Proof. Recall Proposition 11(i) and (79). The first line of (82) follows directly from (81) by setting $\tau\equiv T$, which implies further the second line of (82). We shall emphasize that (82) involves the term $V^P_\infty(\tilde\theta;\tau^{\theta;t,R^\theta_t,\eta,Z})$, not $V^P_\infty(\tilde\theta;\tau^{\tilde\theta;t,R^{\tilde\theta}_t,\eta,Z})$.

Moreover, if $(\eta^*,Z^*,\zeta^*)\in\mathcal{A}^{\infty,\theta}_{t,R^\theta_t}$ is optimal for $V^P_\infty(\theta;t)$, namely,

$$V^P_\infty(\theta;t)=\mathbb{E}^{\zeta^*,Z^*}\left[\int_t^{T}[I^h(\zeta^*_r;r,Z^*_r)-\eta^*_r]dr-\sum_{i=1}^{\infty}c^P_{\tau^*_i}\mathbf{1}_{\{\tau^*_i<T\}}\right],$$

then $\tilde\theta=\zeta^*_{\tau^*_1}$ is optimal in (82), and (82) becomes

$$V^P_\infty(\theta;t)=\mathbb{E}^{\zeta^*,Z^*}\left[\int_t^{\tau^*_1}[I^h(\theta;r,Z^*_r)-\eta^*_r]dr+V^P_\infty(\zeta^*_{\tau^*_1};\tau^*_1)-c^P_{\tau^*_1}\mathbf{1}_{\{\tau^*_1<T\}}\right].$$

This implies that

$$\mathbb{E}^{\zeta^*,Z^*}[V^P_\infty(\zeta^*_{\tau^*_1};\tau^*_1)]=\mathbb{E}^{\zeta^*,Z^*}\left[\int_{\tau^*_1}^{T}[I^h(\zeta^*_r;r,Z^*_r)-\eta^*_r]dr-\sum_{i=2}^{\infty}c^P_{\tau^*_i}\mathbf{1}_{\{\tau^*_i<T\}}\right].$$

On the other hand, by (75), it is clear that, on $\{\tau^*_1<T\}$,

$$V^P_\infty(\zeta^*_{\tau^*_1};\tau^*_1)\ge\mathbb{E}^{\zeta^*,Z^*}_{\mathcal{F}_{\tau^*_1}}\left[\int_{\tau^*_1}^{T}[I^h(\zeta^*_r;r,Z^*_r)-\eta^*_r]dr-\sum_{i=2}^{\infty}c^P_{\tau^*_i}\mathbf{1}_{\{\tau^*_i<T\}}\right].$$

Thus, equality holds a.s. at above. That is, $(\eta^*,Z^*,\zeta^*)|_{[\tau^*_1,T]}$ remains optimal for the problem $V^P_\infty(\zeta^*_{\tau^*_1};\tau^*_1)$. Repeat the arguments; we see that $(\eta^*,Z^*,\zeta^*)|_{[\tau^*_i,T]}$ remains optimal for the problem $V^P_\infty(\zeta^*_{\tau^*_i};\tau^*_i)$ on $\{\tau^*_i<T\}$ for all $i\ge1$. □

Remark 11. Theorem 4 shows that the principal's problem $V^P_\infty(t)$ is time consistent at the times τ^*_i when the agents quit the job. We should note, though, if the principal reconsiders the contract within the time period

(τ_i^*, τ_{i+1}^*), during which the agent is staying for the job, the principal may find (η^*, Z^*, ζ^*) not optimal. However, because the principal is required to commit to the contract, the principal is not allowed to fire the agent and sign a better contract during those time periods, so such time inconsistency for the principal is irrelevant in our model.

7. Comparison with the Standard Contracts

In this section, we investigate briefly the standard Principal-Agent Problem (22) with full commitment. In particular, the agent is not allowed to quit, rather than having no incentive to quit, as in (23). We shall compare the principal's optimal values under different settings.

7.1. The Dynamic Value Function for the Standard Principal-Agent Problem

We first introduce the principal's dynamic value function corresponding to (22):

$$u_0^S(\theta;t,x) := \sup_{(\eta,Z)\in\mathcal{A}^\theta_{t,x}} J_P(\theta;t,\eta), \quad 0 \le t < T, x \le \overline{L}_t^\theta. \tag{83}$$

We then have the following results. Because the proofs are similar, actually easier than those in Sections 3 and 4, we omit them.

Theorem 5. *Let Assumption 1 hold.*

i. *For each $\theta \in \Theta$ and $t < T$, $u_0^S(\theta;t,\cdot)$ is decreasing in $x \in (-\infty, \overline{L}_t^\theta]$. Consequently,*

$$V_0^{S,P}(\theta;t) = u_0^S(\theta;t,R_t^\theta), \quad \textit{provided} \quad R_t^{\theta_0} \le \overline{L}_t^{\theta_0}. \tag{84}$$

ii. *For any $t_1, t_2 < T$ and $x_1 \le \overline{L}_{t_1}^\theta < 0$, $x_2 \le \overline{L}_{t_2}^\theta$, we have*

$$|u_0^S(\theta;t_1,x_1) - u_0^S(\theta;t_2,x_2)| \le C\left[\sqrt{|x_1-x_2|} + \sqrt{\rho(|t_1-t_2|) - x_1|t_1-t_2|}\,\right]. \tag{85}$$

iii. *$u_0^S(\theta;\cdot,\cdot)$ has boundary conditions*

$$u_0^S(\theta;t,\overline{L}_t^\theta) = -C_0(T-t), \quad \forall t \in [0,T); \qquad u_0^S(\theta;T-,x) = 0, \quad \forall x < 0. \tag{86}$$

Moreover, for $x \to -\infty$, u_0 satisfies the following growth condition: for $\forall t < T$, $x \le \overline{L}_t^\theta$,

$$-C_0(T-t) \le u_0^S(\theta;t,x) \le C\sqrt{\rho(T-t) - x(T-t)}. \tag{87}$$

iv. *$u_0^S(\theta;\cdot,\cdot)$ is the unique viscosity solution of the following HJB equation with Boundary Conditions (86) and Growth Conditions (87):*

$$\mathcal{L}^\theta u_0^S(\theta;t,x) = 0, \quad 0 \le t < T, x \le \overline{L}_t^\theta. \tag{88}$$

Similar to (66) and (67), we now consider the principal-agent problem where the agent is allowed to quit; however, after seeing n agents quitting, the principal will offer only committed contracts so that the new agent cannot quit anymore: for $n = 1, 2, \ldots,$

$$\begin{aligned}
\overline{u}_{n-1}^S(t) &:= \sup_{\theta\in\Theta} u_{n-1}^S(\theta;t,R_t^\theta) - c_t^P;\\
u_n^S(\theta;t,x) &:= \sup_{(\eta,Z)\in A^\theta_{t,x}} \mathbb{E}^{\theta,Z}\left[\int_t^{\tau^{\theta;t,x,\eta,Z}} [I^h(\theta;s,Z_s) - \eta_s]ds + \overline{u}_{n-1}^S(\tau^{\theta;t,x,\eta,Z})\mathbf{1}_{\{\tau^{\theta;t,x,\eta,Z}<T\}}\right];\\
V_n^{S,P}(\theta;t) &:= \sup_{\eta\in A_t^P} \mathbb{E}^{\theta,Z^{\theta;t,\eta}}\left[\int_t^{\tau^{\theta;t,\eta}} [I^h(\theta;s,Z_s^{\theta;t,\eta}) - \eta_s]ds + \overline{u}_{n-1}^S(\tau^{\theta;t,\eta})\mathbf{1}_{\{\tau^{\theta;t,\eta}<T\}}\right],\\
&\qquad \text{subject to} \quad V_1^A(\theta;t,\eta) \ge R_t^\theta.
\end{aligned} \tag{89}$$

Then, Theorem 2 remains true for $u_n^S, \overline{u}_n^S$, and $V_n^{S,P}$ after obvious modifications. Moreover, for the same function u_∞ in (75), following similar arguments as in Proposition 11, we have

Proposition 12. *Under Assumptions 1 and 3, we have $|u_n^S - u_\infty| \le \frac{C}{n}$.*

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Remark 12. We may also define a value function u_0 to characterize V_0^P. Denote

$$\overline{\mathcal{A}}^{\theta}_{t,x} := \{(\eta, Z) \in \mathcal{A}^{\theta}_{t,x} : \mathcal{X}_s^{\theta;t,x,\eta,Z} \geq R_s^{\theta},\ s \in [t,T],\ \text{a.s.}\};\quad \overline{R}^{\theta}_t := \inf\{x : \overline{\mathcal{A}}^{\theta}_{t,x} \neq \emptyset\};$$

$$u_0(\theta;t,x) := \sup_{(\eta,Z)\in\bar{\mathcal{A}}^{\theta}_{t,x}} J_P(\theta;t,\eta),\quad t < T, x \in [\overline{R}^{\theta}_t, \overline{L}^{\theta}_t]. \tag{90}$$

Then, one can easily show that $u_0 \leq u_0^S$, and u_0 is decreasing in x, and thus, $V_0^P(\theta;t) = u_0(\theta;t,\overline{R}^{\theta}_t)$. However, it seems hard to establish the regularity of u_0 and its boundary condition at $\overline{R}^{\theta}$. Nevertheless, we have the desired regularity for $V_0^P(\theta;t)$ in (26).

7.2. Some Comparisons

In this subsection, we compare the principal's optimal utilities under different settings. First, recall (24); it is obvious that

$$V_0^{S,P}(\theta;t) \geq V_0^P(\theta;t),\quad \text{and hence,}\quad u_n^S(\theta;t,x) \geq u_n(\theta;t,x) \text{ for all } \theta \in \Theta, (t,x) \in D_\theta.$$

That is, the principal would endure some loss by providing the agent the incentives to stay in the contract.

We next consider the case that $\Theta = \{\theta_0\}$ is a singleton; namely, there is only one type of agent in the market, as in most publications in the literature. Recall Proposition 8.

Proposition 13. *Let Assumptions 1 and 3 hold, and* $\Theta = \{\theta_0\}$. *Then,* $u_n^S \downarrow u_\infty$ *and* $V_n^{S,P} \downarrow V_\infty^P$, *as* $n \to \infty$.

We again postpone the proof to the appendix.

Remark 13. By Propositions 8 and 13, we see that in the case $\Theta = \{\theta_0\}$, it is most desirable for the principal to offer committed contracts where the agent is not allowed to quit, and the principal receives the optimal utility $V_0^{S,P}$. However, if the principal cannot force the agent to commit, the principal would still prefer to offer a standard contract than a self-enforcing one, being aware that the principal might need to hire a new agent if the current one quits the job (or improve the contract by a retention offer when the agent threatens to quit).

We should note, though, that, in practice, if the agent is asked to commit to the contract for the whole period, the agent might ask for a larger individual reservation R^θ. This will, of course, change our analysis.

We now turn to the case that there are multiple (or even infinitely many) types of agents in the market, which is one of the main features of our model and is often the case in practice. In this case, the principal may have the opportunity to choose a cheaper agent when the current one quits the job. If this gain is larger than the costs c^{θ_0}, c^P induced by the agent's quitting, namely (again, this is impossible when $\Theta = \{\theta_0\}$, as we will see in (A.5) below in the proof of Proposition 13):

$$\overline{u}_0(\tau_1) = \sup_{\theta\in\Theta} u_0(\theta;\tau_1, R^{\theta}_{\tau_1}) - c^P_{\tau_1} > u_0(\theta_0;\tau_1, R^{\theta_0}_{\tau_1} - c^{\theta_0}_{\tau_1}),$$

then the principal may design the contract to induce the agent to quit. The following example shows that this can, indeed, be the case:

Example 1. Set $T = 1$, $\Theta = \{\theta_0, \theta_1\} \subset (0,\infty)$, $f(\theta;t,\eta) = -e^{-\eta}$, $h(\theta;t,\alpha) = \alpha_t^2/2\theta$, and $R_0^{\theta_0} = R_0^{\theta_1} = x_0 := -2e^{-C_0}$, where $C_0 > 0$ is the fixed constant for $\mathcal{A}^P$. These determine the value $u_0^S(\theta;0,x_0)$ and assume, without loss of generality, that $u_0^S(\theta_0;0,x_0) \geq u_0^S(\theta_1;0,x_0)$. Set further that $c^\theta \equiv e^{-C_0}$, $c^P \equiv 0$, and for some constant $n \geq 1$,

$$R_t^{\theta_0} := [x_0 + 3e^{-C_0}t]\mathbf{1}_{[0,\frac{1}{2})}(t) - e^{-C_0}[1-t]\mathbf{1}_{[\frac{1}{2},1]}(t);$$
$$R_t^{\theta_1} := [x_0 - nt]\mathbf{1}_{[0,\frac{1}{2})}(t) + 2\left[x_0 - \frac{n}{2}\right](1-t)\mathbf{1}_{[\frac{1}{2},1](t).}$$

Then, $\lim_{n\to\infty} u_1^S(\theta_0;0,x_0) = \infty$, and consequently, when n is large enough, we have

$$\max_{\theta\in\Theta} V_0^{S,P}(\theta;0) = u_0^S(\theta_0;0,x_0) < u_1^S(\theta_0;0,x_0) \leq \max_{\theta\in\Theta} V_1^{S,P}(\theta;0). \tag{91}$$

Proof. First, it is clear that $R_t^{\theta_1} \leq R_t^{\theta_0} \leq -e^{-C_0}(1-t) = \overline{L}_t^{\theta_0} = \overline{L}_t^{\theta_1}$. Note that

$$\underline{L}_t^{\theta_0} = [-2e^{-C_0} + 3e^{-C_0}t]\mathbf{1}_{[0,\frac{1}{2})}(t) - e^{-C_0}[1-t]\mathbf{1}_{[\frac{1}{2},1]}(t) - e^{-C_0}.$$

Set

$$\eta_t := C_0\mathbf{1}_{[0,\frac{1}{2})}(t) + [C_0 - \ln 3]\mathbf{1}_{[\frac{1}{2},1]}(t),\quad \tilde{\eta} \equiv -\ln(n + 4e^{-C_0}).$$

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Then, one can easily see that $(\eta,0)\in\mathcal{A}^{\theta_0}_{0,x_0}$ with $\tau^{\theta_0;0,x_0,\eta,0}=\frac{1}{2}$ and $(\tilde{\eta},0)\in\mathcal{A}^{\theta_1}_{\frac{1}{2},x_0-\frac{n}{2}}$. Thus,

$$u_1(\theta_0;0,x_0)\geq J_1(\theta_0;0,x_0,\eta,Z)=\int_0^{\frac{1}{2}}[\theta_0 Z_s-\eta_s]ds+\overline{u}_0\left(\frac{1}{2}\right)$$
$$\geq -\frac{1}{2}C_0+u_0\left(\theta_1;\frac{1}{2},R^{\theta_1}_{\frac{1}{2}}\right)=-\frac{1}{2}C_0+u_0\left(\theta_1;\frac{1}{2},x_0-\frac{n}{2}\right);$$
$$u_0\left(\theta_1;\frac{1}{2},x_0-\frac{n}{2}\right)\geq J_0\left(\theta_1;\frac{1}{2},\tilde{\eta},\tilde{Z}\right)=\int_{\frac{1}{2}}^1[\ln(n+4e^{-C_0})]ds=\frac{1}{2}\ln(n+4e^{-C_0}).$$

Therefore,

$$u_1(\theta_0;0,x_0)\geq\frac{1}{2}\ln(n+4e^{-C_0})-\frac{1}{2}C_0\to\infty,\quad\text{as}\quad n\to\infty.\quad\square$$

Remark 14. In Example 1, the agent θ_1 becomes cheaper when time t evolves (before $\frac{1}{2}$). Whereas it is desirable for the principal to hire agent θ_0 at initial time 0, it becomes obvious that at a later time, agent θ_1 is more desirable. Because, in our model, the principal is not allowed to fire the agent θ_0, the principal may design the contract to induce the agent θ_0 to quit. This will provide the principal the opportunity to hire the cheaper agent θ_1 at a later time and obtain a larger utility.

8. Summary

In this paper, we study a principal-agent problem with moral hazard in a finite time horizon where only the principal is committed to the contract, but the agent is allowed to quit. Instead of offering self-enforcing contracts, the principal allows the agent to quit. When the current agent quits the job, both the principal and the agent would incur some costs, and the principal would hire a new agent, satisfying the agent's individual rationality constraint at that time. One feature of our model is that there is a family of agents with different types in the market. However, unlike the third-best case with adverse selection, here, the principal knows the types of agents, and at any time, the principal hires only one agent.

The principal's optimal utility is a function of the agent's continuation utility, which is continuous in the interior of the domain but could be discontinuous at the boundary. Mathematically, our main goal is to characterize the principal's dynamic optimal utility function through a system of HJB equations, parameterized by the agent's type. Our results verify that the self-enforcing contracts can only be suboptimal for the principal. That is, it is not desirable or, say, too expensive for the principal to disincentivize the agent from quitting. It is somewhat surprising, though, that in some markets, the standard optimal contract for committed agents may also be suboptimal. That is, the principal may prefer the agent to quit so that the principal can hire a cheaper agent from the market. In this case, the principal would design the contract to induce the current agent to quit. This is mainly because of the presence of various types of agents. Moreover, by assuming a uniform lower bound of the agent's quitting cost, the agents won't quit too frequently, and the principal will only see finitely many quittings, which is consistent with common practice. We would also like to mention that the principal's problem is time consistent in the sense that the optimal contract remains optimal at any optimal quitting time of the agents.

We should note that for technical reasons, in this paper, each agent's dynamic individual rationality is given exogenously and is deterministic. It will be very interesting to provide endogenous models for this individual rationality through the agent's past performance and/or multiple principals' competition. Another interesting problem is that the principal can also fire the agent, and thus, it becomes a Dynkin game problem mixed with stochastic controls. We shall leave these important issues to future research.

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Appendix. Proofs Not Presented in the Main Text

In this appendix, we present the postponed technical proofs.

Proof of Lemma 1. Denote $(Y,Z):=(Y^{\theta,\eta},Z^{\theta,\eta})$ for notational simplicity. Note that

$$Y_t=\int_t^T[f(\theta;s,\eta)+\tilde{H}(\theta;s,Z_s)]ds-\int_t^T Z_s dB^{\theta,Z}_s.$$

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Then, by (17),

$$\overline{L}_t^\theta - Y_t = \mathbb{E}_t^{\theta,Z}\left[\int_t^T [[f(\theta;s,C_0) - f(\theta;s,\eta_s)] - \tilde{H}(\theta;s,Z_s)]ds\right].$$

Because $\eta \le C_0$ and $\partial_\eta f \ge 1/\Lambda_0$, and by (12), we have

$$f(\theta;s,C_0) - f(\theta;s,\eta_s) \ge \frac{1}{\Lambda_0}(C_0 - \eta_s) \ge 0; \quad -\tilde{H}(\theta;s,Z_s) \ge \frac{1}{C}|Z_s|^2 \ge 0.$$

Then,

$$\mathbb{E}_t^{\theta,Z}\left[\int_t^T |Z_s|^2 ds\right] \le C\mathbb{E}_t^{\theta,Z}\left[\int_t^T [-\tilde{H}(\theta;s,Z_s)]ds\right] \le C[\overline{L}_t^\theta - Y_t];$$

$$\mathbb{E}_t^{\theta,Z}\left[\int_t^T [C_0 - \eta_s]ds\right] \le C\mathbb{E}_t^{\theta,Z}\left[\int_t^T [f(\theta;s,C_0) - f(\theta;s,\eta_s)]ds\right] \le C[\overline{L}_t^\theta - Y_t].$$

Moreover, by (12), we have $|\eta_s|^2 \le C[1 - f(\theta;s,\eta_s)]$. Then,

$$\begin{aligned}\mathbb{E}_t^{\theta,Z}\left[\int_t^T |\eta_s|^2 ds\right] &\le C\mathbb{E}_t^{\theta,Z}\left[\int_t^T [1 - f(\theta;s,\eta_s)]ds\right] \\ &\le C\mathbb{E}_t^{\theta,Z}\left[\int_t^T [1 - f(\theta;s,\eta_s) - \tilde{H}(\theta;s,Z_s)]ds\right] = C[T - t - Y_t]. \quad \square\end{aligned}$$

Proof of Proposition 2. Without loss of generality, we estimate only $|V_0^P(\theta;0) - V_0^P(\theta;\delta)|$ for $\delta \le T$.

Step 1. Let $\eta \in \mathcal{A}^{\mathbb{P}} = \mathcal{A}_0^P$ satisfy the constraint in (23), and simplify the notation $(Y,Z) := (Y^{\theta,\eta}, Z^{\theta,\eta})$. Clearly, $\eta_{[\delta,T]}$ also satisfies the constraint in (25) with $t = \delta$; then,

$$\mathbb{E}_{\mathcal{T}_\delta}^{\theta,Z}\left[\int_\delta^T [I^h(\theta;s,Z_s) - \eta_s]ds\right] \le V_0^P(\theta;\delta), \quad \text{a.s.}$$

Thus, by (12) and (19),

$$\begin{aligned}J_P(\theta;0,\eta) - V_0^P(\theta;\delta) &\le \mathbb{E}^{\theta,Z}\left[\int_0^\delta [I^h(\theta;s,Z_s) - \eta_s]ds\right] \\ &\le \mathbb{E}^{\theta,Z}\left[\int_0^\delta [C|Z_s| + |\eta_s|]ds\right] \le C\sqrt{\delta}\left(\mathbb{E}^{\theta,Z}\left[\int_0^T [|Z_s|^2 + |\eta_s|^2]ds\right]\right)^{\frac{1}{2}} \\ &\le C\sqrt{\delta}(\overline{L}_0^\theta - Y_0 + T - Y_0)^{\frac{1}{2}} \le C\sqrt{\delta}(\overline{L}_0^\theta + T - 2R_0^\theta)^{\frac{1}{2}} \le C\sqrt{\delta}.\end{aligned}$$

Because η is arbitrary, we obtain $V_0^P(\theta;0) - V_0^P(\theta;\delta) \le C\sqrt{\delta}$.

Step 2. Let $\eta \in \mathcal{A}_\delta^{\mathbb{P}}$ satisfies the constraint in (25). It suffices to prove

$$J_P(\theta;\delta,\eta) - V_0^P(\theta;0) \le C\sqrt{\rho(\delta)}. \tag{A.1}$$

Then, by the arbitrariness of η, we obtain $V_0^P(\theta;\delta) - V_0^P(\theta;0) \le C\sqrt{\rho(\delta)}$.

Denote $\delta' := \rho(\delta) + \Lambda_0\delta$ and $(Y,Z) := (Y^{\theta,\eta}, Z^{\theta,\eta})$. We prove (A.1) in two cases.

Case 1. $Y_\delta \ge \overline{L}_\delta^\theta - \delta'$. Then, by (12), (19), and (24),

$$\begin{aligned}J_P(\theta;\delta,\eta) &= \mathbb{E}^{\theta,Z}\left[\int_\delta^T [I^h(\theta;s,Z_s) - \eta_s]ds\right] \\ &\le \mathbb{E}^{\theta,Z}\left[\int_\delta^T [C|Z_s| + C_0 - \eta_s]ds\right] - C_0(T - \delta) \\ &\le C[\sqrt{\overline{L}_\delta^\theta - Y_\delta} + \overline{L}_\delta^\theta - Y_\delta] + V_0^P(\theta;0) + C_0\delta \\ &\le C[\sqrt{\delta'} + \delta'] + V_0^P(\theta;0) + C_0\delta \le V_0^P(\theta;0) + C\sqrt{\rho(\delta)}.\end{aligned}$$

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Case 2. $Y_\delta < \overline{L}^\theta_\delta - \delta'$. Construct

$$\tau_\delta := \inf\{s \geq \delta : Y_s = \overline{L}^\theta_s - \delta'\} < T,\ s \in [\delta, T],$$
$$\eta^\delta_s := C_0 \mathbf{1}_{[0,\delta)}(s) + \eta_s \mathbf{1}_{[\delta,\tau_\delta)}(s) + C_0 \mathbf{1}_{[\tau_\delta,T]}(s).$$

It is clear that $\eta^\delta \in \mathcal{A}^P$. Denote $(Y^\delta, Z^\delta) := (Y^{\theta,\eta^\delta}, Z^{\theta,\eta^\delta})$. One can easily verify that

$$(Y^\delta_s, Z^\delta_s) = (Y_s + \delta', Z_s)\mathbf{1}_{[\delta,\tau_\delta]}(s) + (\overline{L}^\theta_s, 0)\mathbf{1}_{(\tau_\delta,T]}(s), \quad s \in [\delta, T].$$

In particular, $Y^\delta_s \geq Y_s \geq R^\theta_s$ for $s \in [t, T]$. Moreover, for $s \in [0, \delta]$, we have

$$Y^\delta_s = Y^\delta_\delta + \int_t^\delta [f(\theta; s, \eta^\delta) + H(\theta; s, Z^\delta_s)]ds - \int_t^\delta Z^\delta_s dB_s.$$

Because $\eta \in \mathcal{A}^P_\delta$ is $\mathbb{F}^{B^\delta}$-progressively measurable, then Y_δ is deterministic, and hence, so is Y^δ_δ. This implies $Z^\delta_s = 0$ for $s \in [0, \delta]$. Therefore, by (7) and the regularity of R^θ,

$$Y^\delta_s = Y^\delta_\delta + \int_s^\delta f(\theta; r, C_0)dr \geq Y^\delta_\delta - \Lambda_0\delta = Y_\delta + \rho(\delta) \geq R^\theta_\delta + \rho(\delta) \geq R^\theta_s, \quad s \in [0, \delta].$$

That is, η^δ satisfies the constraints in (23). Then,

$$\begin{aligned} J_P(\theta; \delta, \eta) - V^P_0(\theta; 0) &\leq J_P(\theta; \delta, \eta) - J_P(\theta; 0, \eta^\delta) \\ &= \mathbb{E}^{\theta,Z}\left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s]ds\right] - \mathbb{E}^{\theta,Z^\delta}\left[\int_\delta^{\tau_\delta} [I^h(\theta; s, Z_s) - \eta_s]ds + C_0[T - \tau_\delta - \delta]\right]. \end{aligned}$$

Because $Z^\delta = Z$ on $[\delta, \tau_\delta]$, then $\mathbb{P}^{\theta,Z^\delta} = \mathbb{P}^Z$ on $\mathcal{F}^\delta_{\tau_\delta}$. Thus, by (19),

$$\begin{aligned} &J_P(\theta; \delta, \eta) - V^P_0(\theta; 0) \\ &\leq \mathbb{E}^{\theta,Z}\left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s]ds\right] - \mathbb{E}^{\theta,Z}\left[\int_\delta^{\tau_\delta} [I^h(\theta; s, Z_s) - \eta_s]ds + C_0[T - \tau_\delta - \delta]\right] \\ &\leq \mathbb{E}^{\theta,Z}\left[\int_{\tau_\delta}^T [I^h(\theta; s, Z_s) + C_0 - \eta_s]ds\right] \leq \mathbb{E}^{\theta,Z}\left[\int_{\tau_\delta}^T [C|Z_s| + C_0 - \eta_s]ds\right] \\ &\leq C\mathbb{E}^{\theta,Z}\left[\sqrt{\overline{L}^\theta_{\tau_\delta} - Y_{\tau_\delta}} + \overline{L}^\theta_{\tau_\delta} - Y_{\tau_\delta}\right] = C[\sqrt{\delta'} + \delta'] \leq C\sqrt{\rho(\delta)}. \quad \square \end{aligned}$$

To prove Proposition 3, we shall first prove Lemma 2.

Proof of Lemma 2. (i) Recall (32), and fix $n \geq 1$ sufficiently large. Clearly, η^n is bounded from below, and $\partial_y I^f > 0$ implies $\eta^n_s \leq C_0$; thus, $\eta^n \in \mathcal{A}^P_t$. Denote

$$\mathcal{X}^n_s := x - \int_t^s f(\theta; r, \eta^n_r)dr = x - \int_t^s [f(\theta; r, C_0) - n]dr = x - \overline{L}^\theta_t + \overline{L}^\theta_s + n(s - t)$$

for $s \in [t, t + \varepsilon_n]$. Then, it is clear that $\mathcal{X}^n_s < \overline{L}^\theta_s$ for $s \in [t, t + \varepsilon_n)$, and $\mathcal{X}^n_{t+\varepsilon_n} = \overline{L}^\theta_{t+\varepsilon_n}$. Because $\eta^n_s = C_0$ for $s > t + \varepsilon_n$, this implies further that $\mathcal{X}^n_s = \overline{L}^\theta_s$ for $s \in [t + \varepsilon_n, T]$. In particular, $\mathcal{X}^n_T = \overline{L}^\theta_T = 0$, and thus, $(\eta^n, 0) \in \mathcal{A}^\theta_{t,x}$.

Moreover, for $s \in [t, t + \varepsilon_n]$, by (12) and (7), we have

$$|\eta^n_s|^2 \leq C[1 - f(\theta; s, \eta^n_s)] = C[1 - f(\theta; s, C_0) + n] \leq C[1 + \Lambda_0 + n] \leq Cn.$$

Then,

$$\int_t^T [C_0 - \eta^n_s]ds = \int_t^{t+\varepsilon_n} [C_0 - \eta^n_s]ds \leq \int_t^{t+\varepsilon_n} [C_0 + |\eta^n_s|]ds \leq \left[\frac{C_0}{n} + \frac{C}{\sqrt{n}}\right](\overline{L}^\theta_t - x).$$

This verifies (31) immediately by choosing n large enough.

(ii) For $s \in [t, t + \varepsilon_n]$, by (7) and Assumption 3(iii), we have, whenever $n \geq \Lambda_1$,

$$\mathcal{X}^n_s - \underline{L}^\theta_s > \underline{L}^\theta_t - \int_t^s [f(\theta; r, C_0) - n]dr - \underline{L}^\theta_s \geq n(s - t) - |\underline{L}^\theta_s - \underline{L}^\theta_t| \geq 0.$$

Moreover, for $s \in (t + \varepsilon_n, T]$, we have $\mathcal{X}^n_s = \overline{L}^\theta_s > \underline{L}^\theta_s$. So $(\mathcal{X}^n, 0, 0)$ satisfies RBSDE (29) with η^n, and thus, $\tau^{\theta;t,\eta^{\theta;t,x}} = T$. $\square$

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Proof of Proposition 3. (i) First, for fixed $\tau \in \mathcal{T}_t$, the agent's optimal effort can be solved through a BSDE:

$$Y_s^{\theta;t,\eta,\tau} = \underline{L}_\tau^\theta \mathbf{1}_{\{\tau<T\}} + \int_s^\tau [f(\theta;r,\eta_r) + H(\theta;r,Z_r^{\theta;t,\eta,\tau})]dr - \int_s^\tau Z_r^{\theta;t,\eta,\tau} dB_r. \tag{A.2}$$

Similar to Proposition 1, we see that $V_1^A(\theta;t,\eta) = \sup_{\tau\in\mathcal{T}_t} Y_t^{\theta;t,\eta,\tau}$, and once τ is given, the agent's optimal effort is $\alpha^* = I^h(\theta;s,Z^{\theta;t,\eta,\tau})$ on $[t,\tau]$. Then, it follows from the standard RBSDE theory that $V_1^A(\theta;t,\eta) = Y_t^{\theta;t,\eta}$ and $\tau^{\theta;t,\eta}$ is the smallest optimal stopping time. Notice further that $K_s^{\theta;t,\eta} = 0$ for $t \le s \le \tau^{\theta;t,\eta}$; then, (29) coincides with (A.2) on $[t,\tau^{\theta;t,\eta}]$. Thus, $Z_s^{\theta;t,\eta} = Z_s^{\theta;t,\eta,\tau^{\theta;t,\eta}}$, and $s \in [t,\tau^{\theta;t,\eta}]$. Because $\underline{L}_\tau^\theta \mathbf{1}_{\{\tau<T\}}$ is bounded, by (A.2), we see that $Z^{\theta;t,\eta,\tau} \in \mathcal{A}_t^A$ (see, e.g., Zhang [41, chapter 7]), and hence, $\alpha^{\theta;t,\eta} = I^h(\theta;s,Z^{\theta;t,\eta})\mathbf{1}_{[t,\tau^{\theta;t,\eta}]} \in \mathcal{A}_t^A$.

(ii) First, by definition, $Y_t^{\theta;t,\eta} \ge \underline{L}_t^\theta$. Next, because $\overline{L}^\theta > \underline{L}^\theta$, then $(\overline{L}^\theta, 0, 0)$ is the unique solution of the RBSDE (29) with $\eta \equiv C_0$, and thus, it follows from the comparison principle of RBSDEs that $Y_t^{\theta;t,\eta} \le \overline{L}_t^\theta$. On the other hand, for any $x \in (\underline{L}_t^\theta, \overline{L}_t^\theta)$, by Lemma 2, we see that $\eta^{\theta;t,x} \in \mathcal{A}_t^P$ and $x = Y^{\theta;t,\eta^{\theta;t,x}}$. Moreover, $x = \underline{L}_t^\theta$; from the arguments in Lemma 2, we can easily see that we still have $x = Y_t^{\theta;t,\eta^{\theta;t,x}}$. However, in this case, $Y_s^{\theta;t,\eta^{\theta;t,x}} > \underline{L}_s^\theta$ holds only for $s \in (t,T]$. Then, we obtain the claimed equality. □

Proof of Lemma 3. First, by Proposition 2 and Assumption 3, clearly, $\overline{u}_0(t)$ is uniformly continuous in $[0,T)$ and hence on $[0,T]$. Next, for any $t < T$ and $(\eta,Z) \in \mathcal{A}_{t,R_t^\theta}^\theta$, by (19), we have

$$\begin{aligned}|J_P(\theta;t,\eta)| &\le \mathbb{E}^{\theta,Z}\left[\int_t^T [C|Z_s| + |\eta_s|]ds\right] \le C\sqrt{T-t}\left(\mathbb{E}^{\theta,Z}\left[\int_t^T [|Z_s|^2 + |\eta_s|^2]ds\right]\right)^{\frac{1}{2}} \\ &\le C\sqrt{T-t}(\overline{L}_t^\theta - R_t^\theta + T - t - R_t^\theta)^{\frac{1}{2}}.\end{aligned}$$

Then,

$$|V_0^p(\theta;t)| \le C\sqrt{T-t}(\overline{L}_t^\theta - R_t^\theta + T - t - R_t^\theta)^{\frac{1}{2}}.$$

This implies $\overline{u}_0$ is bounded, and $\lim_{t\uparrow T}|\overline{u}_0(t)| = 0$. □

Proof of Lemma 4. Denote

$$(\underline{\eta},\underline{Z}) := (\underline{\eta}^{\theta;t,x}, \underline{Z}^{\theta;t,x}), \quad \mathcal{X} := \mathcal{X}^{\theta;t,x,C_0,1}, \quad \tau := \tau^{\theta;t,x,C_0,1}, \quad \underline{\tau} := \tau^{\theta;t,x,\underline{\eta}^{\theta;t,x},\underline{Z}^{\theta;t,x}}.$$

By Lemma 2, one can easily verify that $(\underline{\eta},\underline{Z}) \in \mathcal{A}_{t,x}^\theta$ and $\underline{\tau} = \tau\mathbf{1}_{\{\tau\le T_\delta\}} + T\mathbf{1}_{\{\tau>T_\delta\}}$. Then,

$$\begin{aligned}\overline{u}_0(t) - J_1(\theta;t,x,\underline{\eta},\underline{Z}) = \overline{u}_0(t) - \mathbb{E}^{\theta,1}[\underline{u}_0(\tau)\mathbf{1}_{\{\tau\le T_\delta\}} + \mathbf{1}_{\{\tau\le T_\delta\}}\int_t^\tau [I^h(\theta;s,1) - C_0]ds \\ + \mathbf{1}_{\{\tau>T_\delta\}}\left[\int_t^{T_\delta}[I^h(\theta;s,1) - C_0]ds - \int_{T_\delta}^T \eta_s^{\theta;T_\delta,\mathcal{X}_{T_\delta}} ds\right]\Big].\end{aligned}$$

Note that $t \le T_{2\delta}$; then, $\{\tau > T_\delta\} \subset \{\tau > t+\delta\}$. Then, by considering the two cases $\{\tau \le t+\delta\}$ and $\{\tau > t+\delta\}$, it follows from (31) and Lemma 3 that

$$\begin{aligned}&\overline{u}_0(t) - J_1(\theta;t,x,\underline{\eta},\underline{Z}) \le \rho_0(\delta) + \delta + C\mathbb{P}^{\theta,1}(\tau > t+\delta) \\ &\le C\rho_0(\delta) + \mathbb{P}^{\theta,1}\left(\inf_{t\le s\le t+\delta}[\mathcal{X}_s - \underline{L}_s^\theta] > 0\right) \\ &= C\rho_0(\delta) + \mathbb{P}^{\theta,1}\left(\inf_{t\le s\le t+\delta}\left[x - \int_t^s [f(\theta;r,C_0) + \tilde{H}(\theta;r,1)]dr + [B_s^{\theta,1} - B_t^{\theta,1}] - \underline{L}_s^\theta\right] > 0\right).\end{aligned}$$

Note that

$$-\int_t^s [f(\theta;r,C_0) + \tilde{H}(\theta;r,1)]dr \le C\delta, \quad x - \underline{L}_s^\theta = x - \underline{L}_t^\theta + \underline{L}_t^\theta - \underline{L}_s^\theta \le \delta + \Lambda_1\delta.$$

Thus,

$$\begin{aligned}\overline{u}_0(t) - J_1(\theta;t,x,\underline{\eta},\underline{Z}) &\le C\rho_0(\delta) + \mathbb{P}^{\theta,1}\left(\inf_{t\le s\le t+\delta}[B_s^{\theta,1} - B_t^{\theta,1}] > -C\delta\right) \\ &\le C\rho_0(\delta) + C\sqrt{\delta} \le C\rho_0(\delta).\end{aligned}$$

□

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Proof of Proposition 9. The result is trivial when $n = 1$. We shall prove (74) only for $n = 2$. The general case, $n > 2$, can be proved following similar but slightly more involved arguments. Let $\tilde{u}_2(\theta;t,x)$; denote the right side of the first line in (74) with $n = 2$.

Step 1. We first prove $\tilde{u}_2(\theta;t,x) \le u_2(\theta;t,x)$. For any (η, Z, ζ) satisfying the required properties in (74), and for any $\delta > 0$ small, denote

$$(\eta^0_s, Z^0_s) := (\eta_s, Z_s)\mathbf{1}_{[t,\tau_1\wedge T_\delta)}(s) + (\eta_s^{\theta;\tau_1\wedge T_\delta, \mathcal{X}^{\theta;t,x,\eta,Z}_{\tau_1\wedge T_\delta}}, 0)\mathbf{1}_{[\tau_1\wedge T_\delta, T)}(s);$$
$$(\eta^1_s, Z^1_s) := (\eta_s, Z_s)\mathbf{1}_{[\tau_1,\tau_2\wedge T_\delta)}(s) + (\eta_s^{\zeta_{\tau_1};\tau_2\wedge T_\delta, \mathcal{X}^{\zeta_{\tau_1};\tau_1,R^{\zeta_{\tau_1}}_{\tau_1},\eta,Z}_{\tau_2\wedge T_\delta}}, 0)\mathbf{1}_{[\tau_2\wedge T_\delta, T)}(s), \text{ on } \{\tau_1 \le T_\delta\}.$$

Then,

$$(\eta^0, Z^0) \in \mathcal{A}^\theta_{t,x}, \quad \tau^\delta_1 := \tau^{\theta;t,x,\eta^0,Z^0} = \tau_1\mathbf{1}_{\{\tau_1\le T_\delta\}} + T\mathbf{1}_{\{\tau_1>T_\delta\}};$$
$$(\eta^1, Z^1) \in \mathcal{A}^{\zeta_{\tau_1}}_{\tau_1,R^{\zeta_{\tau_1}}_{\tau_1}}, \quad \tau^\delta_2 := \tau^{\zeta_{\tau_1};\tau_1,R^{\zeta_{\tau_1}}_{\tau_1},\eta^1,Z^1} = \tau_2\mathbf{1}_{\{\tau_2\le T_\delta\}} + T\mathbf{1}_{\{\tau_2>T_\delta\}}, \text{ on } \{\tau_1 \le T_\delta\}.$$

Thus, by (66),

$$\begin{aligned}
u_2(\theta;t,x) &\ge \mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau^\delta_1}[I^h(\theta;s,Z^0_s) - \eta^0_s]ds + \overline{u}_1(\tau_1)\mathbf{1}_{\{\tau_1\le T_\delta\}}\right]\\
&\ge \mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau^\delta_1}[I^h(\theta;s,Z^0_s) - \eta^0_s]ds + [u_1(\zeta_{\tau_1};\tau_1,R^{\zeta_{\tau_1}}_{\tau_1}) - c^P_{\tau_1}]\mathbf{1}_{\{\tau_1\le T_\delta\}}\right]\\
&\ge \mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau^\delta_1}[I^h(\theta;s,Z^0_s) - \eta^0_s]ds\right.\\
&\quad\left. + \left(\mathbb{E}^{\zeta_{\tau_1},Z^1}_{\tau_1}\left[\int_{\tau_1}^{\tau^\delta_2}[I^h(\theta;s,Z^1_s) - \eta^1_s]ds + \overline{u}_0(\tau_2)\mathbf{1}_{\{\tau_2\le T_\delta\}}\right] - c^P_{\tau_1}\right)\mathbf{1}_{\{\tau_1\le T_\delta\}}\right]\\
&= \mathbb{E}^{\zeta,Z}\left[\mathbf{1}_{\{\tau_2\le T_\delta\}}\left[\int_t^{\tau_2}[I^h(\zeta_s;s,Z_s) - \eta_s]ds - c^P_{\tau_1} + \overline{u}_0(\tau_2)\right]\right.\\
&\quad + \mathbf{1}_{\{\tau_1\le T_\delta<\tau_2\}}\left[\int_t^{T_\delta}[I^h(\zeta_s;s,Z_s) - \eta_s]ds - \int_{T_\delta}^T \eta_s^{\zeta_{\tau_1};T_\delta,X^{\zeta_{\tau_1};\tau_1,R^{\zeta_{\tau_1}}_{\tau_1},\eta,Z}_{T_\delta}}ds - c^P_{\tau_1}\right]\\
&\quad\left. + \mathbf{1}_{\{\tau_1>T_\delta\}}\left[\int_t^{T_\delta}[I^h(\zeta_s;s,Z_s) - \eta_s]ds - \int_{T_\delta}^T \eta_s^{\theta;T_\delta,X^{\theta;t,x,\eta,Z}_{T_\delta}}ds\right]\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
&J_2(t,x,\eta,Z,\zeta) - u_2(\theta;t,x)\\
&\le \mathbb{E}^{\zeta,Z}\left[\mathbf{1}_{\{\tau_2>T_\delta\}}\int_{T_\delta}^{\tau_2}[I^h(\zeta_s;s,Z_s) - \eta_s]ds + \overline{u}_0(\tau_2)\mathbf{1}_{\{T_\delta<\tau_2<T\}} - c^P_{\tau_1}\mathbf{1}_{\{T_\delta<\tau_1<T\}}\right.\\
&\quad\left. + \mathbf{1}_{\{\tau_1\le T_\delta<\tau_2\}}\int_{T_\delta}^T \eta_s^{\zeta_{\tau_1};T_\delta,X^{\zeta_{\tau_1};\tau_1,R^{\zeta_{\tau_1}}_{\tau_1},\eta,Z}_{T_\delta}}ds + \mathbf{1}_{\{\tau_1>T_\delta\}}\int_{T_\delta}^T \eta_s^{\theta;T_\delta,X^{\theta;t,x,\eta,Z}_{T_\delta}}ds\right]\\
&\le \mathbb{E}^{\zeta,Z}\left[\mathbf{1}_{\{\tau_2>T_\delta\}}\int_{T_\delta}^{\tau_2}[I^h(\zeta_s;s,Z_s) - \eta_s]ds + \overline{u}_0(\tau_2)\mathbf{1}_{\{T_\delta<\tau_2<T\}} - c^P_{\tau_1}\mathbf{1}_{\{T_\delta<\tau_1<T\}}\right.\\
&\quad\left. + C_0\delta\left[\mathbf{1}_{\{\tau_1\le T_\delta<\tau_2\}} + \mathbf{1}_{\{\tau_1>T_\delta\}}\right]\right].
\end{aligned}$$

Send $\delta \to 0$; we obtain $J_2(t,x,\eta,Z,\zeta) \le u_2(\theta;t,x)$. Because (η, Z, ζ) are arbitrary, we have $\tilde{u}_2(\theta;t,x) \le u_2(\theta;t,x)$.

Step 2. We next show that $u_2(\theta;t,x) \le \tilde{u}_2(\theta;t,x)$. Let $\tilde{\rho}$ be a generic modulus of continuity function. Fix a small $\delta > 0$. First, by (66), there exists $(\eta^0, Z^0) \in \mathcal{A}^\theta_{t,x}$ such that

$$u_2(\theta;t,x) \le \mathbb{E}^{\theta,Z^0}\left[\int_t^{\tau_0}[I^h(\theta;s,Z^0_s) - \eta^0_s]ds + \overline{u}_1(\tau_0)\mathbf{1}_{\{\tau_0<T\}}\right] + \delta,$$

where $\tau_0 := \tau^{\theta;t,x,\eta^0,Z^0}$. Let $t = t_0 < \cdots < t_n = T$ be a partition of $[t,T]$ such that $t_{n-1} = T_\delta$ and $t_i - t_{i-1} \le \delta$, $i = 1,\dots,n-1$. Now, fix an $i = 1,\dots,n-1$. Choose θ_i such that

$$\overline{u}_1(t_i) \le u_1(\theta_i;t_i,R^\theta_{t_i}) - c^P_{t_i} + \delta.$$

Denote $x_i := \inf_{t_{i-1}\le s\le t_i} R^{\theta_i}_{t_i}$. Because $R^\theta_t - \underline{L}^\theta_t = c^\theta_t \ge \frac{1}{\Lambda_0}$, by the uniform regularity of R^θ_t, we have $x_i > \underline{L}^\theta_t + \Lambda_0\delta$ for $t_{i-1}\le t\le t_i$, provided δ is small enough. Let $(\eta^i,Z^i) \in \mathcal{A}^{\theta_i}_{t_i,x_i}$ be such that, denoting $\tau_i := \tau^{\theta_i;t_i,x_i,\eta^i,Z^i}$,

$$u_1(\theta_i;t_i,x_i) \le \mathbb{E}^{\theta_i,Z^i}\left[\int_{t_i}^{\tau_i} [I^h(\theta_i;s,Z^i_s) - \eta^i_s]ds + \overline{u}_0(\tau_i)\mathbf{1}_{\{\tau_i<T\}}\right] + \delta.$$

Denote, on $\{t_{i-1} < \tau_0 \le t_i\}$,

$$x'_i := \mathcal{X}^{\theta_i;\tau_0,R^{\theta_i}_{\tau_0},C_0,0}_{t_i} = R^{\theta_i}_{\tau_0} - \int_{\tau_0}^{t_i} f(\theta_i;s,C_0)ds.$$

We shall note that x'_i is random here, and it is obvious that $0 \le x'_i - x_i \le \tilde{\rho}(\delta)$. Following the arguments in Proposition 5, Step 1, we can construct $(\tilde{\eta}^i,\tilde{Z}^i) \in \mathcal{A}^{\theta_i}_{t_i,x'_i}$ such that

$$\begin{aligned}&\mathbb{E}^{\theta_i,Z^i}\left[\int_{t_i}^{\tau_i} [I^h(\theta_i;s,Z^i_s) - \eta^i_s]ds + \overline{u}_0(\tau_i)\mathbf{1}_{\{\tau_i<T\}}\right]\\ &\le \mathbb{E}^{\theta_i,\tilde{Z}^i}_{t_i}\left[\int_{t_i}^{\tilde{\tau}_i} [I^h(\theta_i;s,\tilde{Z}^i_s) - \tilde{\eta}^i_s]ds + \overline{u}_0(\tilde{\tau}_i)\mathbf{1}_{\{\tilde{\tau}_i<T\}}\right] + \tilde{\rho}(\delta),\end{aligned}$$

where $\tilde{\tau}_i := \tau^{\theta_i;t_i,x'_i,\tilde{\eta}^i,\tilde{Z}^i}$, and $\tilde{\eta}^i$ has a lower bound. Now, extend $(\tilde{\eta}^i,\tilde{Z}^i)$ to $[\tau_0,T]$ with $(\tilde{\eta}^i,\tilde{Z}^i) = (C_0,0)$ on $[\tau_0,t_i]$. Note that, for $\delta > 0$ small enough, $\mathcal{X}^{\theta_i;\tau_0,R^{\theta_i}_{\tau_0},C_0,0}_s > \underline{L}^\theta_s$, $\tau_0 \le s \le t_i$. Then, we have $(\tilde{\eta}^i,\tilde{Z}^i) \in \mathcal{A}^{\theta_i}_{\tau_0,R^{\theta_i}_{\tau_0}}$, and $\tau^{\theta_i;\tau_0,R^{\theta_i}_{\tau_0},\tilde{\eta}^i,\tilde{Z}^i} = \tilde{\tau}_i$, and thus, we can easily check that, again, on $\{t_{i-1} < \tau_0 \le t_i\}$,

$$\begin{aligned}&\mathbb{E}^{\theta_i,Z^i}\left[\int_{t_i}^{\tau_i} [I^h(\theta_i;s,Z^i_s) - \eta^i_s]ds + \overline{u}_0(\tau_i)\mathbf{1}_{\{\tau_i<T\}}\right]\\ &\le \mathbb{E}^{\theta_i,\tilde{Z}^i}_{\tau_0}\left[\int_{\tau_0}^{\tilde{\tau}_i} [I^h(\theta_i;s,\tilde{Z}^i_s) - \tilde{\eta}^i_s]ds + \overline{u}_0(\tilde{\tau}_i)\mathbf{1}_{\{\tilde{\tau}_i<T\}}\right] + \tilde{\rho}(\delta).\end{aligned}$$

We now define

$$\begin{aligned}(\eta_s,Z_s,\zeta_s) := (\eta^0_s,Z^0_s,\theta)\mathbf{1}_{[t,\tau_0\wedge T_\delta]}(s) + \mathbf{1}_{[\tau_0\wedge T_\delta,T]}(s)\\ \times\left[\sum_{i=1}^{n-1}(\tilde{\eta}^i,\tilde{Z}^i,\theta_i)\mathbf{1}_{\{t_{i-1}<\tau_0\le t_i\}} + (\eta^{\theta;T_\delta,\mathcal{X}^{\theta;t,x,\eta^0,Z^0}_{T_\delta}}_s,0,\theta)\mathbf{1}_{\{\tau_0>t_{n-1}\}}\right].\end{aligned}$$

Recall Remark 4, and because the above expression involves only finitely many η^i, the above η has a uniform lower bound, and then, one can verify that $(\eta,Z,\zeta) \in \mathcal{A}^{\infty,\theta}_{t,x}$ and

$$u_2(\theta;t,x) \le J_2(t,x,\eta,Z,\zeta) + \tilde{\rho}(\delta) \le \tilde{u}_2(\theta;t,x) + \tilde{\rho}(\delta).$$

Send $\delta \to 0$; we obtain $u_2(\theta;t,x) \le \tilde{u}_2(\theta;t,x)$. □

Proof of Proposition 10. (i) Note that, for each $i \ge 1$, on $\{\tau_i < T\}$, we have

$$\begin{aligned}&\underline{L}^{\zeta_{\tau_i}}_{\tau_{i+1}}\mathbf{1}_{\{\tau_{i+1}<T\}} + \mathcal{X}_{T-}\mathbf{1}_{\{\tau_{i+1}=T\}} = \mathcal{X}_{\tau_{i+1}-}\\ &= R^{\zeta_{\tau_i}}_{\tau_i} - \int_{\tau_i}^{\tau_{i+1}} [f(\zeta_{\tau_i};s,\eta_s) + \tilde{H}(\zeta_{\tau_i};s,Z_s)]ds + \int_{\tau_i}^{\tau_{i+1}} Z_s dB^{\zeta_{\tau_i},Z}_s\\ &\ge \underline{L}^{\zeta_{\tau_i}}_{\tau_i} + \frac{1}{\Lambda_0} + \int_{\tau_i}^{\tau_{i+1}} Z_r dB^{\zeta_{\tau_i},Z}_r.\end{aligned}$$

Then,

$$\mathbb{E}^{\zeta,Z}_{\tau_i}[\underline{L}^{\zeta_{\tau_i}}_{\tau_{i+1}}\mathbf{1}_{\{\tau_{i+1}<T\}} + \mathcal{X}_{T-}\mathbf{1}_{\{\tau_{i+1}=T>\tau_i\}}] \ge \mathbb{E}^{\zeta,Z}_{\tau_i}\left[\left[\underline{L}^{\zeta_{\tau_i}}_{\tau_i} + \frac{1}{\Lambda_0}\right]\mathbf{1}_{\{\tau_i<T\}}\right],$$

and thus, noting that $\underline{L}$ and $\mathcal{X}$ are uniformly bounded,

$$\begin{aligned}\frac{1}{\Lambda_0}\mathbb{P}^{\zeta,Z}(\tau_i<T) &\le \mathbb{E}^{\zeta,Z}[\underline{L}^{\zeta_{\tau_i}}_{\tau_{i+1}}\mathbf{1}_{\{\tau_{i+1}<T\}} - \underline{L}^{\zeta_{\tau_i}}_{\tau_i}\mathbf{1}_{\{\tau_i<T\}} + \mathcal{X}_{T-}\mathbf{1}_{\{\tau_{i+1}=T>\tau_i\}}]\\ &\le \mathbb{E}^{\zeta,Z}[\Lambda_1[\tau_{i+1}-\tau_i]\mathbf{1}_{\{\tau_i<T\}} + [\mathcal{X}_{T-} - \underline{L}^{\zeta_{\tau_i}}_T]\mathbf{1}_{\{\tau_{i+1}=T>\tau_i\}}]\\ &\le C\mathbb{E}^{\zeta,Z}[\tau_{i+1}-\tau_i + \mathbf{1}_{\{\tau_{i+1}=T>\tau_i\}}].\end{aligned}$$

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Take summation over i; we have

$$\frac{n}{\Lambda_0}\mathbb{P}^{\zeta,Z}(\tau_n<T)\le\frac{1}{\Lambda_0}\sum_{i=1}^{n}\mathbb{P}^{\zeta,Z}(\tau_i<T)\le C\sum_{i=1}^{n}\mathbb{E}^{\zeta,Z}[\tau_{i+1}-\tau_i+\mathbf{1}_{\{\tau_{i+1}=T>\tau_i\}}]\le C.$$

This implies (77) immediately.

(ii) Step 1. First, for any $(\eta,Z,\zeta)\in\mathcal{A}_{t,x}^{\infty,\theta}$ as in (74) and $\delta>0$ small, construct

$$\tau_n^\delta:=\tau_n\wedge T_\delta,\quad (\tilde\eta,\tilde Z,\tilde\zeta)_s:=(\eta,Z,\zeta)_s\mathbf{1}_{[0,\tau_n^\delta)}(s)+(\eta_s^{\zeta_{\tau_n^\delta};\tau_n^\delta,R^{\zeta_{\tau_n^\delta}}_{\tau_n^\delta}},0,\zeta_{\tau_n^\delta})\mathbf{1}_{[\tau_n^\delta,T)}(s). \tag{A.3}$$

Then, clearly, $(\tilde\eta,\tilde Z,\tilde\zeta)\in\mathcal{A}_{t,x}^{\infty,\theta}$ satisfies the constraints in (75), with corresponding $\tilde\tau_i=\tau_i$ on $\{\tau_i\le T_\delta\}$ for $i\le n$, and $\tilde\tau_i=T$ otherwise. Then,

$$\begin{aligned}
&J_n(t,x,\eta,Z,\zeta)-u_\infty(\theta;t,z)\le J_n(t,x,\eta,Z,\zeta)-J_\infty(t,x,\tilde\eta,\tilde Z,\tilde\zeta)\\
&=\mathbb{E}^{\zeta,Z}\left[\int_{\tau_n^\delta}^{\tau_n}[I^h(\zeta_s;s,Z_s)-\eta_s]ds+\int_{\tau_n^\delta}^{T}\eta_s^{\zeta_{\tau_n^\delta};\tau_n^\delta,R^{\zeta_{\tau_n^\delta}}_{\tau_n^\delta}}ds\right.\\
&\quad\left.-\sum_{i=1}^{n}c^P_{\tau_i}\mathbf{1}_{\{T_\delta<\tau_i<T\}}+\sup_\theta V_0^P(\theta;\tau_n)\mathbf{1}_{\{\tau_n<T\}}\right]\\
&\le\mathbb{E}^{\zeta,Z}\left[\int_{\tau_n^\delta}^{\tau_n}[I^h(\zeta_s;s,Z_s)-\eta_s]ds+C_0(T-\tau_n^\delta)+C\mathbf{1}_{\{\tau_n<T\}}\right].
\end{aligned}$$

Send $\delta\to0$; one can easily see that

$$J_n(t,x,\eta,Z,\zeta)-u_\infty(\theta;t,z)\le\mathbb{E}^{\zeta,Z}[C_0(T-\tau_n)+C\mathbf{1}_{\{\tau_n<T\}}]\le C\mathbb{P}^{\zeta,Z}(\tau_n<T)\le\frac{C}{n},$$

and thus, $u_n(\theta;t,x)-u_\infty(\theta;t,z)\le C/n$.

Step 2. On the other hand, for any $(\eta,Z,\zeta)\in\mathcal{A}_{t,x}^{\infty,\theta}$ satisfying the constraints in (75), again let $(\tilde\eta,\tilde Z,\tilde\zeta)$ be constructed by (A.3). Then, $(\tilde\eta,\tilde Z,\tilde\zeta)\in\mathcal{A}_{t,x}^{\infty,\theta}$ satisfies the constraints in (74) with $\tilde\tau_i=\tau_i$ on $\{\tau_i\le T_\delta\}$, $i=0,\dots,n$. Thus, by (31),

$$\begin{aligned}
&J_\infty(t,x,\eta,Z,\zeta)-u_n(\theta;t,x)\le J_\infty(t,x,\eta,Z,\zeta)-J_n(t,x,\tilde\eta,\tilde Z,\tilde\zeta)\\
&=\mathbb{E}^{\zeta,Z}\left[\int_{\tau_n^\delta}^{T}\left[I^h(\zeta_s;s,Z_s)-\eta_s+\eta_s^{\zeta_{\tau_n^\delta};\tau_n^\delta,R^{\zeta_{\tau_n^\delta}}_{\tau_n^\delta}}\right]ds\right.\\
&\quad\left.-\sum_{i=1}^{n}c^P_{\tau_i}\mathbf{1}_{\{T_\delta<\tau_i<T\}}-\sum_{i=n+1}^{\infty}c^P_{\tau_i}\mathbf{1}_{\{\tau_i<T\}}-\sup_\theta V_0^P(\theta;\tau_n)\mathbf{1}_{\{\tau_n<T\}}\right]\\
&\le\mathbb{E}^{\zeta,Z}\left[\int_{\tau_n^\delta}^{T}[C|Z_s|+C_0-\eta_s]ds+C_0\mathbf{1}_{\{\tau_n<T\}}\right]\\
&\le C\mathbb{E}^{\zeta,Z}\left[\sqrt{(T-\tau_n^\delta)(\overline{L}^{\zeta_{\tau_n^\delta}}_{\tau_n^\delta}-X_{\tau_n^\delta})}+(\overline{L}^{\zeta_{\tau_n^\delta}}_{\tau_n^\delta}-X_{\tau_n^\delta})+\mathbf{1}_{\{\tau_n<T\}}\right]\\
&\le C\mathbb{P}^{\zeta,Z}(\tau_n<T)+C\mathbb{E}^{\zeta,Z}\left[\mathbf{1}_{\{\tau_n=T\}}\sqrt{\delta(\overline{L}^{\zeta_{T_\delta}}_{T_\delta}-X_{T_\delta})}+(\overline{L}^{\zeta_{T_\delta}}_{T_\delta}-X_{T_\delta})\right],
\end{aligned}$$

where the third inequality is thanks to (20). Send $\delta\to0$, and note that $\mathcal{X}_{T-}=0$; we have

$$J_\infty(t,x,\eta,Z,\zeta)-u_n(\theta;t,x)\le C\mathbb{P}^{\zeta,Z}(\tau_n<T)\le\frac{C}{n}.$$

This implies $u_\infty(\theta;t,x)-u_n(\theta;t,z)\le\frac{C}{n}$, and hence, $|u_n(\theta;t,x)-u_\infty(\theta;t,z)|\le\frac{C}{n}$. □

Proof of Proposition 13. We first show that

$$u_1^S(\theta_0;t,x)\le u_0^S(\theta_0;t,x). \tag{A.4}$$

This, together with (89) and Proposition 12, implies recursively $u_n^S\downarrow u_\infty$ and $V_n^{S,P}\downarrow V_\infty^P$.

Indeed, in this case, by (89), we have

$$u_1^S(\theta_0;t,x)=\sup_{(\eta,Z)\in A^{\theta_0}_{t,x}}\mathbb{E}^{\theta_0,Z}\left[\int_t^{\tau_1}[I^h(\theta_0;s,Z_s)-\eta_s]ds+[u_0^S(\theta_0;\tau_1,R^{\theta_0}_{\tau_1})-c^P_{\tau_1}]\mathbf{1}_{\{\tau_1<T\}}\right].$$

On the other hand, similar to (44), u_0^S satisfies the dynamic programming principle

$$u_0^S(\theta_0;t,x)=\sup_{(\eta,Z)\in A_{t,x}^{\theta_0}}\mathbb{E}^{\theta_0,Z}\left[\int_t^{\tau_1}[I^h(\theta_0;s,Z)-\eta_s]ds+u_0^S(\theta_0;\tau_1,\underline{L}_{\tau_1}^{\theta_0})\mathbf{1}_{\{\tau_1<T\}}\right].$$

Note that $R_{\tau_1}^{\theta_0}>R_{\tau_1}^{\theta_0}-c_{\tau_1}^{\theta_0}=\underline{L}_{\tau_1}^{\theta_0}$ and u_0 is decreasing in x; then,

$$u_0^S(\theta_0;\tau_1,R_{\tau_1}^{\theta_0})-c_{\tau_1}^P\le u_0^S(\theta_0;\tau_1,\underline{L}_{\tau_1}^{\theta_0}),\tag{A.5}$$

which implies (A.4) immediately, where the loss is because of the costs $c_{\tau_1}^{\theta_0}$ and $c_{\tau_1}^P$. □

Endnotes

[1] We should emphasize that the time inconsistency is not because of the randomness or asymmetric information: even the first-best contract in a deterministic setting is typically time inconsistent. Indeed, the optimal contract over period $[t,T]$ should naturally depend on the IR constraint at t, which is not involved at all for the problem over $[0,T]$, and thus, the optimal contracts over different periods typically do not agree. This exogenously given dynamic IR constraint is the main source of the time inconsistency. In fact, as we see in Cvitanić et al. [7, 8] and Sannikov [31], one may choose the (random) IR constraint carefully so as to make the problem time consistent. However, there is no reason to expect this mathematically chosen IR constraint would match the real market values of the agent, and thus, the problem, in reality, is typically time inconsistent.

[2] It is called nonlapsing constraint in Gottlieb and Zhang [12], and the so-called front-loading constraint is also studied there.

[3] In Capponi and Frei [4], the agent chooses a density function for a certain stopping time, not the stopping time itself.

[4] Let's consider an extreme case: the IR constraint is a real number at $t=0$ but is $-\infty$ at $t>0$. Then, the standard optimal contract over $[0,T]$ is self-enforcing. However, it may not be optimal over $[t,T]$ for $t>0$, which has no IR constraint now. We should point out, though, that in this case, the desire for renegotiation is from the principal. When the principal is committed to the contract, as we assume in this paper, then this time inconsistency is irrelevant.

[5] Some other principals may still prefer agent θ at time t, and that's why R_t^θ could remain high in the market. But we do not model competition among principals in this paper. We refer again to Ai et al. [2] and Krueger and Uhlig [22] for endogenous models for R_t^θ. It will be very interesting to combine our work with those models, and we shall leave this for future research.

[6] In this work. both the principal and the agent have time-inconsistent preferences, and they consider certain optimal renegotiation-proof contracts.

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